

Preface

In his thesis [74], J.-P. Serre initiated the use of spectral sequences in algebraic topology. Serre equipped his spectral sequence with a theory of products that greatly enhanced its effectiveness in computing the cohomology of a fiber space. Then it was natural to ask whether Steenrod’s squaring operations could be introduced into spectral sequences as well. For example, in a collection of problems in algebraic topology [46] published in 1955, William Massey wrote: “Problem 6. Is it possible to introduce Steenrod squares and reduced p ’th powers into the spectral sequence of a fibre mapping so that, for the term E_2 , they behave according to the usual rules for squares and reduced p ’th powers in a product space?”

The problem was first addressed by S. Araki [6] and R. Vazquez-Garcia [87], who both developed a theory of Steenrod operations in the Serre spectral sequence. L. Kristensen [34] also wrote down such a theory, and used it to calculate the cohomology rings of some two-stage Postnikov systems. Later, in the papers [77, 78], the present writer developed a theory of Steenrod operations for a class of first-quadrant cohomology spectral sequences. The theory applied to the Serre spectral sequence, but also to the change-of-rings spectral sequence for the cohomology of an extension of Hopf algebras, and to the Eilenberg-Moore spectral sequence for the cohomology of a classifying space.

The present work offers a more detailed exposition of the theory originally set out in [77] and [78], and generalizes its results.

To introduce the context for our theory, we recall the context in which Steenrod operations were originally defined, first by Steenrod [83], and then more generally by Dold [16]. One starts with a commutative, simplicial coalgebra R over the ground ring $\mathbb{Z}/2$. Associated to R is a chain complex CR , defined by $(CR)_n = R_n$, with differential the sum of the face operators of R . Steenrod and Dold showed how to use the “cup- i ” products, $D_i : C(R \times R) \rightarrow CR \otimes CR$, together with the coproduct on R , to define the Steenrod squaring operations on the homology of the cochain complex dual to CR :

$$(0.1) \quad Sq^k : H^n(\mathrm{Hom}(CR, \mathbb{Z}/2)) \rightarrow H^{n+k}(\mathrm{Hom}(CR, \mathbb{Z}/2)).$$

To get a general theory of Steenrod operations in spectral sequences we generalize this construction in several directions. We begin by changing the ground ring from $\mathbb{Z}/2$ to any $\mathbb{Z}/2$ -bialgebra Θ with commutative coproduct. This generalization is useful in dealing with Steenrod operations on the cohomology rings of Hopf algebras. Because we wish to discuss spectral sequences, we replace the simplicial coalgebra R by a bisimplicial object X over the category of Θ -coalgebras. This bisimplicial object determines a “total” chain complex CX of Θ -modules by the rule $(CX)_n = \bigoplus_{p+q=n} (X_{p,q})$. The differential is defined as a sum of the horizontal and vertical face operators in the usual way. We define Steenrod operations on the

homology of the “dual” cochain complex, replacing the coefficient ring $\mathbb{Z}/2$ of (0.1) by an arbitrary Θ -algebra N :

