# Groundwork

In this chapter we lay the foundation for the computation of the trace of $T_{\mathtt{n}}$. First we show that $S_{\mathtt{k}}(N,\omega')$ embeds isometrically into $L_0^2(\omega)$ for an appropriately chosen Hecke character $\omega$. This having been done, the main result of this chapter is the construction of a function $f$ on $G(\mathbf{A})$ for which the following diagram commutes:

$$\begin{array}{ccc} L^2(\omega) & \xrightarrow{\ \mathtt{n}^{\frac{\mathtt{k}}{2}-1}R(f)\ } & L^2(\omega) \\ {\scriptstyle \text{orthog. proj.}}\downarrow & & \uparrow \\ S_{\mathtt{k}}(N,\omega') & \xrightarrow{\ T_{\mathtt{n}}\ } & S_{\mathtt{k}}(N,\omega') \end{array}$$

In particular, although $L^2(\omega)$ is infinite-dimensional, $\mathtt{n}^{\frac{\mathtt{k}}{2}-1}R(f)$ will be an operator of finite rank (with $\operatorname{rank}_{\mathbf{C}} R(f) \le \dim_{\mathbf{C}} S_{\mathtt{k}}(N,\omega')$) and having the same eigenvalues as $T_{\mathtt{n}}$.

## 12. Cusp forms as elements of $L_0^2(\omega)$

**12.1. From Dirichlet characters to Hecke characters.** Let $\omega'$ be a Dirichlet character modulo $N$ satisfying (3.12):

$$\omega'(-1) = (-1)^{\mathtt{k}}. \tag{12.1}$$

Using strong approximation for the ideles

$$\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*),$$

we use $\omega'$ to define a Hecke character of $\mathbf{A}^*$ (trivial on $\mathbf{Q}^*$ and $\mathbf{R}_+^*$):

$$\omega : \mathbf{A}^* \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*, \tag{12.2}$$

where the first arrow is the canonical projection, the second arrow is the quotient map, and the last arrow is given by $\omega'$. Let

$$\pi_N : \prod_{p|N} \mathbf{Z}_p \longrightarrow \mathbf{Z}/N\mathbf{Z}$$

be the canonical surjection. For any idele $x \in \mathbf{A}^*$, let $x_N$ be the idele which agrees with $x$ at the places $p|N$, and which is 1 at all other places. Then

$$\text{for } x \in \mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*, \quad \omega(x) = \omega(x_N) = \omega'(\pi_N(\prod_{p|N} x_p)). \tag{12.3}$$

If $d$ is an integer coprime to $N$, then $d_N \in \mathbf{R}_+^* \times \widehat{\mathbf{Z}}^*$, so by the above,

$$\omega(d_N) = \omega'(d). \tag{12.4}$$

(However $\omega(d) = 1$ since $d \in \mathbf{Q}^*$). More generally, if $d$ is an arbitrary nonzero integer, it is not hard to check that

$$\omega(d_N) = \omega'(\frac{d}{\prod_{p|N} p^{v_p(d)}}).$$

The above procedure can be reversed. A Hecke character $\omega$ has **finite order** if there exists an integer $\ell$ such that $\omega(x)^\ell = 1$ for all $x \in \mathbf{A}^*$. Such a character is necessarily unitary.

LEMMA 12.1. *A Hecke character has finite order if and only if it is unitary and trivial on* $\mathbf{R}_+^*$.

PROOF. Suppose $\omega$ has order $\ell \geq 1$. Define $\omega_\infty^+ : \mathbf{R}_+^* \to \mathbf{C}^*$ by $\omega_\infty^+(x) = \omega(x_\infty \times 1_{\text{fin}})$. Such a character must be of the form $\omega_\infty^+(x) = x^s$ for some $s \in \mathbf{C}$ by Proposition 11.6. Now $\omega_\infty^+(x)^\ell = x^{s\ell} = 1$ for all $x \in \mathbf{R}_+^*$, so we must have $s = 0$. Thus $\omega$ is trivial on $\mathbf{R}_+^*$.

Conversely, suppose $\omega$ is a (unitary) Hecke character which is trivial on $\mathbf{R}_+^*$. Then $\omega$ defines a continuous homomorphism $\omega : \widehat{\mathbf{Z}}^* \to \mathbf{C}^*$. If $O \subset \mathbf{C}^*$ is a small open neighborhood of 1, then $\omega^{-1}(O)$ is open, and hence contains $U_M \subset \widehat{\mathbf{Z}}^*$ for some $M > 0$. Then $\omega(U_M) \subset O$ is a subgroup of $\mathbf{C}^*$, which must be trivial if $O$ is sufficiently small. Thus each such $\omega$ factors through $\widehat{\mathbf{Z}}^*/U_M \cong (\mathbf{Z}/M\mathbf{Z})^*$ for some positive integer $M$. □

If in the above proof $M > 0$ is chosen to be as small as possible, we set $N_\omega = M$ and call this integer the **conductor** of $\omega$. In this way, there is a natural bijection

$$\left\{ \begin{array}{c} \text{Dirichlet characters} \\ \text{of conductor } M \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{finite order Hecke} \\ \text{characters of conductor } M \end{array} \right\}.$$

We remark that for the character defined in (12.2),

$$N_\omega = N_{\omega'}.$$

The character $\omega'$ may not be primitive, i.e. $N$ may not be minimal, so we can only say that $N_\omega | N$.

