Chapter 2

Dynamical Systems and Crossed Products

A C^*-dynamical system is a locally compact group G acting by automorphisms on a C^*-algebra A. A crossed product is a C^*-algebra built out of a dynamical system. In abelian harmonic analysis (cf., Section 1.4), an important principle is to recover information about a function f from its Fourier coefficients $\hat{f}(\omega)$, and $f \mapsto \hat{f}(\omega)$ is a representation of the group algebra on a one-dimensional Hilbert space. We are going to study dynamical systems and their crossed products via classes of representations called covariant representations. We define dynamical systems, their crossed products and covariant representations in the first two sections. Just as characters are in one-to-one correspondence with complex homomorphisms (that is, one-dimensional representations) of the L^1-algebra of a locally compact abelian group, covariant representations of a dynamical system are in one-to-one correspondence with representations of the associated crossed product. This correspondence is developed in Sections 2.3 and 2.4. This correspondence is crucial to understanding crossed products and suggests that it is very profitable to view a crossed product as the C^*-algebra generated by a universal covariant representation. We validate this approach in Section 2.6.

2.1 Dynamical Systems

The study of dynamical systems is a subject unto itself, and good references abound. A nice summary of the connections to physics and operator algebras can be found in [136]. In this treatment, we'll just concentrate on what we need to get started.

Definition 2.1. A group G *acts* on the left of a set X if there is a map

$$(s, x) \mapsto s \cdot x \tag{2.1}$$

from $G \times X \to X$ such that for all $s, r \in G$ and $x \in X$

$$e \cdot x = x \quad \text{and} \quad s \cdot (r \cdot x) = sr \cdot x.$$

If G is a topological group and X a topological space, then we say the action is continuous if (2.1) is continuous from $G \times X$ to X.[1] In this case, X is called a *left G-space* and the pair (G, X) is called a *transformation group*. If both G and X are locally compact, then (G, X) is called a *locally compact transformation group*, and X is called a locally compact G-space. A right G-space is defined analogously.

Since we are most concerned with left G-spaces, we will assume here that group actions are on the left unless indicated otherwise.[2]

Example 2.2 ([136, §1]). Consider an ordinary autonomous differential equation of the form

$$\begin{cases} \mathbf{x}' = f(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{2.2}$$

for a function $f : \mathbf{R}^n \to \mathbf{R}^n$. Assuming f satisfies some mild smoothness conditions, there exists a unique solution $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ to (2.2) which depends continuously on the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. If $t \in \mathbf{R}$ and $\mathbf{z} \in \mathbf{R}^n$, then we can define

$$t \cdot \mathbf{z} := \mathbf{x}(t), \tag{2.3}$$

where $\mathbf{x}$ is the unique solution with $\mathbf{x}_0 = \mathbf{z}$. So we trivially have $0 \cdot \mathbf{z} = \mathbf{z}$. Now $s \cdot (t \cdot \mathbf{z}) = \mathbf{y}(s)$ where $\mathbf{y}'(s) = f\big(\mathbf{y}(s)\big)$ and $\mathbf{y}(0) = t \cdot \mathbf{z} = \mathbf{x}(t)$. Now it is straightforward to see that $\mathbf{y}(s) = \mathbf{x}(t+s)$. In other words, $s \cdot (t \cdot \mathbf{z}) = (s+t) \cdot \mathbf{z}$. The continuity of the map $(t, \mathbf{z}) \mapsto t \cdot \mathbf{z}$ is an consequence of the continuous dependence of the solution to (2.2) on its initial conditions, and we get a transformation group $(\mathbf{R}, \mathbf{R}^n)$.

Example 2.3. Let $h \in \operatorname{Homeo}(X)$. Then $\mathbf{Z}$ acts on X by $n \cdot x := h^n(x)$ and $(\mathbf{Z}, X)$ is a transformation group.

Example 2.4. Suppose that G is a locally compact group and H a closed subgroup. Then H is locally compact and acts on G by left translation:

$$h \cdot s := hs.$$

Then (H, G) is a locally compact transformation group.

Now let (G, X) be a locally compact transformation group. Then for each $s \in G$, $x \mapsto s \cdot x$ is in $\operatorname{Homeo}(X)$. (In particular, every $\mathbf{Z}$-space arises from a single homeomorphism as in Example 2.3.) Therefore we obtain a homomorphism

$$\alpha : G \to \operatorname{Aut} C_0(X) \tag{2.4}$$

defined by

$$\alpha_s(f)(x) := f(s^{-1} \cdot x). \tag{2.5}$$

[1] Often we say that the action is *jointly continuous* to emphasize the continuity on the product $G \times X$.

[2] Preferring left over right actions is pure prejudice. If X is a left G-space, then we can view X as a right G-space by defining $x \cdot s := s^{-1} \cdot x$. The same formula may be used to convert from right to left actions.

We certainly have $\alpha_s^{-1} = \alpha_{s^{-1}}$ and

$$\begin{aligned}\alpha_{sr}(f)(x) &= f(r^{-1}s^{-1} \cdot x) = \alpha_r(f)(s^{-1} \cdot x) \\ &= \alpha_s\big(\alpha_r(f)\big)(x).\end{aligned} \tag{2.6}$$

Therefore $\alpha_{sr} = \alpha_s \circ \alpha_r$ as required. (The computation in (2.6) explains the inverse in (2.5).)

Lemma 2.5. *Suppose that* (G, X) *is a locally compact transformation group and that* $\operatorname{Aut} C_0(X)$ *is given the point-norm topology. Then the associated homomorphism* (2.4) *of* G *into* $\operatorname{Aut} C_0(X)$ *is continuous.*

Proof. It suffices to see that $\|\alpha_s(f) - f\|_\infty \to 0$ as $s \to e$. If this were to fail, then there would be an $\epsilon > 0$, $s_i \to e$ and $x_i \in X$ such that

$$|f(s_i^{-1} \cdot x_i) - f(x_i)| \geq \epsilon \quad \text{for all } i. \tag{2.7}$$

Since f vanishes at infinity, $K := \{\, x \in X : |f(x)| \geq \epsilon/2 \,\}$ is compact. In order for (2.7) to hold, we must have either $x_i \in K$ or $s_i^{-1} \cdot x_i \in K$. Since we must eventually have the s_i in a compact neighborhood V of e, the x_i eventually lie in the compact set $V \cdot K = \{\, s \cdot x : s \in V \text{ and } x \in K \,\}$. Then we can assume that $x_i \to x_0$, where $x_0 \in V \cdot K$. But then $s_i^{-1} \cdot x_i \to x_0$ and we eventually contradict (2.7). □

Definition 2.6. A *C^*-dynamical system* is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a continuous homomorphism $\alpha : G \to \operatorname{Aut} A$. We say that (A, G, α) is *separable* if A is separable and G is second countable.

We'll usually shorten C^*-dynamical system to just "dynamical system". Notice that the continuity condition on α in Definition 2.6 amounts to the statement that $s \mapsto \alpha_s(a)$ is continuous for all $a \in A$. Lemma 2.5 simply states that a locally compact transformation group (G, X) gives rise to a dynamical system with A commutative. It turns out that all dynamical systems with A commutative arise from locally compact transformation groups.

Proposition 2.7. *Suppose that* $\big(C_0(X), G, \alpha\big)$ *is a dynamical system (with* X *locally compact). Then there is a transformation group* (G, X) *such that*

$$\alpha_s(f)(x) = f(s^{-1} \cdot x). \tag{2.8}$$

Proof. We saw in Lemma 1.33 on page 8 that there is a $h_s \in \operatorname{Homeo}(X)$ such that

$$\alpha_s(f)(x) = f\big(h_s(x)\big), \tag{2.9}$$

and that the map $s \mapsto h_s$ is continuous from G to $\operatorname{Homeo}(X)$ with the topology described in Definition 1.31 on page 8. Clearly $h_e = \mathrm{id}_X$ and $h_{sr} = h_r \circ h_s$. Therefore we get an action of G on X via

$$s \cdot x := h_s^{-1}(x) = h_{s^{-1}}(x).$$

The map $(s, x) \mapsto s \cdot x$ is continuous in view of Remark 1.32 on page 8. Therefore (G, X) is a transformation group such that (2.8) holds. □

Recall that the irreducible representations of $C_0(X)$ correspond exactly to the point-evaluations ev_x where $\mathrm{ev}_x(f) := f(x)$. Furthermore, the map $x \mapsto [\mathrm{ev}_x]$ is a homeomorphism of X onto $C_0(X)^\wedge$. If $\alpha : G \to \operatorname{Aut} C_0(X)$ is a dynamical system, then (2.8) amounts to

$$\mathrm{ev}_x \circ \alpha_s^{-1} = \mathrm{ev}_{s \cdot x},$$

and the G-action on $C_0(X)^\wedge$ is given by

$$s \cdot [\mathrm{ev}_x] = [\mathrm{ev}_x \circ \alpha_s^{-1}].$$

Notice that if $\alpha : G \to \operatorname{Aut} A$ is any dynamical system and $[\pi] \in \hat{A}$, then $[\pi \circ \alpha_s^{-1}] \in \hat{A}$, and depends only on the class of π.

Proposition 2.7 on the preceding page admits a significant generalization which will be used repeatedly in the sequel. A complete proof is given in [139], and we will not repeat that proof here.

Lemma 2.8 ([139, Lemma 7.1])**.** *Suppose that (A, G, α) is a dynamical system. Then there is a jointly continuous action of G on the spectrum $\hat{A}$ of A given by $s \cdot [\pi] := [\pi \circ \alpha_s^{-1}]$ called the* action induced by α*.*

Since $[\pi] \mapsto \ker \pi$ is a continuous, open surjection of $\hat{A}$ onto $\operatorname{Prim} A$ and $\ker(\pi \circ \alpha_s^{-1}) = \alpha_s(\ker \pi)$, there is jointly continuous action of G on $\operatorname{Prim} A$ given by

$$s \cdot P = \alpha_s(P) := \{\, \alpha_s(a) : a \in P \,\}. \tag{2.10}$$

Remark 2.9 (Degenerate Examples). It will be helpful to keep in mind that groups and C^*-algebras are by themselves degenerate examples of dynamical systems. Since the only (algebra) automorphism of $\mathbf{C}$ is the identity, every locally compact group G gives rise to a dynamical system $(\mathbf{C}, G, \mathrm{id})$. Similarly, every C^*-algebra A is associated to a dynamical system with G trivial: $(A, \{\, e \,\}, \mathrm{id})$.

2.2 Covariant Representations

From our point of view in these notes, C^*-dynamical systems are a natural algebraic framework in which to view and to generalize classical dynamical systems. The physical significance of these systems and their representations is described in [136]. Here we will limit the motivation to the idea that C^*-algebras and groups are profitably studied via representations on Hilbert space. The next definition gives a reasonable way to represent a dynamical system on a Hilbert space.

Definition 2.10. Let (A, G, α) be a dynamical system. Then a *covariant representation* of (A, G, α) is a pair (π, U) consisting of a representation $\pi : A \to B(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ on the same Hilbert space such that

$$\pi\bigl(\alpha_s(a)\bigr) = U_s \pi(a) U_s^*. \tag{2.11}$$

We say that (π, U) is a possibly degenerate covariant representation if π is a possibly degenerate representation.[3]

[3] Recall that by convention, all representations of C^*-algebras are presumed to be nondegenerate.

Example 2.11 (Degenerate examples). Obviously, covariant representations of degenerate dynamical systems such as $(A, \{e\}, \mathrm{id})$ correspond exactly to representations of A. Covariant representations of dynamical systems $(\mathbf{C}, G, \mathrm{id})$ correspond to unitary representations of G.

Example 2.12. Let G act on itself by left translation, and let $\mathrm{lt} : G \to \operatorname{Aut} C_0(G)$ be the associated dynamical system. Let $M : C_0(G) \to B\big(L^2(G)\big)$ be given by pointwise multiplication:

$$M(f)h(s) := f(s)h(s),$$

and let $\lambda : G \to U\big(L^2(G)\big)$ be the left-regular representation. Then (M, λ) is a covariant representation of $\big(C_0(G), G, \mathrm{lt}\big)$.

Example 2.13. Let $h \in \mathrm{Homeo}(\mathbf{T})$ be "rotation by θ": that is,

$$h(z) := e^{2\pi i\theta} z,$$

and let $\big(C(\mathbf{T}), \mathbf{Z}, \alpha\big)$ be the associated dynamical system:

$$\alpha_n(f)(z) = f(e^{-2\pi i n\theta} z).$$

(Although it might seem natural to think of h as "rotation through the angle $2\pi\theta$", the crucial feature is that h is rotation through θ of the circle. This action, as we shall see, has a very different character depending on whether θ is rational or irrational.)

(a) Let $M : C(\mathbf{T}) \to B\big(L^2(\mathbf{T})\big)$ be the representation given by pointwise multiplication:

$$M(f)h(z) := f(z)h(z).$$

Let $U : \mathbf{Z} \to U\big(L^2(\mathbf{T})\big)$ be the unitary representation given by

$$U_n h(z) := h(e^{-2\pi i n\theta} z).$$

Then it is not hard to check that (M, U) is a covariant representation of $\big(C(\mathbf{T}), \mathbf{Z}, \alpha\big)$.

(b) Now fix $w \in \mathbf{T}$ and let λ be the left-regular representation of $\mathbf{Z}$ on $L^2(\mathbf{Z})$. Define $\pi_w : C(\mathbf{T}) \to B\big(L^2(\mathbf{Z})\big)$ to be the representation

$$\pi_w(f)\xi(n) = f(e^{2\pi i n\theta} w)\xi(n).$$

Then (π_w, λ) is a covariant representation for each $w \in \mathbf{T}$.

In general, it is not obvious that there are any covariant representations of a given dynamical system. However, we know from the GNS theory (see [139, Appendix A.1]) that C^*-algebras have lots of representations. Given this, we can produce covariant representations of any system.

Example 2.14. Let $\rho : A \to B(\mathcal{H}_\rho)$ be any (possibly degenerate) representation of A on $\mathcal{H}_\rho$. Then define $\mathrm{Ind}_e^G \rho$ to be the pair $(\tilde{\rho}, U)$ of representations on the Hilbert space $L^2(G, \mathcal{H}_\rho) \cong L^2(G) \otimes \mathcal{H}_\rho$,[4] where

$$\tilde{\rho}(a)h(r) := \rho\big(\alpha_r^{-1}(a)\big)\big(h(r)\big) \quad \text{and} \quad U_s h(r) := h(s^{-1}r).$$

[4] See Appendix I.4.

Now compute:

$$\begin{aligned} U_s\tilde\rho(a)U_s^*h(r) &= \tilde\rho(a)U_s^*h(s^{-1}r) \\ &= \rho\big(\alpha_{s^{-1}r}^{-1}(a)\big)\big(U_s^*h(s^{-1}r)\big) \\ &= \rho\big(\alpha_r^{-1}\big(\alpha_s(a)\big)\big)\big(h(r)\big) \\ &= \tilde\rho\big(\alpha_s(a)\big)h(r). \end{aligned}$$

Thus, $\operatorname{Ind}_e^G \rho := (\tilde\rho, U)$ is a (possibly degenerate) covariant representation.

Example 2.15. The representation in part (b) of Example 2.13 on the previous page is $\operatorname{Ind}_e^G \mathrm{ev}_w$.

Remark 2.16. The representations constructed in Example 2.14 on the preceding page are called *regular representations* of (A, G, α). The notation Ind_e^G is meant to suggest that a regular representation is induced from the system $(A, \{e\}, \alpha|_{\{e\}})$. We will discuss a general theory of induced representations in Section 5.1 and make this suggestion formal in Remark 5.7 on page 156.

