CHAPTER 2

Some Examples

Lie groups and linear algebraic groups are the most interesting and useful topological groups. Without going into any structure theory here, we summarize some basic facts on linear Lie groups and then turn to a number of important examples. Those examples include general linear groups $GL(n;\mathbb{F})$, special linear groups $SL(n;\mathbb{F})$, unitary groups $U(p,q;\mathbb{F})$ of various signatures over the real, complex and quaternion fields, real and complex symplectic groups $Sp(n;\mathbb{F})$, $\mathfrak{p}$–adic completions of linear algebraic groups defined over the rationals, and various types of Heisenberg groups. Other examples are concerned with application to harmonic analysis on spheres, projective spaces and Grassmann manifolds. All of these examples will be used later in the book.

2.1. General and Special Linear Groups

Let $\mathbb{F}$ be the real number field $\mathbb{R}$, the complex number field $\mathbb{C}$, or the quaternion division algebra $\mathbb{H}$. Let V be a finite dimensional vector space over $\mathbb{F}$. Since we'll be concentrating on groups of linear transformations of V it will be convenient to have linear transformations act from the left, so that the correspondence between matrices and linear transformations does not reverse order of products. Thus we want scalars to act from the right, that is, we take V to be a right vector space over $\mathbb{F}$. Now a choice $\beta = \{e_1, \dots, e_n\}$ of basis of V gives an isomorphism of V with the right vector space $\mathbb{F}^n$ of $n \times 1$ (column) vectors with entries from $\mathbb{F}$, and here every $\mathbb{F}$–linear transformation of V corresponds to an $n \times n$ matrix with entries from $\mathbb{F}$ acting on $\mathbb{F}^n$ by $g : v \mapsto gv$. Now we have the **general linear group**

$$(2.1.1) \qquad GL(V) \cong GL(n;\mathbb{F}) : \text{ invertible } \mathbb{F}\text{–linear transformations of } V \cong \mathbb{F}^n.$$

Here we think of $GL(V)$ as a group of linear transformations and $GL(n;\mathbb{F})$ as a multiplicative group of matrices.

Recall the notation $\mathbb{F}^{n\times n}$ for the space of $n \times n$ matrices over $\mathbb{F}$. If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ then $\mathbb{F}^{n\times n}$ is a vector space of dimension n^2 over $\mathbb{F}$. If $\mathbb{F} = \mathbb{H}$ then it is a vector space of dimension $4n^2$ over $\mathbb{F}$.

We can view V as a vector space of dimension nr over $\mathbb{R}$ where $r = \dim_{\mathbb{R}} \mathbb{F}$. Then $GL(V)$ consists of all $\mathbb{R}$–linear transformations $g : V \to V$ with $\det_{\mathbb{R}}(g) \neq 0$ that also are $\mathbb{F}$–linear, i.e. that commute with the (right) scalar action of $\mathbb{F}$. This exhibits $GL(n;\mathbb{F})$ as an open subset of the real vector space $\mathbb{F}^{n\times n} \cong_{\mathbb{R}} \mathbb{R}^{rn\times rn}$. In the subspace topology the matrix group $GL(n;\mathbb{F})$ now is a locally compact topological group, and $GL(V)$ acquires the same structure from the isomorphism of (2.1.1).

We can also view $GL(V)$ as a closed submanifold of $\mathbb{F}^{2n\times 2n}$, where

$$a \in GL(V) \text{ corresponds to } \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathbb{F}^{2n\times 2n}. \tag{2.1.2}$$

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, so that one has the determinant, it is more usual to use

$$GL(V) \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & (\det a)^{-1} \end{pmatrix} \middle| a \in \mathbb{F}^{n\times n} \right\}$$

with the group isomorphism given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & \det a^{-1} \end{pmatrix}$.

The general linear group has some obvious and useful subgroups, such as

(2.1.3) $GL'(V) \cong GL'(n;\mathbb{F})$: all elements that preserve Lebesgue measure on V.

