Introduction

Chapter I. The pioneering work of Euler on the zeta function and Dirichlet's subsequent introduction of L-functions provided a major conceptual advance in our understanding of the set of all prime numbers. The identity

$$\prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which exhibits clearly the multiplicative and the additive properties of the ordinary integers, is a cornerstone of the modern analytic theory of numbers, a theory which evolved in the course of the last century into a vast and powerful enterprize. Within this new framework, formal infinite products indexed by the primes, often called "*product formulas*", are intended to be the carriers of both algebraic and geometric as well as analytic and spectral information. The genesis of these ideas can be traced back to Hilbert's formulation of his **reciprocity law** for the residue symbol, as well as to the work done by Herbrand, Chevalley and Weil who forged the old arithmetic concepts of unique factorization with the topological concepts present in Tychonoff's Theorem on infinite products of locally compact spaces to produce the basic tools of **harmonic analysis** on the groups of ideles and adeles.

On the analytic side, the study of zeta and L-functions dates back to one of Riemann's three proofs of the functional equation for $\zeta(s)$ - the one based on the transformation formula for the elliptic theta function:

(**Theta Transformation**) $$\sum_{n\in\mathbf{Z}} e^{-(n+\alpha)^2\pi/x} = \sqrt{x}\sum_{n\in\mathbf{Z}} e^{-n^2\pi x+2\pi i n\alpha}.$$

To gain a better understanding of the origin of these ideas, we recall that a proof of the theta transformation formula is based on the solution to the following[1].

"**Initial value problem for heat conduction**":

> *On a closed linearly extended heat conductor (a wire for instance) of length 1, find a solution $u(x,t)$ to the heat equation*
>
> $$u_{xx} - u_t = 0,$$
>
> *with continuous derivatives up to the second order for all values of the variable x and for all $t > 0$, having a prescribed set of*

[1]See Courant-Hilbert [**37**]: *Methods of Mathematical Physics* , vol. II, p. 197

values

$$u(x,0) = \psi(x) \qquad at \qquad t = 0.$$

The function $\psi(x)$ is assumed to be everywhere continuous and bounded. It being assumed that both u and ψ as functions of x are periodic of period 1.

From physical considerations based on the superposition principle and the idea of separation of variables one is lead to the two solutions

$$u(x,t) = \int_0^1 \psi(\xi)\Big\{1 + 2\sum_{\nu=1}^{\infty} e^{-4\pi^2\nu^2 t} \cos 2\pi\nu(x-\xi)\Big\}d\xi,$$

and

$$u(x,t) = \int_0^1 \psi(\xi)\Big\{\frac{1}{\sqrt{\pi t}} \sum_{\nu=-\infty}^{\infty} e^{-(x-\xi-\nu)^2/4t}\Big\}d\xi.$$

Using an elementary lemma from the calculus of variations, the uniqueness of the solution u, itself a consequence of the energy estimate $\frac{d}{dt}\int_0^1 u^2 dx \leq 0$, leads immediately to an equivalent form of the theta formula. The functional equation of the Riemann zeta function in its usual form is then obtained by applying a suitable Mellin transform to both sides of the theta identity.

New ideas in functional analysis and the rise of the theory of distributions made it possible to develop the above elementary argument into a powerful technique capable of new applications to two closely related branches of mathematics: to zeta and L-functions and to infinite dimensional group representations.

