Introduction

The *fundamental theorem of arithmetic* describes the structure of the multiplicative group $\mathbb{Q}^\times$ of the field $\mathbb{Q}$ of rational numbers as a direct sum

$$\mathbb{Q}^\times \cong (\mathbb{Z}/2) \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}.$$

Namely, a non-zero rational number a has a unique decomposition $a = \pm \prod_p p^{v_p(a)}$, where the exponents $v_p(a)$ are integers and are zero for all but finitely many primes p. This very basic fact brings together the three main objects studied in this book: *multiplicative groups of fields*, *valuations*, and *orderings*. In fact, as we shall see later on, the maps v_p are all non-trivial valuations on $\mathbb{Q}$, and the $\pm$ sign corresponds to its unique ordering.

The attempts to generalize the fundamental theorem of arithmetic to arbitrary number fields F led to the creation of algebraic number theory. Of course, to make such a generalization possible, one had to modify the mathematical language used. The right generalization of both the notion of a prime number as well as of the $\pm$ sign turned out to be that of an *absolute value*: a map $|\cdot|$ from F to the non-negative real numbers such that $|x| = 0$ if and only if $x = 0$, and such that

$$|x \cdot y| = |x| \cdot |y| \quad \text{and} \quad |x + y| \leq |x| + |y|$$

for all x, y in F. For instance, on $\mathbb{Q}$ the usual ordering gives an absolute value $|\cdot|_\infty$ in the standard way, and each map v_p as above gives the *p-adic* absolute value $|x|_p = 1/p^{v_p(x)}$. For the p-adic absolute value $|\cdot| = |\cdot|_p$ the triangle inequality can be strengthened to the so-called *ultrametric inequality*

$$|x + y| \leq \max\{|x|, |y|\}.$$

Absolute values having this stronger property are called *non-Archimedean*, the rest being referred to as *Archimedean*. Using these concepts it was possible to develop one of the most beautiful branches of algebraic number theory: the so-called *ramification theory*, which describes the behavior of absolute values under field extensions, and especially their reflection in Galois groups.

At this point, it was natural to ask for a generalization of this theory to arbitrary fields F. Unfortunately, the notion of an absolute value, which was satisfactory in the number field case, is inadequate in general, so better concepts had to be found. The right substitute for the notion of an Archimedean absolute value has been systematically developed by E. Artin and O. Schreier in the late 1920s ([**Ar**], [**AS1**], [**AS2**]), following an earlier work by Hilbert: this is the notion of an *ordering* on F, i.e., an additively closed subgroup P of the multiplicative group $F^\times$ of F (standing for the set of "positive" elements) such that $F^\times = P \,\dot\cup\, -P$.

The proper definition in the non-Archimedean case is more subtle, and was introduced by W. Krull in his landmark 1931 paper [**Kru2**]. Roughly speaking, instead of looking at the absolute value $|\cdot|$ itself, Krull focused on the group homomorphism $v = -\log|\cdot|: F^\times \to \mathbb{R}$. Of course, this minor modification cannot change much, and is still insufficient for general fields. However, Krull's conceptual breakthrough was to replace the additive group $\mathbb{R}$ by an arbitrary *ordered abelian group* $(\Gamma, \leq)$. Thus what we now call a *Krull valuation* on the field F is a group homomorphism $v: F^\times \to \Gamma$, where $(\Gamma, \leq)$ is an ordered abelian group, which satisfies the following variant of the ultrametric inequality:

$$v(x + y) \geq \min\{v(x), v(y)\}$$

for $x \neq -y$.