So

$GL'(n;\mathbb{R}) = \{g \in GL(n;\mathbb{R}) \mid \det(g) = \pm 1\}$,

$GL'(n;\mathbb{C}) = \{g \in GL(n;\mathbb{C}) \mid |\det(g)| = 1\} = \{g \in GL(n;\mathbb{C}) \mid \det_{\mathbb{R}} g = 1\}$, *and*

$GL'(n;\mathbb{H}) = \{g \in GL(n;\mathbb{H}) \mid \det_{\mathbb{R}}(g) = 1\}$.

A slightly smaller but more famous subgroup is the **special linear group**

(2.1.4) $SL(V) \cong SL(n;\mathbb{F})$: the derived group of $GL(V) \cong GL(n;\mathbb{F})$.

All these are closed subgroups of general linear groups, hence also are locally compact topological groups.

These groups have an interesting involutive automorphism, the **Cartan involution** given on the matrix level by

$$\theta : g \mapsto \overline{{}^t g^{-1}} \tag{2.1.5}$$

where the bar designates conjugation of $\mathbb{F}$ over $\mathbb{R}$. It has fixed point set

$$\{g \in GL(n;\mathbb{F}) \mid \theta(g) = g\} \quad = \quad U(n;\mathbb{F}), \tag{2.1.6}$$

which will be defined below and which turns out to be a maximal compact subgroup.

2.2. Linear Lie Groups

For our purposes, a **linear Lie group** is a group G of linear transformations of a (real, complex or quaternionic) vector space V defined by some polynomial equations (in the matrix entries for a choice of basis of V). Thus G is a linear algebraic group by definition, and we will see just how G is a Lie group. By definition, a **Lie group** is a C^ω (real analytic) differentiable manifold with a group structure such that the group operations are C^ω.

First, we may always regard V as a real vector space, perhaps at the cost of doubling or quadrupling its dimension, so we may view G as a real linear Lie group perhaps at the additional cost of increasing the number of defining equations. That done, $n = \dim V$, and a basis chosen for V, now G may be viewed as the group of all $n \times n$ real matrices $(g_{i,j})$ that satisfy a collection of polynomial equations $f_k(g_{1,1}, \ldots, g_{n,n}) = 0$. The implicit function theorem now ensures that generically and locally G is a C^ω submanifold of the vector space $\mathbb{R}^{n\times n}$ of $n \times n$ real matrices. Applying group translations G is globally a C^ω submanifold of $\mathbb{R}^{n\times n}$. The group operations are polynomial in $\mathbb{R}^{n\times n}$, thus C^ω on the C^ω submanifold G. Thus, by definition, G is a Lie group.

Equation (2.1.2) shows that the general linear groups $GL(n;\mathbb{F})$ are linear Lie groups. It follows that the $GL'(n;\mathbb{F})$ and the special linear groups $SL(n;\mathbb{F})$ are linear Lie groups because we obtain them by just enlarging the set of $\mathbb{R}$–polynomial equations used to define $GL(n;\mathbb{F})$.

We will want the **Lie algebra** $\mathfrak{g}$ of a linear Lie group G. By definition (only for linear Lie groups!) $\mathfrak{g}$ consists of all linear transformations ξ of V such that $\exp(t\xi) \in G$ for all real t. Here $\exp(t\xi)$ means the exponential series $\sum_{r \geqq 0} \frac{t^r}{r!}\xi^r$, and it is easy to see that this series converges for all real t. One can prove that $\mathfrak{g}$ is closed under linear combination and also under the composition $[\xi, \eta] = \xi\eta - \eta\xi$. Thus $\mathfrak{g}$ is a vector space of linear transformations of V with the Lie algebra composition $[\xi, \eta]$.

Definition 2.2.1. The automorphism $\theta : GL(n;\mathbb{F}) \to GL(n;\mathbb{F})$, given by $\theta(g) = \overline{{}^t g^{-1}}$ as in (2.1.5), is a Cartan involution of $GL(n;\mathbb{F})$. If a linear Lie group $G \subset GL(n;\mathbb{F})$ is stable under θ (or one of its conjugates in the group of automorphisms of $GL(n;\mathbb{F})$) then $\theta|_G$ (or the restriction of the conjugate that stabilizes G) is a **Cartan involution** of G. If ψ is a Cartan involution of G then $d\psi : \mathfrak{g} \to \mathfrak{g}$ is called a Cartan involution of the Lie algebra $\mathfrak{g}$, and the fixed point set $K = G^\psi$ is a maximal compact subgroup of G. ♢

