Preface

This book and its planned sequel(s) are intended to compose an introduction to the Ricci flow in general and in particular to the program originated by Hamilton to apply the Ricci flow to approach Thurston's Geometrization Conjecture. The Ricci flow is the geometric evolution equation in which one starts with a smooth Riemannian manifold $(\mathcal{M}^n, g_0)$ and evolves its metric by the equation

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc},$$

where Rc denotes the Ricci tensor of the metric g. The Ricci flow was introduced in Hamilton's seminal 1982 paper, "Three-manifolds with positive Ricci curvature". In this paper, closed 3-manifolds of positive Ricci curvature are topologically classified as spherical space forms. Until that time, most results relating the curvature of a 3-manifold to its topology involved the influence of curvature on the fundamental group. For example, Gromov–Lawson and Schoen–Yau classified 3-manifolds with positive scalar curvature essentially up to homotopy. Among the many results relating curvature and topology that hold in arbitrary dimensions are Myers' Theorem and the Cartan–Hadamard Theorem.

A large number of innovations that originated in Hamilton's 1982 and subsequent papers have had a profound influence on modern geometric analysis. Here we mention just a few. Hamilton's introduction of a nonlinear heat-type equation for metrics, the Ricci flow, was motivated by the 1964 harmonic heat flow introduced by Eells and Sampson. This led to the renewed study by Huisken, Ecker, and many others of the mean curvature flow originally studied by Brakke in 1977. One of the techniques that has dominated Hamilton's work is the use of the maximum principle, both for functions and for tensors. This technique has been applied to control various geometric quantities associated to the metric under the Ricci flow. For example, the so-called pinching estimate for 3-manifolds with positive curvature shows that the eigenvalues of the Ricci tensor approach each other as the curvature becomes large. (This result is useful because the Ricci flow shrinks manifolds with positive Ricci curvature, which tends to make the curvature larger.) Another curvature estimate, due independently to Ivey and Hamilton, shows that the singularity models that form in dimension three necessarily have nonnegative sectional curvature. (A singularity model is a solution of the flow that arises as the limit of a sequence of dilations of an original solution approaching a singularity.) Since the underlying manifolds of such limit solutions are topologically simple, a detailed analysis of the singularities which arise in dimension three is therefore possible. For example, showing that certain singularity models are shrinking cylinders is one of the cornerstones for enabling geometric-topological surgeries to be performed on singular solutions to the Ricci flow on 3-manifolds. The Li–Yau–Hamilton-type differential Harnack quantity is another innovation; this yields an *a priori* estimate for a certain expression involving the curvature and its first and second derivatives in space. This estimate allows one to compare a solution at different points and times. In particular, it shows that in the presence of a nonnegative curvature operator, the scalar curvature does not decrease too fast. Another consequence of Hamilton's differential Harnack estimate is that slowly forming singularities (those in which the curvature of the original solution grows more quickly than the parabolically natural rate) lead to singularity models that are stationary solutions.

The convergence theory of Cheeger and Gromov has had fundamental consequences in Riemannian geometry. In the setting of the Ricci flow, Gromov's compactness theorem may be improved to obtain C^∞ convergence of a sequence of solutions to a smooth limit solution. For singular solutions, Perelman's recent *No Local Collapsing* Theorem allows one to dilate about sequences of points and times approaching the singularity time in such a way that one can obtain a limit solution that exists infinitely far back in time. The analysis of such limit solutions is important in Hamilton's singularity theory.

A guide for the reader

The reader of this book is assumed to have a basic knowledge of Riemannian geometry. Familiarity with algebraic topology and with nonlinear second-order partial differential equations would also be helpful but is not strictly necessary.

In Chapters 1 and 2, we begin the study of the Ricci flow by considering special solutions which exhibit typical properties of the Ricci flow and which guide our intuition. In the case of an initial homogeneous metric, the solution remains homogenous so that its analysis reduces to the study of a system of ODE. We study examples of such solutions in Chapter 1. Solutions which exist on long time intervals such as those which exist since time $-\infty$ or until time $+\infty$ are very special and can appear as singularity models. In Chapter 2, we both present such solutions explicitly and provide some intuition for how they arise. An important example of a stationary solution is the so-called "cigar soliton" on $\mathbb{R}^2$; intuitive solutions include the neckpinch and degenerate neckpinch. In Chapter 2, we discuss degenerate neckpinches heuristically. We also rigorously construct neckpinch solutions under certain symmetry assumptions.

The short-time existence theorem for the Ricci flow with an arbitrary smooth initial metric is proved in Chapter 3. This basic result allows one to use the Ricci flow as a practical tool. In particular, a number of smoothing results in Riemannian geometry can be proved using the short-time existence of the flow combined with the derivative estimates of Chapter 7. Since the Ricci flow system of equations is only weakly parabolic, the short-time existence of the flow does not follow directly from standard parabolic theory. Hamilton's original proof relied on the Nash–Moser inverse function theorem. Shortly thereafter, DeTurck gave a simplified proof by showing that the Ricci flow is equivalent to a strictly parabolic system.