A continuous character of $\mathbf{Q}_p^*$ is **unramified** if its kernel contains $\mathbf{Z}_p^*$. Every continuous character of $\mathbf{A}^*$ factorizes as a product of local characters, all but finitely many of which are unramified. Let $\omega$ be the character defined in (12.2). We factorize $\omega$ in this way as follows. For $x_p \in \mathbf{Q}_p$ ($p \leq \infty$), define

$$\omega_p(x_p) \stackrel{\text{def}}{=} \omega(1, \ldots, 1, \overset{p^{\text{th}}}{x_p}, 1, 1, \ldots).$$

For $p$ finite, suppose $v_p(x_p) = j$ so that $x_p = p^j u$, where $u \in \mathbf{Z}_p^*$. Then if $p \nmid N$,

$$\omega_p(x_p) = \omega(p^j(p^{-j}, \ldots, p^{-j}, u, p^{-j}, \ldots)) = \omega'(p)^{-j}. \tag{12.5}$$

In particular, if $j = 0$ then $\omega_p(u) = 1$, so $\omega_p$ is unramified when $p \nmid N$. As a result, the following decomposition holds for any $x \in \mathbf{A}^*$:

$$\omega(x) = \prod_{p \le \infty} \omega_p(x_p).$$

Using (12.1) and (12.3), it is easy to show that

$$\omega_\infty(x) = \operatorname{sgn}(x)^{\mathrm{k}}. \tag{12.6}$$

Suppose $d > 0$ and $\gcd(d, N) = 1$. Then $\omega_\infty(d) = 1$ and $\omega_p(d) = 1$ for all $p \nmid dN$ since $\omega_p$ is unramified. Therefore

$$1 = \omega(d) = \prod_{p|d} \omega_p(d) \prod_{p|N} \omega_p(d) = \prod_{p|d} \omega_p(d)\, \omega'(d)$$

by (12.4). Thus

$$\prod_{p|d} \omega_p(d) = \omega'(d)^{-1} \qquad (d > 0,\ (d, N) = 1). \tag{12.7}$$

**12.2. From cusp forms to functions on $G(\mathbf{A})$.** We now review the embedding

$$S_{\mathrm{k}}(N, \omega') \longrightarrow L_0^2(\omega).$$

Recall that we have defined $K_0(N) = \prod_{p<\infty} K_0(N)_p$, where

$$K_0(N)_p = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p \,|\, c \equiv 0 \bmod N\}.$$

By strong approximation for $G(\mathbf{A})$, we have

$$G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_0(N)). \tag{12.8}$$

If $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$, define

$$\omega(k_0) = \omega(d_N). \tag{12.9}$$

Because $\omega(d_N)$ depends only on $d_N$ modulo $N\widehat{\mathbf{Z}}^*$, this defines a character of $K_0(N)$. If $k_0 = \gamma \in \Gamma_0(N) \subset K_0(N)$, then by (12.4) this agrees with $\omega'(\gamma)$ defined in (3.9):

$$\omega(\gamma) = \omega(d_N) = \omega'(d) = \omega'(\gamma).$$

By identifying $Z(\mathbf{A})$ with $\mathbf{A}^*$, we also view $\omega$ as a character of $Z(\mathbf{A})$. Suppose $z \in Z(\mathbf{A})$, and write $z = z_{\mathbf{Q}}(z_\infty \times z_0)$ with $z_\infty \in Z(\mathbf{R})^+$, and $z_0 \in Z(\widehat{\mathbf{Z}}) \subset K_0(N)$. Then by (12.3),

$$\omega(z) = \omega((z_\infty \times z_0)_N) = \omega(z_0), \tag{12.10}$$

where $\omega(z_0)$ is defined as in (12.9).

For $z \in Z(\mathbf{A}_{\text{fin}})$ define $\omega(z) = \omega(1_\infty \times z)$. Then we find by the above that for

$$z \in Z(\mathbf{A}_{\text{fin}}) \cap K_0(N), \tag{12.11}$$

the value of $\omega(z)$ is independent of whether we regard $z$ as belonging to $Z(\mathbf{A}_{\text{fin}})$ or $K_0(N)$.

There is a chart in the Appendix which tabulates the various uses of $\omega$ and $\omega'$.

For $h \in W_{\mathrm{k}}(N, \omega')$, define a left $G(\mathbf{Q})$-invariant function $\phi_h$ on $G(\mathbf{A})$ by

$$\phi_h(\gamma(g_\infty \times k_0)) = \omega(k_0) j(g_\infty, i)^{-\mathrm{k}} h(g_\infty(i)), \tag{12.12}$$

for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$, and $k_0 \in K_0(N)$. The decomposition

$$g = \gamma(g_\infty \times k_0)$$

is not unique, so we will show that $\phi_h(g)$ is well-defined. Suppose $g = \gamma'(g'_\infty \times k'_0)$ is another decomposition. Then

$$\gamma'^{-1}\gamma = g'_\infty g_\infty^{-1} \times k'_0 k_0^{-1} \in (G(\mathbf{R})^+ \times K_0(N)) \cap G(\mathbf{Q}) = \Gamma_0(N).$$

Thus any decomposition of $g$ has the form $g = \gamma\delta^{-1}(\delta_\infty g_\infty \times \delta_{\text{fin}} k_0)$ for some $\delta \in \Gamma_0(N)$. To check that $\phi_h$ is well-defined, we must check that inserting $\delta$ in this manner does not affect the value of $\phi_h$. We have

$$\begin{aligned}\phi_h(\gamma\delta^{-1}(\delta_\infty g_\infty \times \delta_{\text{fin}} k_0)) &= \omega(\delta_{\text{fin}} k_0) j(\delta g_\infty, i)^{-\mathrm{k}} h(\delta g_\infty(i)) \\ &= \omega(\delta_{\text{fin}})\, \omega(k_0)\, j(\delta, g_\infty(i))^{-\mathrm{k}}\, j(g_\infty, i)^{-\mathrm{k}}\, \omega'(\delta)^{-1}\, j(\delta, g_\infty(i))^{\mathrm{k}}\, h(g_\infty(i)) \\ &= \omega(k_0) j(g_\infty, i)^{-\mathrm{k}} h(g_\infty(i)) \\ &= \phi_h(\gamma(g_\infty \times k_0)),\end{aligned}$$

as needed.