2.3. Groups Defined by Bilinear Forms

Let $\mathbf{b}$ be a bilinear form on V. That means $\mathbf{b}$: $V \times V \to \mathbb{F}$ is an $\mathbb{F}$–bilinear map. In general an $\mathbb{F}$–bilinear map $T : U \times V \to W$ factors through an $\mathbb{F}$–linear map $U \otimes V \to W$, so $U \otimes V$ must be defined and must be a (right, in our case) vector space over $\mathbb{F}$. This requires $\mathbb{F}$ to be commutative. So we only consider bilinear forms in the real and complex cases.

Fix a bilinear form $\mathbf{b}$ on V. Then a choice $\beta = \{e_1, \dots, e_n\}$ of basis of V gives a matrix $B = (\mathbf{b}(e_i, e_j))$ such that, identifying a vector $v = \sum v_i e_i \in V$ with the column vector ${}^t(v_1, \dots, v_n)$, we have

$$\mathbf{b}(u, v) = \sum_{1 \leqq i,j \leqq n} u_i b_{i,j} v_j = {}^t u B v. \tag{2.3.1}$$

The bilinear form now defines a group

$$\begin{aligned} O(V, \mathbf{b}) &= \{g \in GL(V) \mid \mathbf{b}(gu, gv) = \mathbf{b}(u, v)\ \forall u, v \in V\} \\ &\cong \{g \in GL(n;\mathbb{F}) \mid {}^t g B g = B\}. \end{aligned} \tag{2.3.2}$$

Each of the groups (2.3.2) is a subgroup of the general linear group defined by a system of quadratic equations in the matrix entries, thus a closed subgroup of the general linear group, and thus a locally compact topological group. The group (2.3.2) is called the **orthogonal group** of $\mathbf{b}$ when $\mathbf{b}$ is symmetric and nondegenerate, called the **symplectic group** of $\mathbf{b}$ when $\mathbf{b}$ is antisymmetric and nondegenerate.

Suppose that $\mathbf{b}$ is symmetric and nondegenerate. In the complex case there is a basis in which the matrix $B = I$, the identity matrix, and the corresponding group is the **complex orthogonal group**

$$O(n;\mathbb{C}) = \{g \in GL(n;\mathbb{C}) \mid {}^t g g = I\}. \tag{2.3.3}$$

In the real case there is a basis in which B is the matrix $I_{p,q} = \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right)$, $0 \leqq p \leqq n$ and $p+q=n$, and the corresponding groups are the **real orthogonal groups**

$$O(p,q) = \left\{g \in GL(n;\mathbb{R}) \mid {}^t g I_{p,q} g = I_{p,q}\right\} \text{ where } I_{p,q} = \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right) \tag{2.3.4}$$

of "signature" (p,q). They are called **indefinite orthogonal groups** when $pq \neq 0$. If $pq = 0$, then $O(p,q)$ is the ordinary orthogonal group $O(n) = O(0,n) = O(n,0)$, and the condition of (2.3.4) is the standard condition ${}^t gg = I$.

Taking determinant on the defining conditions of (2.3.3) and (2.3.4) we see that it forces $\det(g)^2 = 1$. The additional condition $\det(g) = 1$ defines the **special orthogonal groups** $SO(n,\mathbb{C}) \subset O((n,\mathbb{C})$ and $SO(p,q) \subset O(p,q)$. In all cases the special orthogonal group is a subgroup of index 2 in the orthogonal group. $SO(n,\mathbb{C})$ and $SO(n)$ are connected, but $SO(p,q)$ has 2 components when $pq \neq 0$.

Now suppose that $\mathbf{b}$ is antisymmetric and nondegenerate. Then $n = \dim_{\mathbb{F}} V$ is even, $n = 2m$, and there is a basis in which B is the matrix $J = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$. The corresponding groups are the **complex symplectic group**

$$Sp(m;\mathbb{C}) = \{g \in GL(n;\mathbb{C}) \mid {}^t gJg = J\} \text{ where } J = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right). \tag{2.3.5}$$

and the **real symplectic group**

$$Sp(m;\mathbb{R}) = \{g \in GL(n;\mathbb{R}) \mid {}^t gJg = J\} \text{ where } J = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right). \tag{2.3.6}$$

The groups $Sp(m,\mathbb{F})$ are connected. Note that $Sp(m;\mathbb{R})$ simply consists of the real matrices in $Sp(m;\mathbb{C})$.

