Preface

Self-similar groups (groups generated by automata) appeared in the early eighties as interesting examples. It was discovered that very simple automata generate groups with complicated structure and exotic properties which are hard to find among groups defined by more "classical" methods.

For example, the Grigorchuk group [**Gri80**] can be defined as a group generated by an automaton with five states over an alphabet of two letters. This group is a particularly simple example of an infinite finitely generated torsion group and is the first example of a group whose growth is intermediate between polynomial and exponential. Another interesting example is a group discovered in [**GŻ02a**], which is generated by a three-state automaton over the alphabet of two letters. This group can be defined as the *iterated monodromy group* of the polynomial $z^2 - 1$ (see Chapter 5 of this book). It is the first example of an amenable group (see [**BV**]), which cannot be constructed from groups of sub-exponential growth by the group-theoretical operations preserving amenability.

Many other interesting groups were constructed using self-similar actions and automata. This includes, for instance, groups of finite width, groups of non-uniform exponential growth, new just-infinite groups, etc.

The definition of a self-similar group action is as follows. Let X be a finite alphabet and let X^* denote the set of all finite words over X. A faithful action of a group G on X^* is said to be *self-similar* if for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that

$$g(xw) = yh(w)$$

for all words $w \in \mathsf{X}^*$. Thus, self-similar actions agree with the self-similarity of the set X^* given by the shift map $xw \mapsto w$.

One of the aims of these notes is to show that self-similar groups are not just isolated examples, but that they have close connections with dynamics and fractal geometry.

We will show, for instance, that self-similar groups appear naturally as *iterated monodromy groups* of self-coverings of topological spaces (or *orbispaces*) and encode combinatorial information about the dynamics of such self-coverings. Especially interesting is the case of a post-critically finite rational function $f(z)$. We will see that iterated monodromy groups give a convenient algebraic way of characterizing combinatorial (Thurston) equivalence of rational functions and that the Julia set of f can be reconstructed from its iterated monodromy group.

In the other direction, we will associate a *limit dynamical system* to every *contracting* self-similar action. The limit dynamical system consists of the *limit (orbi)space* $\mathcal{J}_G$ and of a continuous finite-to-one surjective map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$, which becomes a partial self-covering if we endow $\mathcal{J}_G$ with a natural orbispace structure.

Since the main topics of these notes are geometry and dynamics of self-similar groups and algebraic interpretation of self-similarity, we do not go deep into the rich and various algebraic aspects of groups generated by automata such as just-infiniteness, branch groups, computation of spectra, Lie methods, etc. A reader interested in these topics may read the surveys [**BGŠ03, Gri00, BGN03**].

The first chapter, "Basic definitions and examples", serves as an introduction. We define the basic terminology used in the study of self-similar groups: automorphisms of rooted trees, automata and wreath products. We define the notion of a self-similar action by giving several equivalent definitions and conclude with a sequence of examples illustrating different aspects of the subject.

The second chapter, "Algebraic theory", studies self-similarity of groups from the algebraic point of view. We show that self-similarity can be interpreted as a *permutational bimodule*, i.e., a set with two commuting (left and right) actions of the group. The bimodule associated to a self-similar action is defined as the set $\mathfrak{M}$ of transformations $v \mapsto xg(v)$ of the set of words X^*, where $x \in \mathsf{X}$ is a letter and $g \in G$ is an element of the self-similar group. It follows from the definition of a self-similar action that for every $m \in \mathfrak{M}$ and $h \in G$ the compositions $m \cdot h$ and $h \cdot m$ are again elements of $\mathfrak{M}$. We get in this way two commuting (left and right) actions of the self-similar group G on $\mathfrak{M}$. The bimodule $\mathfrak{M}$ is called the *self-similarity bimodule*. The self-similarity bimodules can be abstractly described as bimodules for which the right action is free and has a finite number of orbits. A self-similarity bimodule together with a choice of a *basis* (an orbit transversal) of the right action uniquely determines the self-similar action. Change of a basis of the bimodule changes the action to a conjugate one.