If $g = \gamma(g_\infty \times k_0) \in G(\mathbf{A})$ and $z = z_{\mathbf{Q}}(z_\infty \times z_0)$, then

$$\begin{aligned}\phi_h(zg) &= \omega(z_0 k_0) j(z_\infty g_\infty, i)^{-\mathrm{k}} h(z_\infty g_\infty(i)) \\ &= \omega(z)\phi_h(g),\end{aligned} \tag{12.13}$$

by equations (12.10), (3.7) and (3.8). Thus $\phi_h$ has central character $\omega$, and our goal is to show that $\phi_h \in L_0^2(\omega)$ when $h$ is a cusp form.

**12.3. Comparison of classical and adelic Fourier coefficients.** Let $h \in W_{\mathrm{k}}(N, \omega')$. Fix any $g \in G(\mathbf{A})$ and consider the map $\mathbf{A} \to \mathbf{C}$ defined by

$$x \mapsto \phi_h(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g).$$

By the $G(\mathbf{Q})$-invariance of $\phi_h$ this defines a continuous function on $\mathbf{Q}\backslash\mathbf{A}$. Therefore by Proposition 8.10 it has a Fourier expansion

$$\phi_h(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) = \sum_{\beta \in \mathbf{Q}} W_\beta(g)\, \theta(-\beta x), \tag{12.14}$$

assuming absolute convergence of the series. The Fourier coefficient $W_\beta(g)$ is called the $\theta_\beta$-**Whittaker function** of $\phi_h$. Our goal in the next proposition is to compute $W_\beta(g)$ explicitly. In fact the above Fourier expansion is closely related to the Fourier expansion of $h$ at a certain cusp determined by $g$. Consequently, we will see that (12.14) is justified, and we will be able to prove that $\phi_h \in L_0^2(\omega)$ when $h$ is a cusp form.

First we claim that for computing $W_\beta(g)$, it suffices to consider the case where $\det g_\infty > 0$. In fact, suppose $g' = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g$. Then $W_\beta(g) = W_{-\beta}(g')$. Indeed, because $\phi_h$ is $G(\mathbf{Q})$-invariant,

$$\phi_h(ng) = \phi_h(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g') = \phi_h(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} g') = \phi_h(n^{-1}g').$$

Therefore

$$W_\beta(g) = \int\limits_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi_h(ng)\overline{\theta_\beta(n)}dn = \int\limits_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi_h(n^{-1}g')\,\overline{\theta_{-\beta}(n^{-1})}dn$$
$$= W_{-\beta}(g'),$$

as claimed. Here (and below) we regard $\theta_\beta$ as a character on $N(\mathbf{Q})\backslash N(\mathbf{A})$ in the obvious way.

Proposition 12.2. *Let $h \in W_{\mathbf{k}}(N, \omega')$, and let $\phi_h$ be the associated function defined in* (12.12). *For $\delta \in G(\mathbf{Q})^+$ write*

$$h_\delta(z) = \sum_{n \in \mathbf{Z}} a_n(\delta) q^n,$$

*where $q = e^{\frac{2\pi i z}{M}}$ and $M = M_\delta(\Gamma_1(N))$ is given in Lemma 3.7. Fix $g \in G(\mathbf{R})^+ \times G(\mathbf{A}_{\text{fin}})$, and consider the Fourier expansion* (12.14). *Then there exists $\delta \in G(\mathbf{Q})^+$, determined by $g_{\text{fin}}$ in* (12.17) *below, such that for any $\beta \in \mathbf{Q}$,*

$$W_\beta(g) = \left[\begin{array}{ll} j(g_\infty, i)^{-\mathbf{k}} e^{\frac{2\pi i n z}{M}} a_n(\delta) & \text{if } \beta = \frac{n}{M} \in \frac{1}{M}\mathbf{Z} \\ 0 & \text{if } \beta \notin \frac{1}{M}\mathbf{Z}, \end{array}\right.$$

*where $z = g_\infty(i)$. (If $g_{\text{fin}} \in G(\mathbf{Q})^+$, then $\delta = g_{\text{fin}}^{-1}$.) In particular, taking $\beta = 0$, we see that*

$$W_0(g) = (\phi_h)_N(g) = j(g_\infty, i)^{-\mathbf{k}} a_0(\delta) \tag{12.15}$$

*is the constant term of $\phi_h$.*

Before proving the proposition, we highlight two consequences.