In all cases, we have the Cartan involution θ of G given by the restriction of the Cartan involution (2.1.5) of the general linear group. It has fixed point sets

$$\{g \in O(n;\mathbb{C}) \mid \theta(g) = g\} = O(n;\mathbb{C}) \cap GL(n;\mathbb{R}) = O(n), \tag{2.3.7}$$

$$\{g \in O(p,q) \mid \theta(g) = g\} = O(p,q) \cap O(n) \cong O(p) \times O(q), \tag{2.3.8}$$

$$\{g \in Sp(m;\mathbb{C}) \mid \theta(g) = g\} = Sp(m;\mathbb{C}) \cap U(2m) \cong Sp(m), \tag{2.3.9}$$

$$\{g \in Sp(m;\mathbb{R}) \mid \theta(g) = g\} = Sp(m;\mathbb{R}) \cap O(2m) \cong U(m), \tag{2.3.10}$$

respective maximal compact subgroups of $O(n;\mathbb{C})$, $O(p,q)$, $Sp(m;\mathbb{C})$ and $Sp(m;\mathbb{R})$. Here $Sp(m)$ will be defined later.

2.4. Groups Defined by Hermitian Forms

Let $\mathbf{h}$ be an hermitian form on V. This will mean that $\mathbf{h} : V \times V \to \mathbb{F}$ is linear in the first variable, conjugate–linear in the second variable, and satisfies $\mathbf{h}(v,u) = \overline{\mathbf{h}(u,v)}$. This is the usual (for mathematicians – not for physicists) rôle of the first and second variables. It is slightly inconvenient in the quaternionic case, where conjugation reverses the order of products, but we meet that problem with the formulation (2.4.1) for the matrix expressions of hermitian forms.

Now, as for bilinear forms, a choice $\beta = \{e_1, \dots, e_n\}$ of basis of V gives a matrix $H = (h_{i,j}) = (\mathbf{h}(e_j, e_i))$ such that, identifying a vector $v = \sum e_i v_i \in V$ with the column vector ${}^t(v_1, \dots, v_n)$, we have

$$\mathbf{h}(u,v) = \sum_{1 \leqq i,j \leqq n} \overline{v_i} h_{i,j} u_j = v^* H u, \qquad v^* = {}^t\overline{v}. \tag{2.4.1}$$

The hermitian form now defines a group

$$U(V,\mathbf{h}) = \{g \in GL(V) \mid \mathbf{h}(gu,gv) = \mathbf{h}(u,v)\ \forall u,v \in V\} \\ \cong \{g \in GL(n;\mathbb{F}) \mid g^*Hg = H\},\ g^* = {}^t\overline{g}. \tag{2.4.2}$$

As mentioned above, there is a delicate point here in the quaternionic case, where it uses the calculation

$$\mathbf{h}(gu,gv) = \sum_{i,j}\left(\overline{\sum_\ell g_{i,\ell}v_\ell}\right) h_{i,j}\left(\sum_m g_{j,m}u_m\right) \\ = \sum_{\ell,m}\overline{v_\ell}\left(\sum_{i,j}(g^*)_{\ell,i}h_{i,j}g_{j,m}\right)u_m = \sum_{\ell,m}\overline{v_\ell}(g^*hg)_{\ell,m}u_m.$$

Each of the groups (2.4.2) is a subgroup of the general linear group defined by a system of quadratic equations in the real components of the matrix entries, thus is a closed subgroup of the general linear group, so it is a locally compact topological group and a linear Lie group.