Virtual endomorphisms are another convenient tool used to construct permutational bimodules and hence self-similar actions. A virtual endomorphism ϕ of a group G is a homomorphism from a subgroup of finite index $\mathrm{Dom}\,\phi \le G$ to G. We show that the set of formal expressions of the form $\phi(g)h$ (with natural identifications) is a permutational bimodule and that one gets a self-similar action in this way. If we start from a self-similar action, then the *associated virtual endomorphism* ϕ is defined on the stabilizer G_x of a letter $x \in \mathsf{X}$ in G by the condition that

$$g\,(xw) = x\phi(g)(w)$$

for every $w \in \mathsf{X}^*$ and $g \in \mathrm{Dom}\,\phi = G_x$.

For example, the *adding machine* action, i.e., the natural action of $\mathbb{Z}$ on the ring of diadic integers $\mathbb{Z}_2 \ge \mathbb{Z}$, where $\mathbb{Z}_2$ is encoded in the usual way by infinite binary sequences, is the self-similar action defined by the virtual endomorphism $\phi : n \mapsto n/2$. In this sense self-similar actions may be viewed as generalizations of *numeration systems*. In Section 2.9 of Chapter 2, we apply the developed technique to describe self-similar actions of the free abelian groups $\mathbb{Z}^n$, making the relation between self-similar actions and numeration systems more explicit.

Section 2.11 introduces the main class of self-similar actions for these notes. It is the class of the so-called *contracting actions*. An action is called contracting if the associated virtual endomorphism ϕ asymptotically shortens the length of the elements of the group. Contraction of a self-similar action corresponds to the condition of expansion of a dynamical system. We show in the next chapters that if a self-covering of a Riemannian manifold (or orbifold) is expanding, then its iterated monodromy group is contracting with respect to a *standard* self-similar action.

The *limit spaces* and the *limit dynamical systems* of contracting self-similar actions are constructed and studied in Chapter 3. If $\mathfrak{M}$ is the permutational bimodule associated to a self-similar action of a group G, then its tensor power $\mathfrak{M}^{\otimes n}$ is defined in a natural way. It describes the action of G on the set of words of length n and is interpreted as the nth iteration of the self-similarity of the group. Passing to the (appropriately defined) limits as n goes to infinity, we get the left G-module (G-space) $\mathfrak{M}^{\otimes\omega} = \mathfrak{M} \otimes \mathfrak{M} \otimes \ldots$ and the right G-module $\mathfrak{M}^{\otimes -\omega} = \ldots \otimes \mathfrak{M} \otimes \mathfrak{M}$. The left G-space $\mathfrak{M}^{\otimes\omega}$ is naturally interpreted as the action of G on the space of infinite words $\mathsf{X}^\omega = \{x_1 x_2 \ldots : x_i \in \mathsf{X}\}$.

The right G-space $\mathcal{X}_G = \mathfrak{M}^{\otimes -\omega}$ (if the action is contracting) is a finite-dimensional metrizable locally compact topological space with a proper co-compact right action of G on it. The limit space $\mathcal{X}_G$ can also be described axiomatically as the unique proper co-compact G-space with a *contracting self-similarity* (Theorem 3.4.13). A right G-space $\mathcal{X}$ is called *self-similar* if the actions $(\mathcal{X}, G)$ and $(\mathcal{X} \otimes_G \mathfrak{M}, G)$ are topologically conjugate. For the notion of a contracting self-similarity see Definition 3.4.11.

Another construction is the quotient (orbispace) $\mathcal{J}_G$ of $\mathcal{X}_G$ by the action of G (Section 3.6). The limit space $\mathcal{J}_G$ can be alternatively defined as the quotient of the space of the left-infinite sequences $\mathsf{X}^{-\omega} = \{\ldots x_2 x_1 : x_i \in \mathsf{X}\}$ by the equivalence relation, which identifies two sequences $\ldots x_2 x_1$ and $\ldots y_2 y_1$ if there exists a bounded sequence $g_k \in G$ such that $g_k(x_k \ldots x_1) = y_k \ldots y_1$ for all k. Here a sequence is called bounded if it takes a finite set of values. One can prove that this equivalence is described by a finite graph labeled by pairs of letters and that equivalence classes are finite. This gives us a nice symbolic presentation of the space $\mathcal{J}_G$.

