Traces of Hecke operators

1. Introduction

A modular form of level 1 and weight $\mathtt{k}$ is a holomorphic function $h(z)$ on the complex upper half-plane $\mathbf{H}$ which satisfies

$$h(\frac{az+b}{cz+d}) = (cz+d)^{\mathtt{k}} h(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. Taking $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives in particular

$$h(z+1) = h(z).$$

Therefore h defines a holomorphic function of $q = e^{2\pi i z}$. The mapping $z \mapsto q$ takes $\mathbf{H}$ onto the open unit disk with the origin removed. The origin corresponds to the cusp $z = i\infty$. Modular forms are required to be holomorphic at the cusps, i.e. as a function of q, h has a power series expansion

$$h(z) = \sum_{n=0}^{\infty} a_n q^n.$$

If $a_0 = 0$, then h is a **cusp form**. The Fourier coefficients a_n of modular forms contain a great deal of arithmetic information. For instance the following quantities are encoded in the Fourier coefficients of appropriately chosen modular forms:

- The number of ways of representing an integer by a given quadratic form, e.g. as a sum of four squares ([**Iw1**], Ch. 10, 11.)
- The number of points on a $\mathbf{Q}$-rational elliptic curve over the field with p elements. (See the survey [**Da**].)

One way to access the Fourier coefficients is as follows. For each prime number p (not dividing the level N) there is a linear Hecke operator T_p acting on the vector space of cusp forms of a given level and weight. T_p is diagonalizable, and its eigenvalues are the p^{th} Fourier coefficients of the elements of a certain basis of eigenvectors. There is a well-known formula for the trace of T_p from which these Fourier coefficients can be recovered. This formula was originally given in the level 1 case by Selberg without proof in his pioneering 1956 paper [**S**] on the trace formula for $\mathrm{SL}_2(\mathbf{R})$. Subsequent improvements were made by various authors, notably Eichler [**E**], who developed a different technique allowing $\mathtt{k} = 2$ and square-free level, and Hijikata [**H**], who gave the trace of $T_{\mathtt{n}}$, with no restriction on the level N, for $(\mathtt{n}, N) = 1$. Hijikata's computation builds on work of Shimizu ([**Sh**], which applies Selberg's ideas to the Hilbert modular setting) and Saito ([**Sa**], which generalizes Eichler's work). The most general formula for the trace of $T_{\mathtt{n}}$ on $S_{\mathtt{k}}(N, \omega)$, valid for all $\mathtt{n}$ and N, was given in 1977 by Oesterlé in his thesis ([**Oe**]; see [**Coh**] for a description). This explicit formula is known as the **Eichler-Selberg trace formula**. A statement of the formula is given on page 370.

The first goal of these notes is to provide a reference with a comprehensive self-contained proof of this fundamental formula, using the more modern methods provided by the Arthur-Selberg trace formula for the adelic group $\mathrm{GL}_2(\mathbf{A})$. We evaluate the trace formula using a function $f : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ which is constructed from double cosets at the finite places in the same way as the classical Hecke operator $T_{\mathtt{n}}$, and whose infinite component f_∞ is a matrix coefficient for the weight $\mathtt{k}$ discrete series representation of $\mathrm{GL}_2(\mathbf{R})$. Because this matrix coefficient is not integrable when $\mathtt{k} = 2$, we need to require $\mathtt{k} > 2$. We also assume $(\mathtt{n}, N) = 1$.

This technique is basic in the theory of automorphic forms. For example, it is used in Langlands' general strategy for computing the Hasse-Weil zeta function of a Shimura variety in terms of automorphic L-functions. Roughly, an analytic expression coming from the trace formula for a function like our f (which can be evaluated in terms of automorphic L-factors) is compared with a geometric expression involving the traces of Frobenius elements acting on the cohomology of the variety (in terms of which the zeta function can be evaluated). See [**L1**], [**L2**] and [**Ro2**].