V has a basis that is orthonormal in the sense that $H = I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $0 \leqq p \leqq n$ and $p+q=m$, the matrix we met while describing the real orthogonal groups. The corresponding groups are the **(indefinite) unitary groups** over $\mathbb{F}$,

$$U(p,q;\mathbb{F}) = \{g \in GL(n;\mathbb{F}) \mid g^*I_{p,q}g = I_{p,q}\}, 0 \leqq p,q \text{ and } p+q=n. \tag{2.4.3}$$

When $\mathbf{h}$ is definite, i.e. when $pq=0$, these are the usual **unitary groups** over $\mathbb{F}$,

$$U(n;\mathbb{F}) = U(n,0;\mathbb{F}) = U(0,n;\mathbb{F}).$$

This notation is standard, but unfortunately there are several standards. Another standard notation is to write

$$O(p,q) \text{ for } U(p,q;\mathbb{R}),\ U(p,q) \text{ for } U(p,q;\mathbb{C}), \text{ and } Sp(p,q) \text{ for } U(p,q;\mathbb{H}), \tag{2.4.4}$$

reflecting the respective names **orthogonal**, **unitary**, and **symplectic unitary**, of "signature" (p,q), for these groups. In the definite case one has the standard notation

$$O(n) \text{ for } U(n;\mathbb{R}),\ U(n) \text{ for } U(n;\mathbb{C}), \text{ and } Sp(n) \text{ for } U(n;\mathbb{H})$$

and one usually refers to these groups as the **orthogonal**, **unitary** and **symplectic unitary** groups.

The corresponding **(indefinite) special unitary groups** are the

$$SU(p,q;\mathbb{F}) = U(p,q;\mathbb{F}) \cap SL(n;\mathbb{F}) \text{ where } 0 \leqq p,q \text{ and } p+q=n. \tag{2.4.5}$$

In the real and complex cases $SU(p,q;\mathbb{F}) = \{g \in U(p,q;\mathbb{F}) \mid \det(g) = 1\}$. There is no distinction in the quaternion case: $SU(p,q;\mathbb{H}) = U(p,q;\mathbb{H})$. Except in the case where $\mathbb{F} = \mathbb{R}$ and $n=2$, $SU(p,q;\mathbb{F})$ is the derived group of $U(p,q;\mathbb{F})$. As before, when $\mathbf{h}$ is definite these are the usual **special unitary groups** over $\mathbb{F}$,

$$SU(n;\mathbb{F}) = SU(n,0;\mathbb{F}) = SU(0,n;\mathbb{F}),$$

and one has the other standard notation

$$SO(p,q) = SU(p,q;\mathbb{R}),\ SU(p,q) = SU(p,q;\mathbb{C}),\ Sp(p,q) = SU(p,q;\mathbb{H}) \tag{2.4.6}$$

reflecting the respective names **special orthogonal**, **special unitary**, and **symplectic unitary**, of "signature" (p, q), for these groups. And of course in the definite case we have the notation

$$SO(n) \text{ for } SU(n;\mathbb{R}),\ SU(n) \text{ for } SU(n;\mathbb{C}), \text{ and } Sp(n) \text{ for } SU(n;\mathbb{H}),$$

called the **special orthogonal**, **special unitary** and **symplectic unitary** groups.

Again the Cartan involutions of our groups are of the form (2.1.5), that is, $\theta(g) = (g^*)^{-1}$. It has fixed point set

$$\{g \in U(p, q; \mathbb{F}) \mid \theta(g) = g\} = U(p, q; \mathbb{F}) \cap U(n; \mathbb{F}) \cong U(p; \mathbb{F}) \times U(q; \mathbb{F}) \tag{2.4.7}$$

in the full unitary case, but is slightly more complicated in the special unitary case. Of course nothing new happens in the quaternionic setting, but

$$\begin{aligned}\{g \in SU(p, q; \mathbb{F}) \mid \theta(g) = g\} &= SU(p, q; \mathbb{F}) \cap U(n; \mathbb{F}) \\ &\cong S\left(U(p; \mathbb{F}) \times U(q; \mathbb{F})\right), \quad \mathbb{F} = \mathbb{R}, \mathbb{C}.\end{aligned} \tag{2.4.8}$$

The fixed point set of θ is a maximal compact subgroup in all cases.

