CHAPTER 5

Energy, Monotonicity, and Breathers

Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things. – Sir Isaac Newton

The most beautiful thing we can experience is the mysterious. It is the source of all true art and science. – Albert Einstein

Much of the 'classical' study of the Ricci flow is based on the maximum principle. In large part, this is the point of view we have taken in Volume One. As we have seen in Section 8 in Chapter 5 of Volume One, a notable exception to this is Hamilton's entropy estimate, which holds for closed surfaces with positive curvature.[1] Even in this case, the time-derivative of the entropy is the space integral of Hamilton's trace Harnack quantity, which satisfies a partial differential inequality amenable to the maximum principle.[2] Indeed, this fact is the basis for Hamilton's original proof by contradiction of the entropy estimate which uses the global in time existence of the Ricci flow on surfaces.[3] Originally, Hamilton's entropy was a crucial component of the proofs for the convergence of the Ricci flow on surfaces and the classification of ancient solutions on surfaces. Via dimension reduction, the latter result has applications to singularity analysis in Hamilton's program on 3-manifolds.

An interesting direction is that of finding monotonicity formulas for integrals of local geometric quantities. Beautiful recent examples of this are Perelman's energy and entropy estimates in all dimensions. We briefly touched upon these estimates in Section 8 of Chapter 1 (Theorems 1.72 and 1.73) to motivate the study of gradient Ricci solitons. Perelman's energy is the time-derivative of a classical entropy ((5.64) in Section 4 below). Observe how the resulting calculation in Perelman's proof of the upper bound for the maximum time interval of existence of the gradient flow (Proposition 5.34) is reminiscent of Hamilton's proof of his entropy formula. In fact this upper bound says that a modified classical entropy is increasing (see (5.67)).

Monotonicity formulas usually have geometric applications. In particular, Perelman proved that any breather on a closed manifold is a Ricci soliton of the same type. This statement includes the shrinking case which remained open until his work; previously, we have seen the proofs of the

[1]See [**108**], Proposition 5.44, for the case of curvature changing sign.

[2]See (5.70).

[3]See Theorem 5.38.

expanding and steady cases in Proposition 1.13. To prove the nonexistence of nontrivial breathers, Perelman needed to do a separate study of each type of breather. However, in each case, the method is the same: introduce a new functional, study its properties, and apply them to the proof that there are no nontrivial breathers of each type. All such functionals have three basic characteristics:

- they are nondecreasing along systems of equations including the Ricci flow,
- they are invariant under diffeomorphisms and/or homotheties,
- their critical points are gradient Ricci solitons (of a different type in each case).

Moreover, Perelman's functionals are successive modifications of his initial functional $\mathcal{F}$ and are motivated by the consideration of gradient Ricci solitons of each type. So it is important to study the cases of the proofs successively in order to see how the evolutions of the functionals are used and how to modify the functionals gradually to define the entropy functional, which is the key to proving the shrinking case and where the proof follows essentially the same steps as the other two cases but uses the new functional.

In this chapter, we shall discuss in detail the energy functional, its geometric applications and its relation with classical entropy; in the next chapter we study Perelman's entropy and some of its geometric applications. The style of this chapter is that of filling in the details of §§1–2 of Perelman [**297**] in the hopes of aiding the reader in their perusal of [**297**]. Throughout this chapter $\mathcal{M}^n$ is a closed n-manifold.

1. Energy, its first variation, and the gradient flow

The Ricci flow is not a gradient flow of a functional on the space $\mathfrak{Met}$ of smooth metrics on a manifold $\mathcal{M}^n$ with respect to the standard L^2-inner product.[4] On the other hand, variational methods have played major roles in geometric analysis, partial differential equations, and mathematical physics. It was unusual that the Ricci flow, a natural geometric partial differential equation, should appear to be an exception to this. Perelman's introduction of the $\mathcal{F}$ functional (defined below) solved the important question of whether the Ricci flow can be seen as a gradient flow. More precisely, as we shall see in this and the following section, the Ricci flow is a *gradient-like flow*; it is a gradient flow when we enlarge the system. The key to solving the question above is to look for functionals whose critical points are Ricci solitons, that is, fixed points of the Ricci flow modulo diffeomorphisms and homotheties (so that the ambient space in which we consider Ricci flow is $\mathfrak{Met}/\mathfrak{Diff} \times \mathbb{R}_+$ instead of $\mathfrak{Met}$). This is consistent with the point of view we adopted in Chapter 1 on Ricci solitons.

[4] An exception is when $n = 2$ (see Appendix B of [**111**]), and more generally, for the Kähler-Ricci flow.

1.1. The energy functional $\mathcal{F}$. Let $C^\infty(\mathcal{M})$ denote the set of all smooth functions on a closed manifold $\mathcal{M}^n$. We define the **energy functional** $\mathcal{F} : \mathfrak{Met} \times C^\infty(\mathcal{M}) \to \mathbb{R}$ by

$$\mathcal{F}(g, f) \doteqdot \int_{\mathcal{M}} \left(R + |\nabla f|^2\right) e^{-f} d\mu. \tag{5.1}$$

Note, in addition to the metric, the introduction of a function f. This embeds the space of metrics in a larger space. We shall sometimes follow the physics literature and call f the **dilaton**.

Since $\Delta\left(e^{-f}\right) = \left(-\Delta f + |\nabla f|^2\right) e^{-f}$, we see from $\int_{\mathcal{M}} \Delta\left(e^{-f}\right) d\mu = 0$ that

$$\int_{\mathcal{M}} |\nabla f|^2 e^{-f} d\mu = \int_{\mathcal{M}} \Delta f e^{-f} d\mu. \tag{5.2}$$

So we have two other expressions for the energy:

$$\mathcal{F}(g, f) = \int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \tag{5.3}$$

$$= \int_{\mathcal{M}} (R + 2\Delta f - |\nabla f|^2) e^{-f} d\mu. \tag{5.4}$$

The second way of expressing the energy is motivated by the pointwise formula (5.43) in subsection 2.3.2 below.

LEMMA 5.1 (Elementary properties of $\mathcal{F}$).

(1) Dirichlet-type energy. *The geometric aspect of $\mathcal{F}$ is reflected by $\mathcal{F}(g, 0) = \int_{\mathcal{M}} R d\mu$ being the total scalar curvature and the function theory aspect of $\mathcal{F}$ is reflected by expressing it as*

$$\mathcal{F}(g, f) = \int_{\mathcal{M}} \left(4|\nabla w|^2 + R w^2\right) d\mu \doteqdot \mathcal{G}(g, w), \tag{5.5}$$

where $w = e^{-f/2}$, which is a Dirichlet energy with a potential term.

(2) Diffeomorphism invariance. *For any diffeomorphism φ of $\mathcal{M}$, we have*

$$\mathcal{F}(\varphi^* g, f \circ \varphi) = \mathcal{F}(g, f).$$

(3) Scaling. *For any $c > 0$ and b*

$$\mathcal{F}\left(c^2 g, f + b\right) = c^{n-2} e^{-b} \mathcal{F}(g, f).$$

EXERCISE 5.2. Prove the properties for the energy in the lemma above.

1.2. The first variation of $\mathcal{F}$. We use the symbol δ to denote the variation of a tensor. We shall denote the variations of the metric and dilaton as $\delta g = v \in C^\infty(T^*\mathcal{M} \otimes_S T^*\mathcal{M})$ and $\delta f = h \in C^\infty(\mathcal{M})$, and we define $V \doteqdot g^{ij}v_{ij}$. Routine calculations give

$$\delta_v \Gamma_{ij}^k(g) = \frac{1}{2}g^{kl}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right), \tag{5.6}$$

$$\delta_v \Gamma_{pj}^p = \frac{1}{2}\nabla_j V, \tag{5.7}$$

$$\delta_{(v,h)}\left(e^{-f}d\mu\right) = \left(\frac{V}{2} - h\right)e^{-f}d\mu. \tag{5.8}$$

We calculate the last one, for example,

$$\delta_{(v,h)}\left(e^{-f}d\mu\right) = -e^{-f}h\ d\mu + e^{-f}\frac{1}{2}g^{ij}v_{ij}\ d\mu = \left(\frac{V}{2} - h\right)e^{-f}d\mu. \tag{5.9}$$

LEMMA 5.3 (First variation of $\mathcal{F}$). *Then the* ***first variation of*** $\mathcal{F}$ *can be expressed as*

$$\begin{aligned}\delta_{(v,h)}\mathcal{F}(g,f) = &-\int_{\mathcal{M}} v_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f}d\mu \\ &+\int_{\mathcal{M}}\left(\frac{V}{2} - h\right)\left(2\Delta f - |\nabla f|^2 + R\right)e^{-f}d\mu,\end{aligned} \tag{5.10}$$

where $\delta_{(v,h)}\mathcal{F}(g,f)$ *denotes the variation of* $\mathcal{F}$ *at* (g,f) *in the direction* (v,h), *i.e.,*

$$\delta_{(v,h)}\mathcal{F}(g,f) \doteqdot \left.\frac{d}{ds}\right|_{s=0}\mathcal{F}(g+sv, f+sh).$$

PROOF. Recall (V1-p. 92), i.e.,

$$R_{ij} = R_{pij}^p = \partial_p\Gamma_{ij}^p - \partial_i\Gamma_{pj}^p + \Gamma_{ij}^q\Gamma_{pq}^p - \Gamma_{pj}^q\Gamma_{iq}^p,$$

so that

$$\delta R_{ij} = \nabla_p\left(\delta\Gamma_{ij}^p\right) - \nabla_i\left(\delta\Gamma_{pj}^p\right).$$

Since $\nabla_i\nabla_j = \partial_i\partial_j - \Gamma_{ij}^k\partial_k$ as an operator acting on functions, we have

$$\delta(\nabla_i\nabla_j f) = \nabla_i\nabla_j(\delta f) - \left(\delta\Gamma_{ij}^p\right)\nabla_p f.$$

Hence, using (5.7),

$$\begin{aligned}\delta(R_{ij} + \nabla_i\nabla_j f) &= \nabla_p\left(\delta\Gamma_{ij}^p\right) - \left(\delta\Gamma_{ij}^p\right)\nabla_p f + \nabla_i\left(\nabla_j(\delta f) - \delta\Gamma_{pj}^p\right) \\ &= e^f\nabla_p\left(e^{-f}\delta\Gamma_{ij}^p\right) + \nabla_i\nabla_j\left(h - \frac{V}{2}\right).\end{aligned}$$

We then compute

$$\begin{aligned}&\delta\left[(R_{ij} + \nabla_i\nabla_j f)e^{-f}d\mu\right] \\ &= \left[\begin{array}{c}\nabla_p\left(e^{-f}\delta\Gamma_{ij}^p\right) + e^{-f}\nabla_i\nabla_j\left(h - \frac{V}{2}\right) \\ +(R_{ij} + \nabla_i\nabla_j f)e^{-f}\left(\frac{V}{2} - h\right)\end{array}\right]d\mu.\end{aligned} \tag{5.11}$$

So using (5.8),

$$\begin{aligned}
&\delta\left[(R+\Delta f)\, e^{-f} d\mu\right] \\
&= g^{ij}\delta\left[(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu\right] - \delta g_{ij}\cdot(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu \\
&= \left[\nabla_p\left(e^{-f} g^{ij}\delta\Gamma_{ij}^p\right) + e^{-f}\Delta\left(h-\frac{V}{2}\right) + (R+\Delta f)\, e^{-f}\left(\frac{V}{2}-h\right)\right] d\mu \\
&- v_{ij}\cdot(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu.
\end{aligned}$$

Note that $\delta\Gamma_{ij}^p$ is a tensor and we do not need an explicit formula for it in the rest of the proof.

