Preface

Trinity: It's the question that drives us, Neo. It's the question that brought you here. You know the question just as I did.
Neo: What is the Matrix? ...
Morpheus: Do you want to know what it is? The Matrix is everywhere.... Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.
– From the movie "The Matrix".

What this book is about

This is the sequel to the book "The Ricci Flow: An Introduction" by two of the authors [**108**]. In the previous volume (henceforth referred to as Volume One) we laid some of the foundations for the study of Richard Hamilton's Ricci flow. The Ricci flow is an evolution equation which deforms Riemannian metrics by evolving them in the direction of minus the Ricci tensor. It is like a heat equation and tries to smooth out initial metrics. In some cases one can exhibit global existence and convergence of the Ricci flow. A striking example of this is the main result presented in Chapter 6 of Volume One: Hamilton's topological classification of closed 3-manifolds with positive Ricci curvature as spherical space forms. The idea of the proof is to show, for any initial metric with positive Ricci curvature, the normalized Ricci flow exists for all time and converges to a constant curvature metric as time approaches infinity. Note that on any closed 2-dimensional manifold, the normalized Ricci flow exists for all time and converges to a constant curvature metric. Many of the techniques used in Hamilton's original work in dimension 2 have influenced the study of the Ricci flow in higher dimensions. In this respect, of special note is Hamilton's 'meta-principle' of considering *geometric quantities which either vanish or are constant on gradient Ricci solitons.*

It is perhaps generally believed that the Ricci flow tries to make metrics more homogeneous and isotropic. However, for general initial metrics on closed manifolds, singularities may develop under the Ricci flow in dimensions as low as 3.[1] In Volume One we began to set up the study of singularities by discussing curvature and derivative of curvature estimates, looking at how generally dilations are done in all dimensions, and studying

[1] For noncompact manifolds, finite time singularities may even occur in dimension 2.

aspects of singularity formation in dimension 3. In this volume, we continue the study of the fundamental properties of the Ricci flow with particular emphasis on their application to the study of singularities. We pay particular attention to dimension 3, where we describe some aspects of Hamilton's and Perelman's nearly complete classification of the possible singularities.[2]

As we saw in Volume One, Ricci solitons (i.e., self-similar solutions), differential Harnack inequalities, derivative estimates, compactness theorems, maximum principles, and injectivity radius estimates play an important role in the study of the Ricci flow. The maximum principle was used extensively in the 3-dimensional results we presented. Some of the other techniques were presented only in the context of the Ricci flow on surfaces. In this volume we take a more detailed look at these general topics and also describe some of the fundamental new tools of Perelman which almost complete Hamilton's partial classification of singularities in dimension 3. In particular, we discuss Perelman's energy, entropy, reduced distance, and some applications. Much of Perelman's work is independent of dimension and leads to a new understanding of singularities. It is difficult to overemphasize the importance of the reduced distance function, which is a space-time distance-like function (not necessarily nonnegative!) which is intimately tied to the geometry of solutions of the Ricci flow and the understanding of forming singularities. We also discuss stability and the linearized Ricci flow. Here the emphasis is not just on one solution to the Ricci flow, but on the dependence of the solutions on their initial conditions. We hope that this direction of study may have applications to showing that certain singularity types are not generic.

This volume is divided into two parts plus appendices. For the most part, the division is along the lines of whether the techniques are geometric or analytic. However, this distinction is rather arbitrary since the techniques in Ricci flow are often a *synthesis of geometry and analysis*. The first part is intended as an introduction to some basic geometric techniques used in the study of the singularity formation in general dimensions. Particular attention is paid to finite time singularities on closed manifolds, where the spatial maximum of the curvature tends to infinity in finite time. We also discuss some basic 3-manifold topology and reconcile this with some classification results for 3-dimensional finite time singularities. The partial classification of such singularities is used in defining Ricci flow with surgery. In particular, given a good enough understanding of the singularities which can occur in dimension 3, one can perform topological-geometric surgeries on solutions to the Ricci flow either right before or at the singularity time. One would then like to continue the solution to the Ricci flow until the next singularity and iterate this process. In the end one hopes to infer the existence of a geometric decomposition on the underlying 3-manifold. This is what Hamilton's program aims to accomplish and this is the same framework on which

[2]Not all singularity models have been classified, even for finite time solutions of the Ricci flow on closed 3-manifolds. Apparently this is independent of Hamilton's program for Thurston geometrization.

Perelman's work is based. In view of the desired topological applications of Ricci flow, in Chapter 9 we give a more detailed review of 3-manifold topology than was presented in Volume One. We hope to discuss the topics of nonsingular solutions (and their variants), where one can infer the existence of a geometric decomposition, surgery techniques, and more advanced topics in the understanding of singularities elsewhere.[3]

The second part of this volume emphasizes analytic and geometric techniques which are useful in the study of Ricci flow, again especially in regards to singularity analysis. We hope that the second part of this volume will not only be helpful for those wishing to understand analytic and geometric aspects of Ricci flow but that it will also provide tools for understanding certain technical aspects of Ricci flow. The appendices form an eclectic collection of topics which either further develop or support directions in this volume.

We have endeavored to make each of the chapters as self-contained as possible. In this way it is hoped that this volume may be used not only as a text for self-study, but also as a reference for those who would like to learn any of the particular topics in Ricci flow. To aid the reader, we have included a detailed guide to the chapters and appendices of this volume and in the first appendix we have also collected the most relevant results from Volume One for handy reference.