Lemma 2.17. *$\operatorname{Ind}_e^G \rho$ is nondegenerate if ρ is nondegenerate.*

Proof. Let $(\tilde\rho, U) = \operatorname{Ind}_e^G \rho$ as above. Let $\{e_i\}$ be an approximate identity in A. It suffices to see that $\tilde\rho(e_i)\xi \to \xi$ for all $\xi \in L^2(G, \mathcal{H})$. Since $\tilde\rho$ is norm decreasing and ρ is nondegenerate, we can assume that $\xi(s) = (f \otimes \rho(a)h)(s) := f(s)\rho(a)h$ for $f \in C_c(G)$, $a \in A$ and $h \in \mathcal{H}$. Since

$$\tilde\rho(e_i)\xi(r) = f(r)\rho\big(\alpha_r^{-1}(e_i)a\big)h,$$

it suffices to see that given $\epsilon > 0$ we eventually have

$$\|\alpha_r^{-1}(e_i)a - a\| < \epsilon \quad \text{for all } r \in \operatorname{supp} f.$$

If this is not true, then there is an $\epsilon_0 > 0$ and a subset of $\{e_i\}$ and $r_i \in \operatorname{supp} f$ such that after relabeling

$$\|\alpha_{r_i}^{-1}(e_i)a - a\| \ge \epsilon_0. \tag{2.12}$$

Since $\operatorname{supp} f$ is compact, we can assume that $r_i \to r$. But then

$$\begin{aligned} \|\alpha_{r_i}^{-1}(e_i)a - a\| &= \|\alpha_{r_i}^{-1}\big(e_i\alpha_{r_i}(a)\big) - \alpha_{r_i}^{-1}\big(\alpha_{r_i}(a)\big)\| \\ &= \|e_i\alpha_{r_i}(a) - \alpha_{r_i}(a)\| \\ &\le \|e_i\big(\alpha_{r_i}(a) - \alpha_r(a)\big)\| + \\ &\qquad \|e_i\alpha_r(a) - \alpha_r(a)\| + \|\alpha_r(a) - \alpha_{r_i}(a)\|. \end{aligned} \tag{2.13}$$

But (2.13) goes to 0 since a subnet of $\{e_i\}$ is still an approximate identity. This contradicts (2.12) and finishes the proof. □

Definition 2.18. Suppose that (A, G, α) is a dynamical system and that (π, U) and (ρ, V) are covariant representations on $\mathcal{H}$ and $\mathcal{V}$ respectively. Their *direct sum* $(\pi, U) \oplus (\rho, V)$ is the covariant representation $(\pi \oplus \rho, U \oplus V)$ on $\mathcal{H} \oplus \mathcal{V}$ given by

$(\pi \oplus \rho)(a) := \pi(a) \oplus \rho(a)$ and $(U \oplus V)_s := U_s \oplus V_s$. A subspace $\mathcal{H}' \subset \mathcal{H}$ is invariant for (π, U) if $\pi(a)(\mathcal{H}') \subset \mathcal{H}'$ and $U_s(\mathcal{H}') \subset \mathcal{H}'$ for all $a \in A$ and $s \in G$. If $\mathcal{H}'$ is invariant, then the restrictions π' of π to $\mathcal{H}'$ and U' of U to $\mathcal{H}'$ are representations and the covariant representation (π', U') on $\mathcal{H}'$ is called a *subrepresentation* of (π, U). We call (π, U) irreducible if the only *closed* invariant subspaces are the trivial ones: $\{ 0 \}$ and $\mathcal{H}$. Finally, we say that (π, U) and (ρ, V) are *equivalent* if there is a unitary $W : \mathcal{H} \to \mathcal{V}$ such that

$$\rho(a) = W\pi(a)W^* \quad \text{and} \quad V_s = WU_sW^* \quad \text{for all } a \in A \text{ and } s \in G.$$

Remark 2.19. If $\mathcal{V} \subset \mathcal{H}$ is invariant for (π, U), then it is not hard to check that $\mathcal{V}^\perp$ is also invariant. Thus if (π', U') and (π'', U'') are the subrepresentations corresponding to $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively, then $(\pi, U) = (\pi', U') \oplus (\pi'', U'')$. In particular, (π, U) is irreducible if and only if it is not equivalent to the direct sum of two nontrivial representations.

2.3 The Crossed Product

When G is abelian, we used the $*$-algebra $L^1(G)$ in Section 1.4 to recover the characters — that is, the irreducible representations of G. Here we want to construct a $*$-algebra — a C^*-algebra called the crossed product of A by G — from which we can recover exactly the covariant representations of a given dynamical system (A, G, α). This will include a special case, the group C^*-algebra as described in [139, Appendix C.3].

Example 2.20. Suppose $1_{\mathcal{H}} \in A \subset B(\mathcal{H})$ and that $u \in U(\mathcal{H})$ is such that $uAu^* \subset A$. For example, we could start with any automorphism of A, take a faithful representation of A and proceed as in Example 2.14 on page 45. Then $\alpha(a) = uau^*$ is an automorphism of A and $(A, \mathbf{Z}, \alpha)$ is a dynamical system. (Here $\alpha_n := \alpha^n$.) In this case, it would be natural to associate the C^*-subalgebra $C^*(A, \{ u \})$ of $B(\mathcal{H})$ generated by A and u to $(A, \mathbf{Z}, \alpha)$. I claim that

$$\mathcal{B} := \Big\{ \sum_{i \in \mathbf{Z}} a(i)u^i : a \in C_c(\mathbf{Z}, A) \Big\}$$

is a $*$-subalgebra of $B(\mathcal{H})$ and that $C^*(A, \{ u \})$ is the closure of $\mathcal{B}$.[5] For example, keeping in mind that all the sums in sight are finite, we can consider the product

$$\begin{aligned}\Big(\sum_i a(i)u^i\Big)\Big(\sum_j b(j)u^j\Big) &= \sum_{i,j} a(i)u^i b(j)u^j \\ &= \sum_i \sum_j a(i)\alpha_i\big(b(j)\big)u^{i+j}\end{aligned}$$

[5] If $1 \notin A$, then $\mathcal{B}$ is the subalgebra generated by *products* au with $a \in A$.

which, letting $k = j + i$, equals

$$= \sum_i \sum_k a(i)\alpha_i\big(b(k-i)\big)u^k$$
$$= \sum_k a * b(k)u^k,$$

where we have defined $a * b \in C_c(\mathbf{Z}, A)$ by

$$a * b(k) := \sum_i a(i)\alpha_i\big(b(k-i)\big). \tag{2.14}$$

This shows that $\mathcal{B}$ is closed under multiplication. Since a similar computation shows that the adjoint of $\sum_i a(i)u^i$ is given by $\sum_k a^*(k)u^k$ where

$$a^*(k) := \alpha_k\big(a(-k)^*\big), \tag{2.15}$$

$\mathcal{B}$ is closed under taking adjoints, and $\mathcal{B}$ is a $*$-algebra as claimed. Since $1_{\mathcal{H}} \in A$, it follows easily that $C^*(A, \{\, u \,\})$ is simply the closure of $\mathcal{B}$.

The previous example, and some experience with the group C^*-algebra construction, suggests we might be able to construct a crossed product by starting with a $*$-algebra based on $C_c(G, A)$ with the multiplication and involution compatible with (2.14) and (2.15). If $f, g \in C_c(G, A)$, then $(s, r) \mapsto f(r)\alpha_r\big(g(r^{-1}s)\big)$ is in $C_c(G \times G, A)$, and Corollary 1.104 on page 37 guarantees that

$$f * g(s) := \int_G f(r)\alpha_r\big(g(r^{-1}s)\big)\, d\mu(r) \tag{2.16}$$

defines an element of $C_c(G, A)$ called the *convolution* of f and g. Using the properties of Haar measure, our "Poor Man's" vector-valued Fubini Theorem (Proposition 1.105 on page 37) and Lemma 1.92 on page 32 it is not hard to check that (2.16) is an associative operation: $f * (g * h) = (f * g) * h$. Similarly, a little more computation shows that

$$f^*(s) := \Delta(s^{-1})\alpha_s\big(f(s^{-1})^*\big) \tag{2.17}$$

is an involution on $C_c(G, A)$ making $C_c(G, A)$ a $*$-algebra. For example:

$$\begin{aligned}
f^* * g^*(s) &= \int_G f^*(r)\alpha_r\big(g^*(r^{-1}s)\big)\, d\mu(r) \\
&= \Delta(s^{-1}) \int_G \alpha_r\big(f(r^{-1})^*\big)\alpha_s\big(g(s^{-1}r)^*\big)\, d\mu(r) \\
&= \Delta(s^{-1}) \int_G \alpha_{sr}\big(f(r^{-1}s^{-1})^*\big)\alpha_s\big(g(r)^*\big)\, d\mu(r) \\
&= \Delta(s^{-1})\alpha_s\Big(\int_G g(r)\alpha_r\big(f(r^{-1}s^{-1})\big)\, d\mu(r)\Big)^* \\
&= \Delta(s^{-1})\alpha_s\big(g * f(s^{-1})\big)^* \\
&= (g * f)^*(s).
\end{aligned}$$

Furthermore
$$\|f\|_1 := \int_G \|f(s)\|\,d\mu(s) \tag{2.18}$$
is a norm on $C_c(G,A)$, and the properties of Haar measure guarantee that $\|f^*\|_1 = \|f\|_1$ and $\|f*g\|_1 \le \|f\|_1\|g\|_1$.

Definition 2.21. A $*$-homomorphism $\pi : C_c(G,A) \to B(\mathcal{H})$ is called a *$*$-representation* of $C_c(G,A)$ on $\mathcal{H}$. We say π is nondegenerate if
$$\{\,\pi(f)h : f \in C_c(G,A) \text{ and } h \in \mathcal{H}\,\}$$
spans a dense subset of $\mathcal{H}$. If $\|\pi(f)\| \le \|f\|_1$, then π is called L^1-norm decreasing.

Example 2.22. Since a locally compact group G is a degenerate dynamical system, $C_c(G)$ is a $*$-algebra with operations
$$f*g(s) = \int_G f(r)g(r^{-1}s)\,d\mu(s) \quad \text{and} \quad f^*(s) = \Delta(s^{-1})\overline{f(s^{-1})}.$$

Proposition 2.23. *Suppose that (π,U) is a (possibly degenerate) covariant representation of (A,G,α) on $\mathcal{H}$. Then*
$$\pi \rtimes U(f) := \int_G \pi\big(f(s)\big)U_s\,d\mu(s) \tag{2.19}$$
defines a L^1-norm decreasing $*$-representation of $C_c(G,A)$ on $\mathcal{H}$ called the integrated form *of (π,U). Furthermore, $\pi \rtimes U$ is nondegenerate if π is nondegenerate. We call $\pi \rtimes U$ the* integrated form *of (π,U).*

Proof. Notice that $s \mapsto U_s$ is strictly continuous by Corollary 1.99 on page 34, and $s \mapsto \pi\big(f(s)\big)$ is in $C_c\big(G,B(\mathcal{H})\big)$. Consequently, the integrand in (2.19) is in $C_c\big(G,B_s(\mathcal{H})\big)$, and $\pi \rtimes U(f)$ is defined by Lemma 1.101 on page 35. If h and k are unit vectors in $\mathcal{H}$, then (1.35) implies that $|\big(\pi \rtimes U(f)h \mid k\big)| \le \int_G |\big(\pi\big(f(s)\big)U_sh \mid k\big)|\,d\mu(s)$. The Cauchy-Schwarz inequality implies that $|\big(\pi\big(f(s)\big)U_sh \mid k\big)| \le \|f(s)\|$. Thus $|\big(\pi \rtimes U(f)h \mid k\big)| \le \|f\|_1$. Since h and k are arbitrary, $\|\pi \rtimes U(f)\| \le \|f\|_1$.

To see that $\pi \rtimes U$ is a $*$-homomorphism, we compute using Lemma 1.101 on page 35:
$$\begin{aligned}
\pi \rtimes U(f)^* &= \int_G \big(\pi(f(s))U_s\big)^*\,d\mu(s)\\
&= \int_G U_{s^{-1}}\pi\big(f(s)^*\big)\,d\mu(s)\\
&= \int_G U_s\pi\big(f(s^{-1})^*\big)\Delta(s^{-1})\,d\mu(s)\\
&= \int_G \pi\big(\alpha_s\big(f(s^{-1})^*\Delta(s^{-1})\big)\big)U_s\,d\mu(s)\\
&= \pi \rtimes U(f^*).
\end{aligned}$$

Similarly,

$$\begin{aligned}\pi \rtimes U(f * g) &= \int_G \int_G \pi\big(f(r)\alpha_r\big(g(r^{-1}s)\big)\big)U_s \,d\mu(r)\,d\mu(s)\\ &= \int_G \int_G \pi\big(f(r)\big)U_r\pi\big(g(r^{-1}s)\big)U_{r^{-1}s}\,d\mu(r)\,d\mu(s),\end{aligned}$$

which, by Fubini (Proposition 1.105 on page 37) and left-invariance, is

$$\begin{aligned}&= \int_G \int_G \pi\big(f(r)\big)U_r\pi\big(g(s)\big)U_s\,d\mu(s)\,d\mu(r)\\ &= \pi \rtimes U(f) \circ \pi \rtimes U(g).\end{aligned}$$

Now assume that π is nondegenerate. Let $h \in \mathcal{H}$ and $\epsilon > 0$. Then if $\{ e_i \}$ is an approximate identity for A, we have $\pi(e_i)h \to h$ in $\mathcal{H}$. Thus we can choose an element u of norm one in A such that $\|\pi(u)h - h\| < \epsilon/2$. Let V be a neighborhood of e in G such that $\|U_s h - h\| < \epsilon/2$ if $s \in V$. Let $\varphi \in C_c(G)$ be nonnegative with $\operatorname{supp}\varphi \subset V$ and integral one. Let $f(s) = (\varphi \otimes u)(s) := \varphi(s)u$. Then $f \in C_c(G, A)$ and if k is an element of norm one in $\mathcal{H}$,

$$\begin{aligned}\big|\big(\pi \rtimes U(f)h \mid k\big) - (h \mid k)\big| &= \Big|\int_G \varphi(s)\big(\pi(u)U_s h - h \mid k\big)\,ds\Big|\\ &\le \int_G \varphi(s)\big|\big(\pi(u)(U_s h - h) \mid k\big)\big|\,ds +\\ &\qquad \int_G \varphi(s)\big|\big(\pi(u)h - h \mid k\big)\big|\,ds\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since k is arbitrary, it follows that $\|\pi \rtimes U(f)h - h\| \le \epsilon$ and therefore $\pi \rtimes U$ is nondegenerate. □

Example 2.24. Notice that the map in Example 2.20 on page 47 sending $a \in C_c(\mathbf{Z}, A)$ to $\sum_i a(i)u^i$ is simply the integrated form of (id, u) where id is the identity representation of A and u is viewed as a representation of $\mathbf{Z}$ in the obvious way: $u_k := u^k$.

Example 2.25. In the case of a locally compact group G and a unitary representation U, the integrated form is:

$$\mathrm{id} \rtimes U(z) := \int_G z(s)U_s\,d\mu(s) \quad \text{for all } z \in C_c(G).$$

Traditionally, $\mathrm{id} \rtimes U$ is shortened to just U. That is, the same letter is used both for a unitary representation and its integrated form.