By the Divergence Theorem, we have

$$\begin{aligned}
\delta_{(v,h)}\mathcal{F}(g,f) &= \int_{\mathcal{M}} \delta\left[(R+\Delta f)\, e^{-f} d\mu\right] \\
&= \int_{\mathcal{M}} \left(-\Delta\left(e^{-f}\right) + (R+\Delta f)\, e^{-f}\right)\left(\frac{V}{2}-h\right) d\mu \\
&- \int_{\mathcal{M}} v_{ij}\cdot(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu,
\end{aligned}$$

from which the lemma follows. □

REMARK 5.4. By (5.11), the variation of $(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu$ is a divergence when $h = \frac{V}{2}$:

$$\delta\left[(R_{ij}+\nabla_i\nabla_j f)\, e^{-f} d\mu\right] = \nabla_p\left(e^{-f}\delta\Gamma_{ij}^p\right) d\mu.$$

Note also the factor $\frac{V}{2}-h$ in front of the second term in the RHS of (5.10). The significance of when this factor vanishes will be seen in subsection 1.4 below. By (5.8) we have

LEMMA 5.5. *Define the measure*

$$dm \doteqdot e^{-f} d\mu.$$

If the variations of g *and* f *keep the measure* dm *fixed, that is,* $\delta_{(v,h)}(dm) = 0$, *then*

$$V = 2h. \tag{5.12}$$

As a consequence of Lemma 5.3, we have

COROLLARY 5.6 (Measure-preserving first variation of $\mathcal{F}$). *For variations* (v,h) *with* $\delta_{(v,h)}\left(e^{-f} d\mu\right) = 0$, *we have*

$$\delta_{(v,h)}\mathcal{F}(g,f) = -\int_{\mathcal{M}} v_{ij}\left(R_{ij}+\nabla_i\nabla_j f\right) e^{-f} d\mu. \tag{5.13}$$

Notice in formula (5.10) for $\delta_{(v,h)}\mathcal{F}(g,f)$ the occurrence of the terms

$$R_{ij}^m \doteqdot \left(\mathrm{Rc}^m\right)_{ij} \doteqdot R_{ij}+\nabla_i\nabla_j f, \tag{5.14}$$

$$R^m \doteqdot R + 2\Delta f - |\nabla f|^2. \tag{5.15}$$

The first quantity vanishes on steady gradient solitons flowing along ∇f, whereas the second appeared in (5.4).[5] We call R^m_{ij} and R^m the **modified Ricci curvature and modified scalar curvature**, respectively; they are natural quantities from the perspective of the Ricci flow. We can rewrite

$$\mathcal{F}(g,f) = \int_{\mathcal{M}} g^{ij} R^m_{ij} e^{-f} d\mu = \int_{\mathcal{M}} R^m e^{-f} d\mu$$

and

$$\delta_{(v,h)} \mathcal{F}(g,f) = -\int_{\mathcal{M}} v_{ij} R^m_{ij} e^{-f} d\mu$$

when $V = 2h$.

1.3. The modified Ricci and scalar curvatures. In this subsection we digress by showing R^m_{ij} and R^m are natural quantities. Consider a closed Riemannian manifold $(\mathcal{M}^n, g)$ and a metric $\bar{g} = e^{-\frac{2}{n}f} g$ conformal to g. Let $\bar{R}_{ij} = \operatorname{Rc}(\bar{g})_{ij}$, $R_{ij} = \operatorname{Rc}(g)_{ij}$, $\bar{R} = R(\bar{g})$, and $R = R(g)$. The Ricci and scalar curvatures are related by (see for example subsection 7.2 of Chapter 1 in [**111**] or (A.2) and (A.3) in this volume)

(5.16)
$$\bar{R}_{ij} = R_{ij} + \left(1 - \frac{2}{n}\right) \nabla_i \nabla_j f + \frac{1}{n} \Delta f g_{ij} + \frac{n-2}{n^2} \nabla_i f \nabla_j f - \frac{n-2}{n^2} |\nabla f|^2 g_{ij}.$$

Tracing this yields

$$\bar{R} = e^{\frac{2}{n}f} \left(R + \frac{2(n-1)}{n} \Delta f - \frac{(n-1)(n-2)}{n^2} |\nabla f|^2 \right). \tag{5.17}$$

The volume forms are related by $d\mu_{\bar{g}} = e^{-f} d\mu$ and the total scalar curvature of $\bar{g}$ is given by

$$\int_{\mathcal{M}} \bar{R} d\mu_{\bar{g}} = \int_{\mathcal{M}} e^{-\frac{n-2}{n}f} \left(R + \frac{(n-1)(n-2)}{n^2} |\nabla f|^2 \right) d\mu,$$

where we integrated by parts, i.e., we used

$$\int_{\mathcal{M}} e^{-\frac{n-2}{n}f} \Delta f d\mu = \frac{n-2}{n} \int_{\mathcal{M}} e^{-\frac{n-2}{n}f} |\nabla f|^2 d\mu.$$

Now consider the Riemannian product $(\mathcal{M}^n, g) \times (T^q, h_q)$, where (T^q, h_q) is a flat unit volume q-dimensional torus. The formulas for the Ricci curvature and scalar curvature of metric $e^{-\frac{2}{n+q}f}(g + h_q)$ are given by (5.16) and (5.17), respectively, where we replace n by $n+q$. If we take the limit as $q \to \infty$ while fixing $(\mathcal{M}^n, g)$, then we obtain Perelman's modified Ricci tensor:

$$\lim_{q\to\infty} \operatorname{Rc}\left(e^{-\frac{2}{n+q}f}(g + h_q)\right) = \operatorname{Rc} + \nabla\nabla f \tag{5.18}$$

and Perelman's modified scalar curvature:

$$\lim_{q\to\infty} R\left(e^{-\frac{2}{n+q}f}(g + h_q)\right) = R + 2\Delta f - |\nabla f|^2, \tag{5.19}$$

[5] Earlier we also encountered these quantities in Chapter 1.

where we think of $\operatorname{Rc}\left(e^{-\frac{2}{n+q}f}(g+h_q)\right)$ and $R\left(e^{-\frac{2}{n+q}f}(g+h_q)\right)$ as quantities on $\mathcal{M}$ since they are independent of the point in T^q. The total scalar curvatures of $\left(\mathcal{M}\times T^q, e^{-\frac{2}{n+q}f}(g+h_q)\right)$ limit to Perelman's $\mathcal{F}$ functional:

$$\begin{aligned}
&\lim_{q\to\infty}\int_{\mathcal{M}\times T^q} R\left(e^{-\frac{2}{n+q}f}(g+h_q)\right) d\mu_{e^{-\frac{2}{n+q}f}(g+h_q)}\\
&=\lim_{q\to\infty}\int_{\mathcal{M}}\int_{T^q} R\left(e^{-\frac{2}{n+q}f}(g+h_q)\right) e^{-2f} d\mu_{h_q} d\mu_g\\
&=\int_{\mathcal{M}}\left(R+|\nabla f|^2\right) e^{-f} d\mu\\
&=\mathcal{F}(g,f).
\end{aligned}$$

Note that

$$g^{ij}R_{ij}^m = R+\Delta f = R^m - \Delta f + |\nabla f|^2.$$

There is an analogue of the contracted second Bianchi identity for R_{ij}^m and R^m. In particular we compute

$$\nabla_i R_{ij}^m = \nabla_i R_{ij} + \nabla_i\nabla_j\nabla_i f = \frac{1}{2}\nabla_j R + \nabla_j\Delta f + R_{jk}\nabla_k f$$

and

$$\frac{1}{2}\nabla_j R^m = \nabla_j\Delta f - \frac{1}{2}\nabla_j|\nabla f|^2 + \frac{1}{2}\nabla_j R = \frac{1}{2}\nabla_j R + \nabla_j \Delta f - \nabla_j\nabla_k f\nabla_k f,$$

which imply

$$\nabla_i R_{ij}^m = \frac{1}{2}\nabla_j R^m + R_{jk}^m \nabla_k f. \tag{5.20}$$

To understand this formula further, we define

$$\nabla^{*m} : C^\infty(T^*\mathcal{M}\otimes_S T^*\mathcal{M}) \to C^\infty(T^*\mathcal{M})$$

by

$$\left(\nabla^{*m} a\right)_j \doteqdot \nabla_i a_{ij} - a_{ji}\nabla_i f.$$

LEMMA 5.7. *The operator ∇^{*m} is the adjoint of $-\nabla$ with respect to the measure $dm = e^{-f}d\mu$.*

PROOF. For any symmetric 2-tensor a_{ij} and 1-form b_i,

$$\begin{aligned}
\int_{\mathcal{M}} a_{ij}(-\nabla_i) b_j e^{-f} d\mu &= \int_{\mathcal{M}} b_j \nabla_i\left(a_{ij}e^{-f}\right) d\mu\\
&= \int_{\mathcal{M}} b_j\left(\nabla_i a_{ij} - a_{ij}\nabla_i f\right) e^{-f} d\mu\\
&= \int_{\mathcal{M}} b_j\left(\nabla^{*m} a\right)_j e^{-f} d\mu.
\end{aligned}$$

□

Thus (5.20) implies the following, which is the analogue of the contracted second Bianchi identity.

LEMMA 5.8 (Modified contracted second Bianchi identity).

$$\nabla_i^{*m} R_{ij}^m \doteqdot (\nabla^{*m} \operatorname{Rc}^m)_j = \frac{1}{2} \nabla_j R^m. \tag{5.21}$$

1.4. The functional $\mathcal{F}^m$ and its gradient flow. Unlike $\mathcal{F}(g, f)$, we can obtain a functional of just the metric g by fixing a measure dm on a closed manifold $\mathcal{M}^n$; by a **measure** we mean a positive n-form on $\mathcal{M}$.[6] Define $\mathcal{F}^m : \mathfrak{Met} \to \mathbb{R}$ by

$$\mathcal{F}^m(g) \doteqdot \mathcal{F}(g, f) = \int_{\mathcal{M}} (R + |\nabla f|^2) dm, \tag{5.22}$$

where

$$f \doteqdot \log\left(\frac{d\mu}{dm}\right). \tag{5.23}$$

REMARK 5.9. The expression (5.23) makes sense because, given a fixed measure dm on $\mathcal{M}^n$, we can define the bijection

$$C^\infty(\Lambda^n T^* \mathcal{M}) \to C^\infty(\mathcal{M}),$$
$$\omega \mapsto \varphi,$$

where φ is defined so that $\omega = \varphi dm$ (here we have used the fact that $\Lambda^n T_x^* \mathcal{M} \cong \mathbb{R}$). Thanks to this, it is possible to define the quotient of two n-forms; e.g., if $\omega_1 = \varphi_1 dm$ and $\omega_2 = \varphi_2 dm$, where $\varphi_2 > 0$, then we set

$$\frac{\omega_1}{\omega_2} \doteqdot \frac{\varphi_1}{\varphi_2}.$$

Without using the notation f, we can write the energy of the metric g as

$$\mathcal{F}^m(g) = \int_{\mathcal{M}} \left(R + \left|\nabla \log\left(\frac{d\mu}{dm}\right)\right|^2\right) dm.$$

Using the modified Ricci and scalar curvatures, we can rewrite

$$\mathcal{F}^m(g) = \int_{\mathcal{M}} g^{ij} R_{ij}^m dm = \int_{\mathcal{M}} R^m dm.$$

REMARK 5.10. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism. Note that in general

$$\mathcal{F}^m(\varphi^* g) \neq \mathcal{F}^m(g).$$

That is, by fixing the measure dm, we get $\mathcal{F}^m(g)$, which breaks the diffeomorphism invariance of $\mathcal{F}(g, f)$. In subsection 3.1 of this chapter we shall solve this problem by considering a functional $\lambda(g)$ which is diffeomorphism-invariant.

[6] For a calculational motivation for fixing the measure, see the notes and commentary at the end of this chapter.

From (5.13) we have

$$\delta_v \mathcal{F}^m(g) = -\int_{\mathcal{M}} v_{ij}\left(R_{ij} + \nabla_i \nabla_j f\right) dm, \tag{5.24}$$

where f is given by (5.23). The L^2-inner product on $\mathfrak{Met}$, using the metric g and the measure dm, is defined by

$$\langle a_{ij}, b_{ij}\rangle_m (g) \doteqdot \int_{\mathcal{M}} \langle a_{ij}, b_{ij}\rangle_g \, dm.$$

Then by (5.24) we have

$$\nabla \mathcal{F}^m(g) = -(R_{ij} + \nabla_i \nabla_j f),$$

where f is given by (5.23). Hence (twice) the positive **gradient flow** of $\mathcal{F}^m$ is

$$\frac{\partial}{\partial t} g_{ij} = -2\left(R_{ij} + \nabla_i \nabla_j f\right), \tag{5.25}$$

$$f = \log\left(\frac{d\mu}{dm}\right). \tag{5.26}$$

We can also write the above system as

$$\frac{\partial}{\partial t} g_{ij} = -2\left[R_{ij} + \nabla_i \nabla_j \log\left(\frac{d\mu}{dm}\right)\right]. \tag{5.27}$$

We shall call an equation of the form (5.25) by itself, for some function f, a **modified Ricci flow**.