One can also consider groups defined by skew–hermitian forms $\mathbf{s}$ on V. So $\mathbf{s}: V \times V \to \mathbb{F}$ is linear in the second variable, conjugate linear in the first variable, and satisfies $\mathbf{s}(u, v) + \mathbf{s}(v, u) = 0$. In the real case, skew–hermitian is the same as antisymmetric, so we get the groups $Sp(n; \mathbb{R})$. In the complex case, a matrix S is skew–hermitian if and only if it is of the form $S = \sqrt{-1}H$ where H is hermitian. In that case, $g^*Sg = S$ if and only if $g^*Hg = H$, so we get the groups $U(p, q)$ and $SU(p, q)$. But in the quaternionic case we get something new,

$$SO^*(V, \mathbf{s}) = \{g \in GL(V) \mid \mathbf{s}(gu, gv) = \mathbf{s}(u, v) \text{ for all } u, v \in V\}. \tag{2.4.9}$$

We can choose a basis of V in which $\mathbf{s}$ has matrix iI_n, and then this group is written

$$SO^*(2n) = \{g \in GL(n; \mathbb{H}) \mid g^*Sg = S\} \text{ where } S = iI_n. \tag{2.4.10}$$

We embed $GL(n; \mathbb{H}) \hookrightarrow GL(4n; \mathbb{R})$ by the usual map

$$\Psi : g_1 + ig_2 + jg_3 + kg_4 \mapsto \begin{pmatrix} g_1 & -g_2 & -g_3 & -g_4 \\ g_2 & g_1 & -g_4 & g_3 \\ g_3 & g_4 & g_1 & -g_2 \\ g_4 & -g_3 & g_2 & g_1 \end{pmatrix}. \tag{2.4.11}$$

That is induced from

$$\mathbb{H} \to \mathbb{R}^4 \text{ by } u_1 + u_2 i + u_3 j + u_4 k \mapsto \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Similarly $GL(2n; \mathbb{C}) \hookrightarrow GL(4n; \mathbb{R})$ by $\Phi : a + ib \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. One can check that

$$\Psi(SO^*(2n)) = \Psi(GL(n; \mathbb{H})) \cap \Phi(U(n, n)), \tag{2.4.12}$$

which is another standard way of describing $SO^*(2n)$.

2.5. Degenerate Forms

Let $\mathbf{b}$ be a symmetric bilinear form on V, not necessarily nondegenerate. Thus the subspace

$$U = \mathrm{Ker}(\mathbf{b}) = \{v \in V \mid \mathbf{b}(v, V) = 0\} \subset V \tag{2.5.1}$$

can be nonzero. In any case $\mathbf{b}$ induces a nondegenerate symmetric bilinear form $\overline{\mathbf{b}}$ on the quotient space V/U. Let W be any vector space complement to U in V, so $V = U \oplus W$. Then $\mathbf{b}$ is nondegenerate on W. Let $O(V, \mathbf{b})$ denote the group of (2.3.2) and choose a basis $\beta = \{u_1, \dots, u_r; w_1, \dots, w_s\}$ of V that starts with a basis of U and finishes with a basis of W. Relative to β the matrix group $O(V, \mathbf{b})$ is

$$O(V, \mathbf{b}) = \left\{ g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \;\middle|\; a \in GL(U) \text{ and } b \in O(W, \mathbf{b}|_{W \times W}) \right\}. \tag{2.5.2}$$

Let $X = \mathbb{F}^{r \times s} \cong \mathrm{Hom}_{\mathbb{F}}(W, U)$. A quick computation with (2.5.2) shows that

$$\begin{aligned} &O(V, \mathbf{b}) = X \rtimes (GL(U) \times O(W, \mathbf{b}|_{W \times W})), \text{ semidirect product for} \\ &\qquad \text{the action } \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : x \mapsto axb^{-1} \text{ of } GL(U) \times O(W, \mathbf{b}|_{W \times W}) \text{ on } X \end{aligned} \tag{2.5.3}$$

