Preface

Billiards are mathematical models for many physical phenomena where one or more particles move in a container and collide with its walls and/or with each other. The dynamical properties of such models are determined by the shape of the walls of the container, and they may vary from completely regular (integrable) to fully chaotic. The most intriguing, though least elementary, are chaotic billiards. They include the classical models of hard balls studied by L. Boltzmann in the nineteenth century, the Lorentz gas introduced to describe electricity in 1905, as well as modern dispersing billiard tables due to Ya. Sinai.

Mathematical theory of chaotic billiards was born in 1970 when Ya. Sinai published his seminal paper [**Sin70**], and now it is only 35 years old. But during these years it has grown and developed at a remarkable speed and has become a well-established and flourishing area within the modern theory of dynamical systems and statistical mechanics.

It is no surprise that many young mathematicians and scientists attempt to learn chaotic billiards in order to investigate some of these and related physical models. But such studies are often prohibitively difficult for many novices and outsiders, not only because the subject itself is intrinsically quite complex, but to a large extent because of the lack of comprehensive introductory texts.

True, there are excellent books covering general mathematical billiards [**Ta95, KT91, KS86, GZ90, CFS82**], but these barely touch upon chaotic models. There are surveys devoted to chaotic billiards as well (see [**DS00, HB00, CM03**]) but those are expository; they only sketch selective arguments and rarely get down to 'nuts and bolts'. For readers who want to look 'under the hood' and become professional (and we speak of graduate students and young researchers here), there is not much choice left: either learn from their advisors or other experts by way of personal communication or read the original publications (most of them very long and technical articles translated from Russian). Then students quickly discover that some essential facts and techniques can be found only in the middle of long dense papers. Worse yet, some of these facts have never even been published – they exist as folklore.

This book attempts to present the fundamentals of the mathematical theory of chaotic billiards in a systematic way. We cover all the basic facts, provide full proofs, intuitive explanations and plenty of illustrations. Our book can be used by students and self-learners. It starts with the most elementary examples and formal definitions and then takes the reader step by step into the depth of Sinai's theory of hyperbolicity and ergodicity of chaotic billiards, as well as more recent achievements related to their statistical properties (decay of correlations and limit theorems).

The reader should be warned that our book is designed for active learning. It contains plenty of exercises of various kinds: some constitute small steps in the proofs of major theorems, others present interesting examples and counterexamples, yet others are given for the reader's practice (some exercises are actually quite challenging). The reader is strongly encouraged to do exercises when reading the book, as this is the best way to grasp the main concepts and eventually master the techniques of billiard theory.

The book is restricted to two-dimensional chaotic billiards, primarily dispersing tables by Sinai and circular-arc tables by Bunimovich (with some other planar chaotic billiards reviewed in the last chapter). We have several compelling reasons for such a confinement. First, Sinai's and Bunimovich's billiards are the oldest and best explored (for instance, statistical properties are established only for them and for no other billiard model). The current knowledge of other chaotic billiards is much less complete; the work on some of them (most notably, hard ball gases) is currently under way and should perhaps be the subject of future textbooks. Second, the two classes presented here constitute the core of the entire theory of chaotic billiards. All its apparatus is built upon the original works by Sinai and Bunimovich, but their fundamental works are hardly accessible to today's students or researchers, as there have been no attempts to update or republish their results since the middle 1970s (after Gallavotti's book [**Ga74**]). Our book makes such an attempt. We do not cover polygonal billiards, even though some of them are mildly chaotic (ergodic). For surveys of polygonal billiards see [**Gut86, Gut96**].

We assume that the reader is familiar with standard graduate courses in mathematics: linear algebra, measure theory, topology, Riemannian geometry, complex analysis, probability theory. We also assume knowledge of ergodic theory. Although the latter is not a standard graduate course, it is absolutely necessary for reading this book. We do not attempt to cover it here, though, as there are many excellent texts around [**Wa82, Man83, KH95, Pet83, CFS82, DS00, BrS02, Dev89, Sin76**] (see also our previous book, [**CM03**]). For the reader's convenience, we provide basic definitions and facts from ergodic theory, probability theory, and measure theory in the appendices.

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