It is clear from taking $v_{ij} = -2\left(R_{ij} + \nabla_i \nabla_j f\right)$ in (5.13) that we obtain the following.

PROPOSITION 5.11 ($\mathcal{F}^m$ evolution under modified Ricci flow). *Suppose* $g(t)$ *is a solution of* (5.25)–(5.26). *Then*

$$\frac{d}{dt}\mathcal{F}^m(g(t)) = 2\int_{\mathcal{M}} \left|R_{ij} + \nabla_i \nabla_j f\right|^2 e^{-f} d\mu. \tag{5.28}$$

This is Perelman's **monotonicity formula for the gradient flow** of $\mathcal{F}^m$. We may rewrite (5.28) as

$$\frac{d}{dt}\mathcal{F}^m = \frac{d}{dt}\int_{\mathcal{M}} R^m dm = 2\int_{\mathcal{M}} \left|R^m_{ij}\right|^2 dm.$$

Note that for a general measure dm, solutions to the initial-value problem for the gradient flow may **not** exist even for a short time; however, as we shall see, this will not cause us problems in applications.

2. Monotonicity of energy for the Ricci flow

For monotonicity formula (5.28) to be useful, we need a corresponding version for solutions of the Ricci flow. In this section we show that solutions to equations (5.25) and (5.26), if they exist, differ from solutions of the Ricci flow by the pullback by time-dependent diffeomorphisms. Thus this gives a monotonicity formula for the energy of the Ricci flow.

2.1. A coupled system equivalent to the gradient flow of $\mathcal{F}^m$. There is a coupled system, i.e., (5.29)–(5.30), induced from the gradient flow (5.25)–(5.26) obtained simply by computing the evolution equation for $f = \log(d\mu/dm)$. As we shall see, this coupled system is equivalent to the gradient flow.

LEMMA 5.12 (Measure-preserving evolution of f under modified RF). *The function $f(t)$ in a solution $(g(t), f(t))$ of the gradient flow of $\mathcal{F}^m$* (5.25) *and* (5.26) *satisfies the following equation:*

$$\frac{\partial f}{\partial t} = -\Delta f - R.$$

PROOF. We calculate

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \log\left(\frac{d\mu}{dm}\right) = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial t} = -g^{ij}\left(R_{ij} + \nabla_i \nabla_j f\right).$$

□

Related to the above calculation, we have the following.

EXERCISE 5.13. Show that if $\omega_1(t)$ and $\omega_2(t)$ are time-dependent n-forms, then

$$\frac{\partial}{\partial t} \log\left(\frac{\omega_1}{\omega_2}\right) = \frac{\frac{\partial}{\partial t}\omega_1}{\omega_1} - \frac{\frac{\partial}{\partial t}\omega_2}{\omega_2},$$

where the quotient of two n-forms is defined as in Remark 5.9.

Hence we consider the **coupled modified Ricci flow**

$$\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f), \tag{5.29}$$

$$\frac{\partial f}{\partial t} = -\Delta f - R. \tag{5.30}$$

Note that the first equation is a modified Ricci flow equation whereas the second equation is a **backward heat equation**.

LEMMA 5.14. *The coupled modified Ricci flow equations* (5.29)–(5.30) *are equivalent to the gradient flow* (5.27).

PROOF. If $g(t)$ is a solution to (5.27), then by Lemma 5.12, $(g(t), f(t))$, where $f = \log(d\mu/dm)$, is a solution to the system (5.29)–(5.30).

Conversely, if $(g(t), f(t))$ is a solution to the system (5.29)–(5.30), then $dm \doteqdot e^{-f} d\mu$ satisfies

$$\frac{\partial}{\partial t}(dm) = \left(-\frac{\partial f}{\partial t} - R - \Delta f\right) e^{-f} d\mu = 0;$$

that is, $g(t)$ is a solution to (5.27) with dm as defined above. □

Hence, by (5.28), if $(g(t), f(t))$ is a solution to (5.29)–(5.30), then

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_{\mathcal{M}} |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu. \tag{5.31}$$

2.2. Correspondence between solutions of the gradient flow and solutions of the Ricci flow.

2.2.1. *Converting a solution of the gradient flow to a solution of Ricci flow.* We first show that solutions of the gradient flow, if they exist, give rise to solutions of the Ricci flow with the same initial data (Lemma 5.15). In particular, suppose we have a solution $\left(\bar{g}(t), \bar{f}(t)\right)$ of the flow (5.25) and (5.26) on $[0,T]$; then we can obtain a solution $g(t)$ of the Ricci flow on $[0,T]$ by modifying $\bar{g}(t)$ by diffeomorphisms generated by the gradient of $\bar{f}(t)$.

LEMMA 5.15 (Perelman's coupling for Ricci flow). *Let $\left(\bar{g}(t), \bar{f}(t)\right)$ be a solution of* (5.25) *and* (5.26) *on* $[0,T]$. *We define a* 1*-parameter family of diffeomorphisms $\Psi(t) : \mathcal{M} \to \mathcal{M}$ by*

$$\frac{d}{dt}\Psi(t) = \nabla_{\bar{g}(t)}\bar{f}(t), \tag{5.32}$$

$$\Psi(0) = \mathrm{id}_{\mathcal{M}}. \tag{5.33}$$

Then the pullback metric $g(t) = \Psi(t)^*\bar{g}(t)$ and the dilaton $f(t) = \bar{f} \circ \Psi(t)$ satisfy the following system:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc}, \tag{5.34}$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \tag{5.35}$$

REMARK 5.16. Basically we can see this from the facts that $L_{\nabla f}g = 2\nabla\nabla f$ and $L_{\nabla f}f = |\nabla f|^2$. For the sake of completeness we give the detailed calculations below.

PROOF. First note that by Lemma 3.15 of Volume One the system of ODE (5.32)–(5.33) is always solvable. We compute

$$\frac{\partial g}{\partial t} = \Psi^*\left(\frac{\partial \bar{g}}{\partial t}\right) + \Psi^*\left(L_{\nabla_{\bar{g}}\bar{f}}\bar{g}\right) = -2\Psi^*\left(\mathrm{Rc}\left(\bar{g}\right)\right) = -2\,\mathrm{Rc}\left(g\right).$$

To obtain the equation for $\frac{\partial f}{\partial t}$, we compute

$$\begin{aligned}\frac{\partial f}{\partial t} = \frac{\partial\left(\bar{f}\circ\Psi\right)}{\partial t} &= \frac{\partial \bar{f}}{\partial t}\circ\Psi + \left\langle\left(\bar{\nabla}\bar{f}\right)\circ\Psi, \frac{\partial\Psi}{\partial t}\right\rangle_{\bar{g}} \\ &= \left(-\bar{\Delta}\bar{f} - \bar{R}\right)\circ\Psi + \left|\left(\bar{\nabla}\bar{f}\right)\circ\Psi\right|^2_{\bar{g}} \\ &= -\Delta f - R + |\nabla f|^2,\end{aligned}$$

where barring a quantity indicates that it corresponds to $\bar{g}(t)$. □

So a solution to the gradient flow (5.25)–(5.26) yields a solution to the Ricci flow-backward heat equation system (5.34)–(5.35). Note that we can first solve the Ricci flow (5.34) forward in time and then solve (5.35) backward in time to get a solution of (5.34)–(5.35); this will be useful in applications.

2.2.2. *Converting a solution of Ricci flow to a solution of the gradient flow.* Now we show the converse of Lemma 5.15 by reversing the procedure of the last subsection. Given a solution $g(t)$ of the Ricci flow (5.34) on $[0,T]$, we can construct a solution $\left(\bar{g}(t), \bar{f}(t)\right)$ of the gradient flow (5.25) and (5.26) on $[0,T]$ by modifying the solution $g(t)$ by diffeomorphisms. In doing so, we also need to solve a backward heat equation with initial data at time T.

LEMMA 5.17. *Let $g(t)$ be a solution of the Ricci flow $\frac{\partial g}{\partial t} = -2\,\mathrm{Rc}$ on $[0,T]$ and let f_T be a function on $\mathcal{M}$.*

(i) We can solve the backward heat equation backwards in time

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R, \qquad t \in [0,T],$$
$$f(T) = f_T.$$

(ii) Given a solution $f(t)$ to the equation above, define the 1-parameter family of diffeomorphisms $\Phi(t) : \mathcal{M} \to \mathcal{M}$ by

$$\frac{d}{dt}\Phi(t) = -\nabla_{g(t)} f(t), \qquad \Phi(0) = \mathrm{id}_{\mathcal{M}}, \tag{5.36}$$

which is a system of ODE *and hence is solvable on $[0,T]$.*[7] *Then the pulled-back metrics $\bar{g}(t) = \Phi(t)^* g(t)$ and the pulled-back dilaton $\bar{f}(t) = f \circ \Phi(t)$ satisfy* (5.29) *and* (5.30)*.*

PROOF. (i) Let $\tau = T - t$. To get the existence of solutions to equation (5.35), we simply set

$$u \doteqdot e^{-f} \tag{5.37}$$

and compute that

$$\frac{\partial u}{\partial \tau} = \Delta u - Ru, \tag{5.38}$$

which is a linear parabolic equation and has a solution on $[0,T]$ with initial data at $\tau = 0$. Indeed, (5.38) follows from

$$\frac{\partial u}{\partial \tau} = -\frac{\partial u}{\partial t} = u\frac{\partial f}{\partial t} = u\left(-\Delta f + |\nabla f|^2 - R\right) = \Delta u - Ru.$$

(ii) Let $g(t)$ be a solution of the Ricci flow and let $f(t)$ be a solution of equation (5.35). One can verify that they satisfy (5.29) and (5.30) as in the proof of Lemma 5.15. □

2.2.3. *The adjoint heat equation.* Let $g(t)$ be a solution of Ricci flow and let $\square \doteqdot \frac{\partial}{\partial t} - \Delta$ be the heat operator acting on functions on $\mathcal{M} \times [0,T]$, where $\mathcal{M} \times [0,T]$ is endowed with the volume form $d\mu dt$. Its adjoint is

$$\square^* \doteqdot -\frac{\partial}{\partial t} - \Delta + R \tag{5.39}$$

[7] Again see Lemma 3.15 of Volume One.

since

$$\begin{aligned}\int_0^T \int_{\mathcal{M}} b\square a d\mu dt &= \int_0^T \int_{\mathcal{M}} b\left(\frac{\partial}{\partial t} - \Delta\right) a d\mu dt \\ &= \int_0^T \int_{\mathcal{M}} \left[a\left(-\frac{\partial}{\partial t} - \Delta\right) b d\mu - ab\frac{\partial}{\partial t} d\mu\right] dt \\ &= \int_0^T \int_{\mathcal{M}} a\square^* b d\mu dt\end{aligned}$$

for C^2 functions a and b on $\mathcal{M} \times [0, T]$ with compact support in $\mathcal{M} \times (0, T)$, where we used $\frac{\partial}{\partial t} d\mu = -R d\mu$.

By (5.38), if $(g(t), f(t))$ is a solution to (5.34)–(5.35), then $u = e^{-f}$ satisfies the **adjoint heat equation** (also known as the **conjugate heat equation**)

$$\square^* u = \left(-\frac{\partial}{\partial t} - \Delta + R\right) u = 0. \tag{5.40}$$

It is often better to think in terms of u than in terms of f since u satisfies the adjoint heat equation. In particular, the fundamental solution to the adjoint heat equation is important.

2.3. Monotonicity of $\mathcal{F}$ for the Ricci flow. In this subsection we give two proofs of the monotonicity of energy for Ricci flow. In the next section we give an application of this formula to the nonexistence of nontrivial breather solutions.

2.3.1. *Deriving the monotonicity of $\mathcal{F}$ from the monotonicity of $\mathcal{F}^m$.* By the diffeomorphism invariance of all the quantities under consideration, the monotonicity formula for the gradient flow implies a **monotonicity formula for the Ricci flow**. This involves a function $f(t)$ obtained by solving the backward heat equation (5.35).