In Sections 3 through 11 we have attempted to assemble the necessary background from representation theory and number theory in one place for anyone who wishes to understand the whole story without having to jump between too many sources. This includes detailed treatments of modular forms and Hecke operators, adeles and ideles, structure theory and strong approximation for GL(2), integration theory, Poisson summation for functions on the adeles, adelic zeta functions, representation theory for locally compact groups, and the unitary representations of $\mathrm{GL}_2(\mathbf{R})$.

The heart of the text begins in Section 12 where we give a thorough account of the passage from the classical setting to the adelic one. In the sections that follow, we essentially reprove the convergence of the truncated terms on the geometric side of the trace formula for GL(2). This discussion is quite general and overlaps significantly with the articles [**G2**] and [**GJ**], however we have tried to include more detail than these sources, particularly on convergence issues. Some extra care is required since our test function is not compactly supported.

Lastly, we hope that the explicit computations of orbital integrals for GL(2) over $\mathbf{R}$ and p-adic fields in Sections 24-26 will be interesting for anyone studying the trace formula or local harmonic analysis. We will not discuss zeta functions further (and indeed the most natural application in this direction would be to compute the zeta functions of modular curves, which would require the $\mathtt{k} = 2$ case), but we include some more modest applications and examples in the last chapter. These include the dimension formula for $S_{\mathtt{k}}(N, \omega')$, the integrality of Hecke eigenvalues, and the asymptotic equidistribution of eigenvalues of T_p as $\mathtt{k} + N \to \infty$.

Other references for the traces of Hecke operators include Duflo and Labesse [**DL**], who used the trace formula for $\mathrm{GL}_2(\mathbf{A})$ to sketch a derivation of the formula for the traces of Hecke operators. Miyake's book [**Mi**] contains a proof (for $\mathtt{k} > 2$) using the trace formula for $\mathrm{SL}_2(\mathbf{R})$. Miyake's exposition is based on [**Sh**] and [**H**], and includes the case of cusp forms attached to the unit groups of indefinite quaternion algebras due to Hijikata. Zagier gave a proof for level 1 and weight $\mathtt{k} > 3$ (also using the classical language) in [**Z1**] and [**Z2**]. In [**Oe**], Oesterlé removed the condition $(\mathtt{n}, N) = 1$, and allowed for half-integer weights by building on work of Shimura. We adopt Oesterlé's notation for the final form of the trace formula.

Acknowledgements. We would like to thank J. Rogawski for his encouragement and helpful advice, without which these notes could not have been written. We also thank the anonymous referees for their detailed comments. This work was supported in part by NSA grants H98230-05-1-0028 and H98230-06-1-0039. The second author thanks the UCLA math department for support when most of this work was done, as well as the Taiwan National Center for Theoretical Sciences which provided travel support for a visit to Maine.

2. The Arthur-Selberg trace formula for GL(2)

We begin with a review of the trace formula for GL(2) for a compactly supported function. Although we will not use it explicitly, this formula provides the framework for the trace formula derived in these notes. Nearly all of the definitions and concepts which are mentioned briefly in this section will be discussed in greater detail later on. A good introduction to the trace formula is given in Lecture 1 of Gelbart's book [**G2**].

Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$, and let $\mathbf{A}^*$ be the idele group (see Section 5.2 below for definitions and topology).

Let G be the group GL_2. Thus for any ring R (we always assume rings are commutative with 1), $G(R)$ is the group of 2×2 invertible matrices with entries in R. We use this notation for any linear group. For example let $B \subset G$ denote the Borel subgroup of invertible upper triangular matrices. Then $B(R) = M(R)N(R)$ where $M(R)$ is the group of diagonal matrices with invertible entries in R, and $N(R)$ is the group of unipotent matrices $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \in R$. The **Iwasawa decomposition** of $G_p = G(\mathbf{Q}_p)$ (or $G_\infty = G(\mathbf{R})$) is

$$G_p = B_p K_p \ \text{ (or } G_\infty = B_\infty K_\infty),$$