Weil was the first to formulate in the local to global language of **Tate's Thesis** the relation between functional equations and the uniqueness of certain zeta distributions. At the heart of his new interpretation is an old idea of Weil concerning the determination of *relative invariant measures*, which had already appeared in his classic work on integration over topological groups. The subsequent full development of these ideas by Weil himself, and the light they shed on Siegel's insights into the Poisson Summation Formula, established beyond doubt the fruitfulness of this new point of view. **Chapter I** is an elementary introduction to this circle of ideas. It includes a detailed presentation of Tate's Thesis together with Weil's ideas about distributions and zeta functions. In this framework, the local calculations at the infinite prime are simply an elaboration of the well known results of Hadamard and M. Riesz on homogeneous distributions. Our elementary presentation can serve as an introduction to the theories of Sato concerning hyperfunctions defined by complex powers of polynomials and zeta functions on prehomogeneous spaces and to the theory of the Bernstein polynomial. We have also given a brief outline of the Jacquet-Godement theory of principal L-functions which is the natural generalization to GL(n) of Hecke's L-functions viewed as automorphic objects on GL(1). The principal results of Chapter I are those of Tate and Weil concerning the uniqueness of local and global zeta distributions (Lemma 9, Theorem 22) and the functional equation (Theorem 23).

Chapter II. The essential nature of Artin's reciprocity law, in Chevalley's formulation, is the identification of two seemingly distinct objects of interest in the arithmetic of number fields:

(i) *The Pontrjagin dual of the Galois group of the maximal abelian extension of a number field k,*

(ii) *The characters of finite order of the idele class group* $\mathcal{C}\ell(k) = k_{\mathbf{A}}^{\times}/k^{\times}$,

together with the attendant functoriality properties describing the behavior of the objects in (i) and (ii) under finite extensions (restriction) and the norm map (induction).

As is well known, Artin constructed L-functions that generalize those of Dirichlet in the non-abelian case. The main gluing ingredient being the Artin-Brauer theorem, which describes how the characters of a finite group can be expressed as linear combinations of those arising by induction from abelian groups. This, and the fact that Hecke had constructed L-functions associated to arbitrary characters of the idele class group, lead to the possibility of amalgamating these two types of L-functions (Artin's and Hecke's) into a single analytic package. This challenge was taken up by Weil, who based his construction on the existence and uniqueness of the group extension

$$1 \longrightarrow \mathcal{C}\ell(K) \longrightarrow W(K/k) \longrightarrow Gal(K/k) \longrightarrow 1$$

associated to the fundamental class of class field theory. Weil's construction of the Artin-Hecke L-functions associated to finite dimensional representations of $W(K/k)$ requires a generalization of the Artin-Brauer theorem, which works for infinite non-abelian locally compact groups like $W(K/k)$. This theory was complemented by work of Tamagawa on conductors and archimedean L-factors. The main results of this theory are explained in **Chapter II** which includes the functional equation for the L-functions of representations of $W(K/k)$ (Lemma 3 and the Main Theorem in §13). We also include a brief description of the Dwork-Langlands' theorem on the decomposition of the root number into local factors, a result of fundamental significance for the modern theory of automorphic L-functions on GL(n), and the new non-abelian reciprocity laws of Langlands. A brief survey of the basic Langlands functoriality principle as it applies to $GL(n)$ is included. This principle implies among other things that the Galois class of Artin-Hecke L-functions studied in Chapter II coincides with a subset of the class of principal L-functions studied in Chapter I.

Chapter III. The finite primes in a number field, that is to say, those associated to non-archimedean valuations, are to some extent determined by the well known theorem of Wedderburn concerning the commutativity of finite division rings. In contrast, the infinite primes in a number field, those corresponding to archimedean valuations, are controlled by the fundamental theorem of Gelfand, which identifies the field of complex numbers, up to topological isomorphism, as the only complex commutative Banach division ring with identity. In this context, the main characteristic property which distinguishes number fields from function fields of positive characteristic is the existence of archimedean primes. An old saying in analytic number theory declares that the central questions will remain open until we have gained a better understanding of the nature of all the primes including the archimedean ones. It is in this spirit that the verification of the integrality of L-functions remains a difficult problem intimately related to the presence of gamma factors. A similar instance is the location and distribution of the zeros of L-functions, a problem which rests ultimately on bounds for the local archimedeam L-factors. An important result in this connection is Riemann's product formula:

(**Product Formula**) $$\pi^{-s/2}\Gamma(\frac{s}{2})\prod_p \frac{1}{1-\frac{1}{p^s}} = \frac{1}{s(s-1)}\prod_\rho{}' (1-\frac{s}{\rho}),$$

where the product on the right-hand side is taken over all the zeros of $\zeta(s)$ in the strip $0 < Re(s) < 1$, it being understood that ρ and $1-\rho$ go together.