LEMMA 5.18 ($\mathcal{F}$ energy monotonicity). *If $(g(t), f(t))$ is a solution to (5.34)–(5.35) on a closed manifold $\mathcal{M}^n$, then*

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_{\mathcal{M}} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu. \tag{5.41}$$

PROOF. Since $(g(t), f(t))$ is a solution to (5.34)–(5.35), $(\bar{g}(t), \bar{f}(t))$, defined by $\bar{g}(t) \doteqdot \Phi^*(t) g(t)$ and $\bar{f}(t) = f(t) \circ \Phi(t)$, where $\Phi(t)$ satisfies (5.36), is a solution to (5.29)–(5.30). Now $\mathcal{F}(g, f) = \mathcal{F}(\bar{g}, \bar{f})$, so that by (5.31), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(g(t), f(t)) &= \frac{d}{dt}\mathcal{F}(\bar{g}(t), \bar{f}(t)) \\ &= 2\int_{\mathcal{M}} \left|\bar{R}_{ij} + \bar{\nabla}_i\bar{\nabla}_j\bar{f}\right|_{\bar{g}}^2 e^{-\bar{f}} d\bar{\mu} \\ &= 2\int_{\mathcal{M}} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu.\end{aligned}$$

□

2.3.2. *Deriving the monotonicity of $\mathcal{F}$ from a pointwise estimate.* This second approach to the energy monotonicity formula is based on the pointwise formula (5.43), which is a simpler version of the evolution equation for Perelman's backward Harnack quantity (6.22).

Let $(g(t), f(t))$ be a solution to (5.34)–(5.35). Let $u = e^{-f}$ and

$$V \doteqdot (2\Delta f - |\nabla f|^2 + R)u = R^m u, \tag{5.42}$$

where R^m is the modified scalar curvature defined by (5.15),[8] so that

$$\mathcal{F} = \int_{\mathcal{M}} V d\mu.$$

Lemma 5.19 (Bochner-type formula for V). *If $(g(t), f(t))$ is a solution to (5.34)–(5.35) and if $u = e^{-f}$, then we have the pointwise differential equality:*

$$\Box^* V = -2|R_{ij} + \nabla_i \nabla_j f|^2 u. \tag{5.43}$$

This calculation, which we carry out below, is in a similar spirit to that of the calculations for the differential Harnack quantities considered in §10 of Chapter 5 in Volume One and Part II of this volume. To obtain (5.41) from the lemma, we compute

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(g(t), f(t)) &= \frac{d}{dt}\int_{\mathcal{M}} V d\mu \\ &= \int_{\mathcal{M}} \left(\frac{\partial}{\partial t}V - RV\right) d\mu \\ &= \int_{\mathcal{M}} 2|R_{ij} + \nabla_i \nabla_j f|^2 u d\mu.\end{aligned}$$

Proof of the lemma. Using definition (5.42) and $g^{ij}\frac{\partial}{\partial t}\Gamma_{ij}^k = 0$, a direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}R^m &= \frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R) \\ &= 4R_{ij}\nabla_i\nabla_j f + 2\Delta\left(\frac{\partial f}{\partial t}\right) - 2R_{ij}\nabla_i f\nabla_j f - 2\nabla\left(\frac{\partial f}{\partial t}\right)\cdot\nabla f + \frac{\partial R}{\partial t} \\ &= 4R_{ij}\nabla_i\nabla_j f - \Delta(2\Delta f - |\nabla f|^2 + R) + \Delta|\nabla f|^2 + 2\nabla\Delta f\cdot\nabla f \\ &\quad - 2R_{ij}\nabla_i f\nabla_j f - 2\nabla(|\nabla f|^2 - R)\nabla f + \frac{\partial R}{\partial t} - \Delta R.\end{aligned}$$

From the above we have

$$\left(\frac{\partial}{\partial t} + \Delta\right) R^m = 2|R_{ij} + \nabla_i\nabla_j f|^2 + 2\nabla R^m \cdot \nabla f.$$

[8] The above V is not to be confused with our earlier V, which was the trace of the variation v of g.

On the other hand,

$$\frac{\partial V}{\partial t}+\Delta V-RV=\left(\frac{\partial R^m}{\partial t}+\Delta R^m\right)u+\left(\frac{\partial u}{\partial t}+\Delta u-Ru\right)R^m+2\nabla R^m\cdot\nabla u.$$

Plugging in the equation for $\left(\frac{\partial}{\partial t}+\Delta\right)R^m$ and using (5.40), we have

$$\frac{\partial V}{\partial t}+\Delta V-RV=2|R_{ij}+\nabla_i\nabla_j f|^2u+2u\nabla R^m\cdot\nabla f+2\nabla R^m\cdot\nabla u.$$

The last two terms cancel each other since $\nabla f=-\nabla u/u$, which yields the lemma. □

REMARK 5.20 (Backward heat-type equation for modified scalar curvature). From the proof of the lemma, we have

$$\frac{\partial}{\partial t}R^m=-\Delta R^m+2\nabla R^m\cdot\nabla f+2|R^m_{ij}|^2. \tag{5.44}$$

Note the similarity to the equation $\frac{\partial R}{\partial t}=\Delta R+2\left|\mathrm{Rc}\right|^2$, except now we have a backward heat-type equation.

3. Steady and expanding breather solutions revisited

A solution $g(t)$ of the Ricci flow on a manifold $\mathcal{M}^n$ is called a **Ricci breather** if there exist times $t_1<t_2$, a constant $\alpha>0$ and a diffeomorphism $\varphi:\mathcal{M}\to\mathcal{M}$ such that

$$g(t_2)=\alpha\varphi^*g(t_1).$$

When $\alpha=1$, $\alpha<1$, or $\alpha>1$, we call $g(t)$ a **steady**, **shrinking**, or **expanding Ricci breather**, respectively. Recall that $g(t)$ is a Ricci soliton (or **trivial Ricci breather**) if for *each* pair of times $t_1<t_2$ there exist $\alpha>0$ and a diffeomorphism $\varphi:\mathcal{M}\to\mathcal{M}$ (α and φ will in general depend on t_1 and t_2) such that $g(t_2)=\alpha\varphi^*g(t_1)$.

Note that if we consider the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and homotheties, the Ricci breathers correspond to the periodic orbits whereas the Ricci solitons correspond to the fixed points. Since the Ricci flow is a heat-type equation, we expect that there are no periodic orbits except fixed points.

A nice application of the energy monotonicity formula is the nonexistence of nontrivial steady or expanding breather solutions on closed manifolds (§2 of [**297**]). This was first proved by one of the authors in [**218**] (see Proposition 1.66 in this volume). In the next chapter we shall see the application of Perelman's entropy formula to prove shrinking breather solutions on closed manifolds are gradient Ricci solitons (§3 of [**297**]). Hence we confirm the above expectation.

3.1. The infimum λ of $\mathcal{F}$. Suppose we have a steady breather solution to the Ricci flow with $g(t_2) = \varphi^* g(t_1)$ for some $t_1 < t_2$ and diffeomorphism φ. One drawback of the energy monotonicity formula is that in general the solution f to (5.35) has $f(t_2) \neq f(t_1) \circ \varphi$, so that in general, $\mathcal{F}(g(t_2), f(t_2)) \neq \mathcal{F}(g(t_1), f(t_1))$. By taking the infimum of $\mathcal{F}$ among f, we obtain an invariant of the Riemannian metric g which avoids this trouble.

DEFINITION 5.21 (λ-invariant). Given a metric g on a closed manifold $\mathcal{M}^n$, we define the functional $\lambda : \mathfrak{Met} \to \mathbb{R}$ by

$$\lambda(g) \doteqdot \inf \left\{ \mathcal{F}(g,f) : f \in C^\infty(\mathcal{M}),\ \int_{\mathcal{M}} e^{-f} d\mu = 1 \right\}. \tag{5.45}$$

Taking $w = e^{-f/2}$, we have

$$\lambda(g) = \inf \left\{ \mathcal{G}(g,w) : \int_{\mathcal{M}} w^2 d\mu = 1,\ w > 0 \right\}, \tag{5.46}$$

where, as in (5.5),[9]

$$\mathcal{G}(g,w) \doteqdot \int_{\mathcal{M}} \left(4 |\nabla w|^2 + R w^2 \right) d\mu. \tag{5.47}$$

Thus, when we fix g and minimize $\mathcal{F}(g,f)$ among f, we are minimizing a Dirichlet-type functional and we get an eigenfunction-type equation for w. Aspects of this point of view are discussed in the next two lemmas.

Note that the variation of $\mathcal{G}(g,\cdot)$ is given by

$$\frac{1}{2}\delta_{(0,h)}\mathcal{G}(g,w) = \int_{\mathcal{M}} (4\nabla w \cdot \nabla h + Rwh)\, d\mu = \int_{\mathcal{M}} (-4\Delta w + Rw)\, h d\mu,$$

where $h = \delta w$. Hence the Euler–Lagrange equation for (note that we dropped the positivity condition on w)

$$\lambda(g) \doteqdot \inf \left\{ \mathcal{G}(g,w) : \int_{\mathcal{M}} w^2 d\mu = 1 \right\}$$

is

$$Lw \doteqdot -4\Delta w + Rw = \lambda(g)\, w. \tag{5.48}$$

LEMMA 5.22 (Existence and regularity of minimizer of $\mathcal{G}$). *There exists a unique minimizer w_0 (up to a change in sign) of*

$$\inf \left\{ \mathcal{G}(g,w) : \int_{\mathcal{M}} w^2 d\mu = 1 \right\}. \tag{5.49}$$

The minimizer w_0 is positive and smooth. Moreover,

[9] In view of Lemma 5.1(1), the monotonicity of $\mathcal{F}$ exhibits a dichotomy, it is analogous to both the monotonicty of the total scalar curvature under its gradient flow, $\frac{\partial}{\partial t} g = -2\left(\mathrm{Rc} - \frac{1}{2} g\right)$, and the monotonicity of the Dirichlet energy under its gradient flow, the backward heat equation $\frac{\partial}{\partial t} w = -\Delta w$. In this sense, the monotonicity of $\mathcal{F}$ exhibits a beautiful synthesis of geometry and analysis.

(1) *the minimum value* $\lambda(g)$ *of* $\mathcal{G}(g,w)$ *is equal to* $\lambda_1(g)$, *where* $\lambda_1(g)$ is the lowest eigenvalue of the elliptic operator $-4\Delta + R$, *and*

(2) w_0 *is the unique positive eigenfunction of*

$$-4\Delta w_0 + Rw_0 = \lambda_1(g)\, w_0 \tag{5.50}$$

with L^2*-norm equal to* 1.

Proof. To establish the existence of a minimizer w_0 of (5.46), one takes a minimizing sequence $\{w_i\}_{i=1}^{\infty}$ of (5.46) in $W^{1,2}(\mathcal{M})$. There then exists a subsequence $\{w_i\}_{i=1}^{\infty}$ which converges to $w_0 \in W^{1,2}(\mathcal{M})$ weakly in $W^{1,2}(\mathcal{M})$ and strongly in $L^2(\mathcal{M})$ (by the Sobolev embedding theorem). Since

$$\begin{aligned} 0 &\le \int_{\mathcal{M}} |\nabla (w_i - w_0)|^2 \, d\mu \\ &= \int_{\mathcal{M}} |\nabla w_i|^2 \, d\mu + \int_{\mathcal{M}} |\nabla w_0|^2 \, d\mu - 2\int_{\mathcal{M}} \langle \nabla w_i, \nabla w_0 \rangle \, d\mu, \end{aligned}$$

by the weak convergence in $W^{1,2}$, we have $\lim_{i\to\infty} \int_{\mathcal{M}} \langle \nabla w_i, \nabla w_0\rangle \, d\mu = \int_{\mathcal{M}} |\nabla w_0|^2 \, d\mu$ exists, hence

$$\int_{\mathcal{M}} |\nabla w_0|^2 \, d\mu \le \liminf_{i\to\infty} \int_{\mathcal{M}} |\nabla w_i|^2 \, d\mu.$$

On the other hand, by the strong convergence of $\{w_i\}_{i=1}^{\infty}$ in L^2, we have

$$\lim_{i\to\infty} \int_{\mathcal{M}} R w_i^2 d\mu = \int_{\mathcal{M}} R w_0^2 d\mu,$$

$$\int_{\mathcal{M}} w_0^2 d\mu = \lim_{i\to\infty} \int_{\mathcal{M}} w_i^2 d\mu = 1.$$

Hence w_0 is a minimizer of (5.46) in $W^{1,2}(\mathcal{M})$, and w_0 is a weak solution to the eigenfunction equation (5.48). By standard regularity theory, $w_0 \in C^\infty$. We also have that any minimizer is either nonnegative or nonpositive, since otherwise $\pm |w_0|$ is a distinct smooth minimizer which agrees with w_0 on an open set, contradicting the unique continuation property of solutions to second-order linear elliptic equations.