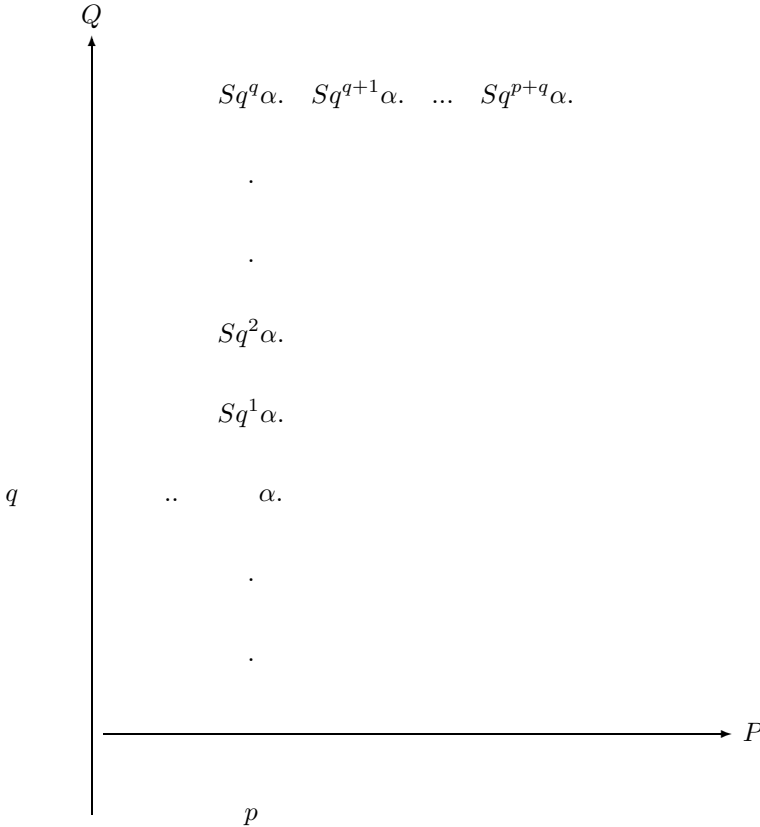
$$(0.2) \quad Sq^k : H^n(\text{Hom}_\Theta(CX, N)) \rightarrow H^{n+k}(\text{Hom}_\Theta(CX, N)).$$

This construction is completely analogous to (0.1). But the bisimplicial object X also determines a double chain complex CX , with $(CX)_{p,q} = X_{p,q}$, and vertical and horizontal differentials defined as sums, respectively, of vertical and horizontal face operators. So a naturally defined first-quadrant spectral sequence (“the spectral sequence of a bisimplicial coalgebra”) converges to the groups $H^*(\text{Hom}_\Theta(CX, N))$ that appear in (0.2). We show how to put Steenrod operations into this spectral sequence. That is, for all $r \geq 2$ and all $p, q, k \geq 0$ we define Steenrod squares:

$$(0.3a) \quad Sq^k : E_r^{p,q} \rightarrow E_r^{p,q+k} \quad \text{if } 0 \leq k \leq q,$$

$$(0.3b) \quad Sq^k : E_r^{p,q} \rightarrow E_t^{p+k-q, 2q} \quad \text{if } q \leq k.$$

Here t is an integer that must in general be chosen larger than r , in order that Sq^k be well-defined (the exact value of t is given in Theorem 2.15). However, if $r = 2$ then $t = 2$. All Steenrod squares defined on E_2 take values in E_2 , and they carry a typical class $\alpha \in E_2^{p,q}$ to positions on the $P - Q$ plane as shown by the diagram:



The squaring operations on E_r are determined by the operations on E_2 , by passage to the subquotient (Theorem 2.16). For reasons made obvious by the diagram, we refer to the family (0.3a) as “vertical” operations, and the family (0.3b) as “diagonal”. The Steenrod operations commute with the differentials of the spectral sequence (Theorem 2.17 and the diagram that follows it). The operations (0.3) are defined when $r = t = \infty$, and agree with those defined on the graded vector space associated to the filtration on the target groups $H^*(\text{Hom}_\Theta(CX, N))$ of the spectral sequence (Theorem 2.22).

We apply the general theory in three cases. A. Dress observed [17] that the Serre spectral sequence can be obtained as the spectral sequence of a bisimplicial coalgebra. So our theory applies and we get Steenrod operations in the Serre spectral sequence, as in [6, 34, 87]. We sketch the application to the Serre spectral sequence at the end of Chapter 2, but do not develop it fully.