The following maps will be of considerable importance in Section 2.4 and in Proposition 2.34 on page 54 in particular. For each $r \in G$ let $i_G(r) : C_c(G, A) \to C_c(G, A)$ be defined by

$$i_G(r)f(s) := \alpha_r\big(f(r^{-1}s)\big). \tag{2.20}$$

Note that if (π, U) is a covariant representation of (A, G, α), then

$$\begin{aligned}\pi \rtimes U\big(i_G(r)f\big) &= \int_G \pi\big(i_G(r)f(s)\big)U_s\,d\mu(s)\\ &= \int_G \pi\big(\alpha_r\big(f(r^{-1}s)\big)\big)U_s\,d\mu(s)\\ &= \int_G \pi\big(\alpha_r\big(f(s)\big)\big)U_{rs}\,d\mu(s)\\ &= U_r \circ \pi \rtimes U(f).\end{aligned} \tag{2.21}$$

Lemma 2.26. *Let ρ be a faithful representation of A on $\mathcal{H}$ and let $\operatorname{Ind}_e^G \rho = (\tilde\rho, U)$ be the corresponding regular representation. Suppose that $f \in C_c(G, A)$ and $f \neq 0$. Then $\tilde\rho \rtimes U(f) \neq 0$. That is, the integrated form of a regular representation corresponding to a faithful representation of A is faithful on $C_c(G, A)$.*

Proof. There is a $r \in G$ such that $f(r) \neq 0$. Since (2.21) implies that $\|\tilde\rho \rtimes U(f)\| = \|\tilde\rho \rtimes U\big(i_G(r^{-1})f\big)\|$, we can replace f by $i_G(r^{-1})f$ and assume that $r = e$. Since ρ is faithful, there are vectors h and k in $\mathcal{H}$ such that

$$\big(\rho\big(f(e)\big)h \mid k\big) \neq 0.$$

We can find a neighborhood V of e such that $s, r \in V$ implies

$$\big|\big(\rho\big(\alpha_r^{-1}\big(f(s)\big)\big)h \mid k\big) - \big(\rho\big(f(e)\big)h \mid k\big)\big| < \frac{|\big(\rho\big(f(e)\big)h \mid k\big)|}{3}.$$

Choose $\varphi \in C_c^+(G)$ with support contained in symmetric neighborhood W of e such that $W^2 \subset V$, and such that

$$\int_G \int_G \varphi(s^{-1}r)\varphi(r)\,d\mu(r)\,d\mu(s) = 1.$$

Now define ξ and η in $L^2(G, \mathcal{H})$ by

$$\xi(s) := \varphi(s)h \quad \text{and} \quad \eta(s) := \varphi(s)k.$$

Then

$$\begin{aligned}&|\big(\tilde\rho \rtimes U(f)\xi \mid \eta\big) - \big(\rho\big(f(e)\big)h \mid k\big)|\\ &\quad= \Big|\int_G \int_G \big(\rho\big(\alpha_r^{-1}\big(f(s)\big)\big)\xi(s^{-1}r) \mid \eta(r)\big)\,d\mu(s)\,d\mu(r)\\ &\qquad\qquad - \big(\rho\big(f(e)\big)h \mid k\big)\Big|\\ &\quad\le \int_G \int_G \varphi(s^{-1}r)\varphi(r)\Big|\big(\big(\rho\big(\alpha_r^{-1}\big(f(s)\big)\big)h \mid k\big) -\\ &\qquad\qquad \big(\rho\big(f(e)\big)h \mid k\big)\big)\Big|\,d\mu(s)\,d\mu(r)\\ &\quad< \frac{|\big(\rho\big(f(e)\big)h \mid k\big)|}{2}.\end{aligned}$$

Now it follows that $\tilde\rho \rtimes U(f) \neq 0$. □

We define the crossed product associated to (A, G, α) as a completion of $C_c(G, A)$ as described in Section 1.5.1.

Lemma 2.27. *Suppose that (A, G, α) is a dynamical system and that for each $f \in C_c(G, A)$ we define*

$$\|f\| := \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \textit{ is a (possibly degenerate) covariant representation of } (A, G, \alpha)\}. \tag{2.22}$$

Then $\|\cdot\|$ is a norm on $C_c(G, A)$ called the universal norm. *The universal norm is dominated by the $\|\cdot\|_1$-norm, and the completion of $C_c(G, A)$ with respect to $\|\cdot\|$ is a C^*-algebra called the* crossed product *of A by G and is denoted by $A \rtimes_\alpha G$.*

Remark 2.28. Some might find the supremum in (2.22) suspicious because the collection of covariant representations is not clearly a set. Various finesses for this apparent defect are employed in the literature. For example, it would be possible to consider only covariant representations on a fixed Hilbert space of suitably large dimension and argue that all representations are equivalent to a (possibly degenerate) representation on this space. However, the collection of values in (2.22) is a subclass of the set $\mathbf{R}$ of real numbers and the separation axioms of set theory guarantee that a subclass of a set is a set and that we are taking the supremum of a bounded *set* of real numbers [84, §1.1]. Thus here, and in the sequel, we will not worry about taking such supremums.

Proof. Once we are satisfied that we have made sense of the supremum in (2.22), Proposition 2.23 on page 49 and Lemma 2.26 on the previous page imply that $0 < \|f\| \leq \|f\|_1 < \infty$ (provided $f \neq 0$). Now it easy to see that $\|\cdot\|$ is a norm on $C_c(G, A)$ such that $\|f^* * f\| = \|f\|^2$. Therefore the completion is a C^*-algebra. □

Remark 2.29. Since the universal norm is a norm on $C_c(G, A)$, we can view $C_c(G, A)$ as a $*$-subalgebra of $A \rtimes_\alpha G$. Therefore we will rarely distinguish between an element of $C_c(G, A)$ and its image in $A \rtimes_\alpha G$.

Remark 2.30. Let $\mathcal{B} \subset C_c(G, A)$ be a $*$-subalgebra which is dense in the inductive limit topology. This means that given $f \in C_c(G, A)$ there is a compact set K such that for all $\epsilon > 0$ there exists a $b \in \mathcal{B}$ such that $\operatorname{supp} b \subset K$ and $\|b - f\|_\infty < \epsilon$. This immediately implies that $\mathcal{B}$ is $\|\cdot\|_1$-norm dense in $C_c(G, A)$ and therefore dense in $A \rtimes_\alpha G$ as well.

Lemma 2.31. *Let (π, U) be a (possibly degenerate) covariant representation of (A, G, α) on $\mathcal{H}$. Let*

$$\mathcal{V} := \overline{\operatorname{span}}\{\, \pi(a)h : a \in A \textit{ and } h \in \mathcal{H} \,\}$$

be the essential subspace *of π, and let $\operatorname{ess}\pi$ be the corresponding subrepresentation. Then $\mathcal{V}$ is also invariant for U, and if U' is the corresponding subrepresentation, then $(\operatorname{ess}\pi, U')$ is a nondegenerate covariant representation on $\mathcal{V}$. For all $f \in C_c(G, A)$,*

$$\|(\operatorname{ess}\pi) \rtimes U'(f)\| = \|\pi \rtimes U(f)\|.$$

In particular,

$$\|f\| = \sup\{ \|\pi \rtimes U(f)\| : \\ (\pi, U) \text{ is a nondegenerate covariant representation of } (A, G, \alpha).\}$$

Proof. Since $U_s\pi(a)h = \pi\big(\alpha_s(a)\big)U_sh$, it is clear that $\mathcal{V}$ is invariant for U, and that $(\operatorname{ess}\pi, U')$ is a nondegenerate covariant representation. Note that $\pi = \operatorname{ess}\pi \oplus 0$. Thus if $U = U' \oplus U''$, then $\pi \rtimes U = (\operatorname{ess}\pi) \rtimes U' \oplus 0$, and the rest is straightforward. □

It should come as no surprise that crossed products in which the algebra A is commutative are considerably more tractable than the general case. Since there is a one-to-one correspondence between dynamical systems with A commutative and transformations groups, such crossed products are called *transformation group C^*-algebras.* Moreover, it is possible to describe the $*$-algebra structure on $C_c\big(G, C_0(X)\big)$ in terms of functions on $G \times X$. If we agree to identify a function on $G \times X$ with the obvious function from G to functions on X, then it is not hard to prove that we have inclusions

$$C_c(G \times X) \subset C_c\big(G, C_c(X)\big) \subset C_c\big(G, C_0(X)\big). \tag{2.23}$$

If $f \in C_c\big(G, C_0(X)\big)$, then since point evaluation is a homomorphism from $C_0(X)$ to $\mathbf{C}$,

$$\int_G f(s)\,d\mu(s)(x) = \int_G f(s)(x)\,d\mu(s). \tag{2.24}$$

Using (2.24) and the formulas for convolution and involution on $C_c\big(G, C_0(X)\big)$, it is not hard to see that $C_c(G \times X)$ is a $*$-subalgebra. It follows easily (using the first inclusion in (2.23) and Lemma 1.87 on page 29) that $C_c(G \times X)$ is dense in $C_c\big(G, C_0(X)\big)$ and therefore in $C_0(X) \rtimes_\alpha G$. The formulas for convolution and involution on $C_c(G \times X)$ are

$$f * g(s, x) = \int_G f(r, x) g(r^{-1}s, r^{-1} \cdot x)\,d\mu(r), \text{ and} \tag{2.25}$$

$$f^*(s, x) = \Delta(s^{-1})\overline{f(s^{-1}, s^{-1} \cdot x)}. \tag{2.26}$$

(In particular, the formula for convolution is a scalar valued integral.)

Remark 2.32. We mentioned the inclusion of $C_c(G \times X)$ into $C_c\big(G, C_c(X)\big)$ only to make applying Lemma 1.87 on page 29 a bit easier. It would be tempting to guess that this inclusion is always an equality. But this is not the case. Let $G = \mathbf{T}$ and $X = \mathbf{R}$. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be defined by

$$\varphi(x) := \begin{cases} 0 & \text{if } |x| \geq 1, \\ 1 + x & \text{if } -1 \leq x \leq 0, \text{ and} \\ 1 - x & \text{if } 0 \leq x \leq 1. \end{cases}$$

Then it is not hard to check that $f(x+iy)(r) := |y|\varphi(|y|r)$ is in $C_c\big(\mathbf{T}, C_c(\mathbf{R})\big)$. Since

$$\operatorname{supp} f(x+iy) = \begin{cases} \{\,0\,\} & \text{if } y = 0 \text{, and} \\ [-1/|y|, 1/|y|] & \text{otherwise,} \end{cases}$$

the support of f as a function on $\mathbf{T} \times \mathbf{R}$ is unbounded and therefore not compact.

Example 2.33. In the degenerate case where our dynamical system reduces to $(A, \{\,e\,\}, \mathrm{id})$, then the crossed product gives us back A. In the case $(\mathbf{C}, G, \mathrm{id})$, then covariant representations correspond exactly to unitary representations of G, and the crossed product is the *group C^*-algebra* $C^*(G)$ as defined in [139, C.3]. (We took a slightly different approach in [139, C.3], but [139, Corollary C.18] shows the approaches arrive at the same completion of $C_c(G)$.)

2.4 Representations of the Crossed Product

Except in special cases, the crossed product $A \rtimes_\alpha G$ does not contain a copy of either A or G. However, the multiplier algebra $M(A \rtimes_\alpha G)$ does. Recall that if A is a C^*-algebra, then $M(A)$ is $\mathcal{L}(A_A)$ — that is, the collection of adjointable operators from A to itself [139, §2.3]. When convenient, we will view A as a subalgebra of $M(A)$ — $a \in A$ is identified with the operator $b \mapsto ab$. The unitary group of $M(A)$ is denoted by $UM(A)$.

If we view $C_c(G, A)$ as a $*$-subalgebra of $M(A \rtimes_\alpha G)$, then a $T \in M(A \rtimes_\alpha G)$ may, or may not, map $C_c(G, A)$ into itself. In practice however, a multiplier T is usually defined by first defining it as a map from $C_c(G, A)$ to itself, and then showing it is bounded with respect to the universal norm so that it extends to a map also called T from $A \rtimes_\alpha G$ to itself. It defines a multiplier provided we can find an adjoint T^* which is characterized by

$$T(a)^*b = \big\langle T(a)\,,\, b\big\rangle_{A\rtimes_\alpha G} = \big\langle a\,,\, T^*(b)\big\rangle_{A\rtimes_\alpha G} = a^*T^*(b)$$

for all $a, b \in A$. The next proposition is case in point.

Proposition 2.34. *Suppose that $\alpha : G \to \operatorname{Aut} A$ is a dynamical system. Then there is a nondegenerate faithful homomorphism*

$$i_A : A \to M(A \rtimes_\alpha G)$$

and an injective strictly continuous unitary valued homomorphism

$$i_G : G \to UM(A \rtimes_\alpha G)$$

such that for $f \in C_c(G, A)$, $r, s \in G$ and $a \in A$ we have

$$i_G(r)f(s) = \alpha_r\big(f(r^{-1}s)\big) \quad \text{and} \quad i_A(a)f(s) = af(s).$$

Moreover (i_A, i_G) is covariant in that

$$i_A\big(\alpha_r(a)\big) = i_G(r)i_A(a)i_G(r)^*.$$

If (π, U) is nondegenerate, then

$$(\pi \rtimes U)^{-}\big(i_A(a)\big) = \pi(a) \quad \textit{and} \quad (\pi \rtimes U)^{-}\big(i_G(s)\big) = U_s.$$

Proof. We already considered $i_G(r)$ in Section 2.3 and it follows from (2.21) that for all $r \in G$

$$\|i_G(r)f\| = \|f\|.$$

Thus we can extend $i_G(r)$ to a map of $A \rtimes_\alpha G$ to itself, and

$$\big(i_G(r)f\big)^* * g(t) = \int_G \alpha_{sr}\big(f(r^{-1}s^{-1})^*\big)\Delta(s^{-1})\alpha_s\big(g(s^{-1}t)\big)\,d\mu(s)$$

which, since $\Delta(s^{-1})\,d\mu(s)$ is a right Haar measure on G, is

$$\begin{aligned} &= \int_G \alpha_s\big(f(s^{-1})^*\big)\Delta(s^{-1})\alpha_{sr^{-1}}\big(g(rs^{-1}t)\big)\,d\mu(s) \\ &= \int_G f^*(s)\alpha_s\big(i_G(r^{-1})g(s^{-1}t)\big)\,d\mu(s) \\ &= f^* * i_G(r^{-1})g(t). \end{aligned}$$

Thus each $i_G(r)$ is adjointable with adjoint $i_G(r)^* = i_G(r^{-1})$. Since we certainly have $i_G(rs) = i_G(r) \circ i_G(s)$ and $i_G(r)^{-1} = i_G(r^{-1})$, i_G is a unitary-valued homomorphism into $UM(A \rtimes_\alpha G)$. To show that i_G is strictly continuous, fix $f \in C_c(G, A)$, a compact neighborhood W of e in G, and let $K := \operatorname{supp} f$. Notice that as long as $r \in W$, then

$$\operatorname{supp} i_G(r)f \subset WK.$$

If $\epsilon > 0$, then the uniform continuity of f implies that we can choose $V \subset W$ such that $r \in V$ implies

$$\|f(r^{-1}s) - f(s)\| < \frac{\epsilon}{2\mu(WK)} \quad \text{for all } s \in G.$$

Since f has compact support, we can shrink V if need be so that $r \in V$ also implies

$$\|\alpha_r\big(f(s)\big) - f(s)\| < \frac{\epsilon}{2\mu(WK)} \quad \text{for all } s \in G.$$

Since

$$\|i_G(r)f(s) - f(s)\| \le \|\alpha_r\big(f(r^{-1}s) - f(s)\big)\| + \|\alpha_r\big(f(s)\big) - f(s)\|,$$

it follows that $\|i_G(r)f - f\| \le \|i_G(r)f - f\|_1 < \epsilon$. Therefore $r \mapsto i_G(r)$ is strongly continuous, and i_G is strictly continuous by Corollary 1.99 on page 34.

The argument for i_A is similar. If (π, U) is a covariant representation of (A, G, α), then for $f \in C_c(G, A)$,

$$\pi \rtimes U\big(i_A(a)f\big) = \pi(a) \circ \pi \rtimes U(f), \tag{2.27}$$

and it follows that $\|i_A(a)f\| \leq \|a\|\|f\|$. Therefore $i_A(a)$ extends to map from $A \rtimes_\alpha G$ to itself. An easy computation shows that

$$\big(i_A(a)f\big)^* * g = f^* * i_A(a^*)g.$$

Therefore $i_A(a)$ is adjointable with adjoint $i_A(a^*)$, and i_A is a homomorphism of A into $M(A \rtimes_\alpha G)$. The covariance condition can be checked by applying both sides to $f \in C_c(G, A)$.

If (π, U) is nondegenerate, then (2.27) implies that $(\pi \rtimes U)^-\big(i_A(a)\big) = \pi(a)$, and the corresponding assertion for $i_G(s)$ follows from (2.21).

To see that i_A and i_G are injective, let $\rho : A \to B(\mathcal{H})$ be a faithful representation of A, and set $(\tilde{\rho}, U) = \operatorname{Ind}_e^G \rho$. Then $\tilde{\rho}$ is faithful and certainly $U_s \neq \mathrm{id}$ if $s \neq e$. Since $\tilde{\rho}(a) = (\tilde{\rho} \rtimes U)^- \circ i_A(a)$, it follows that $i_A(a)$ is faithful. Similarly $U_s = (\tilde{\rho} \rtimes U)^- \circ i_G(s)$ shows $i_G(s) \neq \mathrm{id}$ if $s \neq e$.

To see that i_A is nondegenerate, note that elementary tensors of the form $\varphi \otimes ab$ span a dense subalgebra of $A \rtimes_\alpha G$ by Lemma 1.87 on page 29, and are also in $i_A(A) \cdot A \rtimes_\alpha G$. ∎

Lemma 2.35. *There is a homomorphism $\tilde{\imath}_G : C^*(G) \to M(A \rtimes_\alpha G)$ such that*

$$\tilde{\imath}_G(z) = \int_G z(s) i_G(s)\, d\mu(s) \quad \text{for all } z \in C_c(G).$$

Just as for unitary representations (c.f., Example 2.25 on page 50), we write $i_G(z)$ in place of $\tilde{\imath}_G(z)$.