We now prove w_0 is unique up to a sign. Without loss of generality, we may assume below that w_0 is nonnegative. Call a minimizer w of $\mathcal{G}$ with $\int_{\mathcal{M}} w^2 d\mu = 1$ a *normalized* minimizer. If the nonnegative normalized minimizer is not unique, then there exist two normalized minimizers $w_0 \ge 0$ and $w_1 \ge 0$ with $\int_{\mathcal{M}} w_0 w_1 d\mu = 0$. Then $w_2 = aw_0 + bw_1$ is also a normalized minimizer for all $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Indeed, since w_0 and w_1 satisfy the linear equation (5.50), so does $w_2 = aw_0 + bw_1$, and $\int_{\mathcal{M}} w_2^2 d\mu = 1$.

Now it not hard to see that there exist a and b such that w_2 changes sign. In particular, if there are points x and y such that $w_1(x) = cw_0(x)$ and $w_1(y) = dw_0(y)$, where $c \ne d$ and $w_0(x) > 0 < w_0(y)$, then by choosing a and b with $a^2 + b^2 = 1$ such that $a + bc$ and $a + bd$ have opposite signs, we have that $w_2(x) = (a + bc)\, w_0(x)$ and $w_2(y) = (a + bd)\, w_0(y)$ have opposite signs, which is a contradiction. Hence w_0 is unique.

Finally we show $w_0 > 0$. By the Hopf boundary point lemma (see Lemma 3.4 of Gilbarg and Trudinger [**155**]), if $w_0 = 0$ somewhere, then there exists a point $x_0 \in \partial\Omega$, where $\Omega = \{x \in \mathcal{M} : w_0(x) > 0\}$, such that $\partial\Omega$ satisfies the interior sphere condition at x_0, so that $w(x_0) = 0$ and $|\nabla w(x_0)| \neq 0$, which is a contradiction to $w_0 \geq 0$.

Finally, properties (1) and (2) follow easily. □

The existence of a unique positive smooth minimizer w_0 of $\mathcal{G}(g, w)$ under the constraint $\int_{\mathcal{M}} w^2 d\mu = 1$ implies the existence of a unique smooth minimizer f_0 of $\mathcal{F}(g, \cdot)$ under the constraint $\int_{\mathcal{M}} e^{-f} d\mu = 1$. From (5.50) we see the following.

LEMMA 5.23 (Euler–Lagrange equation for minimizer of $\mathcal{F}$). *The minimizer $f_0 = -2\log w_0$ of $\mathcal{F}(g, \cdot)$ is unique, C^∞, and a solution to*

$$\lambda(g) = 2\Delta f_0 - |\nabla f_0|^2 + R. \tag{5.51}$$

That is, the modified scalar curvature is a constant, i.e., $R^m \equiv \lambda(g)$. Note that from setting $v = 0$ in (5.10), for the minimizer f of (5.45), we have

$$\delta_{(0,h)}\mathcal{F}(g, f) = -\int_{\mathcal{M}} h\left(2\Delta f - |\nabla f|^2 + R\right) e^{-f} d\mu$$

for all h such that $\int_{\mathcal{M}} he^{-f} d\mu_g = 0$. We can also obtain (5.51) directly from this.

We summarize the properties of the functional λ on a closed manifold $\mathcal{M}^n$.

(i) (*Lower bound for* λ) $\lambda(g)$ is well defined (i.e., finite) since

$$\mathcal{F}(g, f) \geq \min_{x\in\mathcal{M}} R(x) \cdot \int_{\mathcal{M}} e^{-f} d\mu = \min_{x\in\mathcal{M}} R(x) \doteqdot R_{\min}.$$

In particular,

$$\lambda(g) \geq R_{\min}.$$

(ii) (*Diffeomorphism invariance*) If $\varphi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, then

$$\lambda(\varphi^* g) = \lambda(g).$$

(iii) (*Existence of a smooth minimizer*) There exists $f \in C^\infty(\mathcal{M})$ with $\int_{\mathcal{M}} e^{-f} d\mu = 1$ such that $\lambda(g) = \mathcal{F}(g, f)$, i.e.,

$$\lambda(g) = \int_{\mathcal{M}} (R + |\nabla f|^2) e^{-f} d\mu. \tag{5.52}$$

(iv) (*Upper bound for* λ) We have

$$\lambda(g) \leq \frac{1}{\mathrm{Vol}(\mathcal{M})} \int_{\mathcal{M}} R d\mu. \tag{5.53}$$

This can be seen by choosing $f = \log \mathrm{Vol}(\mathcal{M})$, which satisfies

$$\int_{\mathcal{M}} e^{-f} d\mu_g = 1 \quad \text{and} \quad \lambda(g) \leq \int_{\mathcal{M}} (R + |\nabla f|^2) e^{-f} d\mu.$$

(v) (*Scaling*)

$$\lambda (cg) = c^{-1} \lambda (g) .$$

3.2. The monotonicity of λ. Let $(\mathcal{M}^n, g(t))$, $t \in [0, T]$, be a solution of the Ricci flow on a closed manifold. In this subsection we discuss some properties related to the continuity and monotonicity of $\lambda(g(t))$. Such properties are key to the proof of the nonexistence of nontrivial expanding or steady breathers. First we show that $\lambda(g(t))$ is a continuous function on $[t_1, t_2]$. This is a consequence of the following elementary result (see also Craioveanu, Puta, and Rassias [**118**] or Chapter XII of Reed and Simon [**310**]).[10]

LEMMA 5.24 (Effective estimate for continuous dependence of λ on g). *If g_1 and g_2 are two metrics on $\mathcal{M}$ which satisfy*

$$\frac{1}{1+\varepsilon} g_1 \leq g_2 \leq (1+\varepsilon) g_1 \quad \text{and} \quad R(g_1) - \varepsilon \leq R(g_2) \leq R(g_1) + \varepsilon,$$

then[11]

$$\begin{aligned} &\lambda (g_2) - \lambda (g_1) \\ &\leq \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-n/2} \right) (1+\varepsilon)^{n/2} \left(\lambda (g_1) - \min R_{g_1} \right) \\ &+ \left((1+\delta) \max |R_{g_2} - R_{g_1}| + 2\delta \max |R_{g_1}| \right) (1+\varepsilon)^{n/2} , \end{aligned}$$

where $\delta \to 0$ as $\varepsilon \to 0$.[12] *In particular, $\lambda : \mathfrak{Met} \to \mathbb{R}$ is a continuous function with respect to the C^2-topology.*

PROOF. The proof is straightforward but slightly tedious. First note that $(1+\varepsilon)^{-n/2} d\mu_{g_1} \leq d\mu_{g_2} \leq (1+\varepsilon)^{n/2} d\mu_{g_1}$. If w is a positive function on $\mathcal{M}$, then in view of (5.46), we compute (writing $a \cdot b - c \cdot d = a(b-d) + (a-c)d$)

$$\begin{aligned} &\int_{\mathcal{M}} w^2 d\mu_{g_1} \mathcal{G}(g_2, w) - \int_{\mathcal{M}} w^2 d\mu_{g_2} \mathcal{G}(g_1, w) \\ &= 4 \int_{\mathcal{M}} w^2 d\mu_{g_1} \left(\int_{\mathcal{M}} |\nabla w|^2_{g_2} d\mu_{g_2} - \int_{\mathcal{M}} |\nabla w|^2_{g_1} d\mu_{g_1} \right) \\ &+ 4 \left(\int_{\mathcal{M}} w^2 d\mu_{g_1} - \int_{\mathcal{M}} w^2 d\mu_{g_2} \right) \int_{\mathcal{M}} |\nabla w|^2_{g_1} d\mu_{g_1} \\ &+ \int_{\mathcal{M}} w^2 d\mu_{g_1} \left(\int_{\mathcal{M}} R_{g_2} w^2 d\mu_{g_2} - \int_{\mathcal{M}} R_{g_1} w^2 d\mu_{g_1} \right) \\ &+ \left(\int_{\mathcal{M}} w^2 d\mu_{g_1} - \int_{\mathcal{M}} w^2 d\mu_{g_2} \right) \int_{\mathcal{M}} R_{g_1} w^2 d\mu_{g_1}, \end{aligned}$$

[10] Thanks to [**231**] for this last reference.

[11] To denote the dependence on g_i, we use the subscript $_{g_i}$ instead of (g_i). So $R_{g_1} = R(g_1)$.

[12] See the proof for an explicit dependence of δ on ε.

so that

$$
\begin{aligned}
&\int_{\mathcal{M}} w^2 d\mu_{g_1} \mathcal{G}(g_2, w) - \int_{\mathcal{M}} w^2 d\mu_{g_2} \mathcal{G}(g_1, w) \\
&\leq 4 \left((1+\varepsilon)^{\frac{n}{2}+1} - 1 \right) \int_{\mathcal{M}} w^2 d\mu_{g_1} \int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1} \\
&+ 4 \left(1 - (1+\varepsilon)^{-n/2} \right) \int_{\mathcal{M}} w^2 d\mu_{g_1} \int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1} \\
&+ \int_{\mathcal{M}} w^2 d\mu_{g_1} \int_{\mathcal{M}} w^2 \left(\left| (R_{g_2} - R_{g_1}) \frac{d\mu_{g_2}}{d\mu_{g_1}} \right| + \left| \left(\frac{d\mu_{g_2}}{d\mu_{g_1}} - 1 \right) R_{g_1} \right| \right) d\mu_{g_1} \\
&+ \left| \int_{\mathcal{M}} w^2 \left(1 - \frac{d\mu_{g_2}}{d\mu_{g_1}} \right) d\mu_{g_1} \right| \left| \int_{\mathcal{M}} R_{g_1} w^2 d\mu_{g_1} \right| .
\end{aligned}
$$

(In the above estimates we took into account that R may change sign.) Let $\delta \doteqdot \max \left\{ (1+\varepsilon)^{n/2} - 1, 1 - (1+\varepsilon)^{-n/2} \right\}$, so that $\delta \to 0$ as $\varepsilon \to 0$. Since $\left| 1 - \frac{d\mu_{g_2}}{d\mu_{g_1}} \right| \leq \delta$, we have

$$
\begin{aligned}
&\int_{\mathcal{M}} w^2 d\mu_{g_1} \int_{\mathcal{M}} w^2 d\mu_{g_2} \left(\frac{\mathcal{G}(g_2, w)}{\int_{\mathcal{M}} w^2 d\mu_{g_2}} - \frac{\mathcal{G}(g_1, w)}{\int_{\mathcal{M}} w^2 d\mu_{g_1}} \right) \\
&\leq 4 \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-n/2} \right) \int_{\mathcal{M}} w^2 d\mu_{g_1} \int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1} \\
&+ \left((1+\delta) \max |R_{g_2} - R_{g_1}| + \delta \max |R_{g_1}| \right) \left(\int_{\mathcal{M}} w^2 d\mu_{g_1} \right)^2 \\
&+ \delta \max |R_{g_1}| \left(\int_{\mathcal{M}} w^2 d\mu_{g_1} \right)^2 .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\mathcal{G}(g_2, w)}{\int_{\mathcal{M}} w^2 d\mu_{g_2}} - \frac{\mathcal{G}(g_1, w)}{\int_{\mathcal{M}} w^2 d\mu_{g_1}} \\
&\leq 4 \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-n/2} \right) \frac{\int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1}}{\int_{\mathcal{M}} w^2 d\mu_{g_2}} \\
&+ \left((1+\delta) \max |R_{g_2} - R_{g_1}| + 2\delta \max |R_{g_1}| \right) (1+\varepsilon)^{n/2} .
\end{aligned}
$$

Taking w to be a minimizer for $\mathcal{G}(g_1, \cdot)$, we have

$$
\begin{aligned}
&\lambda(g_2) - \lambda(g_1) \\
&\leq 4 \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-n/2} \right) (1+\varepsilon)^{n/2} \frac{\int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1}}{\int_{\mathcal{M}} w^2 d\mu_{g_1}} \\
&+ \left((1+\delta) \max |R_{g_2} - R_{g_1}| + 2\delta \max |R_{g_1}| \right) (1+\varepsilon)^{n/2} .
\end{aligned}
$$

The result now follows from

$$4\frac{\int_{\mathcal{M}} |\nabla w|_{g_1}^2 \, d\mu_{g_1}}{\int_{\mathcal{M}} w^2 d\mu_{g_1}} = \frac{\mathcal{G}(g_1, w)}{\int_{\mathcal{M}} w^2 d\mu_{g_1}} - \frac{\int_{\mathcal{M}} R_{g_1} w^2 d\mu_{g_1}}{\int_{\mathcal{M}} w^2 d\mu_{g_1}}$$
$$\leq \lambda (g_1) - \min R_{g_1}.$$

□

The monotonicity of $\mathcal{F}(g(t), f(t))$ under the system (5.34)–(5.35) implies the monotonicity of $\lambda(g(t))$ under the Ricci flow.

