

# Contents

## Traces of Hecke Operators

1. Introduction	1
2. The Arthur-Selberg trace formula for $GL(2)$	3
3. Cusp forms and Hecke operators	7
3.1. Congruence subgroups of $SL_2(\mathbf{Z})$	7
3.2. Weak modular forms	10
3.3. Cusps and Fourier expansions of modular forms	12
3.4. Hecke rings	20
3.5. The level $N$ Hecke ring	24
3.6. The elements $T(\mathfrak{n})$	29
3.7. Hecke operators	34
3.8. The Petersson inner product	37
3.9. Adjoints of Hecke operators	42
3.10. Traces of the Hecke operators	45

## Odds and Ends

4. Topological groups	49
5. Adeles and ideles	52
5.1. $p$ -adic Numbers	52
5.2. Adeles and ideles	54
6. Structure theorems and strong approximation for $GL_2(\mathbf{A})$	59
6.1. Topology of $GL_2(\mathbf{A})$	59
6.2. The Iwasawa decomposition	61
6.3. Strong approximation for $GL_2(\mathbf{A})$	63
6.4. The Cartan decomposition	66
6.5. The Bruhat decomposition	69
7. Haar measure	69
7.1. Basic properties of Haar measure	69
7.2. Invariant measure on a quotient space	74
7.3. Haar measure on a restricted direct product	82
7.4. Haar measure on the adeles and ideles	85
7.5. Haar measure on $B$	87
7.6. Haar measure on $GL(2)$	90
7.7. Haar measure on $SL_2(\mathbf{R})$	92
7.8. Haar measure on $\overline{GL(2)}$	94
7.9. Discrete subgroups and fundamental domains	95
7.10. Haar measure on $\mathbf{Q}\backslash\mathbf{A}$ and $\mathbf{Q}^*\backslash\mathbf{A}^*$	102
7.11. Quotient measure on $\overline{GL_2(\mathbf{Q})}\backslash\overline{GL_2(\mathbf{A})}$	103
7.12. Quotient measure on $\overline{B(\mathbf{Q})}\backslash\overline{G(\mathbf{A})}$	105

8.	The Poisson summation formula	108
9.	Tate zeta functions	119
	9.1. Definition and meromorphic continuation	119
	9.2. Functional equation and behavior at $s = 1$	124
10.	Intertwining operators and matrix coefficients	128
	10.1. Linear algebra	129
	10.2. Representation theory	134
	10.3. Orthogonality of matrix coefficients	140
11.	The discrete series of $\mathrm{GL}_2(\mathbf{R})$	151
	11.1. $K$ -type decompositions	151
	11.2. Representations of $\mathrm{O}(2)$	155
	11.3. $K$ -type decomposition of an induced representation	156
	11.4. The $(\mathfrak{g}, K)$ -modules $(V_\chi)_K$	162
	11.5. Classification of irreducible $(\mathfrak{g}, K)$ -modules for $\mathrm{GL}_2(\mathbf{R})$	165
	11.6. A detailed look at the Lie algebra action	181
	11.7. Characterization of the weight $\mathfrak{k}$ discrete series of $\mathrm{GL}_2(\mathbf{R})$	186
<b>Groundwork</b>		
12.	Cusp forms as elements of $L_0^2(\omega)$	195
	12.1. From Dirichlet characters to Hecke characters	195
	12.2. From cusp forms to functions on $G(\mathbf{A})$	197
	12.3. Comparison of classical and adelic Fourier coefficients	198
	12.4. Characterizing the image of $S_{\mathfrak{k}}(N, \omega')$ in $L_0^2(\omega)$	201
13.	Construction of the test function $f$	206
	13.1. The non-archimedean component of $f$	206
	13.2. Spectral properties of $R(f)$	213
14.	Explicit computations for $f_{\mathfrak{k}}$ and $f_\infty$	221
<b>The Trace Formula</b>		
15.	Introduction to the trace formula for $R(f)$	227
16.	Terms that contribute to $K(x, y)$	230
17.	Truncation of the kernel	231
18.	Bounds for $\sum_\gamma  f(g_1^{-1}\gamma g_2) $	240
19.	Integrability of $k_{\mathfrak{o}}^T(g)$	248
20.	The hyperbolic terms as weighted orbital integrals	259
21.	Simplifying the unipotent term	270
22.	The trace formula	276
<b>Computation of the Trace</b>		
23.	The identity term	279
24.	The hyperbolic terms	280
	24.1. A lemma about orbital integrals	280
	24.2. The archimedean orbital integral and weighted orbital integral	281
	24.3. Simplification of the hyperbolic term	283
	24.4. Calculation of the local orbital integrals	283
	24.5. The global hyperbolic result	286
25.	The unipotent term	288
	25.1. Explicit evaluation of the zeta integral at $\infty$	288
	25.2. Computation of the non-archimedean local zeta functions	291

25.3.	The global unipotent result	293
26.	The elliptic terms	294
26.1.	Properties of the elliptic orbital integrals	295
26.2.	The archimedean elliptic orbital integral	300
26.3.	Orders and lattices in an imaginary quadratic field	302
26.4.	Local-global theory for lattices	307
26.5.	From $G(\mathbf{A}_{\text{fin}})$ to lattices in $E$	310
26.6.	The non-archimedean orbital integral for $N = 1$	314
26.7.	The case of level $N$	318
<b>Applications</b>		
27.	Dimension formulas	333
27.1.	The elliptic terms	333
27.2.	The unipotent term	337
27.3.	The dimension of $S_{\mathbf{k}}(N, \omega')$ and some examples	338
28.	Computing Hecke eigenvalues	340
28.1.	Obtaining eigenvalues from knowledge of the traces	340
28.2.	Integrality of Hecke eigenvalues	343
28.3.	The $\tau$ -function	344
28.4.	An example with nontrivial character	347
29.	The distribution of Hecke eigenvalues	351
29.1.	Bounds for the Eichler-Selberg trace formula	352
29.2.	Chebyshev polynomials	355
29.3.	Distribution of eigenvalues	356
29.4.	Further applications and generalizations	359
30.	A recursion relation for traces of Hecke operators	360
	Bibliography	363
	Tables of notation	368
	Statement of the final result	370
	Index	373