Proof. If $z \in C_c(G)$, then $\tilde{\imath}_G(z)$ is a well-defined element of $M(A \rtimes_\alpha G)$ by Lemma 1.101 on page 35. Just as in Proposition 2.23 on page 49, $\tilde{\imath}_G$ is a homomorphism on $C_c(G)$. For example,

$$\begin{aligned}\tilde{\imath}(z * w) &= \int_G z * w(s) i_G(s)\, d\mu(s) \\ &= \int_G \int_G z(r) w(r^{-1}s)\, d\mu(r) i_G(s)\, d\mu(s)\end{aligned}$$

which, by our Fubini result Proposition 1.105 and using Lemma 1.101 to move $i_G(s)$ through the integral, is

$$\begin{aligned}&= \int_G \int_G z(r) w(r^{-1}s) i_G(s)\, d\mu(s)\, d\mu(r) \\ &= \int_G \int_G z(r) w(s) i_G(rs)\, d\mu(r)\, d\mu(s) \\ &= \tilde{\imath}(z)\tilde{\imath}(w).\end{aligned}$$

All that remains to show is that $\tilde{\imath}_G$ is bounded with respect to the universal norm on $C^*(G)$. But if π is a nondegenerate faithful representation of $A \rtimes_\alpha G$, then $\bar{\pi}$ is faithful on $M(A \rtimes_\alpha G)$. Let $U_s := \bar{\pi}\big(i_G(s)\big)$. Then $s \mapsto U_s$ is a unitary-valued homomorphism into $U(\mathcal{H}_\pi)$. To see that U is a unitary representation of G, we need to see that $s \mapsto U_s h$ is continuous for all $h \in \mathcal{H}_\pi$. Since π is nondegenerate, it suffices to consider $h = \pi(f)k$ for $f \in C_c(G, A)$. Then $U_s \pi(f)k = \pi\big(i_G(s)f\big)k$. Since $i_G(s)f$ is continuous in s, it follows that U is strongly continuous and therefore a representation. Therefore Lemma 1.101 on page 35 implies that

$$\begin{aligned}\bar\pi\big(\tilde\imath_G(z)\big) &= \bar\pi\Big(\int_G z(s) i_G(s)\, d\mu(s)\Big)\\ &= \int_G z(s) U_s \, d\mu(s)\\ &= U(z).\end{aligned}$$

Thus

$$\|\tilde\imath_G(z)\| = \|\bar\pi\big(i_G(z)\big)\| = \|U(z)\| \le \|z\|. \qquad \square$$

Corollary 2.36. *Suppose that* $\alpha : G \to \operatorname{Aut} A$ *is a dynamical system. Let* $a \in A$, $z \in C_c(G)$ *and* $g, h \in C_c(G, A)$. *Then* $i_A(a) i_G(z)$, $\int_G i_A\big(g(r)\big) i_G(r)(h)\, d\mu(r)$ *and* $\int_G i_A\big(g(r)\big) i_G(r)\, d\mu(r)$ *are in* $C_c(G, A) \subset A \rtimes_\alpha G \subset M(A \rtimes_\alpha G)$. *In fact,*

$$i_A(a) i_G(z) = z \otimes a, \tag{2.28}$$

$$\int_G i_A\big(g(s)\big) i_G(s)(h)\, d\mu(s) = g * h, \text{ and} \tag{2.29}$$

$$\int_G i_A\big(g(s)\big) i_G(s)\, d\mu(s) = g. \tag{2.30}$$

Proof. Notice that if $T \in M(A \rtimes_\alpha G)$ and $(\pi \rtimes U)^{-}(T) = 0$ for all nondegenerate covariant pairs, then $T = 0$. (For example, $Tf = 0$ for all $f \in A \rtimes_\alpha G$ by Lemma 2.31 on page 52.) But if (π, U) is nondegenerate,

$$(\pi \rtimes U)^{-}\Big(\int_G i_A\big(g(s)\big) i_G(s)\, d\mu(s)\Big) = \int_G \pi\big(g(s)\big) U_s \, d\mu(s) = \pi \rtimes U(g).$$

This proves (2.30), and (2.28) is a special case. Finally,

$$g * h = \int_G i_A\big(g(s)\big) i_G(s)\, d\mu(s) h = \int_G i_A\big(g(s)\big) i_G(s)(h)\, d\mu(s)$$

by Lemma 1.101 on page 35. $\square$

Definition 2.37. Suppose that $\alpha : G \to \operatorname{Aut} A$ is a dynamical system and that X is a Hilbert B-module. Then a *covariant homomorphism* of (A, G, α) into $\mathcal{L}(\mathsf{X})$ is a pair (π, u) consisting of a homomorphism $\pi : A \to \mathcal{L}(\mathsf{X})$ and a strongly continuous unitary-valued homomorphism $u : G \to U\mathcal{L}(\mathsf{X})$ such that

$$\pi\big(\alpha_s(a)\big) = u_s \pi(a) u_s^*. \tag{2.31}$$

We say that (π, u) is nondegenerate if π is nondegenerate.

Remark 2.38. If X is a Hilbert space $\mathcal{H}$, then a (nondegenerate) covariant homomorphism into $\mathcal{L}(\mathcal{H}) = B(\mathcal{H})$ is just a (nondegenerate) covariant representation. Corollary 1.99 on page 34 implies that the strong continuity of u in Definition 2.37 on the previous page is equivalent to strict continuity as a map into $\mathcal{L}(\mathsf{X}) = M(\mathcal{K}(\mathsf{X}))$. In fact, a covariant homomorphism can equally well be thought of as maps into $M(B)$ for a C^*-algebra B. Since $M(B) = \mathcal{L}(B_B)$ and $\mathcal{L}(\mathsf{X}) = M(\mathcal{K}(\mathsf{X}))$ there is no real difference.

Proposition 2.39. *Suppose that $\alpha : G \to \operatorname{Aut} A$ is a dynamical system and that X is a Hilbert B-module. If (π, u) is a covariant homomorphism of (A, G, α) into $\mathcal{L}(\mathsf{X})$, then Lemma 1.101 on page 35 implies that the* integrated form

$$\pi \rtimes u(f) := \int_G \pi\big(f(s)\big) u_s \, d\mu(s)$$

is a well-defined operator in $\mathcal{L}(\mathsf{X})$, and $\pi \rtimes u$ extends to a homomorphism of $A \rtimes_\alpha G$ into $\mathcal{L}(\mathsf{X})$ which is nondegenerate whenever π is nondegenerate. In this case, $(\pi \rtimes u)^-\big(i_A(a)\big) = \pi(a)$ and $(\pi \rtimes u)^-\big(i_G(s)\big) = u_s$.

Conversely, if $L : A \rtimes_\alpha G \to \mathcal{L}(\mathsf{X})$ is a nondegenerate homomorphism, then there is a nondegenerate covariant homomorphism (π, u) of (A, G, α) into $\mathcal{L}(\mathsf{X})$ such that $L = \pi \rtimes u$. In fact, if $\bar{L}$ is the canonical extension of L to $M(A \rtimes_\alpha G)$, then

$$u_s := \bar{L}\big(i_G(s)\big) \quad \text{and} \quad \pi(a) := \bar{L}\big(i_A(a)\big). \tag{2.32}$$

Proof. The map $s \mapsto u_s$ is strictly continuous into $\mathcal{L}(\mathsf{X}) = M(\mathcal{K}(\mathsf{X}))$ by Corollary 1.99 on page 34, and so $s \mapsto \pi\big(f(s)\big)u_s$ is a strictly continuous map for each $f \in C_c(G, A)$. Thus $\pi \rtimes u(f)$ is an operator in $\mathcal{L}(\mathsf{X})$ by Lemma 1.101 on page 35.

Let $\rho : \mathcal{K}(\mathsf{X}) \to B(\mathcal{H}_\rho)$ be a faithful nondegenerate representation of $\mathcal{K}(\mathsf{X})$, and let $\bar{\rho}$ be the canonical (and faithful) extension to $\mathcal{L}(\mathsf{X})$. Let $\Pi(a) := \bar{\rho}\big(\pi(a)\big)$ and $U_s := \bar{\rho}(u_s)$. We claim that $s \mapsto U_s$ is strongly continuous from G into $U(\mathcal{H}_\rho)$. Since ρ is nondegenerate, it suffices to show that $s \mapsto U_s\big(\rho(T)h\big)$ is continuous when $h \in \mathcal{H}_\rho$ and $T \in \mathcal{K}(\mathsf{X})$). But $s \mapsto u_sT$ is continuous since u is strictly continuous, and $U_s\big(\rho(T)h\big) = \rho(u_sT)h$ is continuous in s. It now follows easily that (Π, U) is a covariant representation and Lemma 1.101 on page 35 implies

$$\begin{aligned}
\|\bar{\rho}\big(\pi \rtimes u(f)\big)\| &= \Big\|\bar{\rho}\Big(\int_G \pi\big(f(s)\big)u_s \, d\mu(s)\Big)\Big\| \\
&= \Big\|\int_G \Pi\big(f(s)\big)U_s \, d\mu(s)\Big\| \\
&= \|\Pi \rtimes U(f)\| \\
&\le \|f\|.
\end{aligned}$$

It follows that $\pi \rtimes u = \bar{\rho}^{-1} \circ (\Pi \rtimes U)$ is a homomorphism of $C_c(G, A)$ into $\mathcal{L}(\mathsf{X})$ which is bounded with respect to the universal norm. Therefore $\pi \rtimes u$ extends to a homomorphism of $A \rtimes_\alpha G$ into $\mathcal{L}(\mathsf{X})$ as claimed. The proof that $\pi \rtimes u$ is nondegenerate when π is proceeds exactly as in Proposition 2.23 on page 49, and the statements about $(\pi \rtimes u)^-$ follow as in Proposition 2.34 on page 54.

To prove the converse, suppose that $L : A \rtimes_\alpha G \to \mathcal{L}(\mathsf{X})$ is a nondegenerate homomorphism, and let π and u be defined as in (2.32). Since i_G is strictly continuous, it is straightforward to check that u is strongly continuous. Since i_A and L are nondegenerate it follows that if $\{ e_i \}$ is an approximate identity in A, then $i_A(e_i)$ converges to strictly to 1 in $M(A \rtimes_\alpha G)$. Therefore $\pi(e_i) = \bar{L}\bigl(i_A(e_i)\bigr)$ converges strictly to 1_{X} in $\mathcal{L}(\mathsf{X})$, and π is nondegenerate. Since the covariance condition is straightforward to check, it follows that (π, u) is a nondegenerate covariant homomorphism, and the first part of this proof shows the integrated form $\pi \rtimes u$ is a nondegenerate homomorphism into $\mathcal{L}(\mathsf{X})$. Since both $\pi \rtimes u$ and L are nondegenerate, we can apply Lemma 1.101 to conclude that for each $a \in A$ and $z \in C_c(G)$ that

$$\begin{aligned}
\pi \rtimes u\bigl(i_A(a)i_G(z)\bigr) &= \pi(a) \circ (\pi \rtimes u)^{-}\bigl(i_G(z)\bigr) \\
&= \bar{L}\bigl(i_A(a)\bigr) \int_G z(s) u_s \, d\mu(s) \\
&= \bar{L}\bigl(i_A(a)\bigr) \int_G z(s) \bar{L}\bigl(i_G(s)\bigr) \, d\mu(s) \\
&= \bar{L}\bigl(i_A(a)\bigr) \bar{L}\bigl(i_G(z)\bigr) \\
&= L\bigl(i_A(a) i_G(z)\bigr).
\end{aligned}$$

Now Lemma 1.87 on page 29 and Corollary 2.36 on page 57 imply that the span of elements of the form $i_A(a)i_G(z)$ are dense in $A \rtimes_\alpha G$, and it follows that L and $\pi \rtimes u$ are equal. □

Proposition 2.40. *If $\alpha : G \to \operatorname{Aut} A$ is a dynamical system, then the map sending a covariant pair (π, U) to its integrated form $\pi \rtimes U$ is a one-to-one correspondence between nondegenerate covariant representations of (A, G, α) and nondegenerate representations of $A \rtimes_\alpha G$. This correspondence preserves direct sums, irreducibility and equivalence.*

Proof. Proposition 2.39 on the facing page shows that the map $(\pi, U) \mapsto \pi \rtimes U$ is a surjection. It's one-to-one in view of Equations (2.21) and (2.27).

The statement about equivalence is straightforward. Let (π, U), (ρ, V) and W be as in Definition 2.18 on page 46. Then

$$\begin{aligned}
\bigl(W(\pi \rtimes U)(f)h \mid k\bigr) &= \bigl(\pi \rtimes U(f)h \mid W^* k\bigr) \\
&= \int_G \bigl(\pi\bigl(f(s)\bigr)U_s h \mid W^* k\bigr) \, d\mu(s) \\
&= \int_G \bigl(W\pi\bigl(f(s)\bigr)U_s h \mid k\bigr) \, d\mu(s) \\
&= \int_G \bigl(\rho\bigl(f(s)\bigr)V_s W h \mid k\bigr) \, d\mu(s) \\
&= \bigl(\rho \rtimes V(f) W h \mid k\bigr).
\end{aligned}$$

And it follows that $\pi \rtimes U$ and $\rho \rtimes V$ are equivalent. Conversely, if W intertwines $\pi \rtimes U$ and $\rho \rtimes V$, then

$$\begin{aligned} WU_s\pi \rtimes U(f)h &= W\pi \rtimes U\big(i_G(s)f\big)\big)h \\ &= \rho \rtimes V\big(i_G(s)f\big)Wh \\ &= V_s\rho \rtimes V(f)Wh \\ &= V_sW\pi \rtimes U(f)h. \end{aligned}$$

Since $\pi \rtimes U$ is nondegenerate, it follows that $WU_s = V_sW$ for all $s \in G$. A similar argument shows that $W\pi(a) = \rho(a)W$ for all $a \in A$.

The other assertions will follow once we show that a closed subspace $\mathcal{V}$ is invariant for a nondegenerate (π, U) if and only if $\mathcal{V}$ is invariant for $\pi \rtimes U$. Suppose first that $\mathcal{V}$ is invariant for (π, U). Let $h \in \mathcal{V}$ and $k \in \mathcal{V}^\perp$. Then

$$\big(\pi \rtimes U(f)h \mid k\big) = \int_G \big(\pi\big(f(s)\big)U_sh \mid k\big),$$

and since $\pi\big(f(s)\big)U_sh \in \mathcal{V}$ for all $s \in G$,

$$\big(\pi \rtimes U(f)h \mid k\big) = 0.$$

It follows that $\mathcal{V}$ is invariant for $\pi \rtimes U$.

Now assume $\mathcal{V}$ is invariant for $\pi \rtimes U$. Since $\pi \rtimes U$ is nondegenerate, $\pi \rtimes U(e_i) \to 1_{\mathcal{H}}$ strongly for any approximate identity $\{\, e_i \,\}$ for $A \rtimes_\alpha G$. If $h \in \mathcal{V}$ and $k \in \mathcal{V}^\perp$, then $\pi \rtimes U\big(i_G(s)e_i\big) \in \mathcal{V}$ and

$$\begin{aligned} (U_sh \mid k) &= \lim\big(U_s\big(\pi \rtimes U(e_i)h\big) \mid k\big) \\ &= \lim\big(\pi \rtimes U\big(i_G(s)e_i\big)h \mid k\big) \\ &= 0. \end{aligned}$$

It follows that $\mathcal{V}$ is invariant for U_s and a similar argument works for $\pi(a)$. ☐

Remark 2.41. We can now identify the spectrum of $C^*(G)$ with the set $\widehat{G}$ of irreducible unitary representations of G. We give $\widehat{G}$ the topology coming from the spectrum of $C^*(G)$ and refer to $\widehat{G}$ as the *spectrum of* G. We should check to see that this usage is consistent with that for abelian groups. Since $C^*(G)$ is commutative if G is abelian, its irreducible representations are one-dimensional complex homomorphisms [139, Example A.16], and the characters of G correspond exactly the irreducible representations of G. Since a pure state on $C^*(G)$ must be of the form $f \mapsto \big(\omega(f)h \mid h\big) = \omega(f)$ for the integrated form of a character ω and a unimodular scalar h, the pure states also correspond exactly to the characters. [139, Theorem A.38] implies that the weak-$*$ topology on $\widehat{G}$, viewed as the set of pure states, is the same as the topology obtained by viewing $\widehat{G}$ as the spectrum of $C^*(G)$. The proof of Lemma 1.78 on page 25 implies that this topology is the compact-open topology.