The study in depth of the archimedean primes and their local L-factors starts with the well known characterization theorem of Bohr-Mollerup:

The Euler gamma function $\Gamma(s)$ *is the only function defined for real* $s > 0$*, which is positive, is 1 at s=1, satisfies the functional equation* $s\Gamma(s)=\Gamma(s+1)$*, and is logarithmically convex, that is,* $\log\Gamma(s)$ *is a convex function on the real line.*

The uniqueness of the multiplication formula for the gamma function, viewed as an identity between local L-factors, is the arithmetic manifestation of the basic fact that the real archimedean prime admits a unique non-trivial extension: the complex one. The analytic properties of the gamma function stem from the realization that $\log|\Gamma(s)| = Re\ \log\Gamma(s)$ is a harmonic function, a fact already used by Rademacher (and to some extent also by Siegel) to obtain strong Phragmen-Lindelof estimates for $\Gamma(s)$. This circle of ideas have applications to growth estimates for L-functions with at most a finite number of poles.

The above remarks set the tone for the emphasis given to number fields in this book. **Chapter III** is an exposition of these ideas, classically known as convexity estimates. It includes estimates for the automorphic L-functions of principal type on GL(n), as well as for the Artin-Hecke L-functions. The main results in this chapter are Riemann's Formula in Theorem 3 and the estimates in Theorems 14A, 14B, and 14C.

Chapter IV. The linearization of Riemann's product formula, obtained by differentiating the logarithms of both sides, leads to explicit relations between the primes (finite and archimedean) and the non-trivial zeros of $\zeta(s)$. These relations, when integrated against particular functions $\Phi(s)$ over suitable domains in the s-plane provide new identities that form the core of many applications of zeta functions to questions about number fields. The internal symmetries possessed by these formulas seemed to have been first observed by Riemann himself, but were not developed to any great extent until the middle of the last century when Guinand, then Delsarte, and finally Weil found generalizations which exhibited some kind of duality between zeros and primes. In its modern formulation, Weil's explicit formula is an equality between two distributions, one associated to the zeros of an L-function, and another - its "Fourier Dual"- associated to the primes. It has been a goal of many number theorists, a goal which remains unfulfilled to this day, to formulate a "natural operator problem" whose solution has a well defined trace given by both sides of the explicit formulas. An answer to this problem will certainly be accompanied with new insights into the dual relation between the primes and zeros of $\zeta(s)$.

There is no doubt that the explicit formulas have proven their worth in the study of the distribution of prime numbers. They have also been very suggestive in the study of group representations, as already noted by Selberg [2] in reference to the trace formula for $SL(2)$:

> *"... it has a rather striking analogy to certain formulas that arise in analytic number theory from the zeta and L-functions of algebraic number fields."*

An observation that lead him to the introduction of the Selberg zeta function.

Given the new work of Connes, which adds significance to the general notion of *trace*, we expect that the explicit formulas will continue to exercise an ever increasing role in the study of L-functions.

We remark that in the meantime it is possible, using an observation of Delsarte, to calculate certain invariants of operators, e.g. the regularized expression

$$\exp\big(\sum_{\rho} \log\ \rho\big),$$

an expression that attaches a sense to the notion of determinant associated to zeta and L-functions. Another important application of the explicit formulas, not considered in this book, is Montgomery's work on the *Pair Correlation Hypothesis*, a deep insight on how the zeros of $\zeta(s)$ are jointly distributed. The work of Sarnak and Rudnick has demonstrated that this is a very general phenomenon that applies to the larger class of automorphic L-functions on $GL(n)$. The explicit formulas of Weil are studied in **Chapter IV**. The main results are Theorem 4A and Theorem 4B.