Corollary 12.3. *In the above notation, $\sum |W_\beta(g)| < \infty$, so* (12.14) *is justified.*

Proof. Let $z = g_\infty(i)$ and $q = e^{2\pi i z/M}$. By the proposition,

$$\sum_{\beta \in \mathbf{Q}} |W_\beta(g)| = |j(g_\infty, i)|^{-\mathbf{k}} \sum_{n \in \mathbf{Z}} |a_n(\delta) q^n|,$$

which is finite since $h_\delta(q) = \sum a_n(\delta) q^n$ is absolutely convergent. $\square$

Corollary 12.4. *Let $h \in S_{\mathbf{k}}(N, \omega')$, and write $h(z) = \sum_{n>0} a_n q^n$, where $q = e^{2\pi i z}$. Then for $m \in \mathbf{Q}$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}} \phi_h(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix})\theta(mt)dt = \begin{bmatrix} e^{-2\pi m} a_m & \text{if } m \in \mathbf{Z}^+ \\ \\ 0 & \text{otherwise.} \end{bmatrix}$$

Proof. Apply the proposition with $g = 1$, so $\delta = 1$, $M = 1$, $\beta = m$, and $z = i$. □

Proof of the proposition. Let

$$K_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \,|\, d \equiv 1 \bmod N\widehat{\mathbf{Z}}\}.$$

Using (12.12) it is immediate that

$$\phi_h(gk) = \phi_h(g) \tag{12.16}$$

for all $k \in K_1(N)$.

Note that $\det K_1(N) = \widehat{\mathbf{Z}}^*$, so by strong approximation we can write

$$g_{\text{fin}} = \delta^{-1}k \tag{12.17}$$

for some $\delta \in G(\mathbf{Q})$ and $k \in K_1(N)$. Multiplying both $\delta$ and $k$ by $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ if necessary, we can assume $\delta \in G(\mathbf{Q})^+$. Let $z = g_\infty(i)$. Then using (12.16),

$$\begin{aligned} \phi_h(g_\infty \times g_{\text{fin}}) &= \phi_h(g_\infty \times \delta^{-1}k) = \phi_h(\delta g_\infty \times 1_{\text{fin}}) \\ &= j(\delta g_\infty, i)^{-\mathbf{k}} h(\delta(z)) \\ &= j(g_\infty, i)^{-\mathbf{k}} j(\delta, z)^{-\mathbf{k}} h(\delta(z)) \\ &= j(g_\infty, i)^{-\mathbf{k}} h_\delta(z). \end{aligned} \tag{12.18}$$

Let $M = M_\delta(\Gamma_1(N))$ be the positive rational number given in Lemma 3.7 (page 15). By definition, $M$ is the positive rational number satisfying

$$N(\mathbf{Q}) \cap \delta^{-1}\Gamma_1(N)\delta = N(M\mathbf{Z}) = \left\{ \begin{pmatrix} 1 & tM \\ 0 & 1 \end{pmatrix} \,|\, t \in \mathbf{Z} \right\}.$$

Thus

$$\delta \left\{ \begin{pmatrix} 1 & tM \\ 0 & 1 \end{pmatrix} \,|\, t \in \mathbf{Z} \right\} \delta^{-1} \subset \Gamma_1(N).$$

In particular, the lower left entry, as a linear function of $t$, is congruent to 0 mod $N$, and the lower right entry is congruent to 1 mod $N$. This remains true if we allow $t$ to range through all of $\widehat{\mathbf{Z}}$ instead of $\mathbf{Z}$. Consequently,

$$\delta\, N(M\widehat{\mathbf{Z}})\, \delta^{-1} \subset K_1(N).$$

For $n \in N(\mathbf{A})$ and $n' \in N(M\widehat{\mathbf{Z}})$, we have (identifying $n'$ with $1_\infty \times n'$)

$$\phi_h(nn'g) = \phi_h(ng(g_{\text{fin}}^{-1} n' g_{\text{fin}})) = \phi_h(ng)$$

since by (12.17) $g_{\mathrm{fin}}^{-1}n'g_{\mathrm{fin}} = k^{-1}\delta n'\delta^{-1}k \in K_1(N)$. By strong approximation,

$$N(\mathbf{A}) = N(\mathbf{Q})[N(\mathbf{R}) \times N(M\widehat{\mathbf{Z}})],$$

so $N(\mathbf{Q})\backslash N(\mathbf{A}) = N(M\mathbf{Z})\backslash[N(\mathbf{R}) \times N(M\widehat{\mathbf{Z}})]$. Note that the interval $[0, M]$ is a fundamental domain in $\mathbf{R} \cong N(\mathbf{R})$ for $N(M\mathbf{Z})\backslash N(\mathbf{R})$. Thus by the divorce theorem we have

$$W_\beta(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi_h(ng)\overline{\theta_\beta(n)}dn$$

$$= \int_0^M \int_{N(M\widehat{\mathbf{Z}})} \phi_h(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} n'g)\overline{\theta_{\beta,\infty}(t)}\,\overline{\theta_{\beta,\mathrm{fin}}(n')}dn'\,dt$$

$$= \int_0^M \phi_h(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} g)\theta_\infty(\beta t)dt \int_{M\widehat{\mathbf{Z}}} \theta_{\mathrm{fin}}(\beta a)da.$$