Remark 2.42. If $u : G \to U\mathcal{L}(\mathsf{X})$ is a strongly continuous homomorphism, then we'll use the same letter u for the integrated form of u viewed either as a homomorphism of $C_c(G)$ into $\mathcal{L}(\mathsf{X})$ or $C^*(G)$ into $\mathcal{L}(\mathsf{X})$. If $L = (\pi, U)$ is a covariant representation, we'll often write $L(f)$ in place of $\pi \rtimes U(f)$.

Definition 2.43. A $*$-homomorphism $\pi : C_c(G, A) \to B(\mathcal{H})$ is *continuous in the inductive limit topology* if whenever we're given $h, k \in \mathcal{H}$ and a net $f_i \to f$ in the inductive limit topology on $C_c(G, A)$, then we also have $\big(\pi(f_i)h \mid k\big) \to \big(\pi(f)h \mid k\big)$.

Example 2.44. If π is L^1-norm decreasing, then π is continuous in the inductive limit topology.

The following clever argument is due to Iain Raeburn.

Lemma 2.45. *Suppose that $\pi : C_c(G, A) \to B(\mathcal{H})$ is a $*$-homomorphism which is continuous in the inductive limit topology. Then π is bounded with respect to the universal norm on $C_c(G, A) \subset A \rtimes_\alpha G$. That is, $\|\pi(f)\| \le \|f\|$ for all $f \in C_c(G, A)$.*

Proof. By reducing to the essential subspace of π, we may assume that π is nondegenerate. The equation

$$(f \otimes h \mid g \otimes k) := \big(\pi(g^* * f)h \mid k\big)$$

defines a sesquilinear form on the algebraic tensor product $C_c(G, A) \odot \mathcal{H}$. This form is positive since

$$\begin{aligned}\Big(\sum_i f_i \otimes h_i \mid \sum_i f_i \otimes h_i\Big) &= \sum_{ij} \big(\pi(f_j^* * f_i)h_i \mid h_j\big) \\ &= \sum_{ij} \big(\pi(f_i)h_i \mid \pi(f_j)h_j\big) \\ &= \Big(\sum_i \pi(f_i)h_i \mid \sum_i \pi(f_i)h_i\Big) \\ &\ge 0.\end{aligned}$$

Thus we can complete $C_c(G, A) \odot \mathcal{H}$ to get a Hilbert space $\mathcal{V}$. The map $(f, h) \mapsto \pi(f)h$ is bilinear and therefore extends to a map $U : C_c(G, A) \odot \mathcal{H} \to \mathcal{H}$ which has dense range since π is nondegenerate. Moreover

$$\begin{aligned}\big(U(f \otimes h) \mid U(g \otimes k)\big) &= \big(\pi(f)h \mid \pi(g)k\big) \\ &= \big(\pi(g^* * f)h \mid k\big) \\ &= (f \otimes h \mid g \otimes k).\end{aligned}$$

It follows that U extends to a unitary operator $U : \mathcal{V} \to \mathcal{H}$. Now for each $b \in M(A)$, we define an operator on $C_c(G, A) \odot \mathcal{H}$ by

$$M(b)(f \otimes h) = \bar{\iota}_A(b)f \otimes h.$$

Since an easy calculation shows that

$$\big(M(b)(f \otimes h) \mid g \otimes k\big) = \big(f \otimes h \mid M(b^*)(g \otimes k)\big),$$

we can let $a_0 := (\|a\|^2 1 - a^*a)^{\frac{1}{2}}$ and compute that

$$\|a\|^2(f \otimes h \mid f \otimes h) - \big(M(a)(f \otimes h) \mid M(a)(f \otimes h)\big) = \big(M(a_0)(f \otimes h) \mid M(a_0)(f \otimes h)\big) \geq 0.$$

It follows that $\|M(a)\| \leq \|a\|$ and that M extends to a representation of A on $\mathcal{V}$. Similarly, we define a map from G into operators on $C_c(G, A) \odot \mathcal{H}$ by

$$V_s(f \otimes h) = i_G(s)(f) \otimes h.$$

Easy calculations show that $V_{sr} = V_s \circ V_r$. Since we can also check that $i_G(g)^* * i_G(f) = g^* * f$, it follows that

$$\big(V_s(f \otimes h) \mid V_s(g \otimes k)\big) = (f \otimes h \mid g \otimes k).$$

Therefore it follows that V is a unitary-valued homomorphism from G into $U(\mathcal{V})$. To see that V is strongly continuous, notice that

$$\|V_s(f \otimes h) - f \otimes h\|^2 = 2\big(\pi(f^* * f)h \mid h\big) - 2\operatorname{Re}\big(\pi\big(f^* * i_G(s)(f)\big)h \mid h\big). \quad (2.33)$$

Since $i_G(s)(f) \to f$ in the inductive limit topology as $s \to e$ and since π is continuous in the inductive limit topology, it follows that (2.33) goes to zero if $s \to e$ in G. Thus V is a unitary representation of G, and it is easy to see that (M, V) is covariant. We will prove that π and $M \rtimes V$ are equivalent representations.

Now let $f, g \in C_c(G, A)$. Then $(s, r) \mapsto f(s)i_G(s)(g)(r)$ has support in $(\operatorname{supp} f) \times (\operatorname{supp} f)(\operatorname{supp} g)$. Therefore if U is a pre-compact open neighborhood of the compact set $(\operatorname{supp} f)(\operatorname{supp} g)$, then we can view $q(s) := f(s)i_G(s)(g)$ as defining a function in $C_c\big(G, C_0(U, A)\big)$. Then we can form the $C_0(U, A)$-valued integral

$$\int_G^{C_0(U,A)} f(s)i_G(s)(g)\,d\mu(s). \quad (2.34)$$

Since evaluation at $r \in U$ is a continuous homomorphism from $C_0(U, A)$ to A, we have

$$\int_G^{C_0(U,A)} f(s)i_G(s)(g)\,d\mu(s)(r) = \int_G^{A} f(s)i_G(s)(g)(r)\,d\mu(s) = f * g(r).$$

Thus (2.34) is the restriction of $f * g$ to U. If $h, k \in \mathcal{H}$ and j is the inclusion of $C_0(U, A)$ into $C_c(G, A)$, then the continuity of π and j (Lemma 1.106 on page 38) allows us to define a continuous linear functional

$$L : C_0(U, A) \to \mathbf{C}$$

by

$$L(f) := \big(\pi\big(j(f)\big)h \mid k\big).$$

Now on the one hand,

$$L\Big(\int_G^{C_0(U,A)} f(s)i_G(s)(g)\,d\mu(s)\Big) = \big(\pi(f * g)h \mid k\big) = \big(\pi(f)U(g \otimes h) \mid k\big). \quad (2.35)$$

On the other hand, since L is a continuous linear functional, the left-hand side of (2.35) equals

$$\begin{aligned}\int_G L\big(f(s)i_G(s)(g)\big)\,d\mu(s) &= \int_G \big(\pi\big(f(s)i_G(s)(g)\big)h \mid k\big)\,d\mu(s)\\ &= \int_G \big(U\big(f(s)i_G(s)(g)\otimes h\big) \mid k\big)\,d\mu(s)\\ &= \int_G \big(f(s)i_G(s)(g)\otimes h \mid U^{-1}k\big)\,d\mu(s)\\ &= \int_G \big(M\big(f(s)\big)V_s(g\otimes h) \mid U^{-1}k\big)\,d\mu(s)\\ &= \big(M\rtimes V(f)(g\otimes h) \mid U^{-1}k\big)\\ &= \big(U\circ M\rtimes V(f)(g\otimes h) \mid k\big).\end{aligned}$$

This proves that $\pi(f)\circ U = U\circ M\rtimes V(f)$, and that $\|\pi(f)\| = \|M\rtimes V(f)\| \le \|f\|$ as desired. □

It follows that every $*$-homomorphism $L : C_c(G,A)\to B(\mathcal{H})$ which is L^1-norm bounded or just continuous in the inductive limit topology is bounded with respect to the universal norm and extends to a (possibly degenerate) representation of $A\rtimes_\alpha G$. Thus every such representation is the integrated form of a covariant pair.

Corollary 2.46. *Suppose that (A,G,α) is a dynamical system and that $f\in C_c(G,A)$. Then*

$$\begin{aligned}\|f\| &= \sup\{\,\|L(f)\| : L \textit{ is a } L^1\textit{-norm decreasing representation.}\}\\ &= \sup\{\,\|L(f)\| : L \textit{ is continuous in the inductive limit topology.}\}\end{aligned}$$

Corollary 2.47. *Suppose that (A,G,α) and (B,H,β) are dynamical systems and that $\Phi : C_c(G,A)\to C_c(H,B)$ is a $*$-homomorphism which is continuous in the inductive limit topology. Then Φ is norm-decreasing with respect to the universal norms and extends to an homomorphism of $A\rtimes_\alpha G$ into $B\rtimes_\beta H$.*

Proof. Suppose that L is a representation of $B\rtimes_\beta H$. Then $L\circ\Phi$ is a representation of $C_c(G,A)$ which is continuous in the inductive limit topology. Thus Lemma 2.45 on page 61 implies that $L\circ\Phi$ is bounded for the universal norm, and thus for each $f\in C_c(G,A)$, $\|L(\Phi(f)\|\le\|f\|$. Since L is arbitrary, it follows that $\|\Phi(f)\|\le\|f\|$. □

Corollary 2.48. *Suppose that (A,G,α) and (B,G,β) are dynamical systems and that $\varphi : A\to B$ is an equivariant homomorphism. Then there is a homomorphism $\varphi\rtimes\mathrm{id} : A\rtimes_\alpha G\to B\rtimes_\beta G$ mapping $C_c(G,A)$ into $C_c(G,B)$ such that $\varphi\rtimes\mathrm{id}(f)(s)=\varphi\big(f(s)\big)$.*

Proof. Define $\Phi : C_c(G,A)\to C_c(G,B)$ by $\Phi(f)(s)=\varphi\big(f(s)\big)$. Then Φ is norm decreasing for the L^1-norm and therefore continuous in the inductive limit topology. Now we can apply Corollary 2.47 on the previous page. □

2.5 Comments on Examples

Having worked hard just to define a crossed product it would be natural to want, and to provide, a few illustrative examples other than the degenerate examples in Example 2.33 on page 54. It is a bit frustrating to admit that we still need more technology to do this in a systematic way. (We'll deal with the group C^*-algebras of abelian and compact groups in Section 3.1 and Section 3.2, respectively.)

However, we can work out some simple and provocative examples when we assume that (A, G, α) is a dynamical system with G *finite*. To begin with, suppose that $G = \mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. This means we are given $\alpha \in \operatorname{Aut} A$ with $\alpha^2 = \mathrm{id}$. Then $C_c(\mathbf{Z}_2, A) = C(\mathbf{Z}_2, A)$ and elements of $C(\mathbf{Z}_2, A)$ are simply functions from $\mathbf{Z}_2 = \{0, 1\}$ to A. Let

$$D := \Bigl\{ \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} \in M_2(A) : a, b \in A \Bigr\}.$$

It is easy to check that D is a C^*-subalgebra of $M_2(A)$ and that

$$\Phi(f) := \begin{pmatrix} f(0) & f(1) \\ \alpha\bigl(f(1)\bigr) & \alpha\bigl(f(0)\bigr) \end{pmatrix}$$

defines an injective $*$-homomorphism of $C(\mathbf{Z}_2, A)$ (with the $*$-algebra structure coming from $(A, \mathbf{Z}_2, \alpha)$) into D. In fact, Φ is clearly surjective. Since we can faithfully represent D on Hilbert space, it follows that the universal norm of $f \in C(\mathbf{Z}_2, A)$ satisfies

$$\|f\| \geq \|\Phi(f)\|.$$

On the other hand, if L is any representation of $A \rtimes_\alpha \mathbf{Z}_2$, then $L \circ \Phi^{-1}$ is a $*$-homomorphism of D into $B(\mathcal{H}_L)$. Since $*$-homomorphisms of C^*-algebras are norm reducing,

$$\|L(f)\| = \|L\bigl(\Phi^{-1}\bigl(\Phi(f)\bigr)\bigr)\| \leq \|\Phi(f)\|.$$

Therefore,

$$\|f\| = \|\Phi(f)\|.$$

In particular, $C(\mathbf{Z}_2, A)$ is already complete in its universal norm, and Φ is an isomorphism of $A \rtimes_\alpha \mathbf{Z}_2$ onto D. Although this situation is rather atypical — normally $A \rtimes_\alpha G$ is going to be a genuine completion of $C_c(G, A)$ — it will be useful to record the general principle we used above so that we can use it in some other examples of finite group actions.

Lemma 2.49. *Suppose that (A, G, α) is a dynamical system with G* finite *and that $L : C(G, A) \to D$ is a $*$-isomorphism of $C(G, A)$ onto a C^*-algebra D. Then $A \rtimes_\alpha G \cong D$.*

It is an interesting exercise to use Lemma 2.49 to study $A \rtimes_\alpha G$ for various finite groups G — for example, compare $G = \mathbf{Z}_4$ with $G = \mathbf{Z}_2 \times \mathbf{Z}_2$. Here we specialize to transformation groups.

Lemma 2.50. *Suppose that G is a finite group with $|G| = n$. Then G acts on itself by left translation and*
$$C(G) \rtimes_{\mathrm{lt}} G \cong M_n,$$
where M_n denotes the C^*-algebra of $n \times n$-matrices (with complex entries).

Proof. Of course, we give G the discrete topology and use counting measure for Haar measure on G. Let $L = M \rtimes \lambda$ be the natural representation as in Example 2.12 on page 45. Let $G = \{ s_i \}_{i=1}^n$ with $s_1 := e$. Then $L^2(G)$ is a n-dimensional Hilbert space with orthonormal basis $\{ e_s \}_{s \in G}$ where e_s is the function δ_s which is 1 at s and zero elsewhere. We view operators on $L^2(G)$ as $n \times n$ matrices calculated with respect to $\{ e_s \}$. If $f \in C(G \times G)$ and $h \in L^2(G)$, then
$$L(f)h(s) := \sum_{r \in G} f(r, s)h(r^{-1}s) = \sum_{r \in G} f(sr^{-1}, s)h(r).$$
It follows that $L(f)$ is given by the matrix M^f with $(s, r)^{\text{th}}$ entry $M^f_{s,r} = f(sr^{-1}, s)$. If $M = (m_{s,r})_{s,r \in G}$ is any $n \times n$ matrix, then
$$f(s, r) := m_{r, s^{-1}r}$$
satisfies
$$M^f_{s,r} = f(sr^{-1}, s) = m_{s,r}.$$
Thus L is a surjective $*$-isomorphism of $C(G \times G)$ onto M_n. The result now follows from Lemma 2.49 on the facing page. □

Lemma 2.50 can be extended to arbitrary groups.

Example 2.51. Suppose that G is a locally compact group acting on itself by left translation. Then $C_0(G) \rtimes_{\mathrm{lt}} G$ is isomorphic to the compact operators on $L^2(G)$.

Verifying Example 2.51 is a good exercise if G is discrete. In general, some work is required and there are a number of ways to proceed. In this book, we'll eventually prove this in Theorem 4.24 on page 133 using the Imprimitivity Theorems developed in Section 4.3.

Now we suppose that $A = C(X)$ for a *compact* Hausdorff space X.[6] To get a $\mathbf{Z}_2$-action we need a homeomorphism $\sigma \in \mathrm{Homeo}(X)$ such that $\sigma^2 = \mathrm{id}$. For convenience, we'll assume that the $\mathbf{Z}_2$-action is free so that $\sigma(x) \neq x$ for all $x \in X$. We'll let $\big(C(X), \mathbf{Z}_2, \alpha\big)$ be the associated dynamical system (so that $\alpha_1(f)(x) = f\big(\sigma(x)\big)$). For example, we could let X be the n-sphere $S^n = \{ \mathbf{x} \in \mathbf{R}^{n+1} : \|\mathbf{x}\| = 1 \}$, and σ the antipodal map $\sigma(\mathbf{x}) := -\mathbf{x}$.

