Preface

• **General idea and motivation**. In this book, we discuss realizations and applications of a new concept of approximation procedures, called *approximate approximations*. Most of these procedures, which include approximate quasi-interpolation, interpolation, least square approximation, cubature of integral operators, and wavelet approximations, have one common feature. They are accurate without being convergent in a rigorous sense. In numerical mathematics, such a situation is not exceptional. For instance, non-convergent algorithms are natural in solving overdetermined ill-posed problems. However, for the approximation processes mentioned above, convergence is required.

Needless to say, the engineers and researchers who use numerical methods for solving applied problems do not need the convergence of the method. In fact, they need results which are exact within a prescribed accuracy, determined mainly by the tolerance of measurements and other physical parameters, and always by the precision of the computing system. Their attitude, supported by common sense, was a powerful motivation for the development of our theory.

The lack of convergence in approximate approximations is compensated for, first of all, by the flexibility in the choice of basis functions and by the simplicity of the multi-dimensional generalization. Another, and probably the most important, advantage is the possibility of obtaining explicit formulas for values of various integral and pseudodifferential operators of mathematical physics applied to the basis functions.

The concept of approximate approximations and first related results were published by the first author in [**62**] – [**64**]. Later on, various aspects of a general theory of these approximations were systematically investigated in several joint papers by the authors ([**66**] – [**70**]). The present book is essentially based on the papers just mentioned and on our recent unpublished results. We also report on computational algorithms of the approximate approximations developed together with V. Karlin, T. Ivanov, W. Wendland, F. Lanzara, A. B. Movchan, *et al.*

The theory under consideration is at the very beginning of its development and we wrote this book with the hope of attracting new researchers to this area.

• **Approximate quasi-interpolation**. To give an impression of what we have in mind, recall, for example, that a typical error estimate of spline interpolation $M_h u$ on a uniform grid with size h, for a function $u \in C^N$, has the form

$$\|u - M_h u\|_C \le c h^N \|u\|_{C^N}$$

with some integer N and a constant c independent of u and h. Here C and C^N are the spaces of continuous and N-times continuously differentiable functions.

In contrast to this situation, we fix $\varepsilon > 0$ and construct an *approximate quasi-interpolant* $M_{h,\varepsilon}u$ using the translates of a more or less arbitrary function η_ε instead of piecewise polynomials

$$M_{h,\varepsilon}u(x) = \sum_m u(hm)\eta_\varepsilon(x/h - m) \, .$$

One can show that

$$\|u - M_{h,\varepsilon}u\|_C \le c_1(u)h^N + c_2(u)\,\varepsilon \, .$$

Thus, the error consists of a part converging with order N to zero as $h \to 0$ and a non-convergent part $c_2(u)\varepsilon$ called the *saturation error*. Thus, the procedure provides good approximations up to some prescribed error level, but it does not converge as $h \to 0$.

The approximate quasi-interpolation procedure can be extended to the approximation of functions on domains and manifolds with non-uniformly distributed nodes.

• **Cubature formulas**. The numerical treatment of potentials and other integral operators with singular kernels arises as a computational task in different fields. Since standard cubature methods are very time-consuming, there is ongoing research to develop new effective algorithms like panel clustering, multipole expansions, or wavelet compression based on piecewise polynomial approximations of the density. The effective treatment of integral operators is also one of the main applications of approximate approximation.

The richness of the class of generating functions η makes it easier to find approximations for which the action of a given pseudodifferential operator can be effectively determined. For example, suppose one has to evaluate the convolution with a singular radial kernel as in the case of many potentials in mathematical physics. If the density is replaced by a quasi-interpolant with radial η, then after passing to spherical coordinates, the convolution is approximated by one-dimensional integrals. For many important integral operators $\mathcal{K}$ one can choose η even such that $\mathcal{K}\eta$ is analytically known, which results in semi-analytic cubature formulas for these operators. The special structure of the quasi-interpolation error gives rise to an interesting effect. Since the saturation error is a fast oscillating function and converges weakly to zero, the cubature formulas for potentials converge even in the rigorous sense, although there is no convergence for their densities.