The limit space $\mathcal{J}_G$ comes together with a natural *shift map* $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ and with a Markov partition of the dynamical system $(\mathcal{J}_G, \mathsf{s})$. The shift is induced by the usual shift $\ldots x_2 x_1 \mapsto \ldots x_3 x_2$, and the elements of the Markov partition are the images of the cylindrical sets of the described symbolic presentation of $\mathcal{J}_G$. The elements of the Markov partition are called *(digit) tiles*. Digit tiles can also be defined for the limit G-space $\mathcal{X}_G$, and they are convenient tools for the study of the topology of $\mathcal{X}_G$.

The most well-studied contracting groups are the self-similar groups generated by *bounded automata*. They can be defined as the groups whose digit tiles have finite boundary. We show that this condition is equivalent to a condition studied by S. Sidki in [**Sid00**] and show an iterative algorithm which constructs approximations of the limit spaces $\mathcal{J}_G$ of such groups. Groups generated by bounded automata are defined and studied in Section 3.9; their limit spaces are considered in Section 3.10 and Section 3.11, where we prove that in some cases the limit spaces depend only on the algebraic structure of the group and thus can be used to distinguish the groups up to isomorphisms.

Chapter 4, "Orbispaces", is a technical chapter in which we collect the basic definitions related to the theory of orbispaces. Orbispaces are structures represented locally as quotients of topological spaces by finite homeomorphism groups. They are generalizations of a more classical notion of an *orbifold* introduced by W. Thurston (see [**Thu90**] and [**Sco83**]). A similar notion of a V-manifold was introduced earlier by I. Satake [**Sat56**]. We use in our approach pseudogroups and étale groupoids, following [**BH99**]. Most constructions in this chapter are well known, though we present some new (and we hope natural) definitions, like the definition of an open map between orbispaces and the notion of an open sub-orbispace. We also define the *limit orbispace* $\mathcal{J}_G$ of a contracting self-similar action and show that the shift map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is a covering of the limit orbispace by an open sub-orbispace (is a *partial self-covering*).

The orbispace structure on $\mathcal{J}_G$ comes from the fact that the limit space $\mathcal{J}_G$ is the quotient of the limit space $\mathcal{X}_G = \mathfrak{M}^{\otimes -\omega}$ by the action of the group G. Introduction of this additional structure on $\mathcal{J}_G$ makes it possible to reconstruct the group G itself from the partial self-covering s of $\mathcal{J}_G$ as the iterated monodromy group IMG (s) (see Theorem 5.3.1). Hence, if we want to be able to go back and forth between self-similar groups and dynamical systems, then we need to define iterated monodromy groups in the general setting of orbispace mappings.

One cannot avoid using orbispaces even in more classical situations like iterations of rational functions. W. Thurston associated with every post-critically finite rational function its *canonical orbispace*, playing an important role in the study of dynamics (see [**DH93, Mil99**]).

Chapter 5 defines and studies iterated monodromy groups. If $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a covering of a topological space (or an orbispace) $\mathcal{M}$ by an open subset (an open sub-orbispace) $\mathcal{M}_1$, then the fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally by the monodromy action on the set of preimages $p^{-n}(t)$ of the basepoint under the nth iteration of p. Let us denote by K_n the kernel of the action. Then the *iterated monodromy group* of p (denoted IMG (p)) is the quotient $\pi_1(\mathcal{M}, t) \big/ \bigcap_{n\ge 0} K_n$.

The disjoint union $T = \bigsqcup_{n\ge 0} p^{-n}(t)$ of the sets of preimages has a natural structure of a rooted tree. It is the tree with the root t, where a vertex $z \in p^{-n}(t)$ is connected by an edge with the vertex $p(z) \in p^{-(n-1)}(t)$. The iterated monodromy group acts faithfully on this tree in a natural way.

