# Preface to the Second Edition

In 1977, I gave some lectures at the University of Texas which described the general theory of trace ideals initiated by von Neumann and Schatten (*Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979). Because this theory has many different kinds of applications, the lecture notes I produced at the time were widely used, and I got many requests for information on how to obtain it once it fell out of print.

In 1993, I lectured at a summer school in Vancouver on the theory of applications of rank one perturbations of self-adjoint operators (*Spectral analysis of rank one perturbations and applications* in: Mathematical Quantum Theory. II. Schrödinger Operators, CRM Proc. Lecture Notes, 8, pp. 109–149, American Mathematical Society, Providence, RI, 1995). The two topics have much in common. My interest in each arose in my research in several problems at once. And, of course, rank one perturbations are an extreme case of compact perturbations.

Thus, when I started exploring the possibility of reprinting the *Trace Ideals* book as a second edition, it was natural to combine it with my Vancouver lectures. In preparing this new edition, I had to decide first whether to totally rewrite the material, and I chose not to because the basic theory hasn't changed much. Once I made that decision, I felt it made sense to only lightly edit the material, so, for example, references to theorems or equation numbers would be the same. I fixed typos and made a few references to the addendum, especially where a conjecture had been settled. But I followed the original texts closely so much so that the notation in Chapters 1–10 (the original Texas lectures) and Chapters 11–14 (the Vancouver lectures) are, in a few points, slightly different.

I did add a much better index and an addendum describing some developments since the original notes were written.

It is a pleasure to thank many people who helped on this project: R. Bing, J. Dollard, and J. Gilbert for the invitation to give the original lectures in Texas; and J. Feldman, R. Froese, and L. Rosen for the invitation to lecture in Vancouver. V. Jaksic made useful remarks about the addendum. The proofreading was done while I was a Lady Davis Visiting Professor at the Hebrew University of Jerusalem, and I am grateful for the hospitality of the Mathematics Department there provided by H. Farkas and Y. Last. Because of the form of the original notes, it was necessary to TeX them, a task performed admirably by Cherie Galvez. Finally, I acknowledge Martha's love, which makes it all easier.

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Several years ago, I was working simultaneously on three problems: one concerned scattering of a quantum mechanical particle from a very singular repulsive core [**88**], one involved bounds on the number of negative eigenvalues of $-\Delta + \lambda V$ with the correct behavior as $\lambda \to \infty$ [**296**], and the third involved the structure of the two-dimensional Yukawa quantum field theory [**288, 286**]. The physics and the fundamental mathematical structure of these problems are quite different. But it turned out that the technical tools needed to solve the problems were remarkably similar, so much so that at times I couldn't keep straight which one I was thinking about. Since that time, I have had a great respect and use for a subject that might be called "the hard analysis of compact operators in Hilbert space." I discovered that many of the ideas that I grew so fond of had already been developed by Russian mathematicians and mathematical physicists, particularly the group around M. S. Birman (e.g., [**35, 37, 38, 40, 44, 260**]).

In these lectures, I wish to describe the main ideas and illustrate the tools in a group of specific problems. I am a firm believer in the principle that ideas in analysis should be valued largely by their applicability to other parts of mathematics, so I have included lots of applications chosen from my own specialty of mathematical physics, especially quantum theory. However, I have sufficient faith in these tools that I don't doubt that I would have lots of applications if I worked in some other area of analysis. I warn the reader that there is some overlap with pedagogical presentations I have given elsewhere of bits and pieces of this material (Section VI.5, 6 of [**250**]; the appendix to Section IX.4 of [**251**], the second appendix to Section XI.3 in [**253**], Section XIII.17 in [**254**], and my review article on determinants [**300**]) and that virtually nothing I have to say here is not already in the research literature. For beautiful presentations of some of the material from a somewhat different viewpoint I recommend highly the monograph of Goh'berg-Krein [**134**] and Ringrose [**256**]. In particular, much in Chapters 1–3 follows [**134**]. Many of the results of the Birman school are summarized in the lecture notes of Birman and Solomjak [**45**] which have recently been translated.

Like so much of modern analysis, the material to be described has its roots in the famous paper of Fredholm [**115**] (this deep paper is extremely readable and I recommend it to those wishing a pleasurable afternoon). One of the responses to this paper was a flurry of activity from Hilbert and his school which led eventually to the abstraction of what we now call Hilbert space and the Hilbert-Schmidt operators. In modern notation, this latter is the family of operators, $\mathcal{I}_2$, with $\mathrm{Tr}(A^*A) < \infty$. (I should mention that where I use $\mathcal{I}_p$, one often sees $\mathcal{C}_p$, $\mathfrak{C}_p$, or $\mathcal{B}_p$.) For many years, there were many theorems about operators which are products of two or more Hilbert-Schmidt operators (e.g., [**186**]) until von Neumann and Schatten [**280, 279**] formalized the notion of the trace class, $\mathcal{I}_1$. These two ideals are the analogs in a very real sense of $L^2$ and $L^1$: Below, we will develop the theory of the ideals $\mathcal{I}_p$ and $\mathcal{I}_{p,w}$ which are analogous to the $L^p$ and weak-$L^p$ spaces. (See Stein-Weiss [**319**] for a discussion of weak-$L^p$ spaces.) While this analogy is behind much of what we do, we will not systematically develop it — the theory from this point of view has been developed under the name "non-commutative integration" beginning with pioneering work of Segal [**283**]; see also Kunze [**180**], Gross [**139**], and Nelson [**228**] — this abstract theory is done in the more general context where $\mathcal{B}(\mathcal{H})$, the family of all bounded operators on $\mathcal{H}$, is replaced by a von Neumann algebra having a sufficiently regular trace.

Along the way, we will develop not only the trace on a Hilbert space but also the determinant and thereby methods for solving "explicitly" equations of the form $\phi = \psi + K\phi$ for $\phi$ or $\phi = \lambda K\phi$ for $\lambda$. Another kind of problem which becomes tractable concerns the asymptotic distribution of eigenvalues. For suppose that the $n$-th eigenvalue of a positive self-adjoint operator $B$ goes like $n^\alpha$ for $0 < \alpha < 1$. Then $A = (B+1)^{-1}$ will lie in the weak trace ideal $\mathcal{I}_{p,w}$ where $p = \alpha^{-1}$ and this statement captures most of the fact that the $n$-th eigenvalue of $B$ goes like $n^\alpha$. In Chapter 8, we give a hodge-podge of inequalities which may be useful to the reader who wants to apply those methods.

These notes are based on a series of lectures given at the mathematics department of the University of Texas at Austin in April 1977. It is a pleasure to thank Professors R. Bing, J. Dollard, and J. Gilbert for making possible an enjoyable and profitable visit. As usual, I have profited from numerous discussions with various colleagues at Princeton, most especially M. Aizenman, J. Fröhlich, M. Klaus, E. Lieb, and E. Seiler. Finally, I am indebted to Professors A. Pietsch and J. R. Retherford for valuable correspondence concerning the material in Chapter 10, a subject on which I am something of a novice.

Edison, N.J., 1978 Barry Simon