However, for the moment, we let σ be any period two homeomorphism of a compact space X (without fixed points). We'll view operators on $L^2(\mathbf{Z}_2)$ as 2×2-matrices with respect to the orthonormal basis $\{ e_0, e_1 \}$ where e_i is the function δ_i as above. Let $L^x = \pi \rtimes \lambda = \mathrm{Ind}_e^G \mathrm{ev}_x$ be the regular representation on $L^2(\mathbf{Z}_2)$ coming from evaluation at x. Thus λ is the left-regular representation and
$$\pi(\varphi)h(s) = \mathrm{ev}_x\big(\alpha_s^{-1}(\varphi)\big)h(s).$$

[6] It isn't really necessary to take X compact, but it makes the examples a little easier to digest.

Therefore $\pi(\varphi)$ is given by the matrix

$$\begin{pmatrix} \varphi(x) & 0 \\ 0 & \varphi\big(\sigma(x)\big) \end{pmatrix}$$

If $f \in C(\mathbf{Z}_2 \times X)$, then

$$\begin{aligned} \pi \rtimes \lambda(f) &= \sum_{s=0,1} \pi\big(f(s,\cdot)\big)\lambda(s) \\ &= \begin{pmatrix} f(0,x) & 0 \\ 0 & f\big(0,\sigma(x)\big) \end{pmatrix} + \begin{pmatrix} f(1,x) & 0 \\ 0 & f\big(1,\sigma(x)\big) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f(0,x) & f(1,x) \\ f\big(1,\sigma(x)\big) & f\big(0,\sigma(x)\big) \end{pmatrix}. \end{aligned}$$

In this way, we can view L^x as a $*$-homomorphism of $C(\mathbf{Z}_2 \times X)$ into M_2. Since $\sigma(x) \neq x$, L^x is surjective. Let

$$W := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let

$$A := \big\{\, f \in C(X, M_2) : f\big(\sigma(x)\big) = W f(x) W^* \,\big\}.$$

Then A is a C^*-subalgebra of $C(X, M_2)$. Since each L^x is onto M_2, it is not hard to see that the irreducible representations of A correspond to the point evaluations π_x. Since A is a $C(\mathbf{Z}_2\backslash X)$-algebra, we have π_x equivalent to π_y if and only if $y = \sigma(x)$. Thus the spectrum of A is naturally identified with $\mathbf{Z}_2\backslash X$.[7] Since all the irreducible representations of A are 2-dimensional, A is called a 2-homogeneous C^*-algebra.

Now define $\Phi : C(\mathbf{Z}_2 \times X) \to C(X, M_2)$ by

$$\Phi(f)(x) = L^x(f) = \begin{pmatrix} f(0,x) & f(1,x) \\ f\big(1,\sigma(x)\big) & f\big(0,\sigma(x)\big). \end{pmatrix}$$

Since each L^x is a $*$-homomorphism, so is Φ. Furthermore, Φ is injective and

$$\Phi(f)\big(\sigma(x)\big) = \begin{pmatrix} f\big(0,\sigma(x)\big) & f\big(1,\sigma(x)\big) \\ f(1,x) & f(0,x) \end{pmatrix} = W\Phi(f)(x)W^*.$$

Thus Φ maps into A. On the other hand, if $a \in A$, say

$$a(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

and if we define $f \in C(\mathbf{Z}_2 \times X)$ by $f(i,x) := a_{1,1+i}(x)$, then we have $\Phi(f) = a$. After applying Lemma 2.49 on page 64, we have proved the following.

[7] It is not hard to see that $\mathbf{Z}_2\backslash X$ is compact and Hausdorff. This also follows from Corollary 3.43 on page 100.

Proposition 2.52. *Suppose that X is a compact free $\mathbf{Z}_2$-space determined by a period two homeomorphism σ. Then $C(\mathbf{Z}_2 \times X)$ is complete in its universal norm and*

$$\Phi(f)(x) := \begin{pmatrix} f(0,x) & f(1,x) \\ f\bigl(1,\sigma(x)\bigr) & f\bigl(0,\sigma(x)\bigr) \end{pmatrix}$$

defines an isomorphism of $C(X) \rtimes_\alpha \mathbf{Z}_2 = C(\mathbf{Z}_2 \times X)$ with the 2-homogeneous C^*-algebra with spectrum $\mathbf{Z}_2\backslash X$

$$A := \{\, f \in C(X, M_2) : f\bigl(\sigma(x)\bigr) = Wf(x)W^* \,\},$$

where $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We will have more to say about examples provided by Proposition 2.52 later (cf. Proposition 4.15 on page 127).

As sort of a preview of coming attractions, I also want to mention a few non-trivial examples that we'll take up in due course. An obvious question to ask is what happens if the G-action on A is trivial. It turns out that this isn't too different than allowing the G-action to be implemented by a homomorphism into the unitary group of the multiplier algebra of A.

Example 2.53 (Trivial Dynamical Systems). Suppose that (A, G, α) is a dynamical system and that there is a strictly continuous homomorphism $u : G \to UM(A)$ such that that $\alpha_s(a) = u_s a u_s^*$ for all $s \in G$ and $a \in A$. Then it is a straightforward corollary to Lemma 2.73 on page 76 that $A \rtimes_\alpha G$ is isomorphic to $A \otimes_{\max} C^*(G)$ (see Lemma 2.68 on page 74 and Remark 2.71 on page 75).

Another natural example is to let a closed subgroup H of a locally compact group G act by left translation on the coset space G/H. The imprimitivity theorem (Theorem 4.22 on page 132) implies that $C_0(G/H) \rtimes_{\mathrm{lt}} G$ is Morita equivalent to $C^*(H)$. With considerably more work (Theorem 4.30 on page 138), we can sharpen this to an isomorphism result.

Example 2.54. $C_0(G/H) \rtimes_{\mathrm{lt}} G$ is isomorphic to the tensor product of $C^*(H)$ with the compact operators on $L^2(G/H, \mu)$ where μ is any quasi-invariant measure on G/H.

Example 2.54 is more generally applicable than one might think at first. We'll show later that in many cases — namely when the action of G is "nice" in a sense to made precise later — $C_0(X) \rtimes G$ is fibred over the orbits in X with fibres of the form $C_0(G/H_\omega) \rtimes_{\mathrm{lt}} G$ where H_ω depends on the orbit ω (see Section 8.1).

Example 2.55. Let $\mathbf{Z}$ act on $\mathbf{T}$ by rotation through θ as in Example 2.13 on page 45. The resulting crossed product $A_\theta := C_0(\mathbf{T}) \rtimes_\tau \mathbf{Z}$ is called a *rational or irrational rotation algebra* depending on whether θ is rational or irrational.

The structure of the rotation algebras, both irrational and rational, is surprisingly intricate and was a hot research topic in the early 1980's. These algebras continue to be of interest to this day. For example, just seeing that A_θ and $A_{\theta'}$ are not isomorphic for different irrational θ and θ' in $[0, \frac{1}{2}]$ involves more than we're prepared to undertake in this book.[8]

[8] A proof is given in Davidson's text [21, §VI.5].

Proposition 2.56. *If θ is irrational then the irrational rotation algebra A_θ is simple and is generated by two unitaries u and v such that $uv = \rho vu$ where $\rho = e^{2\pi i\theta}$. Furthermore, if $\mathcal{H}$ is a Hilbert space and if U and V are unitaries in $B(\mathcal{H})$ such that $UV = \rho VU$, then there is a representation $L : A_\theta \to B(\mathcal{H})$ such that $L(u) = U$, $L(v) = V$ and L is an isomorphism of A_θ onto the C^*-algebra $C^*(U, V)$ generated by U and V.*

Remark 2.57. One often summarizes the final assertion in Proposition 2.56 by saying that A_θ is the "universal C^*-algebra generated by two unitaries U and V satisfying $UV = \rho VU$".

Remark 2.58. In the proof, we will need to know that $\{\, \rho^n z \,\}_{n\in\mathbf{Z}}$ is dense in $\mathbf{T}$ for all $z \in \mathbf{T}$. This will be proved in Lemma 3.29 on page 96.

Proof. Recall that $A_\theta = C(\mathbf{T}) \rtimes_\tau \mathbf{Z}$ is the completion of the $*$-algebra $C_c(\mathbf{Z} \times \mathbf{T})$ where the convolution product is given by the *finite* sum

$$f * g(n, z) := \sum_{m=-\infty}^{\infty} f(m, z)g(n - m, \rho^{-m} z),$$

and the involution is given by

$$f^*(n, z) = \overline{f(-n, \rho^{-n} z)}.$$

If $\varphi \in C(\mathbf{T})$ and $h \in C_c(\mathbf{Z})$, then $\varphi \otimes h$ is the element of $C_c(\mathbf{Z} \times \mathbf{T})$ given by $\varphi \otimes h(n, z) = \varphi(z)h(n)$. As usual, let δ_n be the function on $\mathbf{Z}$ which is equal to 1 at n and zero elsewhere. In particular, A_θ has an identity in $C_c(\mathbf{Z} \times \mathbf{T})$ given by $1 \otimes \delta_0$. Furthermore, $u = 1 \otimes \delta_1$ is a unitary in $C_c(\mathbf{Z} \times \mathbf{T})$, as is $v = \iota_{\mathbf{T}} \otimes \delta_0$ where $\iota_{\mathbf{T}}(z) := z$ for all $z \in \mathbf{T}$. If $\varphi \in C(\mathbf{T})$, then let $i_{C(\mathbf{T})}(\varphi) := \varphi \otimes \delta_0$. Then $i_{C(\mathbf{T})}$ is a homomorphism of $C(T)$ into $C_c(\mathbf{Z} \times \mathbf{T}) \subset A_\theta$ (and is therefore bounded). Notice that $i_{C(\mathbf{T})}(\varphi) * u^n = \varphi \otimes \delta_n$ for all $n \in \mathbf{Z}$, and that $u * i_{C(\mathbf{T})}(\varphi) * u^* = i_{C(\mathbf{T})}\big(\tau_1(\varphi)\big)$. It follows from Lemma 1.87 on page 29 that

$$\{\, i_{C(\mathbf{T})}(\varphi) * u^n : \varphi \in C(\mathbf{T}) \text{ and } n \in \mathbf{Z} \,\}$$

spans a dense subalgebra of $C_c(\mathbf{Z} \times \mathbf{T})$.

The Stone-Weierstrass Theorem implies that $\iota_{\mathbf{T}}$ generates $C(\mathbf{T})$ as a C^*-algebra. In particular,

$$\{\, v^n : n \in \mathbf{Z} \,\}$$

spans a dense subalgebra of $i_{C(\mathbf{T})}\big(C(\mathbf{T})\big)$. It follows that u and v generate A_θ, and it is easy to verify that $uv = \rho vu$.

Now suppose that $U, V \in B(\mathcal{H})$ are as in the statement of the proposition. Since U and V are unitaries, their spectrums $\sigma(U)$ and $\sigma(V)$ are subsets of $\mathbf{T}$. I claim that $\sigma(V) = \mathbf{T}$.[9] Note that

$$\begin{aligned}\lambda \in \sigma(V) &\iff V - \lambda I \text{ is not invertible}\\ &\iff U^n(V-\lambda I) \text{ is not invertible}\\ &\iff (\rho^n V - \lambda I)U^n \text{ is not invertible}\\ &\iff V - \rho^{-n}\lambda I \text{ is not invertible}\\ &\iff \rho^{-n}\lambda \in \sigma(V).\end{aligned}$$

Since $\sigma(V)$ must be nonempty, $\sigma(V)$ contains a dense subset of $\mathbf{T}$ by Remark 2.58 on the preceding page. Since $\sigma(V)$ is closed, the claim follows.

Thus the Abstract Spectral Theorem [110, Theorem 2.1.13] implies that there is an isomorphism $\pi : C(\mathbf{T}) \to C^*(V) \subset C^*(U,V)$ taking $\iota_{\mathbf{T}}$ to V. Let $W : \mathbf{Z} \to U(\mathcal{H})$ be given by $W_n := U^n$. Since $UV = \rho VU$, it follows that

$$W_n \pi(\iota_{\mathbf{T}}) W_n^* = \pi\big(\tau_n(\iota_{\mathbf{T}})\big).$$

Since $\iota_{\mathbf{T}}$ generates $C(\mathbf{T})$, (π, W) is a covariant representation of $\big(C(\mathbf{T}), \mathbf{Z}, \tau\big)$. Furthermore,

$$\pi \rtimes W(u) = \sum_{m=-\infty}^{\infty} \pi\big(u(m,\cdot)\big)W_m = \pi(1_{C(\mathbf{T})})W_1 = U, \text{ and}$$

$$\pi \rtimes W(v) = \sum_{m=-\infty}^{\infty} \pi\big(v(m,\cdot)\big)W_m = \pi(\iota_{\mathbf{T}}) = V.$$

Since u and v generate A_θ and since U and V generate $C^*(U,V)$, it follows that $\pi \rtimes W(A_\theta) = C^*(U,V)$ and $L := \pi \rtimes W$ is an homomorphism of A_θ onto $C^*(U,V)$ taking u to U and v to V.

Now it will suffice to see that A_θ is simple. However, proving that is going to require more work.

For each $\omega \in \mathbf{T}$, let $\hat{\tau}_\omega : C_c(\mathbf{Z} \times \mathbf{T}) \to C_c(\mathbf{Z} \times \mathbf{T})$ be defined by $\hat{\tau}_\omega(f)(n,z) = \omega^n f(n,z)$. It is not hard to check that $\hat{\tau}_\omega$ is a $*$-isomorphism which is continuous in the inductive limit topology. Therefore $\hat{\tau}_\omega$ extends to an automorphism of A_θ (with inverse $\hat{\tau}_{\bar{\omega}}$). Since $\omega \mapsto \hat{\tau}_\omega(f)$ is continuous from $\mathbf{T}$ to $C_c(\mathbf{Z} \times \mathbf{T})$ in the inductive limit topology for each $f \in C_c(\mathbf{Z} \times \mathbf{T})$, it is not hard to see that $\omega \mapsto \hat{\tau}_\omega(a)$ is continuous from $\mathbf{T}$ to A_θ for all $a \in A_\theta$.[10] Therefore we can define $\Phi : A_\theta \to A_\theta$ by

$$\Phi(a) = \int_{\mathbf{T}} \hat{\tau}_\omega(a)\, d\omega.$$

It is easy to see that Φ is linear and $\|\Phi\| \le 1$. Since for each $\omega \in \mathbf{T}$, $\hat{\tau}_\omega(v) = v$ and $\hat{\tau}_\omega(u) = \omega u$, we have for each $k, m \in \mathbf{Z}$,

$$\Phi(v^k * u^m) = \int_{\mathbf{T}} \hat{\tau}_\omega(v^k * u^m)\, d\omega = v^k * \Big(\int_{\mathbf{T}} \omega^m \, d\omega\Big) u^m.$$

[9] Of course, we also have $\sigma(U) = \mathbf{T}$.

[10] The automorphism group $\hat{\tau} : \mathbf{T} \to \operatorname{Aut} A_\theta$ is an example of the dual-action which will reappear in Section 7.1.

Therefore,

$$\Phi(v^k * u^m) = \begin{cases} v^k & \text{if } m = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.36}$$

Now let $E_n : A_\theta \to A_\theta$ be defined by

$$E_n(a) := \frac{1}{2n+1} \sum_{j=-n}^{n} v^j * a * v^{-j}.$$

Since v is a unitary, E_n is linear with $\|E_n\| \le 1$. Furthermore,

$$E_n(v^k * u^m) = \sum_{j=-n}^{n} v^j * v^k * u^m * v^{-j} = v^k * u^m \sum_{j=-n}^{n} \rho^{jm}. \tag{2.37}$$

Thus if $m = 0$, (2.37) is equal to v^k. Otherwise, notice that the usual sort of manipulations with geometric series gives

$$\begin{aligned} \sum_{j=-n}^{n} \rho^{jm} &= \frac{\rho^{m(n+1)} - \rho^{-mn}}{\rho^m - 1} \\ &= \frac{\rho^{\frac{m}{2}(2n+1)} - \rho^{-\frac{m}{2}(2n+1)}}{\rho^{\frac{m}{2}} - \rho^{-\frac{m}{2}}} \\ &= \frac{e^{i\pi\theta m(2n+1)} - e^{-i\pi\theta m(2n+1)}}{e^{i\pi m\theta} - e^{-i\pi n m\theta}} \\ &= \frac{\sin\big((2n+1)\pi m\theta\big)}{\sin(\pi m\theta)}. \end{aligned}$$

Thus if $m \ne 0$,

$$\lim_{n\to\infty} E_n(v^k * u^m) = v^k * u^m \lim_{n\to\infty} \sum_{j=-n}^{n} \rho^{jm} = 0.$$

It follows that

$$\Phi(a) = \lim_{n\to\infty} E_n(a) \tag{2.38}$$

for all a in the dense subalgebra $A_0 := \operatorname{span}\{\, u^k * v^m : k, m \in \mathbf{Z}\,\}$.

