Preface

Given a smooth n-dimensional Riemannian manifold (M^n, g), does it admit a smooth isometric embedding in Euclidean space $\mathbb{R}^N$ of some dimension N? This is a long-standing problem in differential geometry. When an isometric embedding in $\mathbb{R}^N$ is possible for sufficiently large N, there arises a further question. What is the smallest possible value for N? Those questions have more classical local versions in which solutions are sought only on a sufficiently small neighborhood of some specific point on the manifold.

In this book we present, in a systematic way, results concerning the isometric embedding of Riemannian manifolds in Euclidean spaces, both local and global, with the focus being on the isometric embedding of surfaces in $\mathbb{R}^3$. The book consists of three parts. In the first, we discuss some fundamental results of the isometric embedding of Riemannian manifolds in Euclidean spaces; these include the Janet-Cartan Theorem and Nash Embedding Theorem. In the second part, we study the local isometric embedding of surfaces in $\mathbb{R}^3$; we discuss metrics with Gauss curvature which is everywhere positive, negative, nonnegative, nonpositive, as well as the case of mixed sign. In the third part, we study the global isometric embedding of surfaces in $\mathbb{R}^3$; the main focus is on metrics on $\mathbb{S}^2$ with positive Gauss curvature and complete metrics in $\mathbb{R}^2$ with negative Gauss curvature. The emphasis of this book is on the *PDE techniques* for proving these results.

Differential geometers might, at first glance, consider the inclination toward analysis to be misplaced in these geometric problems and might even prefer less local coordinate calculations. However, all local calculations are designed to uncover the relevant PDE in the most efficient manner. The goal of this book is then to give a clean exposition of the techniques used in the analysis of these PDEs.

Completely omitted from the book is the local isometric embedding of higher-dimensional Riemannian manifolds in the Euclidean space of least dimension. Works on the higher-dimensional problems have involved much more differential geometry and methods such as exterior differential systems and are therefore far less accessible than the techniques presented in this book.

In integrating the results and techniques of a wide range of literature on the subject, we have tried to accommodate a broad readership as well as experts in the field. It is our objective that this book should provide a good entry into the area for second- or third-year graduate students. With this in mind, we have excluded everything that is technically complicated. Background knowledge is kept to an essential minimum. In Riemannian geometry, we assume only an acquaintance with basic concepts. In analysis, we assume the Cauchy-Kowalewsky theorem and some basic knowledge on elliptic and hyperbolic differential equations. On the other hand, we hope that experts in the field will appreciate the organization of the results, covering the span of more than a century, into a unified whole.

Each chapter ends with bibliographical notes. Attributions are kept to a minimum in the body of the text, and the history and context of the works are expanded in the bibliographical notes.

All works quoted herein are already published.

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