• **Approximate wavelets**. Another example of approximate approximations is the notion of *approximate wavelet* decompositions for spaces generated by smooth functions satisfying refinement equations with a small error. It appears that those *approximate refinement equations* are satisfied by a broad class of scaling functions. This relation allows one to perform an approximate multi-resolution analysis of spaces generated by those functions. Therefore a wavelet basis can be constructed in which elements of fine scale spaces are representable within a given tolerance. The approximate wavelets provide most of the properties utilized in wavelet-based numerical methods and possess additionally simple analytic representations. Therefore the sparse approximation of important integral operators in the new basis can be computed using special functions or simple quadrature. One can give explicit formulas for harmonic and diffraction potentials whose densities are approximate wavelets.

• **Applications in mathematical physics**. The capability of approximate approximations to treat multi-dimensional integral operators enables one to develop new efficient numerical and semi-analytic methods for solving various problems in mathematical physics. First of all, this tool can be effectively used as an underlying approximation method in numerical algorithms for solving problems with integro-differential equations. Another very important application of approximate approximations is in the large field of integral equation methods for solving initial and boundary value problems for partial differential equations.

• **Structure of the book**. We describe briefly the contents of the book. More details are given in the introduction of each chapter. Most of the references to the literature are collected in Notes at the end of Chapters 2–13.

In Chapters 1 and 2 we analyze the approximate quasi-interpolation on uniform lattices. We start with simplest examples of second- and higher-order quasi-interpolants in both the one-dimensional and multi-dimensional cases. Then we turn to pointwise and integral error estimates for quasi-interpolation of functions given on the whole space. We formulate conditions on the generating functions η of quasi-interpolation formulas which ensure the smallness of saturation errors and the convergence with a given order up to the saturation bound.

A variety of basis functions and algorithms for their construction are the subject of Chapter 3. We provide examples giving rise to new classes of simple multivariate quasi-interpolation formulas which behave in numerical computations like high-order approximations.

Chapters 4 and 5 are dedicated to semi-analytic cubature formulas for numerous integral and pseudodifferential operators of mathematical physics, in particular for harmonic, elastic, and diffraction potentials. In Chapter 6 we obtain approximations of the inverse operator of the Cauchy problem for the heat, wave, and plate equations. There we also give formulas for the value of integral operators applied to more general basis functions.

The Gaussian functions possess remarkable approximation properties. Chapter 7 is devoted to quasi-interpolation and interpolation with these basis functions.

In Chapter 8 we perform approximate multi-resolution analysis for spaces generated by functions of the Schwartz class and introduce approximate wavelets. For the example of the Gaussian kernel we give simple analytic formulas of such wavelets first in the one-dimensional case and then in the case of many dimensions. We obtain quadratures of Newton and diffraction potentials acting on these wavelets.

In Chapter 9 the method of cubature of potentials is extended to the computation of these potentials over a bounded domain. Here we use mesh refinement towards the boundary of the domain and construct special boundary layer approximations. Our algorithm relies heavily on approximate refinement equations which, as was mentioned, play a crucial role in the construction of approximate wavelets also.

The approximate quasi-interpolation is extended in Chapter 10 to the approximation of functions on non-cubic grids and on domains and manifolds with non-uniformly distributed nodes.

In Chapter 11 we study approximate quasi-interpolation of scattered data. We show that simple modifications of basis functions provide an approximate partition of unity which allows the construction of high-order approximate quasi-interpolants on scattered centers.

Finally, in Chapters 12 and 13, we treat applications of approximate approximations to numerical algorithms of solving linear and non-linear pseudodifferential equations of mathematical physics. To be more specific, in Chapter 12 we apply the cubature methods developed in Chapter 4 to the solution of Lippmann-Schwinger type equations of scattering theory. We describe the boundary point method, the application of approximate approximations to the solution of boundary integral equations. The same chapter contains formulas for the harmonic single layer potential acting on basis functions given on a surface. In Chapter 13, we describe applications to non-linear evolution equations with local and non-local operators, including the Navier-Stokes, Joseph, Benjamin-Ono, and Sivashinsky equations.

• **Readership**. The book is intended for graduate students and researchers interested in applied approximation theory and numerical methods for solving problems of mathematical physics. No special knowledge is required to read this book, except for conventional university courses on functional analysis and numerical methods.

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