LEMMA 5.25 (λ monotonicity). *If $g(t)$, $t \in [0, T]$, is a solution to the Ricci flow, then*

$$\frac{d}{dt}\lambda(g(t)) \geq \frac{2}{n}\lambda^2(g(t)),$$

and $\lambda(g(t))$ is nondecreasing in $t \in [0, T]$. Here the derivative $\frac{d}{dt}$ is in the sense of the lim inf of backward difference quotients.

REMARK 5.26. See the next subsection for the case where $\lambda(g(t))$ is not strictly increasing.

PROOF. Given $t_0 \in [0, T]$, let f_0 be the minimizer of $\mathcal{F}(g(t_0), f)$, so that $\lambda(g(t_0)) = \mathcal{F}(g(t_0), f(t_0))$. Solve

$$\frac{\partial}{\partial t} f = -R - \Delta f + |\nabla f|^2, \qquad f(t_0) = f_0, \tag{5.54}$$

backward in time on $[0, t_0]$. Then $\frac{d}{dt}\mathcal{F}(g(t), f(t)) \geq 0$ for all $t \leq t_0$. Since the constraint $\int_{\mathcal{M}} e^{-f} d\mu$ is preserved under (5.54), we have $\lambda(g(t)) \leq \mathcal{F}(g(t), f(t))$ for $t \leq t_0$. This, (5.41), and $\lambda(g(t_0)) = \mathcal{F}(g(t_0), f(t_0))$ imply both

$$\lambda(g(t)) \leq \mathcal{F}(g(t), f(t)) \leq \mathcal{F}(g(t_0), f(t_0)) = \lambda(g(t_0)) \tag{5.55}$$

and the following:

$$\begin{aligned} \left.\frac{d}{dt}\lambda(g(t))\right|_{t=t_0} &\geq \left.\frac{d}{dt}\mathcal{F}(g(t), f(t))\right|_{t=t_0} \\ &= 2\int_{\mathcal{M}} |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu_{g(t_0)} \\ &\geq 2\int_{\mathcal{M}} \frac{1}{n}(R + \Delta f)^2 e^{-f} d\mu_{g(t_0)} \\ &\geq \frac{2}{n}\left(\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu_{g(t_0)}\right)^2 \\ &= \frac{2}{n}\lambda^2(g(t_0)), \end{aligned} \tag{5.56}$$

where $f = f_0$ is the minimizer. Hence, from either (5.55) or (5.56), we see that $\lambda(g(t))$ is nondecreasing under the Ricci flow. □

EXERCISE 5.27. Prove (5.56).

SOLUTION TO EXERCISE 5.27. We compute[13]

$$\begin{aligned}\left.\frac{d}{dt_-}\lambda\left(g\left(t\right)\right)\right|_{t=t_0} &\doteqdot \liminf_{h\to 0_+}\frac{\lambda\left(g\left(t_0\right)\right)-\lambda\left(g\left(t_0-h\right)\right)}{h}\\ &\geq \liminf_{h\to 0_+}\frac{\mathcal{F}\left(g\left(t_0\right),f_0\right)-\mathcal{F}\left(g\left(t_0-h\right),f\left(t_0-h\right)\right)}{h},\end{aligned}$$

where f_0 is the minimizer for $\mathcal{F}\left(g\left(t_0\right),\cdot\right)$ and $f\left(t\right)$ is the solution to (5.54). On the other hand, we conclude by (5.41) that the last expression is equal to $2\int_{\mathcal{M}}\left|R_{ij}+\nabla_i\nabla_j f_0\right|^2 e^{-f_0}d\mu_{g(t_0)}$.

3.3. There are no nontrivial steady breathers. As an application of the monotonicity of the diffeomorphism-invariant functional λ we prove the nonexistence of nontrivial steady breathers.

LEMMA 5.28 (No nontrivial steady breathers on closed manifolds). *If $(\mathcal{M}^n, g\left(t\right))$ is a solution to the Ricci flow on a closed manifold such that there exist $t_1 < t_2$ with $\lambda\left(g\left(t_1\right)\right) = \lambda\left(g\left(t_2\right)\right)$, then $g\left(t\right)$ is a steady gradient Ricci soliton, which must be Ricci flat. In particular, a steady Ricci breather on a closed manifold is Ricci flat.*

PROOF. Note that if $g\left(t\right)$ is a steady Ricci breather with $g(t_2) = \varphi^* g(t_1)$ for some $t_1 < t_2$ and diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$, then $\lambda(g\left(t_2\right)) = \lambda(g\left(t_1\right))$. Hence we only need to prove the first part of the lemma.

Suppose that for a solution $g\left(t\right)$ to the Ricci flow there exist times $t_1 < t_2$ such that $\lambda(g\left(t_2\right)) = \lambda(g\left(t_1\right))$. Let f_2 be the minimizer for $\mathcal{F}$ at time t_2 so that $\mathcal{F}\left(g\left(t_2\right), f_2\right) = \lambda\left(g\left(t_2\right)\right)$. Take $f\left(t\right)$ to be the solution to the backward heat equation (5.35) on the time interval $[t_1, t_2]$ with the initial data $f\left(t_2\right) = f_2$. By the monotonicity formula (5.41) and the definition of λ we have[14]

$$\lambda\left(g\left(t_1\right)\right) \leq \mathcal{F}\left(g\left(t_1\right), f\left(t_1\right)\right) \leq \mathcal{F}\left(g\left(t\right), f\left(t\right)\right) \leq \mathcal{F}\left(g\left(t_2\right), f_2\right) = \lambda\left(g\left(t_2\right)\right)$$

for all $t \in \left[t_1, t_2\right]$. Since $\lambda\left(g\left(t_1\right)\right) = \lambda\left(g\left(t_2\right)\right)$ and $\lambda\left(g\left(t\right)\right)$ is monotone, we have

$$\mathcal{F}\left(g\left(t\right), f\left(t\right)\right) = \lambda\left(g\left(t\right)\right) \equiv \text{const}$$

for $t \in \left[t_1, t_2\right]$. Therefore the solution $f\left(t\right)$ is the minimizer for $\mathcal{F}\left(g\left(t\right), \cdot\right)$ and $\frac{d}{dt}\mathcal{F}\left(g\left(t\right), f\left(t\right)\right) \equiv 0$, so by (5.41) we have

$$\int_{\mathcal{M}}\left|R_{ij}+\nabla_i\nabla_j f\right|^2 e^{-f}d\mu\left(t\right) \equiv 0$$

for all $t \in \left[t_1, t_2\right]$. Thus

$$R_{ij} + \nabla_i\nabla_j f = 0 \text{ for } t \in [t_1, t_2]. \tag{5.57}$$

In particular, $g(t)$ is a steady gradient Ricci soliton flowing along $\nabla f\left(t\right)$.[15]

[13]Here $\frac{d}{dt_-}$ denotes the lim inf of backward difference quotients.

[14]This is the same as (5.55).

[15]See (1.9), where a gradient soliton is steady if $\varepsilon = 0$.

Note by (5.51) that f satisfies the equation

$$2\Delta f - |\nabla f|^2 + R = \lambda(g).$$

On the other hand, $R + \Delta f = 0$, so that

$$|\nabla f|^2 + R = -\lambda(g).$$

However, integrating, we have

$$-\lambda(g) = \int_{\mathcal{M}} \left(|\nabla f|^2 + R\right) e^{-f} d\mu = \lambda(g),$$

so that $\lambda(g) = 0$ and $\Delta f = |\nabla f|^2 = -R$. Note that then

$$0 = \int_{\mathcal{M}} \left(\Delta f - |\nabla f|^2\right) e^{f} d\mu = -2 \int_{\mathcal{M}} |\nabla f|^2 e^{f} d\mu$$

implies that f is constant and hence g is Ricci flat by (5.57). Alternatively, we could have argued that since $\Delta f = |\nabla f|^2 \geq 0$, f is subharmonic and hence constant. □

REMARK 5.29. Even when $\mathcal{M}$ is noncompact, we have $|\nabla f|^2 + R$ is constant for gradient Ricci solitons; see Proposition 1.15.

3.4. Nonexistence of nontrivial expanding breathers. Recall that $\lambda(g)$ is not scale-invariant, e.g., $\lambda(cg) = c^{-1}\lambda(g)$. Thus we define the **normalized λ-invariant**:

$$\bar{\lambda}(g) \doteqdot \lambda(g) \cdot \operatorname{Vol}(\mathcal{M})^{2/n}. \tag{5.58}$$

It is easy to see that $\bar{\lambda}(cg) = \bar{\lambda}(g)$ for any $c > 0$, so the invariant $\bar{\lambda}$ is potentially useful for expanding and shrinking breathers. *We shall prove the monotonicity of $\bar{\lambda}(g(t))$ under Ricci flow when it is nonpositive.* For this reason it is most useful for expanding breathers.

Recall that by (5.56), we have

$$\frac{d}{dt}\lambda(g(t)) \geq 2\int_{\mathcal{M}} |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu, \tag{5.59}$$

where $\frac{d}{dt}\lambda(g(t))$ is defined as the lim inf of backward difference quotients.[16] Let $V \doteqdot V(t) \doteqdot \operatorname{Vol}_{g(t)}(\mathcal{M})$. From (5.59), we compute

$$\begin{aligned}
\frac{d}{dt}\bar{\lambda}(g(t)) &= \frac{d}{dt}\left[\lambda(g(t)) \cdot V(t)^{2/n}\right] \\
&= V^{2/n}\frac{d\lambda}{dt} + \frac{2}{n}V^{\frac{2}{n}-1}\lambda\frac{dV}{dt} \\
&\geq 2V^{2/n}\int_{\mathcal{M}} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu + \frac{2}{n}\lambda V^{\frac{2}{n}-1}\int_{\mathcal{M}}(-R)\, d\mu,
\end{aligned}$$

[16]This also applies to the time derivatives below in this argument.

where $f = f(t)$ is the minimizer of $\mathcal{F}(g(t), \cdot)$. From this we obtain

$$\frac{1}{2}V^{-2/n}\frac{d}{dt}\bar{\lambda}(g(t)) \geq \int_{\mathcal{M}} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu - \frac{1}{n}\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \cdot \frac{1}{V}\int_{\mathcal{M}} R d\mu.$$

Hence

$$\frac{1}{2}V^{-2/n}\frac{d}{dt}\bar{\lambda}(g(t)) \geq \int_{\mathcal{M}} \left| R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R + \Delta f) g_{ij}\right|^2 e^{-f} d\mu + \int_{\mathcal{M}} \frac{1}{n}(R + \Delta f)^2 e^{-f} d\mu - \frac{1}{n}\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \cdot \frac{1}{V}\int_{\mathcal{M}} R d\mu.$$

Recall from (5.53) that

$$\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \leq \frac{\int R d\mu}{V}.$$

Assuming $\lambda(t) \leq 0$, so that $\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \leq 0$, we have

$$\frac{1}{2}V^{-2/n}\frac{d}{dt}\bar{\lambda}(g(t)) - \int_{\mathcal{M}} \left| R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R + \Delta f) g_{ij}\right|^2 e^{-f} d\mu \geq \frac{1}{n}\int_{\mathcal{M}} (R + \Delta f)^2 e^{-f} d\mu - \frac{1}{n}\left(\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu\right)^2 \geq 0 \tag{5.60}$$

since $\int_{\mathcal{M}} e^{-f} d\mu = 1$. Hence

LEMMA 5.30. *Let $g(t)$ be a solution to the Ricci flow on a closed manifold $\mathcal{M}^n$. If at some time t, $\bar{\lambda}(t) \leq 0$, then*

$$\frac{d}{dt}\bar{\lambda}(g(t)) \geq 2V^{2/n}\int_{\mathcal{M}} \left| R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R + \Delta f) g_{ij}\right|^2 e^{-f} d\mu \geq 0, \tag{5.61}$$

where $V = \mathrm{Vol}_{g(t)}(\mathcal{M})$, $f(t)$ is the minimizer for $\mathcal{F}(g(t), \cdot)$, and the time-derivative is defined as the lim inf *of backward difference quotients. By (5.61), if $\frac{d}{dt}\bar{\lambda}(g(t)) = 0$, then $g(t)$ is a gradient Ricci soliton.*

This is reminiscent of the fact that under the normalized Ricci flow, the minimum scalar curvature is nondecreasing as long as it is nonpositive, whereas under the unnormalized Ricci flow, the minimum scalar curvature is always nondecreasing (see Lemma A.20). However these two facts appear to be quite different in nature.