We work out our other examples in greater detail. Suppose Λ is a cocommutative Hopf algebra over $\mathbb{Z}/2$; suppose M is a commutative Λ -coalgebra and N a commutative Λ -algebra. In [40] Liulevicius showed how one can use cup- i products in a projective resolution of the Λ -module M to define Steenrod squares:

$$(0.4) \quad Sq^k : \text{Ext}_\Lambda^p(M, N) \rightarrow \text{Ext}_\Lambda^{p+k}(M, N).$$

These operations make $\text{Ext}_\Lambda^*(M, N)$ a module over an algebra we call \mathcal{H} . This is the algebra generated by the Steenrod squares and subject to the Adem relations, with the understanding that Sq^0 is not the unit of the algebra, but an independent generator. Now suppose given Γ , a normal sub-Hopf algebra of Λ , and let $\Omega = \Lambda/\Gamma$ be the quotient Hopf algebra. Then for any right Ω -module P and right Λ -modules Q, N we have the change-of-rings (Cartan-Eilenberg) spectral sequence :

$$(0.5) \quad E_2^{p,q} = \text{Ext}_\Omega^p(P, \overline{\text{Ext}}_{-q}^\Gamma(Q, N)) \implies \text{Ext}_\Lambda^{p+q}(P \otimes Q, N).$$

Here $\overline{\text{Ext}}_*^\Gamma(Q, N)$ is the negatively graded cohomology of Γ , as explained in the text. If in addition P and Q are coalgebras over Ω and Λ respectively, and N is a commutative algebra over Λ , then the target of this spectral sequence supports Steenrod operations, as in (0.4). In this case we describe (0.5) as the spectral sequence of a bisimplicial coalgebra, so that we can apply our theory to give Steenrod squares in the spectral sequence. The work generalizes that of Uehara [86], who considered the case in which Γ is central in Λ and coefficient modules are $P = Q = N = \mathbb{Z}/2$. As in (0.3) we have both vertical and horizontal operations at the E_2 level:

$$(0.6a) \quad Sq^k : \text{Ext}_\Omega^p(P, \overline{\text{Ext}}_{-q}^\Gamma(Q, N)) \rightarrow \text{Ext}_\Omega^p(P, \overline{\text{Ext}}_{-q-k}^\Gamma(Q, N)) \quad (k \leq q)$$

$$(0.6b) \quad Sq^k : \text{Ext}_\Omega^p(P, \overline{\text{Ext}}_{-q}^\Gamma(Q, N)) \rightarrow \text{Ext}_\Omega^{p+k-q}(P, \overline{\text{Ext}}_{-2q}^\Gamma(Q, N)) \quad (q \leq k).$$

We give descriptions of both types of operation in terms of the actions of \mathcal{H} on the cohomologies of Γ and Ω . Thus, the vertical operations (0.6a) come about because $\overline{\text{Ext}}_*^\Gamma(Q, N)$ supports an action of $\overline{\mathcal{H}}$, the negative of \mathcal{H} , that is “compatible” with the action of Ω , in a sense that we make precise in Chapters 3 and 5. We then use relative homological algebra in the category of $\overline{\mathcal{H}} - \Omega$ modules to define these vertical operations. On the other hand we show that, after a shift in indexing, the diagonal operations (0.6b) are the same as those defined on the cohomology of the

Hopf algebra Ω using cup- i products on a projective resolution of the Ω -module P , as in (0.4).

Similar patterns appear in our study of the Eilenberg-Moore spectral sequence. Here we suppose given a simplicial group \mathcal{G} , a principal right \mathcal{G} -space \mathcal{E} that is also a Kan complex, and a Kan complex \mathcal{F} on which \mathcal{G} acts from the left. The spectral sequence converges to the cohomology of the Borel construction:

$$(0.7) \quad E_2^{p,q} = \text{Ext}_{H_*\mathcal{G}}^{p,q}(H_*\mathcal{E}, K_*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{E} \times_{\mathcal{G}} \mathcal{F}).$$