In Chapter 4, the maximum principle for both functions and tensors is presented. This provides the technical foundations for the curvature, gradient of scalar curvature, and Li–Yau–Hamilton differential Harnack estimates proved later in the book.

In Chapter 5, we give a comprehensive treatment of the Ricci flow on surfaces. In this lowest nontrivial dimension, many of the techniques used in three and higher dimensions are exhibited. The concept of stationary solutions is introduced here and used to motivate the construction of various quantities, including Li–Yau–Hamilton (LYH) differential Harnack estimates. The entropy estimate is applied to solutions on the 2-sphere. In addition, derivative and injectivity radius estimates are first proved here. We also discuss the isoperimetric estimate and combine it with the Gromov-type compactness theorem for solutions of the Ricci flow to rule out the cigar soliton as a singularity model for solutions on surfaces. In higher dimensions, the cigar soliton is ruled out by Perelman's *No Local Collapsing* Theorem.

The original topological classification by Hamilton of closed 3-manifolds with positive Ricci curvature is proved in Chapter 6. Via the maximum principle for systems, the qualitative behavior of the curvature tensor may be reduced in this case to the study of a system of three ODE in three unknowns. Using this method, we prove the pinching estimate for the curvature mentioned above, which shows that the eigenvalues of the Ricci tensor are approaching each other at points where the scalar curvature is becoming large. This pinching estimate compares curvatures at the same point. Then we estimate the gradient of the scalar curvature in order to compare the curvatures at different points. The combination of these estimates shows that the Ricci curvatures are tending to constant. Using this fact together with estimates for the higher derivatives of curvature, we prove the convergence of the volume-normalized Ricci flow to a spherical space form.

The derivative estimates of Chapter 7 show that assuming an initial curvature bound allows one to bound all derivatives of the curvature for a short time, with the estimate deteriorating as the time tends to zero. (This deterioration of the estimate is necessary, because an initial bound on the curvature alone does not imply simultaneous bounds on its derivatives.) The derivative estimates established in this chapter enable one to prove the long-time existence theorem for the flow, which states that a unique solution to the Ricci flow exists as long as its curvature remains bounded.

In Chapter 8, we begin the analysis of singularity models by discussing the general procedures for obtaining limits of dilations about a singularity. This involves taking a sequence of points and times approaching the singularity, dilating in space and time, and translating in time to obtain a sequence of solutions. Depending on the type of singularity, slightly different methods must be employed to find suitable sequences of points and times so that one obtains a singularity model which can yield information about the geometry of the original solution near the singularity just prior to its formation.

In Chapter 9, we consider Type I singularities, those where the curvature is bounded proportionally to the inverse of the time remaining until the singularity time. This is the parabolically natural rate of singularity formation for the flow. In this case, one sees the $\mathbb{R} \times \mathcal{S}^2$ cylinder (or its quotients) as singularity models. The results in this chapter provide additional rigor to the intuition behind neckpinches.

In the two appendices, we provide elementary background for tensor calculus and some comparison geometry.

REMARK. At present (December 2003) there has been sustained excitement in the mathematical community over Perelman's recent ground-breaking progress on Hamilton's Ricci flow program intended to resolve Thurston's Geometrization Conjecture. Perelman's results allow one to view some of the material in Chapters 8 and 9 in a new light. We have retained that material in this volume for its independent interest, and hope to present Perelman's work in a subsequent volume.

A guide for the hurried reader

The reader wishing to develop a nontechnical appreciation of the Ricci flow program for 3-manifolds as efficiently as possible is advised to follow the fast track outlined below.

In Chapter 1, read Section 1 for a brief introduction to the Geometrization Conjecture.

In Chapter 2, the most important examples are the cigar, the neckpinch and the degenerate neckpinch. Read the discussion of the cigar soliton in Section 2. Read the statements of the main results on neckpinches derived in Section 5. And read the heuristic discussion of degenerate neckpinches in Section 6.

In Chapter 3, review the variation formulas derived in Section 1.

In Chapter 4, read the proof of the first scalar maximum principle (Theorem 4.2) and at least the statements of the maximum principles that follow.

In Chapter 5, review the entropy estimates derived in Section 8 and the differential Harnack estimates derived in Section 10. The entropy estimates will be used in Section 6 of Chapter 9. The Harnack estimates for surfaces are prototypes of those that apply in higher dimensions.

We suggest that Chapter 6 be read in its entirety.

In Chapter 7, read the statements of the main results. These give precise insight into the smoothing properties of the Ricci flow and its long-time behavior. The Compactness Theorem stated in Section 3 is an essential ingredient in important technique of analyzing singularity formation by taking limits of parabolic dilations.

In Chapter 8, read Sections 1–3.1. These present the classification of maximal-time solutions to the flow, and introduce the method of parabolic dilation.

In Chapter 9, read Section 1 for a heuristic description of singularity formation in 3-manifolds. Then read Sections 2 and 3 to gain an understanding of why positive curvature dominates near singularities in dimension three. Finally, review the results of Sections 4 and 6 to gain insight into the technique of dimension reduction.

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Bennett Chow and Dan Knopf