To apply the above monotonicity to the expanding breather case, we need to produce a time t_0 where $\bar{\lambda}(g(t_0)) < 0$. This is accomplished by looking at the evolution of the volume. Below we also give another proof of Lemma 5.28 using $\bar{\lambda}(g(t))$.

LEMMA 5.31. *Expanding or steady breathers on closed manifolds are Einstein.*

PROOF. Let $(\mathcal{M}^n, g(t))$ be an expanding or steady breather with $g(t_2) = \alpha\varphi^* g(t_1)$ for some $t_1 < t_2$ and $\alpha \geq 1$. We have $\bar{\lambda}(g(t_2)) = \bar{\lambda}(g(t_1))$. Let $V(t) \doteqdot \mathrm{Vol}_{g(t)}(\mathcal{M})$. Since $V(t_2) \geq V(t_1)$, we have for some $t_0 \in (t_1, t_2)$,

$$0 \leq \left.\frac{d}{dt}\right|_{t=t_0} \log V(t) = -\frac{\int_{\mathcal{M}} R d\mu}{V(t_0)}(t_0) \leq -\lambda(g(t_0)).$$

By Lemma 5.30, if $g(t)$ is not a gradient Ricci soliton, then $\frac{d}{dt}\bar{\lambda}(g(t_0)) > 0$ and we have $\bar{\lambda}(g(t_0')) < 0$ for some $t_0' < t_0$. Now since $\bar{\lambda}(g(t))$ is increasing whenever it is negative, we have

$$\bar{\lambda}(g(t_2)) = \bar{\lambda}(g(t_1)) \leq \bar{\lambda}\left(g\left(t_0'\right)\right) < 0,$$

which implies $\lambda(g(t)) \leq \lambda(g(t_2)) < 0$ for all $t \in [t_1, t_2]$. Hence $\bar{\lambda}(g(t))$ is nondecreasing, which implies $\bar{\lambda}(g(t))$ is constant. By (5.61), we have

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{n}(R + \Delta f) g_{ij} \equiv 0,$$

and since we are in the equality case of (5.60), we also have

$$R + \Delta f = C_1(t) = \text{const} \;\; \text{(depending on time)}. \tag{5.62}$$

That is, we still conclude that $g(t)$ is an expanding or steady gradient Ricci soliton.

Now let $(\mathcal{M}^n, g(t))$ be an expanding or steady gradient Ricci soliton. Recall

$$2\Delta f + R - |\nabla f|^2 = C_2(t) = \text{const}.$$

This, combined with (5.62), implies

$$\Delta f - |\nabla f|^2 = \text{const}.$$

Since

$$\int_{\mathcal{M}} \left(\Delta f - |\nabla f|^2\right) e^{-f} d\mu = 0,$$

we have $\Delta f - |\nabla f|^2 \equiv 0$. Thus, by the strong maximum principle (or since now $0 = \int_{\mathcal{M}} \left(\Delta f - |\nabla f|^2\right) e^f d\mu = -2\int_{\mathcal{M}} |\nabla f|^2 e^f d\mu$), we conclude that $f \equiv \text{const}$. Hence $R_{ij} - \frac{1}{n} R g_{ij} \equiv 0$ and g_{ij} is Einstein. (When $n = 2$, our conclusion is vacuous.) $\square$

REMARK 5.32. As a corollary of the above result, we again see that expanding or steady *solitons* on closed manifolds are Einstein. In the case of shrinking solitons on closed manifolds, using the entropy functional, we shall see in the next chapter that they are necessarily *gradient* shrinking solitons.

Note that on a shrinking breather we have $V(t_2) < V(t_1)$ for $t_2 > t_1$. In particular, it is possible that $\lambda(g(t)) > 0$ for all $t \in [t_1, t_2]$ (on the other hand, if $\lambda(g(t_0')) < 0$ for some $t_0' \in [t_1, t_2]$, then the proof above implies that a shrinking breather is Einstein), which causes difficulty in extending the proof above to the shrinking case; in the next chapter this problem is solved by the introduction of Perelman's entropy. (Note that for an Einstein manifold with $R \equiv r = \text{const}$, under the constraint $\int e^{-f} d\mu = 1$ we have

$$\mathcal{F}(g, f) = r + \int_{\mathcal{M}} |\nabla f|^2 e^{-f} d\mu \geq r$$

with equality if and only if $f \equiv \log \operatorname{Vol}(g) = \text{const}$. Hence, if $r > 0$, then $\bar{\lambda}(g) = r \operatorname{Vol}(g)^{2/n} > 0$.)

EXERCISE 5.33 (Behavior of $\bar{\lambda}$ on products). Compute $\bar{\lambda}$ of spheres and products of spheres. Show that $\bar{\lambda}(t)$ of a shrinking $S^2 \times S^1$ under the Ricci flow approaches ∞ as t approaches the singularity time. What happens if we start with $S^2 \times S^2$, where the S^2's have different radii? What is the behavior of $\bar{\lambda}$ for the product of Einstein spaces (or Ricci solitons)?

4. Classical entropy and Perelman's energy

Define the **classical entropy** on a closed manifold $\mathcal{M}^n$ by

$$\mathcal{N} \doteqdot \int_{\mathcal{M}} f e^{-f} d\mu = -\int_{\mathcal{M}} u \log u \, d\mu, \tag{5.63}$$

where $u \doteqdot e^{-f}$. Under the gradient flow (5.29)–(5.30), we have

$$\begin{aligned} \frac{d\mathcal{N}}{dt} &= \int_{\mathcal{M}} \frac{\partial f}{\partial t} e^{-f} d\mu = -\int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu \\ &= -\mathcal{F}. \end{aligned} \tag{5.64}$$

That is, *the classical entropy is the anti-derivative of the negative of Perelman's energy.*

In this section we show that, by an upper bound for $\mathcal{F}$, a modification of $\mathcal{N}$ is monotone. For comparison, we discuss Hamilton's original proof of surface entropy monotonicity, the entropy formula for Hamilton's surface entropy, the fact that the gradient of Hamilton's surface entropy is the matrix Harnack, and Bakry–Emery's logarithmic Sobolev-type inequality.

4.1. Monotonicity of the classical entropy. The following gives us an upper bound for the time interval of existence of the Ricci flow in terms of $\int_{\mathcal{M}} dm$ and the initial value of $\mathcal{F}^m$. Equivalently, it also implies the monotonicity of the classical entropy (see also [**356**], pp. 74–75).

PROPOSITION 5.34 (Upper bound for $\mathcal{F}$ in terms of time to blow up). *Suppose that $(g(t), f(t))$ is a solution on a closed manifold $\mathcal{M}^n$ of the gradient flow for $\mathcal{F}^m$, (5.25)–(5.26), for $t \in [0, T)$. Then we have*

$$\mathcal{F}^m(g(0)) \leq \frac{n}{2T} \int_{\mathcal{M}} dm, \tag{5.65}$$

that is,

$$T \leq \frac{n}{2\mathcal{F}^m(g(0))} \int_{\mathcal{M}} dm.$$

The proposition is a consequence of the following.

LEMMA 5.35 (Monotonicity formula for the classical entropy $\mathcal{N}$). *If* $(g(t), f(t))$, $t \in [0, T)$, *is a solution of the gradient flow* (5.29)–(5.30) *on a closed manifold* $\mathcal{M}^n$*, then*

$$\frac{d}{dt}\mathcal{F}^m(g(t)) \geq \frac{2}{n}\left(\int_{\mathcal{M}} dm\right)^{-1} \mathcal{F}^m(g(t))^2,$$

$$\mathcal{F}^m(g(t)) \leq \frac{n}{2(T-t)} \int_{\mathcal{M}} e^{-f} d\mu. \tag{5.66}$$

By (5.64)*, this implies the following entropy monotonicity formula:*

$$\frac{d}{dt}\left(\mathcal{N} - \left(\frac{n}{2}\int_{\mathcal{M}} e^{-f} d\mu\right)\log(T-t)\right) \geq 0. \tag{5.67}$$

REMARK 5.36. Following §6.5 of [**356**], we may adjust the entropy quantity on the LHS of (5.67) by adding a constant and define

$$\widetilde{\mathcal{N}} \doteqdot \mathcal{N} - \left(\frac{n}{2}\int_{\mathcal{M}} e^{-f} d\mu\right)(\log[4\pi(T-t)] + 1).$$

Then we still have $\frac{d\widetilde{\mathcal{N}}}{dt} \geq 0$, whereas $\widetilde{\mathcal{N}}$ has the property that for a fundamental solution $u = e^{-f}$ limiting to a δ-function as $t \to T$, we have $\widetilde{\mathcal{N}} \to 0$ as $t \to T$.

PROOF OF THE LEMMA. From (5.28), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^m(g(t)) &= 2\int_{\mathcal{M}} |R_{ij} + \nabla_i\nabla_j f|^2 dm \geq \frac{2}{n}\int_{\mathcal{M}} (R + \Delta f)^2 dm \\ &\geq \frac{2}{n}\left(\int_{\mathcal{M}} (R + \Delta f) dm\right)^2 \Big/ \int_{\mathcal{M}} dm \\ &= \frac{2}{n}\left(\int_{\mathcal{M}} dm\right)^{-1} \mathcal{F}^m(g(t))^2.\end{aligned}$$

The solution of the ODE

$$\frac{dx}{dt} = cx^2$$

with $\lim_{t\to T} x(t) = \infty$ is

$$x(t) = \frac{1}{c(T-t)}.$$

Hence, taking $c = \frac{2}{n}\left(\int_{\mathcal{M}} dm\right)^{-1}$, we get

$$\mathcal{F}^m(g(t)) \leq \frac{n}{2(T-t)} \int_{\mathcal{M}} dm.$$

□

REMARK 5.37.

(1) The formula above for $\frac{d}{dt}\mathcal{F}^m$ is somewhat reminiscent of Hamilton's formula for the evolution of the time-derivative dN/dt of his entropy $N(g) \doteqdot \int_{\mathcal{M}^2} R \log R dA$ on a positively curved surface evolving by Ricci flow (see [**180**]).

(2) We can rewrite (5.66) as

$$\int_{\mathcal{M}} \left(2\Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}\right) e^{-f} d\mu \leq 0,$$

where $\tau \doteqdot T - t$.

4.2. Hamilton's surface entropy. Recall that the *normalized* surface entropy for a closed surface $\left(\mathcal{M}^2, g\right)$ with positive curvature is defined by

$$N(g) \doteqdot \int_{\mathcal{M}} \log(RA)\, R d\mu,$$

where A is the area. Let $\left(\mathcal{M}^2, g(t)\right)$, $t \in [0, T)$, be a solution, on a maximal time interval of existence, of the Ricci flow on a closed surface with $R > 0$. In this subsection we give two proofs of the monotonicity of $N(g(t)) \doteqdot N(t)$.