Note that the integral over $M\widehat{\mathbf{Z}}$ is nonzero if and only if $\beta M\widehat{\mathbf{Z}} \subset \widehat{\mathbf{Z}}$, i.e. if and only if $\beta \in \frac{1}{M}\mathbf{Z}$ (Lemma 8.3). Assume this is the case, and write $\beta = \frac{n}{M}$. Then by (12.18), the above is

$$= \mathrm{meas}(M\widehat{\mathbf{Z}}) \int_0^M j(g_\infty, i)^{-\mathtt{k}} h_\delta(z+t)e^{-\frac{2\pi int}{M}}dt$$

$$= j(g_\infty, i)^{-\mathtt{k}} \frac{1}{M} \int_0^M h_\delta(z+t)e^{-2\pi int/M}dt$$

$$= j(g_\infty, i)^{-\mathtt{k}} e^{\frac{2\pi inz}{M}} a_n(\delta)$$

by (3.20) on page 17. ☐

## 12.4. Characterizing the image of $S_{\mathtt{k}}(N, \omega')$ in $L^2_0(\omega)$.

PROPOSITION 12.5. *Let $A_{\mathtt{k}}(N, \omega)$ be the space of all functions $\varphi \in L^2_0(\omega)$ satisfying*

(*a*) $\varphi(gk) = \omega(k)\varphi(g)$ *for all* $k \in K_0(N)$ *and* $g \in G(\mathbf{A})$

(*b*) $\varphi(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{i\mathtt{k}\theta}\varphi(g)$ *for all* $\theta$ *and all* $g \in G(\mathbf{A})$

(*c*) *The function $\varphi$ satisfies*

$$R(E^-)\varphi = 0,$$

*where we take* $R(E^-)\varphi(g) = \frac{d}{dt}\big|_{t=0} R(\exp(tE^-) \times 1_{\mathrm{fin}})\varphi(g)$.

*Then the map $h \mapsto \phi_h$ defines an isometry from $S_{\mathtt{k}}(N, \omega')$ onto $A_{\mathtt{k}}(N, \omega)$.*

*Remarks:* (i) For any function $\varphi$ that transforms under $Z(\mathbf{A})$ by $\omega$, condition ($a$) is equivalent to:

($a'$) $\varphi(gk) = \varphi(g)$ for all $k \in K_1(N)$.

(ii) Condition ($c$) can be replaced by

$$R(\Delta)\varphi = \frac{\mathtt{k}}{2}(1 - \frac{\mathtt{k}}{2})\varphi.$$

This can be seen using Theorem 12.6 below and Theorem 11.44.

PROOF. Let $h \in S_{\mathtt{k}}(N, \omega')$. We begin by showing that $\phi_h$ satisfies conditions $(a), (b)$ and $(c)$. Write $g = \gamma(g_\infty \times k_0)$ for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$, and $k_0 \in K_0(N)$. For any $k \in K_0(N)$,

$$\begin{aligned} \phi_h(gk) &= \phi_h(\gamma(g_\infty \times k_0 k)) \\ &= \omega(k)\omega(k_0) j(g_\infty, i)^{-\mathtt{k}} h(g_\infty(i)) \\ &= \omega(k)\phi_h(g). \end{aligned}$$

Thus $\phi_h$ satisfies condition $(a)$.

Let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Then $k_\theta$ stabilizes $i$, so for $g = \gamma(g_\infty \times k)$,

$$\begin{aligned} \phi_h(gk_\theta) &= \omega(k) j(g_\infty k_\theta, i)^{-\mathtt{k}}\, h(g_\infty k_\theta(i)) \\ &= \omega(k) j(g_\infty, k_\theta(i))^{-\mathtt{k}} j(k_\theta, i)^{-\mathtt{k}}\, h(g_\infty(i)) \\ &= e^{i\mathtt{k}\theta}\phi_h(g). \end{aligned}$$

This proves condition $(b)$ for $\phi_h$.

Let $L$ denote the left regular representation $L(g)f(x) = f(g^{-1}x)$. It is clear that $L$ commutes with the right regular action of the Lie algebra, i.e. $L(g)R(X) = R(X)L(g)$. Therefore for any $g \in G(\mathbf{A})$,

$$R(E^-)\phi_h(g_\infty \times g_{\text{fin}}) = L(\varepsilon_\infty \times g_{\text{fin}})^{-1} R(E^-)\phi_h(\varepsilon_\infty g_\infty \times 1_{\text{fin}}),$$

where $\varepsilon_\infty = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{\operatorname{sgn}(\det g_\infty)}$. Thus in order to verify condition $(c)$ it suffices to show that $R(E^-)\phi_h(g_\infty \times 1_{\text{fin}}) = 0$ for $g_\infty \in G(\mathbf{R})^+$. Write

$$g_\infty = z_\infty \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\theta \in G(\mathbf{R})^+.$$

Note that

$$\phi_h(g_\infty \times 1_{\text{fin}}) = y^{\mathtt{k}/2} e^{i\mathtt{k}\theta} h(x+iy).$$

Recall from Proposition 11.37 that as an operator on $C^\infty(G(\mathbf{R})^+)$,

$$R(E^-) = e^{-2i\theta}\left(-2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + i\frac{\partial}{\partial\theta}\right).$$

Let $z = x+iy$. Then using subscripts to denote partial derivatives,

$$\begin{aligned} R(E^-)\phi_h(g_\infty \times 1_{\text{fin}}) &= R(E^-) y^{\mathtt{k}/2} e^{i\mathtt{k}\theta} h(z) \\ = e^{-2i\theta}(-2iy^{\frac{\mathtt{k}}{2}+1} e^{i\mathtt{k}\theta} h_x(z) &+ 2y\left(y^{\frac{\mathtt{k}}{2}} e^{i\mathtt{k}\theta} h_y(z) + \frac{\mathtt{k}}{2} y^{\frac{\mathtt{k}}{2}-1} e^{i\mathtt{k}\theta} h(z)\right) - \mathtt{k} e^{i\mathtt{k}\theta} y^{\mathtt{k}/2} h(z)) \\ = e^{-2i\theta}\left(-2iy^{\frac{\mathtt{k}}{2}+1} e^{i\mathtt{k}\theta} h_x(z) \right. &\left. + 2y^{\frac{\mathtt{k}}{2}+1} e^{i\mathtt{k}\theta} h_y(z) + \mathtt{k} y^{\mathtt{k}/2} e^{i\mathtt{k}\theta} h(z) - \mathtt{k} e^{i\mathtt{k}\theta} y^{\mathtt{k}/2} h(z)\right) \\ &= -2ie^{(\mathtt{k}-2)i\theta} y^{\frac{\mathtt{k}}{2}+1}\left(h_x(z) + ih_y(z)\right) \end{aligned}$$