Krull's seminal work [**Kru2**] paved the way to modern valuation theory. Starting from this definition, he introduced some of the other key ingredients of the theory: valuation rings, the analysis of their ideals, the convex subgroups of $(\Gamma, \leq)$, and the connections between all these objects and coarsenings of valuations. He adapted for his general setting the (already existent) notions of decomposition, inertia, and ramification subgroups of Galois groups over F. Furthermore, he studied maximality properties of valued fields with respect to field extensions. In a somewhat more implicit way he also studied a notion which will later on become central in valuation theory, namely, *Henselian* valued fields (although he does not give it a name). This notion turned out to be the right algebraic substitute in the setup of Krull valuations for the topological property of completeness. It is analogous to the notion of a *real closed field* introduced by Artin and Schreier in the context of ordered fields. The term "Henselian" is in honor of K. Hensel, who discovered the field $\mathbb{Q}_p$ of p-adic numbers, and proved (of course, under a different terminology) that its canonical valuation is Henselian [**He**]. We refer to [**Ro**] for a comprehensive study of the early (pre-Krull) history of valuation theory.

The classical theory of valuations from the point of view of Krull and his followers is well presented in the already classical books by O. Endler [**En**], P. Ribenboim [**Ri1**], and O.F.G. Schilling [**Schi**]. Yet, over the decades that elapsed since the publication of these books, valuation theory went through several conceptual developments, which we have tried to present in this monograph.

First, the different definitions in the Archimedean and non-Archimedean cases caused a split of the unified theory into two separate branches of field arithmetic: the theory of ordered fields on one hand, and valuation theory on the other hand. While Krull still keeps in [**Kru2**] a relatively unified approach (at least to the extent possible), later expositions on general valuation theory have somewhat abandoned the connections with orderings. Fortunately, the intensive work done starting in the 1970s on ordered fields and quadratic forms (which later evolved into real algebraic geometry) revived the interest in this connection, and led to a reintegration of these two sub-theories. T.Y. Lam's book [**Lam2**] beautifully describes this interplay between orderings and valuations from the more restrictive viewpoint of the reduced theory of quadratic forms, i.e., quadratic forms modulo a preordering (see also [**Lam1**] and [**Jr**]). In the present book we adopt this approach in general, and whenever possible study orderings and valuations jointly, under the common name *localities*.

Second, starting already from Krull's paper [**Kru2**], the emphasis in valuation theory has been on its Galois-theoretic aspects. These will be discussed in detail in

Part III of the book. However, by their mere definitions, valuations and orderings are primarily related to the multiplicative group $F^\times$ of the field F, and much can be said when studying them in this context. This approach has become dominant in the ordered field case (as in [**Lam2**]). However, it is our feeling that in the valuation case this viewpoint has been somewhat neglected in favor of the Galois-theoretic one. Therefore, in addition to presenting the classical theory of Galois groups of valued field extensions, we devote several sections (in Parts II and IV of the book) to developing the theory with emphasis on subgroups S of $F^\times$. In particular, we focus on valuations satisfying a natural condition called *S-compatibility*, which is the analog of Henselity in the multiplicative group context.

Part IV takes this approach one step further, and studies the Milnor K-theory of valued and ordered fields F. We recall that the Milnor K-group of F of degree r is just the tensor product $F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times$ (r times) modulo the simple relations $a_1 \otimes \cdots \otimes a_r = 0$ whenever $a_i + a_j = 1$ for some $i < j$. Several important results (or conjectures) in arithmetic geometry indicate that there should be some kind of parallelism between Milnor's K-theory and Galois theory of fields. For instance, the *Bloch-Kato conjecture* predicts a canonical isomorphism between $K_r^M(F)/n$ and the Galois cohomology group $H^r(F, \mu_n^{\otimes r})$ (where $r \geq 0$ and $n \geq 1$ are integers with $\operatorname{char} F \not| n$, and the cohomology is with respect to the r-times twisted cyclotomic action); this has been proved in several important cases by A.S. Merkurjev, A.A. Suslin, M. Rost, V. Voevodsky, and others (see §24.3). It is therefore not surprising that large parts of the Galois theory of valued and ordered fields have analogs in this natural framework of Milnor's K-theory. These analogs will be presented in Part IV. In some sense, this shift of viewpoint resembles the introduction of the K-theoretic approach to higher class field theory, complementing the earlier Galois-theoretic approach (see [**FV**, Appendix B] and [**FK**]).