Here $K_*\mathcal{F}$ is the negatively graded cohomology of \mathcal{F} : $K_{-n}\mathcal{F} = H^n(\mathcal{F}, \mathbb{Z}/2)$. In Chapters 6 and 7 we obtain this spectral sequence as the spectral sequence of a bisimplicial coalgebra. Thus our general theory applies, and gives Steenrod squares in the spectral sequence. The vertical and horizontal operations on the E_2 term are:

$$(0.8a) \quad Sq^k : \text{Ext}_{H_*\mathcal{G}}^{p,q}(H_*\mathcal{E}, K_*\mathcal{F}) \rightarrow \text{Ext}_{H_*\mathcal{G}}^{p,q+k}(H_*\mathcal{E}, K_*\mathcal{F}) \quad (k \leq q)$$

$$(0.8b) \quad Sq^k : \text{Ext}_{H_*\mathcal{G}}^{p,q}(H_*\mathcal{E}, K_*\mathcal{F}) \rightarrow \text{Ext}_{H_*\mathcal{G}}^{p+k-q,2q}(H_*\mathcal{E}, K_*\mathcal{F}) \quad (k \geq q).$$

We give descriptions of these that are analogous to the descriptions we give for the operations on the change-of-rings spectral sequence. Thus, the vertical operations (0.8a) come about because both $K_*\mathcal{F}$ and $H_*\mathcal{E}$ support actions of $\overline{\mathcal{A}}$, the negative of the Steenrod algebra \mathcal{A} , that are compatible with the actions of $H_*\mathcal{G}$ in particular ways that we describe in Chapters 3 and 7. We then use relative homological algebra in the category of $\overline{\mathcal{A}} - H_*\mathcal{G}$ modules to describe the operations (0.8a). On the other hand we show that, after a shift in indexing, the diagonal operations (0.8b) are the same as those defined on the cohomology of the Hopf algebra $H_*\mathcal{G}$ using cup- i products on a projective resolution of $H_*\mathcal{E}$, as in (0.4).

We discuss some applications that are already in the literature. These cite [6, 77, 87] or [78]. For example, Goerss [25] uses Steenrod squares in spectral sequences to discuss the André-Quillen cohomology of commutative algebras - a topic motivated by a desire to understand the E_2 -term of the unstable Adams spectral sequence. We sketch this application at the end of Chapter 2. Palmieri [64] uses Steenrod operations in the change-of-rings spectral sequence in his recent work describing the cohomology of the Steenrod algebra “modulo nilpotence”. We sketch this application at the end of Chapter 5. Mimura, Kameko, Kuribayashi, Mori, Nishimoto and Sambe [31, 38, 53, 54, 55, 56, 57, 61, 62, 70, 71, 72] have applied Steenrod operations in the Eilenberg-Moore spectral sequence to calculate the cohomology of the classifying spaces of the exceptional Lie groups. We cite their work at the end of Chapter 7.

We also discuss potential applications. There are several problems, or types of problem, to which the methods of this work can almost surely be applied to good effect. For example, at the end of Chapter 2 we discuss possible application to the Serre spectral sequence, in the case in which the cohomology of the fiber is not a simple coefficient system over the base. At the end of Chapter 5 we discuss possible applications of Steenrod squares in the change-of-rings spectral sequence to computing cohomologies of sub and quotient Hopf algebras of the Steenrod algebra,

as well as to computing the cohomology rings of groups. Finally, we note that the operations (0.8) in the Eilenberg-Moore spectral sequence have so far been applied only in cases for which \mathcal{E} is contractible and \mathcal{F} is a point. It might be interesting, for example, to study cases in which \mathcal{G} is a finite group, and \mathcal{E} a product of spheres on which \mathcal{G} acts freely.

Much of this work can be considered expository, but some of our results are new, and in particular go beyond [77, 78]. In [78] our treatment of the change-of-rings spectral sequence (0.5) relied on properties of extensions of Hopf algebras as they were developed in [76]. In particular we assumed that the Hopf algebras were graded-connected and that Γ had commutative multiplication. But in an effort to make the present work self-contained we have included Chapter 4, in which we develop from scratch the necessary properties of Hopf extensions. We are able to do this without assuming that the Hopf algebras are connected, and without assuming Γ commutative. Consequently, our results are more general than those in [78]. The greater generality is necessary, for example, for Palmieri's application [64].