We want to see that (2.38) holds for all $a \in A_\theta$. Fix $\epsilon > 0$ and $a \in A_\theta$. Then there is a $b \in A_0$ such that $\|a - b\| < \epsilon/3$, and a $N \in \mathbf{Z}^+$ such that $n \ge N$ implies $\|\Phi(b) - E_n(b)\| < \epsilon/3$. Since Φ and E_n each have norm 1, it follows that if $n \ge N$,

$$\begin{aligned} \|\Phi(a) - E_n(a)\| &\le \|\Phi(a - b)\| + \|\Phi(b) - E_n(b)\| + \|E_n(b - a)\| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore (2.38) holds for all $a \in A_\theta$.

Now suppose that I is a nonzero ideal in A_θ. Let a be a nonzero positive element in I. Let ρ be a state on A_θ such that $\rho(a) > 0$. Then $\omega \mapsto \rho\big(\hat{\tau}_\omega(a)\big)$ is a nonzero continuous nonnegative function on $\mathbf{T}$. Thus

$$\rho\big(\Phi(a)\big) = \int_{\mathbf{T}} \rho\big(\hat{\tau}_\omega(a)\big)\, d\omega > 0,$$

and $\Phi(a) > 0$.

It follows from (2.36) that $\Phi(a) \in i_{C(\mathbf{T})}\big(C(\mathbf{T})\big)$ and from (2.38) that $\Phi(a) \in I$. Thus there is a nonzero nonnegative function $h \in C(\mathbf{T})$ such that $i_{C(\mathbf{T})}(h) \in I$. Let O be a neighborhood of $z_0 \in \mathbf{T}$ such that $h(z) > 0$ for all $z \in O$. Since $\{\, \rho^k z_0 : k \in \mathbf{Z}\,\}$ is dense in $\mathbf{T}$ (Remark 2.58 on page 68), there is a n such that $\bigcup_{k=-n}^{n} \rho^k V$ covers $\mathbf{T}$. Then

$$g := \sum_{k=-n}^{n} \tau_k(h)$$

is positive and nonzero on $\mathbf{T}$. Hence g is invertible in $C(\mathbf{T})$ and $i_{C(\mathbf{T})}(g)$ is invertible in A_θ (with inverse $i_{C(\mathbf{T})}(g^{-1})$). However,

$$i_{C(\mathbf{T})}(g) = \sum_{k=-n}^{n} i_{C(\mathbf{T})}\big(\tau_k(h)\big) = \sum_{k=-n}^{n} u^k * i_{C(\mathbf{T})}(h) * u^{-k}$$

clearly belongs to I. Thus I contains an invertible element and must be all of A_θ. Thus A_θ is simple. □

Example 2.59. To obtain a more concrete realization of A_θ for irrational θ we can let $\mathcal{H} = L^2(\mathbf{T})$. Define U and V as follows:

$$U(f)(z) = zf(z) \quad \text{and} \quad V(f)(z) = f(\bar{\rho}z).$$

Then its not hard to see that U and V are unitaries on $\mathcal{H}$. Furthermore, $VU(f)(z) = U(f)(\bar{\rho}z) = \bar{\rho}zf(\bar{\rho}z)$ and $UV(f)(z) = zV(f)(z) = zf(\bar{\rho}z)$. Thus, $UV = \rho VU$ and $C^*(U,V) \cong A_\theta$.

Remark 2.60 (The Whole Story). Since $C^*(U,V) = C^*(V,U)$, it is not so hard to see that A_θ is isomorphic to $A_{\theta'}$ whenever *either* $\theta - \theta'$ or $\theta + \theta'$ is an integer; that is, whenever $\theta' = \theta \mod 1$. In fact, $A_\theta \cong A_{\theta'}$ if and only if $\theta' = \theta \mod 1$. This was proved for θ irrational by Rieffel in [147]. The same result for rational rotation C^*-algebras was proved later by Høegh-Krohn and Skjelbred [74] (see also [149]). Rieffel also showed that the rational rotation algebras are all Morita equivalent to $C_0(\mathbf{T}^2)$ [149],[11] and that two irrational rotation C^*-algebras A_θ and $A_{\theta'}$ are Morita equivalent if and only if θ and θ' are in the same $\mathrm{GL}_2(\mathbf{Z})$ orbit [147].[12]

[11] See also footnote 11 on page 255.
[12] If α is an irrational number and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z})$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\alpha) = \frac{a\alpha+b}{c\alpha+d}$.

2.6 Universal Property

The notion of a universal object can be a powerful one. Good examples are the direct limit (see [139, Appendix D.1]), the maximal tensor product (see [139, Theorem B.27]) and graph C^*-algebras (see [6]); it is often easiest to exhibit such objects by verifying a particular representation has the required universal property, rather than working directly with the definition. Starting with [133] it has become apparent that the crossed product constructed in Section 2.3 can often profitably be thought of as a universal object for covariant representations of the dynamical system.

Theorem 2.61 (Raeburn). *Let (A, G, α) be a dynamical system. Suppose that B is a C^*-algebra such that*

(a) *there is a covariant homomorphism (j_A, j_G) of (A, G, α) into $M(B)$,*

(b) *given a nondegenerate covariant representation (π, U) of (A, G, α), there is a (nondegenerate) representation $L = L_{(\pi,U)}$ of B such that $\bar{L} \circ j_A = \pi$ and $\bar{L} \circ j_G = U$, and*

(c) $B = \overline{\operatorname{span}}\{\, j_A(a) j_G(z) : a \in A \text{ and } z \in C_c(G)\,\}$.

Then there is an isomorphism

$$j : B \to A \rtimes_\alpha G$$

such that

$$\bar{j} \circ j_A = i_A \quad \text{and} \quad \bar{j} \circ j_G = i_G, \tag{2.39}$$

where (i_A, i_G) is the canonical covariant homomorphism of (A, G, α) into $M(A \rtimes_\alpha G)$ defined in Proposition 2.34 on page 54.

Remark 2.62. Notice that the crossed product $A \rtimes_\alpha G$ is an example of a C^*-algebra B satisfying (a), (b) and (c) above where $(j_A, j_G) = (i_A, i_G)$ and $L_{(\pi,U)} := \pi \rtimes U$. Property (a) follows from Propositions 2.34 on page 54, and Property (b) from Proposition 2.39 on page 58. Property (c) follows from Lemma 1.87 on page 29 together with Equation (2.28) of Corollary 2.36 on page 57.

Lemma 2.63. *Suppose that (B, j_A, j_G) satisfies* (a), (b) *and* (c) *as in Theorem 2.61. Then $j_A : A \to M(B)$ is nondegenerate, and if $\{\, e_i \,\}$ is an approximate identity in A, $j_A(e_i) \to 1$ strictly in $M(B)$.*

Proof. It is easy to see that A^2 is dense in A.[13] Therefore, $\{\, j_A(ab) j_G(z) : a, b \in A \text{ and } z \in C_c(G)\,\}$ is dense in B by property (c). Thus j_A is nondegenerate. The last assertion follows easily. □

Proof of Theorem 2.61. Suppose that (B, j_A, j_G) satisfies (a), (b) and (c) as above. Let $\rho : A \rtimes_\alpha G \to B(\mathcal{H}_\rho)$ be a faithful representation of $A \rtimes_\alpha G$. Let

$$\pi := \bar{\rho} \circ i_A \quad \text{and} \quad u := \bar{\rho} \circ i_G$$

[13] In fact, $A^2 = A$ by the Cohen Factorization Theorem [139, Proposition 2.33].

Then (π, u) is a nondegenerate covariant representation of (A, G, α) (Proposition 2.39 on page 58). Property (b) implies that there is a nondegenerate representation $L : B \to B(\mathcal{H}_\rho)$ such that $\bar{L} \circ j_A = \pi$ and $\bar{L} \circ j_G = u$. Note that if $a \in A$ and $z \in C_c(G)$ then

$$\begin{aligned} L\big(j_A(a)j_G(z)\big) &= \pi(a) \int_G z(s)u_s \, d\mu(s) \\ &= \bar{\rho}\big(i_A(a)\big) \int_G z(s)\bar{\rho}\big(i_G(s)\big) \, d\mu(s) \\ &= \bar{\rho}\big(i_A(a)\big)\bar{\rho}\big(i_G(z)\big) \\ &= \rho\big(i_A(a)i_G(z)\big) \end{aligned}$$

It follows that $j := \rho^{-1} \circ L$ is a homomorphism from B to $A \rtimes_\alpha G$ mapping generators to generators; thus j is surjective and clearly satisfies (2.39).

To finish the proof, we provide an inverse for j. We could do this by reversing the roles of B and $A \rtimes_\alpha G$ above, and noticing we used nothing about $(A \rtimes_\alpha G, i_A, i_G)$ except that it satisfies (a), (b) and (c). Alternatively, we can invoke Proposition 2.40 on page 59 to conclude that there is a nondegenerate homomorphism $j_A \rtimes j_G : A \rtimes_\alpha G \to M(B) = \mathcal{L}(B_B)$ such that

$$j_A \rtimes j_G\big(i_A(a)i_G(z)\big) = j_A(a)j_G(z).$$

Then it follows that $j_A \rtimes j_G(A \rtimes_\alpha G) \subset B$. To see that $j_A \rtimes j_G$ is the required inverse, just note that $j_A \rtimes j_G \circ j$ and $j \circ j_A \rtimes j_G$ are the identity on generators. □

Definition 2.64. Two dynamical systems (A, G, α) and (D, G, δ) are *equivariantly isomorphic* if there is an isomorphism $\varphi : A \to D$ such that $\varphi\big(\alpha_s(a)\big) = \delta_s\big(\varphi(a)\big)$ for all $s \in G$ and $a \in A$. We call φ an *equivariant isomorphism.*

Lemma 2.65. *Suppose that φ is an equivariant isomorphism of (A, G, α) onto (D, G, δ). Then the map $\varphi \rtimes \mathrm{id} : C_c(G, A) \to C_c(G, D)$ defined by*

$$\varphi \rtimes \mathrm{id}(f)(s) := \varphi\big(f(s)\big)$$

extends to an isomorphism of $A \rtimes_\alpha G$ onto $D \rtimes_\delta G$.

The result follows immediately from Corollary 2.48 on page 63. However, it might also be instructive to see a proof using Theorem 2.61 on the facing page.

Proof. We'll produce an isomorphism $j : A \rtimes_\alpha G \to D \rtimes_\delta G$ using Theorem 2.61 (with $A \rtimes_\alpha G$ playing the role of B). Let (i_A, i_G) and (k_D, k_G) be the canonical covariant homomorphisms for (A, G, α) and (D, G, δ), respectively. Then define (j_D, j_G) from (D, G, δ) into $M(A \rtimes_\alpha G)$ by letting $j_D(d) := i_A\big(\varphi^{-1}(d)\big)$ and $j_G = i_G$. It is easy to see that (j_D, j_G) is covariant. If (π, U) is a covariant representation of (D, G, δ), then $(\pi \circ \varphi, U)$ is a covariant representation of (A, G, α) and we can define $L = L_{(\pi,U)}$ to be $(\pi \circ \varphi) \rtimes U$. Clearly

$$((\pi \circ \varphi) \rtimes U)^{-} \circ j_D(d) = \pi(d) \quad \text{and} \quad ((\pi \circ \varphi) \rtimes U)^{-} \circ j_G = U_s.$$

Thus conditions (a) and (b) of Theorem 2.61 on page 72 are satisfied. Since $j_D(d)j_G(z) = i_A\big(\varphi^{-1}(d)\big)i_G(z)$, it is clear that condition (c) is also satisfied. Therefore there is an isomorphism $j : A \rtimes_\alpha G \to D \rtimes_\delta G$ such that

$$j\big(i_A(a)i_G(z)\big) = j\big(j_D\big(\varphi(a)\big)j_G(z)\big) = k_D\big(\varphi(a)\big)k_G(z).$$

It follows that $j = \varphi \rtimes \mathrm{id}$. □

Definition 2.66. Two dynamical systems (A, G, α) and (A, G, β) are called *exterior equivalent* if there is a strictly continuous unitary-valued function $u : G \to UM(A)$ such that

(a) $\alpha_s(a) = u_s\beta_s(a)u_s^*$ for all $s \in G$ and $a \in A$, and
(b) $u_{st} = u_s\bar{\beta}_s(u_t)$ for all $s, t \in G$.

The map u is called a *unitary* 1*-cocycle.*

Remark 2.67. It should be noted that Definition 2.66 is symmetric in α and β: $v_s := u_s^*$ is a unitary 1-cocycle implementing an equivalence between β and α.

Lemma 2.68. *Suppose that (A, G, α) and (A, G, β) are exterior equivalent via a* 1*-cocycle u as in Definition 2.66. Then the map sending $f \in C_c(G, A)$ to $\varphi(f)$, where*

$$\varphi(f)(s) := f(s)u_s$$

extends to an isomorphism between $A \rtimes_\alpha G$ and $A \rtimes_\beta G$.

Remark 2.69. As with Lemma 2.65 on the preceding page, this result could be proved using either Theorem 2.61 or Corollary 2.47. Instead, we give a proof using only the properties of covariant homomorphisms.

Proof. Let (i_A, i_G) be the canonical covariant homomorphism of (A, G, α) into $M(A \rtimes_\alpha G)$, and let (k_A, k_G) be the canonical covariant homomorphism of (A, G, β) into $M(A \rtimes_\beta G)$. Let

$$j_A(a) := k_A(a) \quad \text{and} \quad j_G(s) := \bar{k}_A(u_s)k_G(s).$$

Since

$$\begin{aligned} j_G(st) &= \bar{k}_A(u_{st})k_G(st) \\ &= \bar{k}_A(u_s)\bar{k}_A\big(\bar{\beta}_s(u_t)\big)k_G(s)k_G(t) \\ &= \bar{k}_A(u_s)k_G(s)\bar{k}_A(u_t)k_G(t) \\ &= j_G(s)j_G(t), \end{aligned}$$

j_G is a strictly continuous unitary-valued homomorphism. Furthermore

$$\begin{aligned} j_G(s)j_A(a)j_G(s)^* &= \bar{k}_A(u_s)k_G(s)k_A(a)k_G(s)^*\bar{k}_A(u_s^*) \\ &= \bar{k}_A(u_s)k_A\big(\beta_s(a)\big)\bar{k}_A(u_s^*) \\ &= k_A\big(\alpha_s(a)\big). \end{aligned}$$

and (j_A, j_G) is covariant. Therefore $j_A \rtimes j_G$ is a nondegenerate homomorphism of $A \rtimes_\alpha G$ into $M(A \rtimes_\beta G)$. Furthermore if $f \in C_c(G, A)$, then

$$\begin{aligned} j_A \rtimes j_G(f) &= \int_G j_A\big(f(s)\big) j_G(s)\, d\mu(s) \\ &= \int_G k_A\big(f(s)u_s\big) k_G(s)\, d\mu(s) \\ &= k_A \rtimes k_G\big(\varphi(f)\big). \end{aligned}$$

This proves that φ extends to a homomorphism of $A \rtimes_\alpha G$ into $A \rtimes_\beta G$. Reversing the roles of α and β shows that $\varphi^{-1}(f)(s) := f(s)u_s^*$ is an inverse for φ, and φ is an isomorphism as claimed. □

Definition 2.70. A dynamical system $\alpha : G \to \operatorname{Aut} A$ is *unitarily implemented* or just *unitary* if there is a strictly continuous homomorphism $u : G \to UM(A)$ such that $\alpha_s(a) = u_s a u_s^*$ for all $a \in A$ and $s \in G$.