For the reader who would like to learn more about details of Perelman's work on Hamilton's program, we suggest the following excellent sources: Kleiner and Lott [**231**], Sesum, Tian, and Wang [**326**], Morgan and Tian [**273**], Chen and Zhu [**81**], Cao and Zhu [**56**], and Topping [**356**]. For further expository accounts, please see Anderson [**5**], Ding [**126**], Milnor [**267**], and Morgan [**272**]. Part of the discussion of Perelman's work in this volume was derived from notes of four of the authors [**102**].

Finally a word about notation; if an unnumbered formula appears on p. $\heartsuit\spadesuit$ of Volume One, we refer to it as (V1-p. $\heartsuit\spadesuit$); if the equation is numbered $\diamondsuit.\clubsuit$, then we refer to it as (V1-$\diamondsuit.\clubsuit$).

Highlights of Part I

In Part I of this volume we continue to lay the foundations of Ricci flow and give more geometric applications. We also discuss some aspects of Perelman's work on the Ricci flow.[4] Some highlights of Part I of this volume are the following:

(1) Proof of the existence of the Bryant steady soliton and rotationally symmetric expanding gradient Ricci solitons. Examples of homogeneous Ricci solitons. Triviality of breather solutions (no nontrivial steady or expanding breathers result). The Buscher duality transformation of gradient Ricci solitons of warped product type. An

[3]Some of these topics will appear in Part II of this volume.

[4]Further treatment of Perelman's work will appear in Part II and elsewhere.

open problem list on the geometry and classification of Ricci solitons.

(2) Introduction to the Kähler–Ricci flow. Long-time existence of the Kähler–Ricci flow on Kähler manifolds with first Chern class having a sign. Convergence of the Kähler–Ricci flow on Kähler manifolds with negative first Chern class. Construction of the Koiso solitons and other $\mathrm{U}(n)$-invariant solitons. Differential Harnack estimates and their applications under the assumption of nonnegative bisectional curvature. A survey of uniformization-type results for complete noncompact Kähler manifolds with positive curvature.

(3) Proof of the global version of Hamilton's Cheeger–Gromov-type compactness theorem for the Ricci flow. We take care to follow Hamilton and prove the compactness theorem for the Ricci flow in the category of pointed solutions with the convergence in C^∞ on compact sets. Outline of the proof of the local version of the aforementioned result. Application to the existence of singularity models.

(4) A unified approach to Perelman's monotonicity formulas for energy and entropy and the expander entropy monotonicity formula. Perelman's λ-invariant and application to the second proof of the no nontrivial steady or expanding breathers result. Other entropy results due to Hamilton and Bakry-Emery.

(5) Proof of the no local collapsing theorem assuming only an upper bound on the scalar curvature. Relation of no local collapsing and Hamilton's little loop conjecture. Perelman's μ- and ν-invariants and application to the proof of the no shrinking breathers result. Discussion of Topping's diameter control result. Relation between the variation of the modified scalar curvature and the linear trace Harnack quadratic. Second variation of energy and entropy.

(6) Theory of the reduced length. Comparison between the reduced length for static metrics and solutions of the Ricci flow. The $\mathcal{L}$-length, L-, $\bar{L}$-, and ℓ-distances and the first and second variation formulas for the $\mathcal{L}$-length. Existence of $\mathcal{L}$-geodesics and estimates for their speeds. Formulas for the gradient and time-derivative of the L-distance function and its local Lipschitz property. Formulas for the Laplacian and Hessian of L and differential inequalities for L, $\bar{L}$, and ℓ including a space-time Laplacian comparison theorem. Upper bound for the spatial minimum of ℓ. Formulas for ℓ on Einstein and gradient Ricci soliton solutions. $\mathcal{L}$-Jacobi fields, the $\mathcal{L}$-Jacobian, and the $\mathcal{L}$-exponential map, and their properties. Estimate for the time-derivative of the $\mathcal{L}$-Jacobian. Bounds for ℓ, its space-derivative, and its time-derivative. Properties of Lipschitz functions applied to ℓ and equivalence of notions of supersolutions in view of differential inequalities for ℓ.

(7) Applications of the reduced distance. Reduced volume of a static metric and its monotonicity. Monotonicity formula for the reduced volume and application to weakened no local collapsing for complete (possibly noncompact) solutions of the Ricci flow with bounded curvature. Certain backward limits of ancient κ-solutions are shrinking gradient Ricci solitons.
(8) A survey of basic 3-manifold topology and a brief description of the role of Ricci flow as an approach to the geometrization conjecture.
(9) Concise summary of the contents of Volume One including some main formulas and results. Formulas for the change in geometric quantities given a variation of the metric, evolution of geometric quantities under Ricci flow, maximum principles, curvature estimates, classical singularity theory including applications of classical monotonicity formulas, ancient 2-dimensional solutions, Hamilton's partial classification of 3-dimensional finite time singularities.
(10) List of some results in the basic theory of Ricci flow and the background Riemannian geometry. Bishop–Gromov volume comparison theorem, Laplacian and Hessian comparison theorems, Calabi's trick, geometry at infinity of gradient Ricci solitons, dimension reduction, properties of ancient solutions, existence of necks using the combination of classical singularity theory in dimension 3 and no local collapsing.
(11) Discussion of some results on the asymptotic behavior of complete solutions of the Ricci flow on noncompact manifolds diffeomorphic to Euclidean space. A brief discussion of the mean curvature flow (MCF) of hypersurfaces in Riemannian manifolds. Huisken's monotonicity formula for MCF of hypersurfaces in Euclidean space, including a generalization by Hamilton to MCF of hypersurfaces in Riemannian manifolds. Short-time existence (Buckland) and monotonicity formulas (Hamilton) for the cross curvature flow of closed 3-manifolds with negative sectional curvature.

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