Similarly, our discussion of the Eilenberg-Moore spectral sequence (0.7) is more detailed and general than the very brief treatment given in [78]. We begin in Chapter 6 by giving a self-contained construction of the spectral sequence. The original construction, in [60], used Serre's filtration of the chains of the total space \mathcal{E} , Serre's computation of the E_1 -term of the resulting (Serre) spectral sequence, and the "comparison theorem for spectral sequences" that is developed in [59] and [14]. Our treatment does not use these results; but instead some properties of semi-simplicial fibrations from [7]. This approach is well suited to our purposes because it presents the Eilenberg-Moore spectral sequence directly as the spectral sequence of a bisimplicial coalgebra. So we can immediately apply the results of Chapter 2 to obtain Steenrod operations in the spectral sequence. These ideas were implicit in [78], but not worked out in detail. The present work goes beyond [78] also in allowing \mathcal{F} to be any left \mathcal{G} -space, not necessarily a point. Our description of the Steenrod operations (0.8a) at the E_2 level, using relative homological algebra in the category of $\overline{\mathcal{A}} - H_*\mathcal{G}$ modules, also seems to be new.

In his papers [61, 62] Mori, influenced in part by [77, 78], has given a treatment of Steenrod operations in the Eilenberg-Moore spectral sequence at an arbitrary prime. Our present treatment has some features not found in [61, 62]. Our derivation of the spectral sequence is self-contained, and we get the theory of Steenrod operations within that setting. Our description of the operations at the E_2 -level is independent of choices of resolutions.

Finally we use the present exposition to correct a mistake in a previous one. In [77] we asserted that the diagonal Steenrod squares (0.3b) are well defined as operations from E_r to E_r . This is not so if $r > 2$; at least, not in the theory we have developed. The mistake was pointed out by Sawka, in his treatment [73] of the odd primary version of [77]. In Chapter 2 of the present work we develop the general theory with great care, showing explicitly the verifications that the operations are well-defined, and giving equal attention to the related issue of their additivity.

All the results in this work have analogues at odd primes. We do not write these down explicitly. But, as we have just mentioned, Sawka's paper [73] develops the general theory of odd primary Steenrod operations in the spectral sequence of a bisimplicial coalgebra: it is the analogue of [77] and Chapter 2 of the present work. A reader interested particularly in change-of-rings or Eilenberg-Moore spectral sequences at odd primes should have no trouble combining Sawka's paper with the

theorems in this book to get the results he needs. To facilitate such a process we write much of the present work over arbitrary characteristic, specializing to $p = 2$ only when the Steenrod operations are discussed. For example, Chapter 3 presents a theory of bialgebra actions on the cohomology of algebras that we need in several contexts. The characteristic is arbitrary and we pay all due attention to signs. The same is true of our treatment of extensions of Hopf algebras in Chapter 4, and of our derivation of the Eilenberg-Moore spectral sequence in Chapter 6.

This work treats only first-quadrant spectral sequences. Theories of Steenrod operations in second-quadrant spectral sequences have been developed by Smith [80, 81], Rector [69], and Dwyer [18]. In particular these theories apply to the second-quadrant Eilenberg-Moore spectral sequence that converges to the cohomology of a loop space; or, more generally, to the cohomology of an induced fibration.

The present work is organized in the following way. Chapter 1 is introductory. This chapter reviews some material concerning bialgebras and their modules, chain and cochain complexes, differential algebras, adjunction isomorphisms, cup- i products, Steenrod operations, relative homological algebra, and spectral sequences. Almost all of this is well-known; but we present the material in the forms in which we will be using it. One possibly novel feature is our notion of an “algebra with coproducts”: a kind of generalized bialgebra. It seems to be the right description of the algebra \mathcal{H} of Steenrod squares that operates on the cohomology rings of Hopf algebras. Further discussion of algebras with coproducts can be found in [79].