Finally, there has been much interest lately in construction of non-trivial valuations on fields. Such constructions emerged in the context of ordered fields (in particular, L. Bröcker's *"trivialization of fans"* theorem [**Br1**]), and later in an elementary and explicit way by B. Jacob, R. Ware, J.K. Arason, R. Elman, and Y.S. Hwang ([**J1**], [**War2**], [**AEJ**], [**HwJ**]). Such constructions became especially important in recent years in connection with the so-called *birational anabelian geometry*. This line of research originated from ideas of A. Grothendieck ([**G1**], [**G2**]) as well as from works of J. Neukirch ([**N1**], [**N2**]). Here one wants to recover the arithmetic structure of a field (if possible, up to an isomorphism) from its various canonical Galois groups. The point is that usually the first step is to recover enough valuations from their cohomological (or K-theoretic) "footprints"; see, e.g., [**BoT**], [**Ef1**], [**Ef7**], [**EfF**], [**NSW**, Ch. XII], [**P1**], [**P2**], [**P3**], [**Sp**], [**Sz**] for more details. In §11 we give a new presentation of the above-mentioned line of elementary constructions, based on the coarsening relation among valuations. While these constructions were considered for some time to be somewhat mysterious, they fit very naturally into the multiplicative group approach as discussed above, especially when one uses the K-theoretic language. In §26 we use this language to prove the main criterion for the existence of "optimal" valuations, as is required in the applications to the birational anabelian geometry. This is further related to the notion of *fans* in the theory of ordered fields, thus closing this fruitful circle of ideas that began with [**Br1**].

The prerequisites of this book are quite minimal. We assume a good algebraic knowledge at a beginning graduate level, including of course familiarity with general field theory and Galois theory. The generalization of finite Galois theory to infinite normal extensions is reviewed for the reader's convenience in §13. Likewise we develop the basic facts and formalisms of Milnor's K-theory in §§23–24 in order not to assume any prior knowledge in this area. On the other hand, we do assume familiarity with the language of homological algebra (exact sequences, commutative diagrams, direct and inverse limits, etc.). The presentation is mostly self-contained, and only very few facts are mentioned without proofs: the "snake lemma" and some basic properties of flatness in §1.1, the structure theory of finitely generated modules over a principal ideal domain and the Nakayama lemma in §17.4, short cohomological discussions in §22.2, §24.3 and Remark 25.1.7, and some facts from local class field theory in §27.1.

Unlike most existing texts on valuation theory, we chose not to develop the theory using commutative algebra machinery, but rather to use the machinery of abelian groups. This simplifies the presentation in many respects. The required results about abelian groups (and in particular ordered abelian groups) are developed in Part I of the book.

Needless to say, we have not pretended to fully describe here the vast research work done on valued and ordered fields throughout the twentieth century and which still goes on today. The choice of material reflects only the author's personal taste (and even more so, his limitations). More material can be found in the texts by Ax [**Ax**], Bourbaki [**Bou1**], Endler [**En**], Jarden [**Jr**], Ribenboim ([**Ri1**], [**Ri3**]), Schilling [**Schi**], and Zariski and Samuel [**ZS**] on valuation theory, as well as those by Knebusch and Scheiderer [**KnS**], Lam ([**Lam1**], [**Lam2**]), Prestel [**Pr**] and Scharlau [**Sch2**] on ordered fields. Likewise, the reference list at the end of this monograph surely covers only a small portion of the possible bibliography. Other and more comprehensive lists of references on valuation theory can be found in [**FV**], [**Ro**], and at the Valuation Theory internet site at http://math.usask.ca/fvk/Valth.html. A comprehensive bibliography on the work done until 1980 on ordered fields is given in [**Lam1**].

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