$$= -4ie^{(\mathtt{k}-2)i\theta} y^{\frac{\mathtt{k}}{2}+1} \frac{\partial h}{\partial \bar z}(z). \tag{12.19}$$

Because $h$ is holomorphic, the above is identically 0. This proves that $\phi_h$ satisfies condition $(c)$.

Let $D_N \subset \mathbf{H}$ be a fundamental domain for $\Gamma_0(N)\backslash\mathbf{H}$. We identify $D_N$ in the usual way with a subset of $\mathrm{SL}_2(\mathbf{R})$ (cf. Proposition 7.43, p. 104). For the square-integrability of $\phi_h$, we use Proposition 7.43 to compute

$$\begin{aligned}\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} |\phi_h(g)|^2 dg &= \int_{D_N K_\infty \times K_0(N)} |\phi_h(g)|^2 dg \\ &= \int_{D_N K_\infty \times K_0(N)} |j(g_\infty, i)^{-\mathrm{k}} h(g_\infty(i))|^2 dg \\ &= \mathrm{meas}(K_0(N)) \iint_{D_N} |y^{\mathrm{k}/2} h(x+iy)|^2 \frac{dx\,dy}{y^2} \qquad (12.20)\\ &= \frac{1}{\psi(N)} \iint_{\Gamma_0(N)\backslash\mathbf{H}} |h(x+iy)|^2 y^{\mathrm{k}} \frac{dx\,dy}{y^2}.\end{aligned}$$

Thus the $L^2$-norm of $\phi_h$ equals the Petersson norm of $h$. (We are using the fact that $\mathrm{meas}(K_0(N)) = 1/\psi(N)$. See the beginning of Section 13.) It follows that $\phi_h$ is square-integrable, and because its constant term vanishes by (12.15) of Proposition 12.2, we see that $\phi_h \in L_0^2(\omega)$. Recall that the fact that $\phi_h(zg) = \omega(z)\phi_h(g)$ was shown in (12.13) above. This completes the proof that $\phi_h \in A_{\mathrm{k}}(N, \omega)$.

Conversely, suppose $\varphi \in A_{\mathrm{k}}(N, \omega)$. Define a function $h$ on the upper half plane in the following way. For $z \in \mathbf{H}$, choose $g_\infty \in G(\mathbf{R})^+$ such that $g_\infty(i) = z$, and let

$$h(z) = h(g_\infty(i)) = j(g_\infty, i)^{\mathrm{k}} \varphi(g_\infty \times 1_{\mathrm{fin}}).$$

Using the fact that the stabilizer of $i \in \mathbf{H}$ in $G(\mathbf{R})^+$ is $Z(\mathbf{R})K_\infty$, it is straightforward to check that $h(z)$ is independent of the choice of $g_\infty$. Using (12.12), define a function $\phi_h$ on $G(\mathbf{A})$. Then $\phi_h = \varphi$ since

$$\phi_h(\gamma(g_\infty \times k)) = \omega(k) j(g_\infty, i)^{-\mathrm{k}} h(g_\infty(i)) = \omega(k)\varphi(g_\infty \times 1_{\mathrm{fin}}) = \varphi(\gamma(g_\infty \times k)).$$

Let

$$g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}.$$

Then $z = g_\infty(i) = x + iy$ and $h(x+iy) = y^{-\mathrm{k}/2}\varphi(g_\infty \times 1_{\mathrm{fin}})$. Using condition $(c)$ and (12.19) with $\theta = 0$, we have

$$\frac{\partial h}{\partial \bar{z}} = -\frac{1}{4iy^{\mathrm{k}/2+1}} R(E^-)\varphi(g_\infty \times 1_{\mathrm{fin}}) = 0.$$

Thus $h$ is holomorphic.

Write $z = g_\infty(i)$ as above, and let $\gamma \in \Gamma_0(N)$. Then

$$\begin{aligned} h(\gamma z) = h(\gamma g_\infty(i)) &= j(\gamma g_\infty, i)^{\mathrm{k}} \varphi(\gamma g_\infty \times 1_{\mathrm{fin}}) \\ &= j(\gamma, g_\infty(i))^{\mathrm{k}} j(g_\infty, i)^{\mathrm{k}} \varphi(g_\infty \times \gamma_{\mathrm{fin}}^{-1}) \\ &= \omega'(\gamma)^{-1} j(\gamma, z)^{\mathrm{k}} h(z). \end{aligned}$$

Thus $h$ is weakly modular.