Remark 2.71. A dynamical system $\alpha : G \to \operatorname{Aut} A$ is unitary if and only if it is exterior equivalent to the trivial system $\iota : G \to \operatorname{Aut} A$ where $\iota_s := \mathrm{id}_A$ for all $s \in G$.

Before continuing further, we need some remarks on tensor products and the tensor product of two dynamical systems.

Remark 2.72 (Tensor Products of C^*-algebras). The theory of the tensor product of C^*-algebras has some subtleties which impact the study of crossed products. All that we need, and more, can be found in [139, Appendix B]. For convenience, we mention some properties we'll need here. Usually, the algebraic tensor product $A \odot B$ of two C^*-algebras is not a C^*-algebra. Instead, one looks for a norm $\|\cdot\|_\alpha$ (called a C^*-norm) on $A \odot B$ so that the completion $A \otimes_\alpha B$ is a C^*-algebra. It is a fact of life that there can be more than one C^*-norm on $A \odot B$. The *spatial norm* $\|\cdot\|_\sigma$ has the property that given two representation $\rho_A : A \to B(\mathcal{H}_A)$ and $\rho_B : B \to B(\mathcal{H}_B)$ there is a representation $\rho_A \otimes \rho_B : A \otimes_\sigma B \to B(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\rho_A \otimes \rho_B(a \otimes b) = \rho_A(a) \otimes \rho_B(b)$. If ρ_A and ρ_B are faithful, then

$$\Big\|\sum a_i \otimes b_i\Big\|_\sigma = \Big\|\sum \rho_A(a_i) \otimes \rho_B(b_i)\Big\|.$$

On the other hand, the *maximal norm* $\|\cdot\|_{\max}$ has the property that if $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ are representations with *commuting ranges* — that is, $\pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$ for all $a \in A$ and $b \in B$ — then there is a representation $\pi_A \otimes_{\max} \pi_B : A \otimes_{\max} B \to B(\mathcal{H})$ such that $\pi_A \otimes_{\max} \pi_B(a \otimes b) = \pi_A(a)\pi_B(b)$. In fact,[14]

$$\Big\|\sum a_i \otimes b_i\Big\|_{\max} = \sup\Big\{ \Big\|\sum \pi_A(a_i)\pi_B(b_i)\Big\| :$$
$$\pi_A \text{ and } \pi_B \text{ have commuting ranges} \Big\}.$$

It is nontrivial to show that if $\|\cdot\|_\alpha$ is any C^*-norm on $A \odot B$, then $\|\cdot\|_\sigma \le \|\cdot\|_\alpha \le \|\cdot\|_{\max}$. For this reason, the spatial norm is often called the *minimal norm.* For a very large class of C^*-algebras A the spatial norm coincides with maximal norm on $A \odot B$ for *every* C^*-algebra B and there is a unique C^*-norm on $A \odot B$. By definition, A is called *nuclear* when $A \odot B$ has a unique C^*-norm for all B. The class of nuclear C^*-algebras includes all GCR C^*-algebras.

As illustrated by our next result and Lemma 2.75 on page 78, it is the maximal tensor product which seems to play the key role in the study of crossed products.[15] In fact, the next two results motivate the assertion that a crossed product can be though of as a twisted maximal tensor product of A and $C^*(G)$.

Lemma 2.73. *If* $\iota : G \to \operatorname{Aut} A$ *is trivial, then*

$$A \rtimes_\iota G \cong A \otimes_{\max} C^*(G).$$

Proof. We'll use Theorem 2.61 on page 72 to produce an isomorphism

$$j : A \otimes_{\max} C^*(G) \to A \rtimes_\iota G.$$

Let $j_A : A \to M(A \otimes_{\max} C^*(G))$ and $j_G : C^*(G) \to M(A \otimes_{\max} C^*(G))$ be the natural commuting homomorphisms defined, for example, in [139, Theorem B.27]. Since j_G is certainly nondegenerate, it is the integrated form of the strictly continuous unitary-valued homomorphism $j_G : G \to UM(A \otimes_{\max} C^*(G))$, where $j_G(s) := \bar{\jmath}_G\big(i_G(s)\big)$. Note that

$$\begin{aligned} j_G(s)j_A(a)(b \otimes z) &= j_G(s)j_G(z)j_A(ab) \\ &= j_G\big(i_G(s)z\big)j_A(a)j_A(b) \\ &= j_A(a)j_G(s)(b \otimes z). \end{aligned}$$

Thus $j_G(s)j_A(a)j_G(s)^* = j_A(a)$ and (j_A, j_G) is a covariant homomorphism of (A, G, ι) into $M(A \otimes_{\max} C^*(G))$.

If (π, U) is a nondegenerate covariant representation of (A, G, ι), then $U_s\pi(a)U_s^* = \pi(a)$ for all $s \in G$ and $a \in A$. But if $z \in C_c(G)$, we also have

$$\pi(a)U(z) = \int_G z(s)\pi(a)U_s \, d\mu(s) = U(z)\pi(a).$$

[14] As with the definition of the universal norm in Lemma 2.27 on page 52, we have to be a bit careful to see that we are taking the supremum over a set. There are ways to define the maximal norm which avoid this subtly (cf. [139, Proposition B.25]), but the supremum definition gives a better flavor of the properties of the maximal norm, and can be justified with a bit of set theory: see Remark 2.28 on page 52.

[15] When we get around to defining reduced crossed products in Section 7.2, it will turn out that the appropriate norm for reduced crossed products is the spatial norm.

Therefore π and U are commuting representations of A and $C^*(G)$, respectively, and we can define a representation of $A \otimes_{\max} C^*(G)$ by

$$L = L_{(\pi,U)} = \pi \otimes_{\max} U$$

as in [139, Theorem B.27(b)]. We also have $\bar{L} \circ j_A(a) = \pi(a)$ and $\bar{L} \circ j_G(z) = U(z)$ for $a \in A$ and $z \in C_c(G)$. Then

$$\begin{aligned}\bar{L}\big(j_G(s)\big)L(a \otimes z) &= L\big(j_G(s)j_G(z)j_A(a)\big)\\ &= L\big(j_G\big(i_G(s)z\big)j_A(a)\big)\\ &= U\big(i_G(s)z\big)\pi(a)\\ &= U_sU(z)\pi(a)\\ &= U_sL(a \otimes z).\end{aligned}$$

Thus $\bar{L} \circ j_G(s) = U_s$, and we've established conditions (a) and (b) of Theorem 2.61 on page 72. But the elementary tensors $a \otimes z = j_A(a)j_G(z)$ are certainly dense in $A \otimes_{\max} C^*(G)$, so we're done. □

Remark 2.74 (Tensor product systems). If A and B are C^*-algebras with $\alpha \in \operatorname{Aut} A$ and $\beta \in \operatorname{Aut} B$, then [139, Lemma B.31] implies that there is a $\alpha \otimes_{\max} \beta \in \operatorname{Aut}(A \otimes_{\max} B)$ such that $\alpha \otimes_{\max} \beta(a \otimes b) = \alpha(a) \otimes \beta(b)$. Similarly, [139, Proposition B.13] implies there is a $\alpha \otimes \beta \in \operatorname{Aut}(A \otimes_\sigma B)$. If (A, G, α) and (B, G, β) are dynamical systems, then we clearly obtain homomorphisms

$$\alpha \otimes_{\max} \beta : G \to \operatorname{Aut}(A \otimes_{\max} B) \quad \text{and} \quad \alpha \otimes \beta : G \to \operatorname{Aut}(A \otimes_\sigma B).$$

To see that these are also dynamical systems, we still have to verify that the actions are strongly continuous. However, since automorphisms are isometric and since $A \odot B$ is dense in any completion, this follows from estimates such as

$$\begin{aligned}&\big\|(\alpha \otimes \beta)_s(t) - (\alpha \otimes \beta)_r(t)\big\|_\sigma\\ &\qquad\le \Big\|(\alpha \otimes \beta)_s\Big(t - \sum a_i \otimes b_i\Big)\Big\|_\sigma\\ &\qquad\qquad + \Big\|\sum \alpha_s(a_i) \otimes \beta_s(b_i) - \alpha_r(a_i) \otimes \beta_r(b_i)\Big\|_\sigma\\ &\qquad\qquad\qquad + \Big\|(\alpha \otimes \beta)_r\Big(\sum a_i \otimes b_i - t\Big)\Big\|_\sigma\\ &\qquad\le 2\Big\|t - \sum a_i \otimes b_i\Big\|_\sigma + \sum \big\|\alpha_s(a_i) \otimes \beta_s(b_i) - \alpha_r(a_i) \otimes \beta_r(b_i)\big\|_\sigma.\end{aligned}$$

Furthermore, if ρ_A and ρ_B are faithful representations of A and B, respectively, then the same is true of $\rho_A \circ \alpha$ and $\rho_B \circ \beta$. As in Remark 2.72 on page 75, if $\kappa : A \otimes_{\max} B \to A \otimes_\sigma B$ is the natural map,

$$\begin{aligned}\ker \kappa &= \ker\big((\rho_A \otimes 1) \otimes_{\max} (1 \otimes \rho_B)\big)\\ &= \ker\big((\rho_A \otimes 1) \otimes_{\max} (1 \otimes \rho_B) \circ (\alpha \otimes_{\max} \beta)\big).\end{aligned}$$

Consequently, $\ker \kappa$ is $\alpha \otimes_{\max} \beta$-invariant and $\alpha \otimes \beta$ is the induced action, $(\alpha \otimes_{\max} \beta)^{\ker \kappa}$, on $A \otimes_\sigma B \cong A \otimes_{\max} B / \ker \kappa$ (see Section 3.4). Except when confusion is likely, it is common practice to simply write $\alpha \otimes \beta$ in place of $\alpha \otimes_{\max} \beta$.

At this point, we only want to consider the product of the trivial action with an arbitrary dynamical system. Although an apparently straightforward example, it has a number of interesting applications such as Corollary 7.18 on page 203.

Lemma 2.75. *Suppose that (C, G, γ) is a dynamical system and that D is a C^*-algebra. Then there is an isomorphism*

$$(C \otimes_{\max} D) \rtimes_{\gamma\otimes\mathrm{id}} G \cong C \rtimes_\gamma G \otimes_{\max} D$$

which carries $(c \otimes_{\max} d) \otimes f \mapsto (c \otimes f) \otimes_{\max} d$, and which intertwines the representation $(\pi_C \rtimes V) \otimes_{\max} \pi_D$ of $C \rtimes_\gamma G \otimes_{\max} D$ with the representation $(\pi_C \otimes_{\max} \pi_D) \rtimes V$ of $(C \otimes_{\max} D) \rtimes_{\gamma\otimes\mathrm{id}} G$.

Proof. We want to apply Theorem 2.61 on page 72 to the system $(C \otimes_{\max} D, G, \gamma \otimes \mathrm{id})$ and the C^*-algebra $C \rtimes_\gamma G \otimes_{\max} D$. For any C^*-algebras A and B, we'll use the letter 'k' for the natural nondegenerate maps $k_A : A \to M(A \otimes_{\max} B)$ and $k_B : B \to M(A \otimes_{\max} B)$ as in [139, Theorem B27]. First we need a covariant homomorphism $(j_{C\otimes_{\max}D}, j_G)$ of $(C \otimes_{\max} D, G, \gamma \otimes \mathrm{id})$ into $M\big((C \rtimes_\gamma G) \otimes_{\max} B\big)$. Let (i_C, i_G) be the canonical covariant homomorphism of (A, G, γ) into $M(C \rtimes_\gamma G)$, and define j_C to be the composition

$$C \xrightarrow{i_C} M(C \rtimes_\gamma G) \xrightarrow{\bar{k}_{C\rtimes_\gamma G}} M(C \rtimes_\gamma G \otimes_{\max} B).$$

Let $j_D := k_D : D \to M\big((C \rtimes_\gamma G) \otimes_{\max} D\big)$. Then j_C and j_D have commuting ranges and we can let $j_{C\otimes_{\max}D} := j_C \otimes_{\max} j_D$. We let j_G be the composition

$$G \xrightarrow{i_G} M(C \rtimes_\gamma G) \xrightarrow{\bar{k}_{C\rtimes_\gamma G}} M\big((C \rtimes_\gamma G) \otimes_{\max} D\big).$$

Since (i_C, i_G) is covariant, it is not hard to check that $(j_{C\otimes_{\max}D}, j_G)$ is covariant and we've established (a) of Theorem 2.61 on page 72.

Now suppose that (π, U) is a nondegenerate covariant representation of $(C \otimes_{\max} D, G, \gamma \otimes \mathrm{id})$. Then $\pi = \pi_C \otimes_{\max} \pi_D$ (where π_C and π_D are characterized by $\pi(c \otimes_{\max} d) = \pi_C(c)\pi_D(d)$). Then it is not hard to verify that (π_C, U) is a covariant representation of (C, G, γ):

$$\begin{aligned} U_s \pi_C(c)\pi(c' \otimes_{\max} d) &= U_s \pi(cc' \otimes_{\max} d) \\ &= \pi\big(\gamma_s(cc') \otimes_{\max} d\big)U_s \\ &= \pi_C\big(\gamma_s(c)\big)\pi\big(\gamma_s(c') \otimes_{\max} d\big)U_s \\ &\pi_C\big(\gamma_s(c)\big)U_s\pi(c' \otimes_{\max} d). \end{aligned}$$

We also want to check that $\pi_C \rtimes U$ and π_D have commuting ranges:

$$\begin{aligned}\pi_C \rtimes U(c\otimes z)\pi_D(d)\pi(c'\otimes_{\max} d') &= \pi_C(c)U(z)\pi(c'\otimes_{\max} dd') \\ &= \pi_C(c)\int_G z(s)U_s\pi(c'\otimes_{\max} dd')\,d\mu(s) \\ &= \pi_C(c)\int_G z(s)\pi\big(\gamma_s(c')\otimes_{\max} dd'\big)U_s\,d\mu(s) \\ &= \pi_C(c)\pi_D(d)\int_G z(s)\pi\big(\gamma_s(c')\otimes_{\max} d'\big)U_s\,d\mu(s) \\ &= \pi_D(d)\pi_C(c)\int_G z(s)U_s\pi(c'\otimes_{\max} d')\,d\mu(s) \\ &= \pi_D(d)\pi_C \rtimes U(c\otimes z)\pi(c'\otimes_{\max} d').\end{aligned}$$

Therefore we can form $L = L_{(\pi,U)} := \pi_C \rtimes U \otimes_{\max} \pi_D$ and then verify that $L \circ j_{C\otimes_{\max} D} = \pi$ and $L\circ j_G = U$. This verifies part (b).

For part (c), consider

$$\begin{aligned}j_{C\otimes_{\max} D}(c\otimes_{\max} d)j_G(z) &= j_C(c)j_D(d)j_G(z) \\ &= j_C(c)j_G(z)j_D(d) \\ &= i_C(c)i_G(z)\otimes_{\max} d \\ &= c\otimes z\otimes_{\max} d.\end{aligned}$$

It follows from Theorem 2.61 on page 72 that $j := j_{C\otimes_{\max} D}\rtimes j_G$ is the required isomorphism, and that $j\circ L_{((\pi_C\otimes_{\max}\pi_D)\rtimes V)} = (\pi_C\otimes_{\max}\pi_D)\rtimes V$. Since $L_{((\pi_C\otimes_{\max}\pi_D)\rtimes V)} = (\pi_C\rtimes V)\otimes_{\max}\pi_D$, the assertion about intertwining representations follows. □

Notes and Remarks

The notion of a crossed product — at least as a purely algebraic object — dates to the beginning of the twentieth century. For example, the basic construction appears in [22] and [166] and a nice survey of this early work can be found in [14]. (These references were pointed out by an anonymous reviewer.) The idea of associating an operator algebra to a group of automorphisms of another operator algebra probably originates with the pioneering work of Murray and von Neumann [111]. A (reduced) C^*-crossed product construction appears in [163], and the connections to physics and covariant representations begins in earnest in [30]. A systematic study of transformation group C^*-algebras can be found in [49] and full formed crossed products appear in [100, 162, 173]. The current notation of $A\rtimes_\alpha G$ and the terminology "C^*-dynamical system" was popularized in the work of Olesen and Pedersen [114–116]. Although viewing the crossed product as a universal object for covariant representations is fundamental even to the early work on C^*-crossed products, the concept was formalized by Raeburn in [133].