Chapter V. The diophantine properties of number fields are intimately connected with the presence of ramification as was early realized by Kronecker. The well known theorem of Minkowski:

$$|disc(k)| > 1, \qquad k \neq \mathbf{Q},$$

has a representation theoretic interpretation, as follows from a result of Hamburger, to the effect that the trivial representation is the only Galois representation

$$r : Gal(\mathbf{Q}^{sep}/\mathbf{Q}) \longrightarrow GL_n(\mathbf{C}),$$

which is unramified everywhere, including at the archimedean primes. Artin noted that the zeta functions of simple algebras over number fields could be calculated explicitly, and that their precise form, including the exact location of their poles, could be used advantageously to prove the central theorem of class field theory:

[2] See A. Selberg, *"Harmonic Analysis and Discontinuous Groups ..."*, Indian Jour., p. 75.

A simple algebra A over a number field k is trivial if and only if it is everywhere unramified, i.e. if and only if $A_v = A \otimes_k k_v$ *is trivial over the completions* k_v *of k at all the places.*

In the opposite direction, as soon as ramification is allowed, as in the case of quaternion algebras, the internal structure of the zeta function is more complicated and the explicit form of the global root number, the holder of ramification information, becomes relevant. Siegel, and in a more explicit way Stark, discovered analytic formulas (relatives of the Poisson summation formula in the additive case and variants of the explicit formulas in the multiplicative case) capable of exhibiting the non-triviality of the discriminant of a simple algebra over a number field. The explicit form of these analytic approaches to the study of ramification, which uses increasingly more information about zeros and poles of zeta functions, was developed by Stark, Odlyzko, Serre and others. A noteworthy refinement of these ideas involves the use of the explicit formulas; for example, Mestre has obtained lower bounds for the conductor of certain automorphic L-functions whose archimedean components are discrete series. We give in **Chapter V** a development of these ideas. The results in this chapter are best viewed as contributions to arithmetic-algebraic geometry in the sense of Arakelov, Faltings, and Szpiro. The principal results are The Main Formula in §4, Lemmas 2 and 3 and Theorem 6.

Chapter VI. Norbert Wiener's main contribution to analytic number theory was his tauberian theorem which establishes the equivalence of prime number theorems and the non-vanishing theorems for zeta functions. Whether one is interested in the proof of a classical prime number theorem, or in one of its modern versions, e.g. Chebotarev's density theorem, Sato-Tate distributions for the eigenvalues of Hecke operators, analytic proofs of strong multiplicity one, the Deligne or Katz-Sarnak monodromy distribution theorems etc., the key problem in the analytic approach is the proof of the non-vanishing of L-functions on the boundary of the region of absolute convergence.

The most significant progress in this direction has been the generalization by Deligne of the **method of Hadamard and de la Vallée Poussin**, which establishes non-vanishing for L-functions of a wide class that includes all the classical ones (Riemann zeta, Dirichlet L-functions, Artin-Hecke, etc.) and has the potential for further applicability to arbitrary automorphic L-functions, once a weak version of Langlands' functoriality principle is available - *analytic continuation of the L-functions of Langlands* $L(s, r, \pi)$ *for* $Re(s) \geq 1$ *for all finite dimensional representations* r *of the L-group, except possibly for a simple pole at* $s = 1$ *when* $r = 1$.

