Advice to the Reader

It is necessary for the reader of this survey to be familiar with the following topics:

(a) He must have a good elementary knowledge of the theory of analytic functions of one complex variable, as contained for instance in L. Ahlfors, *Complex Analysis* [**1**].

(b) He must know the basic definitions in the theory of functions of a real variable at the level of H. Royden, *Real Analysis* [**150**] (see also Rudin [**151**]).

(c) He must have taken a graduate level course in number theory at a level comparable to that in E. Hecke, *Lectures in the Theory of Algebraic Numbers* [**76**] (see also [**104**],[**105**],[**207**]). In particular, he must know the three fundamental results of algebraic number theory: the unique factorization of ideals, the finiteness of the class group, and the Dirichlet unit theorem.

If he wishes to profit from the reading of the proofs in the individual chapters, each written in an increasing order of sophistication, he must also have an acquaintance with certain concepts and results which unfortunately appear scattered throughout the mathematical literature. In the following we suggest a road map that can facilitate his reading of the relevant literature and prepare him for further study and research of the relevant topics.

Chapter I. The rudimentary knowledge of abstract harmonic analysis needed can be acquired by selectively reading those chapters in L. Loomis, *An Introduction to Abstract Harmonic Analysis* [**115**] or in the short and elegant monograph by G. Bachman, *Elements of Abstract Harmonic Analysis* [**8**], which deal specifically with topological groups, Haar measure, character and dual groups, and Fourier analysis on locally compact abelian groups. An exposition of these topics that is still worth reading from a historical point of view can be found in A. Weil, *L'integration dans les groupes topologiques et ses applications* [**209**] and in L. Pontrjagin, *Topological Groups* [**144**].

An excellent introduction to the basic theory of distributions can be gleaned from the first two chapters in L. Hormander, *The Analysis of Linear Partial Differential Operators I* [**80**].

The Appendix in Chapter I, §3 on principal L-functions on $GL(n)$ is meant to serve only as an outline of how the Hecke-Tate theory on $GL_1(k)$ generalizes to $GL_n(k)$, and requires some concepts from representation theory not covered in the earlier sections, but which are essential ingredients in the understanding of the theory of automorphic L-functions on $GL_n(k)$. A more detailed presentation can be found in A. Weil, *Dirichlet Series and Automorphic Forms* [**210**], and in H. Jacquet and R.P. Langlands, *Automorphic Forms on GL(2)* [**86**] for the theory on $GL(2)$; a treatment of the higher dimensional case, $GL(r), r \geq 3$, can be found in H. Jacquet, *Principal L-functions of the Linear Group* [**84**] and in R. Godement and H. Jacquet, *Zeta Functions of Simple Algebras* [**69**].

Chapter II. The reader is expected to be familiar with the basic theory of linear representations of finite groups up to and including knowledge of Brauer's Theorem as contained for instance in J.-P. Serre, *Linear Representations of Finite Groups* [**164**], §§1-10.

For a deeper discussion of Weil and Weil-Deligne groups the reader can supplement his study by consulting J. Tate, *Number Theoretic Background*, [**187**].

The ramification theory needed to understand the properties of conductors from the point of view of the Herbrand distribution is given in C.J. Moreno, *Advanced Analytic Number Theory* [**127**]. The definitions and elementary properties of the absolute Weil group of a number field given in Chapter II, §2.3 are taken from the report in A. Weil, *Sur la theorie de corps de classes* [**211**] and from the detailed presentation in [**212**]. A modern exposition is also given in J. Tate's article referred to above [**187**].

The descriptive survey of the local Langlands correspondence for $GL(n)$ given in Chapter II, §15 uses standard terminology about group representations; for these the reader can consult A. Kirillov, *Elements of the Theory of Representations* [**101**] or the excellent *Encyclopedic Dictionary of Mathematics*, second edition, published by the Mathematical Society of Japan. For a more detailed and rigorous presentation the reader can consult the excellent treatment in A. Knapp, *Representation Theory of Semisimple Groups*, [**100**].

Chapter III. The essential requirements for this chapter have been kept to a minimum. An acquaintance with the classical Hadamard theory of entire functions of order 1 and their associated Weierstrass products would be sufficient. This material is found in L. Ahlfors, *Complex Analysis* [**1**], Chapter IV, §3, and Chapter V, §§1, 2 on harmonic and subharmonic functions. The principal results of the chapter deal directly with the analytic properties of archimedean L-factors, also known as gamma factors; for these the reader cannot do better than consult the classical treatment in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* [**215**], particularly Chapters XII and XIII.