We define a special class of isomorphisms of the tree of preimages T with the tree of words X^* using preimages of paths in $\mathcal{M}$. After conjugation of the natural action of IMG (p) on T by such an isomorphism, we get a *standard* faithful self-similar action of IMG (p) on X^*. The standard action depends on a choice of paths connecting the basepoint to its preimages, but a different choice of paths corresponds to a different choice of a basis of the associated self-similarity bimodule. In particular, two different standard actions of IMG (p) are conjugate, and if the actions are contracting, then the limit spaces $\mathcal{X}_{\mathrm{IMG}(p)}$ and $\mathcal{J}_{\mathrm{IMG}(p)}$ (and the limit dynamical system) depend only on the partial self-covering p.

The main result of the chapter is Theorem 5.5.3, which shows that the limit space $\mathcal{J}_{\mathrm{IMG}(p)}$ of the iterated monodromy group of an expanding partial self-covering $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is homeomorphic to the Julia set of p (to the attractor of the backward orbits) and, moreover, that the limit dynamical system $\mathsf{s} : \mathcal{J}_{\mathrm{IMG}(p)} \longrightarrow \mathcal{J}_{\mathrm{IMG}(p)}$ is topologically conjugate to the restriction of p onto the Julia set. The respective orbispace structures of the Julia set and the limit space also agree.

The last chapter shows different examples of iterated monodromy groups and their applications. We start with the case when a self-covering $p : \mathcal{M} \longrightarrow \mathcal{M}$ is defined on the whole (orbi)space $\mathcal{M}$. The case when $\mathcal{M}$ is a Riemannian manifold and p is expanding was studied by M. Shub, J. Franks and M. Gromov. They showed that $\mathcal{M}$ is in this case an *infra-nil* manifold and that p is induced by an expanding automorphism of a nilpotent Lie group (the universal cover of $\mathcal{M}$). We show how results of M. Shub and J. Franks follow from Theorem 5.5.3, also proving them in a slightly more general setting. A particular case, when $\mathcal{M}$ is a torus $\mathbb{R}^n/\mathbb{Z}^n$, corresponds to numeration systems on $\mathbb{R}^n$ and is related to self-affine *digit tilings* of the Euclidean space, which were studied by many mathematicians.

Another interesting class of examples are the iterated monodromy groups of post-critically finite rational functions. A rational function $f(z) \in \mathbb{C}(z)$ is called *post-critically finite* if the orbit of every critical point under the iterations of f is finite. If P is the union of the orbits of the critical points, then f is a partial self-covering of the punctured sphere $\widehat{\mathbb{C}} \setminus P$. Then the iterated monodromy group of f is, by definition, the iterated monodromy group of this partial self-covering.

The closure of the iterated monodromy group of a rational function f in the automorphism group of the rooted tree is isomorphic to the Galois group of an extension of the field of functions $\mathbb{C}(t)$. This is the extension obtained by adjoining the solutions of the equation $f^{\circ n}(x) = t$ to $\mathbb{C}(t)$ for all n. These Galois groups were considered by Richard Pink, who was the first to define the profinite iterated monodromy groups.

Every post-critically finite rational function is an expanding self-covering of the associated *Thurston orbifold* by an open sub-orbifold, so Theorem 5.5.3 can be applied, and we get a symbolic presentation of the action of the rational function on the Julia set.

Iterated monodromy groups are rather exotic from the point of view of group theory. The only known finitely presented examples are the iterated monodromy groups of functions with "smooth" Julia sets: z^d, Chebyshev polynomials and Lattè examples. Some iterated monodromy groups of rational functions are groups of intermediate growth (for instance $\mathrm{IMG}\left(z^2+i\right)$), while some are essentially new examples of amenable groups (like $\mathrm{IMG}\left(z^2-1\right)$).

Chapter 6 concludes with a complete description of automata generating iterated monodromy groups of polynomials and with an example showing how iterated monodromy groups can be used to construct and to understand plane-filling curves originating from matings of polynomials.

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