To show that $h \in S_{\mathbf{k}}(N,\omega')$, it remains to check that $h$ vanishes at the cusps of $\Gamma_1(N)$. This is a consequence of the square-integrability of $\phi$. By (12.20) we see that the Petersson norm of $h$ is finite. By Proposition 3.39, $h$ is a cusp form. Hence $h \mapsto \phi_h$ is surjective. Because it is a norm-preserving linear map, it must also be injective, so the proposition is proven. □

As mentioned earlier, the right regular representation of $G(\mathbf{A})$ on $L_0^2(\omega)$ decomposes into an orthogonal Hilbert space direct sum of irreducible representations

$$R_0 \cong \overline{\bigoplus} \pi,$$

where $\pi$ are (by definition) the cuspidal automorphic representations of $G(\mathbf{A})$ with central character $\omega$.

Each cuspidal representation $\pi$ is isomorphic to a **restricted tensor product**

$$\pi \cong \bigotimes_{p\leq\infty}{}' \pi_p$$

where $\pi_p$ is an irreducible admissible representation of $G(\mathbf{Q}_p)$. For the proof and a rigorous statement, see Section 3.3.3 of [**GGPS**]. A more general factorization theorem which applies to all irreducible admissible representations of $G(\mathbf{A})$ ($G$ any reductive group) was given by Flath, [**Fl**]. See also Section 3.4 of [**Bu**] for the case of $\mathrm{GL}_n(\mathbf{A})$. For the present purpose, it is enough to know that $\pi$ is a tensor product

$$\pi = \pi_\infty \otimes \pi_{\mathrm{fin}}$$

where $\pi_\infty$ (resp. $\pi_{\mathrm{fin}}$) is an irreducible unitary representation of $G(\mathbf{R})$ (resp. $G(\mathbf{A}_{\mathrm{fin}})$). The isomorphism class of $\pi_\infty$ is called the **infinity type** of $\pi$. When $\pi_\infty \cong \pi_{\mathbf{k}}$, we let $v_{\pi_\infty} \in V_{\pi_\infty}$ denote a lowest weight vector (unique up to scalars). For any representation $\pi_{\mathrm{fin}}$ of $G(\mathbf{A}_{\mathrm{fin}})$ and any subgroup $U$ of $G(\mathbf{A}_{\mathrm{fin}})$, let $\pi_{\mathrm{fin}}^U$ denote the subspace of $U$-fixed vectors in the space of $\pi_{\mathrm{fin}}$.

THEOREM 12.6. *With notation as above, we have*

$$A_{\mathbf{k}}(N,\omega) = \bigoplus_{\pi_\infty\cong\pi_{\mathbf{k}}} \mathbf{C}v_{\pi_\infty} \otimes \pi_{\mathrm{fin}}^{K_1(N)} \tag{12.21}$$

*where the sum taken is over all cuspidal representations in $L_0^2(\omega)$ of the form $\pi = \pi_{\mathbf{k}} \otimes \pi_{\mathrm{fin}}$.*

PROOF. Suppose $\phi = v_\infty \otimes v_{\mathrm{fin}}$ belongs to one of the summands on the right-hand side of (12.21). To show that $\phi \in A_{\mathbf{k}}(N,\omega)$, we check that $\phi$ satisfies conditions $(a')$, $(b)$ and $(c)$ of Proposition 12.5. This is straightforward, using Theorem 11.44:

$(a')$ For $k \in K_1(N)$,

$$R(1_\infty \times k)\phi = \pi_\infty(1)v_\infty \otimes \pi_{\mathrm{fin}}(k)v_{\mathrm{fin}} = v_\infty \otimes v_{\mathrm{fin}} = \phi.$$

(*b*) For $k_\theta \in K_\infty$,

$$R(k_\theta \times 1_{\text{fin}})\phi = \pi_\infty(k_\theta)v_\infty \otimes \pi_{\text{fin}}(1)v_{\text{fin}} = e^{i\mathtt{k}\theta}v_\infty \otimes v_{\text{fin}} = e^{i\mathtt{k}\theta}\phi$$

since $v_\infty$ is a lowest weight vector for $\pi_\infty \cong \pi_{\mathtt{k}}$ (cf. Theorem 11.44).

(*c*) Lastly,

$$\begin{aligned} R(E^-)\phi &= \left.\frac{d}{dt}\right|_{t=0} R(\exp(tE^-) \times 1_{\text{fin}})\phi \\ &= \left.\frac{d}{dt}\right|_{t=0} \pi_\infty(\exp(tE^-))v_\infty \otimes \pi_{\text{fin}}(1)v_{\text{fin}} \\ &= (\pi_\infty(E^-)v_\infty) \otimes v_{\text{fin}} = 0, \end{aligned}$$

again by Theorem 11.44.

Conversely, suppose $\phi \in A_{\mathtt{k}}(N,\omega)$ is nonzero. We need to show that $\phi$ belongs to the right-hand side of (12.21). For any cuspidal $\pi$, let $V_\pi$ be the space of $\pi$, so $L_0^2(\omega)$ is the closure of $\bigoplus V_\pi$. Let $p_\pi : L_0^2(\omega) \to V_\pi$ be the orthogonal projection map. Then $p_\pi$ intertwines the action of $R$ since $V_\pi$ is a closed stable subspace and $R$ is unitary. Using this fact, it is straightforward to show that $p_\pi(A_{\mathtt{k}}(N,\omega)) \subset A_{\mathtt{k}}(N,\omega)$, and hence

$$p_\pi(A_{\mathtt{k}}(N,\omega)) = A_{\mathtt{k}}(N,\omega) \cap V_\pi.$$

It follows that $A_{\mathtt{k}}(N,\omega)$ is the closure of $\bigoplus_\pi(V_\pi \cap A_{\mathtt{k}}(N,\omega))$. However this direct sum is finite-dimensional, hence already closed, so

$$A_{\mathtt{k}}(N,\omega) = \bigoplus_\pi (V_\pi \cap A_{\mathtt{k}}(N,\omega)).$$

By this fact, it suffices to consider the case where

$$\phi \in V_\pi \cap A_{\mathtt{k}}(N,\omega)$$

for some cuspidal $\pi$.