4.2.1. *Hamilton's original proof of surface entropy monotonicity.* The time-derivative of $N(t)$ is given by

$$\frac{dN}{dt} = \int_{\mathcal{M}} QR d\mu, \tag{5.68}$$

where

$$Q \doteqdot \Delta \log R + R - r$$

and r is the average scalar curvature. On the other hand, since $\frac{dr}{dt} = r^2$ and by a similar computation to (V1-5.38),

$$\frac{\partial}{\partial t} Q \geq \Delta Q + 2\langle \nabla \log R, \nabla Q\rangle + Q^2 + 2rQ.$$

By the long-time existence theorem (Proposition 5.19 of Volume One), $r = \frac{1}{T-t}$ and $\operatorname{Area}(g(t)) = 4\pi\chi(T-t)$, where χ denotes the Euler characteristic of $\mathcal{M}$. Differentiating (5.68) with respect to time, we compute that $Z \doteqdot \frac{dN}{dt}$ satisfies

$$\begin{aligned}
\frac{dZ}{dt} &= \int_{\mathcal{M}} \left(\frac{\partial}{\partial t} Q\right) R d\mu + \int_{\mathcal{M}} Q \frac{\partial}{\partial t}(R d\mu) \\
&\geq \int_{\mathcal{M}} \left(Q^2 + 2rQ\right) R d\mu \\
&\geq \frac{1}{\int_{\mathcal{M}} R d\mu} \left(\int_{\mathcal{M}} QR d\mu\right)^2 + 2rZ \\
&= \frac{1}{4\pi\chi} Z^2 + \frac{2}{T-t} Z,
\end{aligned}$$

where we integrated by parts and used Hölder's inequality and the Gauss-Bonnet formula. Thus

$$\frac{d}{ds}\left(s^{-2}Z\right) \geq \frac{1}{4\pi\chi}\left(s^{-2}Z\right)^2, \tag{5.69}$$

where $s \doteqdot \frac{1}{T-t}$. From this we conclude that if $s_0^{-2}Z(s_0) > 0$ for some $s_0 < \infty$, then $s^{-2}Z(s) \to \infty$ as $s \to s_1$ for some $s_1 < \infty$. In other words, if $Z(t_0) > 0$ for some $t_0 < T$, then $Z(t) \to \infty$ as $t \to t_1$ for some $t_1 < T$. This contradicts our assumption that the solution exists on $[0, T)$. Hence $Z(t) \leq 0$ for all t and we have proved the following.

THEOREM 5.38 (Hamilton's surface entropy monotonicity). *For a solution of the Ricci flow on a closed surface with $R > 0$, we have*

$$\frac{dN}{dt}(t) \leq 0$$

for all $t \in [0, T)$.

Note that, from (5.69), we have $t \mapsto (T-t)^2 Z(t)$ is nondecreasing (since $\chi > 0$) and hence there is a constant $C > 0$ such that $(T-t)^2 Z(t) \geq -C$ for all $t \in [0, T)$. By (5.68),

$$Z = \frac{dN}{dt} = -\int_{\mathcal{M}} \frac{|\nabla R|^2}{R}\, dA + \int_{\mathcal{M}} (R-r)^2\, dA,$$

and we have

$$\int_{\mathcal{M}} \frac{|\nabla R|^2}{R}\, dA \leq \int_{\mathcal{M}} (R-r)^2\, dA + C(T-t)^{-2}.$$

REMARK 5.39. An inequality of the above type is often referred to as a reverse Poincaré inequality.

4.2.2. *Entropy formula for Hamilton's surface entropy.* Define the potential function f (up to an additive constant) by $\Delta f = r - R$. In [**97**] the monotonicity of the entropy was proved by relating its time-derivative to Ricci solitons via an integration by parts using the potential function (Proposition 5.39 in Volume One). In particular, we have

$$\begin{aligned}\frac{dN}{dt}(t) = &-2\int_{\mathcal{M}} \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2(T-t)} g_{ij} \right|^2 dA \\ &- \int_{\mathcal{M}} \left| \nabla \log\left(R \cdot e^{-f}\right) \right|^2 R\, dA.\end{aligned}$$

Note that $R_{ij} = \frac{1}{2} R g_{ij}$ and $r = \frac{1}{A}\int_{\mathcal{M}} R\, dA = (T-t)^{-1}$. We have purposely written this formula to more resemble Perelman's formulas (5.41) and (6.17).

4.2.3. *The gradient of Hamilton's surface entropy is the matrix Harnack quantity.* A less well-known fact is that the gradient of Hamilton's entropy in the space of all metrics with the L^2-metric is the matrix Harnack quantity:

$$\delta_v N(g) = \int_{\mathcal{M}} v_{ij} \left(-\Delta \log R \cdot g_{ij} + \nabla_i \nabla_j \log R - \frac{1}{2} R g_{ij} \right) dA, \tag{5.70}$$

where $\delta g = v$ (see Lemma 10.23 of [**111**] and use $N(g) - E(g)$ is a constant). In the space of metrics in a fixed conformal class, the gradient is the trace Harnack quantity. Note that the same relation is true relating the entropy and the trace Harnack quantity for the Gauss curvature flow of convex hypersurfaces in Euclidean space [**96**].

4.3. Bakry–Emery's logarithmic Sobolev-type inequality. The proofs of Hamilton's surface entropy formula and Perelman's energy formulas are formally similar to the proof of Bakry and Emery of their logarithmic Sobolev-type inequality [**18**].

PROPOSITION 5.40. *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold with* $\mathrm{Rc} \geq K$ *for some constant $K > 0$. If u is a positive function on $\mathcal{M}$, then*

$$\int_{\mathcal{M}} u \log u d\mu \leq \frac{1}{2K} \int_{\mathcal{M}} u |\nabla \log u|^2 d\mu + \log \left(\frac{1}{\mathrm{Vol}(\mathcal{M})} \int_{\mathcal{M}} u d\mu \right) \int_{\mathcal{M}} u d\mu.$$

PROOF. (See [**104**] for more details of the computations.) Consider the solution v to the heat equation $\frac{\partial v}{\partial t} = \Delta v$ with $v(0) = u$. The solution v exists for all time and

$$\lim_{t \to \infty} v = \frac{1}{\mathrm{Vol}(\mathcal{M})} \int_{\mathcal{M}} u d\mu.$$

Define $E(t) \doteqdot \int_{\mathcal{M}} v \log v d\mu$. Then

$$\lim_{t \to \infty} E(t) = \int_{\mathcal{M}} u d\mu \cdot \log \left(\frac{1}{\mathrm{Vol}(\mathcal{M})} \int_{\mathcal{M}} u d\mu \right). \tag{5.71}$$

We have

$$\frac{dE}{dt} = -\int_{\mathcal{M}} \langle \nabla v, \nabla \log v \rangle d\mu = -\int_{\mathcal{M}} v |\nabla \log v|^2 d\mu \leq 0.$$

Note that $\lim_{t \to \infty} \frac{dE}{dt}(t) = 0$.

Using $\frac{\partial}{\partial t} \log v = \Delta \log v + |\nabla \log v|^2$, we compute

$$\frac{d^2 E}{dt^2} = 2 \int_{\mathcal{M}} v \left(|\nabla \nabla \log v|^2 + \mathrm{Rc}(\nabla \log v, \nabla \log v) \right) d\mu.$$

Using our assumption $\mathrm{Rc} \geq K$, we find

$$\frac{d^2 E}{dt^2} \geq -2K \frac{dE}{dt}.$$

By $\lim_{t\to\infty} \frac{dE}{dt}(t) = 0$ and (5.71), we have

$$\frac{dE}{dt}(0) = -\int_0^\infty \frac{d^2E}{dt^2}(t)\,dt \leq 2K \int_0^\infty \frac{dE}{dt}(t)\,dt$$
$$= 2K \log\left(\frac{1}{\operatorname{Vol}(\mathcal{M})}\int_{\mathcal{M}} u d\mu\right)\int_{\mathcal{M}} u d\mu - 2KE(0).$$

Hence

$$-\int_{\mathcal{M}} u \left|\nabla \log u\right|^2 d\mu \leq 2K \log\left(\frac{1}{\operatorname{Vol}(\mathcal{M})}\int_{\mathcal{M}} u d\mu\right)\int_{\mathcal{M}} u d\mu$$
$$- 2K \int_{\mathcal{M}} u \log u d\mu$$

and the proposition follows. □

5. Notes and commentary

Subsection 1.1. As we remarked earlier, the function f is also known as the dilaton; in the physics literature there are numerous references to Perelman's energy functional (see Green, Schwarz, and Witten [**162**], Polchinksi [**307**], Strominger and Vafa [**341**] for example), although Perelman is the first to consider it in the context of Ricci flow. The Ricci flow is the 1-loop approximation of the renormalization group flow (see Friedan [**145**]).

Subsection 1.2. For a computational motivation for fixing the measure, see also §4 in Chapter 2 of [**111**], where Perelman's functional is motivated starting from the total scalar curvature functional. In particular, let $\delta g = v$. The variation of the total scalar curvature is

$$\delta \int_{\mathcal{M}} R d\mu = \int_{\mathcal{M}} \left(\operatorname{div}(\operatorname{div} v) - \Delta V - \operatorname{Rc} \cdot v + R\frac{V}{2}\right) d\mu$$
$$= -\int_{\mathcal{M}} \left(\operatorname{Rc} - \frac{R}{2} g\right) \cdot v d\mu.$$

This says that $\nabla\left(\int_{\mathcal{M}} R d\mu\right) = -\operatorname{Rc} + \frac{R}{2} g$, where the gradient is calculated with respect to the standard L^2-metric. To try to find a functional $\mathcal{F}$ with $\nabla\mathcal{F} = -\operatorname{Rc}$, we want to get rid of the $\frac{R}{2} g$ term. Now this term is due to the variation of $d\mu$. So we consider the distorted volume form $e^{-f} d\mu$ and assume its variation is 0. Hence

$$\delta \int_{\mathcal{M}} R e^{-f} d\mu = \int_{\mathcal{M}} (\delta R)\, e^{-f} d\mu = \int_{\mathcal{M}} \left(\operatorname{div}(\operatorname{div} v) - \Delta V - \operatorname{Rc} \cdot v\right) e^{-f} d\mu$$

and now we have the extra terms $\int_{\mathcal{M}} \left(\operatorname{div}(\operatorname{div} v) - \Delta V\right) e^{-f} d\mu$. We compensate for this by considering

$$\delta \int_{\mathcal{M}} \left|\nabla f\right|^2 e^{-f} d\mu = \int_{\mathcal{M}} \left(\delta \left|\nabla f\right|^2\right) e^{-f} d\mu$$
$$= \int_{\mathcal{M}} \left(-v(\nabla f, \nabla f) + \nabla f \cdot \nabla V\right) e^{-f} d\mu,$$

using $\frac{V}{2} = h \doteqdot \delta f$. Integrating by parts yields

$$\delta\mathcal{F} = \delta \int_{\mathcal{M}} Re^{-f} d\mu + \delta \int_{\mathcal{M}} |\nabla f|^2 e^{-f} d\mu = -\int_{\mathcal{M}} v_{ij} \left(R_{ij} + \nabla_i \nabla_j f \right) e^{-f} d\mu.$$

Although the Ricci tensor is not strictly elliptic in g, one can ask if the RHS of equation (5.27)

$$\frac{\partial}{\partial t} g_{ij} = -2 \left[R_{ij} + \nabla_i \nabla_j \log \left(\frac{d\mu}{dm} \right) \right]$$

is elliptic in g. The answer is still no. In particular, if $\delta g = v$, then

$$\delta \left[\log \left(\frac{d\mu}{dm} \right) \right] = \frac{V}{2}.$$

Hence

$$\delta \left[\nabla_i \nabla_j \log \left(\frac{d\mu}{dm} \right) \right] = \frac{1}{2} \nabla_i \nabla_j v - \delta \left(\Gamma_{ij}^k \right) \cdot \nabla_k \log \left(\frac{d\mu}{dm} \right).$$

Since

$$\delta \left(-2R_{ij} \right) = -\nabla_i \nabla^k v_{jk} - \nabla_j \nabla^k v_{ik} + \nabla_i \nabla_j v + \Delta v_{ij},$$

we have

$$\delta \left(-2 \left[R_{ij} + \nabla_i \nabla_j \log \left(\frac{d\mu}{dm} \right) \right] \right) = \Delta v_{ij} - \nabla_i \nabla^k v_{jk} - \nabla_j \nabla^k v_{ik} + \text{ lower-order terms},$$

where the last pair of terms form a Lie derivative of the metric term. However the second-order operator on the RHS is still not elliptic in v.