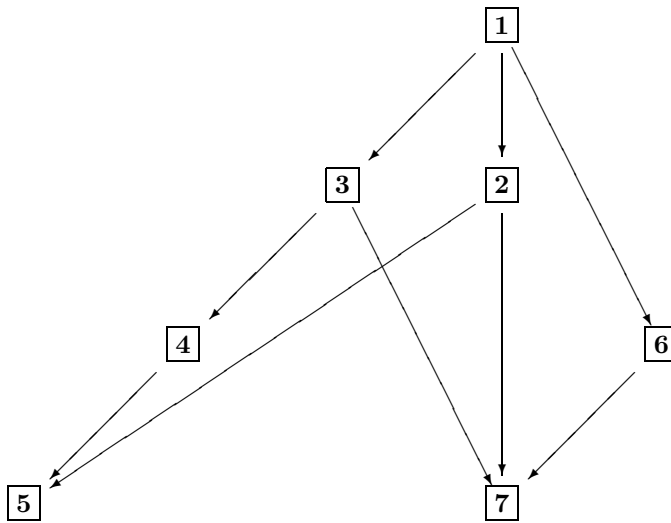
In Chapter 2 we develop the general theory of Steenrod operations in the spectral sequence of a bisimplicial coalgebra, and discuss some applications of this theory.

In Chapter 3 we give a set of axioms under which a bialgebra can operate on the cohomology of an algebra. We study some properties of this action; for example, its relationship to products and Steenrod squares, when these are defined on the cohomology of the algebra. We make several applications of this theory in later chapters. We use it to define the action of the base of an extension of Hopf algebras on the cohomology of the fiber, and to relate this action to the product and Steenrod squares on the cohomology of the fiber. The results play a role in the definitions of both vertical and diagonal operations (0.6a) and (0.6b) in Chapter 5. We use the theory of Chapter 3 also in Chapter 7 to define the vertical operations (0.8a) in the Eilenberg-Moore spectral sequence, and to relate them to the diagonal operations.

In Chapter 4 we develop some properties of extensions of Hopf algebras, and set up the change-of-rings spectral sequence, obtaining it as the spectral sequence of a bisimplicial coalgebra. Then in Chapter 5 we put products and Steenrod operations into the change-of-rings spectral sequence, and describe the operations at the E_2 -level. We discuss some applications of these results.

In Chapter 6 we derive the Eilenberg-Moore spectral sequence “from scratch”, obtaining it as the spectral sequence of a bisimplicial coalgebra. Then in Chapter 7 we define products and Steenrod operations in the spectral sequence, and describe the operations at the E_2 -level. We mention some applications of these results.

The dependence of the various chapters upon one another is indicated by the diagram on the next page. Although the diagram shows Chapter 1 prerequisite for all other chapters, some sections of that chapter are needed only for the change-of-rings spectral sequence, and some are needed only for Eilenberg-Moore. In such cases we have so indicated at the beginning of the section.



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Paul Taylor wrote “Commutative diagrams in $\text{T}_{\text{E}}\text{X}$ ”. Most of the diagrams in this book were created with the aid of this package.

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John Palmieri asked whether the results of [78] could be generalized to extensions of Hopf algebras with non-commutative kernel, thus providing me with one of the motivations for writing this book. William Dwyer, Philip Hirschhorn, and Daniel Kan gave valuable assistance with simplicial and bisimplicial theory. Dagmar Meyer read an early version of Chapter 2 and made many useful suggestions, all of which I have followed. I am grateful to Mamoru Mimura for several conversations and emails about Steenrod operations in the Eilenberg-Moore spectral sequence, and the role they have played in the computation of the cohomology rings of the classifying spaces of Lie groups.

Some forty years ago I began learning algebraic topology from John C. Moore, when he agreed to supervise my doctoral thesis. I have been learning from him ever since. His works on Hopf algebras, relative homological algebra, semi-simplicial fiber bundles, and the Eilenberg-Moore spectral sequence are major influences on this book.

I thank my wife, Eileen, for her patience and steady encouragement over the many months it took to write this book.