It remains to show that $\pi_\infty \cong \pi_{\mathtt{k}}$ and $\phi \in \mathbf{C}v_{\pi_\infty} \otimes \pi_{\text{fin}}^{K_1(N)}$. By linearity, we can assume that $\phi = v_\infty \otimes v_{\text{fin}}$ for some nonzero $v_\infty \in V_{\pi_\infty}$ and $v_{\text{fin}} \in V_{\pi_{\text{fin}}}$. Let

$$V_\infty(\mathtt{k}) = \{v \in V_{\pi_\infty} | \pi_\infty(k_\theta)v = e^{i\mathtt{k}\theta}v\}$$

be the isotypic component in $V_{\pi_\infty}$ of the character $k_\theta \mapsto e^{i\mathtt{k}\theta}$ of $K_\infty$. Note that by property (*b*),

$$\begin{aligned} \pi_\infty(k_\theta)v_\infty \otimes v_{\text{fin}} &= \pi(k_\theta \times 1_{\text{fin}})\phi \\ &= e^{i\mathtt{k}\theta}\phi = e^{i\mathtt{k}\theta}v_\infty \otimes v_{\text{fin}}. \end{aligned}$$

This proves that $v_\infty \in V_\infty(\mathtt{k})$. By a similar argument, we see easily that $v_{\text{fin}} \in \pi_{\text{fin}}^{K_1(N)}$, and hence $\phi \in V_\infty(\mathtt{k}) \otimes \pi_{\text{fin}}^{K_1(N)}$.

Now because $\phi$ satisfies condition (*c*),

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tE^-) \times 1_{\text{fin}})\phi &= \left.\frac{d}{dt}\right|_{t=0} \pi_\infty(\exp(tE^-))v_\infty \otimes v_{\text{fin}} \\ &= (\pi_\infty(E^-)v_\infty) \otimes v_{\text{fin}}. \end{aligned}$$

Thus $\pi_\infty(E^-)v_\infty = 0$. We have now shown that $v_\infty$ satisfies (2) of Theorem 11.44, and hence $\pi_\infty \cong \pi_{\mathtt{k}}$ and $v_\infty$ is a lowest weight vector. □

*Remark:* If $h \in S_{\mathtt{k}}(N, \omega')$ is a Hecke eigenform, the cuspidal representation $\pi$ generated by $\phi_h \in A_{\mathtt{k}}(N, \omega) \subset L^2_0(\omega)$ is irreducible and has $\pi_\infty = \pi_{\mathtt{k}}$. For details about the correspondence (1-1 only at the level of newforms) between $h$ and $\pi$, including a description of the local factors $\pi_p$, see [**G1**] or [**Ro1**].

## 13. Construction of the test function $f$

We now construct a continuous function $f \in L^1(G(\mathbf{A}), \omega^{-1})$ such that the trace of $R(f)$ on $L^2(\omega)$ gives the trace of the Hecke operator $T_{\mathtt{n}}$ on $S_{\mathtt{k}}(N, \omega')$. The function $f$ will be a product of local functions on $G(\mathbf{Q}_p)$, i.e. $f = f_\infty \times f^{\mathtt{n}}$, where $f^{\mathtt{n}} = \prod_{p<\infty} f_p^{\mathtt{n}}$.

**13.1. The non-archimedean component of $f$.** The idea is to define $f^{\mathtt{n}}$ using double cosets as in the construction of $T_{\mathtt{n}}$, using $K_0(N)$ in place of $\Gamma_0(N)$.

LEMMA 13.1. *Suppose $p|N$. Then*

$$K_p = \bigcup_{\delta \in \mathbf{Z}_p/N\mathbf{Z}_p} \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix} K_0(N)_p \cup \bigcup_{\tau \in p\mathbf{Z}_p/N\mathbf{Z}_p} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} K_0(N)_p,$$

*a disjoint union.*

PROOF. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$. If $p|c$, then $a \in \mathbf{Z}_p^*$, so $\begin{pmatrix} a^{-1} & \frac{-b}{ad-bc} \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \in K_0(N)_p$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^{-1} & \frac{-b}{ad-bc} \\ 0 & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}.$$

Because we can further multiply by $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$ to obtain $\begin{pmatrix} 1 & 0 \\ c/a+N & 1 \end{pmatrix}$, the entry $\tau = c/a \in p\mathbf{Z}_p$ is unique modulo $N\mathbf{Z}_p$.

If $c$ is a unit, then multiplying by $\begin{pmatrix} c^{-1} & \frac{d}{ad-bc} \\ 0 & \frac{-c}{ad-bc} \end{pmatrix}$ gives $\begin{pmatrix} a/c & 1 \\ 1 & 0 \end{pmatrix}$. Once again, $\delta = a/c \in \mathbf{Z}_p$ is unique modulo $N\mathbf{Z}_p$.

This proves the decomposition. To see that it is disjoint, note that

$$\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} w & x \\ Ny & z \end{pmatrix} = \begin{pmatrix} * & * \\ \tau w + Ny & * \end{pmatrix},$$

which cannot equal $\begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}$ since $p|(\tau w + Ny)$. □

Define

$$\psi_p(N) = [K_p : K_0(N)_p].$$