There is a second method for proving non-vanishing which rests on the analytic properties of Eisenstein series. In one version it is based on the fact that the **Eisenstein series** $E(s, g)$ of a maximal parabolic subgroup of a connected reductive algebraic group G, as a function of the complex variable s, has analytic continuation to the unitary axis $i\mathbf{R}$, and on the theory of the **Whittaker functional**, which generalizes the theory of Fourier coefficients, and which for a non-trivial additive character ψ of the unipotent group $N_{\mathbf{A}}/N_k$, gives a Fourier coefficient

$$\int_{N_{\mathbf{A}}/N_k} \psi(n)E(s,gn)dn = \varphi(s,g,\psi)\frac{L(s,\pi)}{L(1+s,\pi)},$$

where $\varphi(s,g,\psi) \neq 0$ for $s \in i\mathbf{R}$. This in turn implies that

$$L(1+it,\pi) \neq 0,$$

for all real values of t. An exposition of these ideas, including a brief introduction to the powerful Langlands-Shahidi method is given in **Chapter VI**. In addition, we have included a discussion of a little known result on the non-vanishing of $\zeta(s)$ based on elementary Hilbert space techniques. The reader will also find here a brief discussion to the **generalized Ramanujan conjecture**, a very significant problem, full of implications for the future development of analytic number theory. Some exciting new results have been obtained in this area by Iwaniec and Sarnak and their collaborators ([**155**]) by using methods akin to those presented in **Chapter IV**.

There is another application of the theory of Eisenstein series on metaplectic groups to non-vanishing theorems for L-functions, which is not presented in this book, but because of its non-classical nature deserves to be mentioned in this introductory paragraph. In its simplest manifestation, it arises from the Fourier expansion of Eisenstein series of half integral weight on the Hecke group $\Gamma_0(4N)$. The Fourier coefficients turn out to be essentially Dirichlet L-functions associated to quadratic extensions of the field of rational numbers. By a sieving procedure, average information about these coefficients can be extracted from knowledge of the singularities of the relevant Eisenstein series. In the special case of automorphic L-functions $L(s,\pi)$ on $GL(2)$, a key fundamental idea is Waldspurger's characterization of the values of $L(\frac{1}{2},\pi\otimes\chi)$ in terms of lifts $\tilde{\pi}$ to the metaplectic cover of $GL(2)$. In one of the most striking applications of these ideas to number theory, it has been possible to show the non-vanishing of the twisted Hasse-Weil zeta function $L(s,E\otimes\chi)$ of elliptic curves at the point $s=1$, (the central point of the critical line) for infinitely many quadratic characters χ, as well as for the derivatives. In contrast to the classical situation, this type of non-vanishing has direct applications to problems of diophantine analysis, a situation that is controlled by the well known Birch and Swinnerton-Dyer conjecture. For an excellent exposition of the ideas surrounding this type of non-vanishing, the reader should consult the survey article by Bump, Friedberg, and Hoffstein in [**24**].

Appendix. The functional equation of an Artin L-function, which relates its values at s and at $1-s$, contains an arithmetic factor, known as the root number, whose behavior resembles that of a quadratic Gauss sum

$$\sum_{n\in\mathbf{Z}/N\mathbf{Z}} e^{-2\pi i n^2/N} = \frac{1+i^N}{1+i}\cdot\sqrt{N}.$$

Hasse, who knew well the factorization of the Gaussian sum as a product of similar sums with the integer N replaced by its prime power divisors, suggested that the root number appearing in the functional equation of Artin L-functions should have analogous factorizations into locally defined root numbers. Tate's harmonic interpretation of Hecke's functional equation using ideles established Hasse's conjecture for abelian L-functions. For non-abelian L-functions, the factorization of the root number into a product of local root numbers was achieved by Dwork up to sign, and in full generality by Langlands. Greatly simplified analytic proofs were subsequently produced by Deligne and by Tate.

Langlands' original approach, itself a lengthy and intricate induction which builds on earlier work of Dwork as well as on his own new Gauss sum identities is noteworthy because in it the existence of local root numbers is established by purely local techniques, suggestive of the existence of a non-abelian local class field theory, a theme which is at the heart of Langlands' Functoriality Principle.

The **Appendix** surveys in broad outline the two main components of the local theory of root numbers: (a) a description of the form of the three main local root number identities, (b) the determination of the corresponding three generators for the kernel of Brauer induction for solvable groups. The main goal of this **Appendix** is to introduce the reader to one of the most interesting and profound tools for the study of L-functions and automorphic forms.