Chapter IV. Some acquaintance with the classical Mellin transform as well as knowledge of the conditions under which its inverse exists is needed. The basic theory can be deduced from that of the Fourier transform on the real line. The latter is developed in Y. Katznelson, *An Introduction to Harmonic Analysis* [**98**], Chapter VI (see also [**115**]). A more classical treatment of the Mellin transform is in E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* [**194**].

The integral formulas for the Herbrand distribution used in this chapter are discussed in great detail in the author's monograph [**127**] already cited above, particularly Chapter IX.

Chapter V. Those properties of the gamma and related digamma functions, which are not proved in this chapter, can be found in the treatise of Whittaker and Watson [**215**].

Chapter VI. For an understanding of the geometric example in §1.1, the reader should be acquainted with the elementary theory of zeta and L-functions of algebraic curves over finite fields, particularly as it applies to elliptic curves. The theory of these functions is explained in great detail in the author's monograph [**127**], Chapters III and IV. Some background on the origin and significance of the arithmetic example in §1.1 can be acquired in J.-P. Serre, (i) *A Course in Arithmetic*, [**169**] Chapter VII, and (ii) *Abelian ℓ-adic Representations*, [**165**] Chapter I, (see also G. H. Hardy, *Twelve Lectures on the Work of Ramanujan* [**71**], Chapter XII). In (ii) the reader will also find a useful but brief discussion of the relations between equipartition of conjugacy classes, L-functions, and Chebotarev's density theorem.

The theory of irreducible complex linear representations of a locally compact group G needed in Chapter VI, §1.2, is an extension of the classical theory for compact groups. The reader can find an elementary introduction to the theory of linear complex representations of compact Lie groups, including the unitary group $SU(n)$ and the Peter-Weyl theorem, in W. Fulton and J. Harris, *Representation Theory: A First Course* [**55**], Chapter XX. The reader will also find an elementary treatment of the Bohr compactification in Y. Katznelson, *Harmonic analysis* [**98**], p. 192.

The theory of Eisenstein series for maximal parabolic subgroups of $GL(n)$ used in the proof of the Jacquet-Shalika non-vanishing theorem is a generalization of the Hecke-Riemann method for proving functional equations and uses the properties of theta series developed in Chapter I. The reader who is unfamiliar with these topics or who wishes to acquire a working knowledge of the theory may want to consult the following treatises on the general theory of Eisenstein series: (i) T. Kubota, *Elementary Theory of Eisenstein Series* [**102**], (ii) A. Borel, *Automorphic Forms on* $SL_2(\mathbf{R})$ [**16**], (iii) C. Moeglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series* [**125**], (iv) H.-Chandra, *Automorphic Forms on Semisimple Lie Groups* [**74**]. A third alternative approach to the theory of Eisenstein series is due to A. Selberg (for the group $SL(3)$) and uses the Fredholm theory of operator equations. This has been developed independently by J. Bernstein (unpublished) and in Shek-Tung Wong [**216**].

The reader who wishes to understand the technical aspects behind the powerful Langlands-Shahidi method can consult the original exposition in F. Shahidi, *Functional Equation Satisfied by Certain L-Functions* [**178**]. The approach there is quite general and applies with minor modifications to Chevalley groups. The reader can acquire the necessary knowledge of Whittaker models and Whittaker functions in J. Shalika, *The Multiplicity One Theorem for* GL_n [**170**] and in H. Jacquet, *Fonctions de Whittaker associees aux groupes de Chevalley* [**85**].

Finally we must describe the method followed for cross-references. Theorems have been numbered continuously throughout each chapter; the same is true for lemmas, for definitions, and for the numbered formulas. The few corollaries and propositions appearing in each chapter have not been numbered; all concepts, except those that are assumed to be known, are listed in the index at the end of the book, with reference to the page(s) where they appear. Formulas are numbered for the sole purpose of reference. A reference given as “Chapter IV, §3, Theorem 3” refers to Theorem 3 in §3 of Chapter IV of this book; a reference such as “[16], Chapter IV, §5” refers to Chapter IV, §5 in item [16] of the bibliography. A number of parenthetical remarks appear throughout the text. For the most part, these are meant to amplify some points in the foregoing discussion and can be skipped on a first reading.