Next, let $\mathbf{b}$ be an antisymmetric bilinear form on V, not necessarily nondegenerate. Again, the subspace

$$U = \mathrm{Ker}(\mathbf{b}) = \{v \in V \mid \mathbf{b}(v, V) = 0\} \subset V \tag{2.5.4}$$

can be nonzero, and $\mathbf{b}$ induces a nondegenerate antisymmetric bilinear form $\overline{\mathbf{b}}$ on the quotient space V/U. As before, let W be any vector space complement to U in V, so $V = U \oplus W$. Then $\mathbf{b}$ is nondegenerate on W. Let $Sp(V, \mathbf{b})$ denote the group of (2.3.2) and choose a basis $\beta = \{u_1, \dots, u_r; w_1, \dots, w_s\}$ of V that starts with a basis of U and finishes with a basis of W. Relative to β the matrix group $O(V, \mathbf{b})$ is

$$Sp(V, \mathbf{b}) = \left\{ g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \;\middle|\; a \in GL(U) \text{ and } b \in Sp(W, \mathbf{b}|_{W \times W}) \right\}. \tag{2.5.5}$$

Let $X = \mathbb{F}^{r \times s} \cong \mathrm{Hom}_{\mathbb{F}}(W, U)$. As before, computation with (2.5.5) shows that

$$\begin{aligned} &Sp(V, \mathbf{b}) = X \rtimes (GL(U) \times Sp(W, \mathbf{b}|_{W \times W})), \text{ semidirect product for} \\ &\qquad \text{the action } \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : x \mapsto axb^{-1} \text{ of } GL(U) \times Sp(W, \mathbf{b}|_{W \times W}) \text{ on } X \end{aligned} \tag{2.5.6}$$

Now let $\mathbf{h}$ be an $\mathbb{F}$–hermitian form on V, not necessarily nondegenerate. Yet again, the subspace

$$U = \mathrm{Ker}(\mathbf{h}) = \{v \in V \mid \mathbf{h}(v, V) = 0\} \subset V \tag{2.5.7}$$

can be nonzero, and $\mathbf{h}$ induces a nondegenerate $\mathbb{F}$–hermitian form $\overline{\mathbf{h}}$ on the quotient space V/U. As before, let W be any vector space complement to U in V, so $V = U \oplus W$. Then $\mathbf{h}$ is nondegenerate on W. Let $U(V, \mathbf{h})$ denote the group of (2.4.2) and choose a basis $\beta = \{u_1, \dots, u_r; w_1, \dots, w_s\}$ of V that starts with a basis of U and finishes with a basis of W. Relative to β the matrix group $U(V, \mathbf{h})$ is

$$U(V, \mathbf{h}) = \left\{ g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \;\middle|\; a \in GL(U) \text{ and } b \in U(W, \mathbf{h}|_{W \times W}) \right\}. \tag{2.5.8}$$

Let $X = \mathbb{F}^{r\times s} \cong \mathrm{Hom}_{\mathbb{F}}(W,U)$. Once again, computation with (2.5.8) shows that

$$U(V,\mathbf{h}) = X \rtimes (GL(U) \times U(W,\mathbf{h}|_{W\times W})), \text{ semidirect product for the action } \left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right) : x \mapsto axb^{-1} \text{ of } GL(U)\times U(W,\mathbf{h}|_{W\times W}) \text{ on } X \tag{2.5.9}$$

Finally, let $\mathbf{s}$ be a skew–hermitian form on a vector space V over $\mathbb{H}$, not necessarily nondegenerate. Yet again, the subspace

$$U = \mathrm{Ker}(\mathbf{s}) = \{v \in V \mid \mathbf{s}(v,V) = 0\} \subset V \tag{2.5.10}$$

can be nonzero, and $\mathbf{s}$ induces a nondegenerate skew–hermitian form $\bar{\mathbf{s}}$ on the quotient space V/U. As before, let W be any vector space complement to U in V, so $V = U \oplus W$. Then $\mathbf{s}$ is nondegenerate on W. Let $SO^*(V,\mathbf{s})$ denote the group of (2.4.9) and choose a basis $\beta = \{u_1,\dots,u_r; w_1,\dots,w_s\}$ of V that starts with a basis of U and finishes with a basis of W. Relative to β the matrix group $SO^*(V,\mathbf{s})$ is

$$SO^*(V,\mathbf{s}) = \left\{ g = \begin{pmatrix} a & x \\ 0 & b\end{pmatrix} \;\middle|\; a \in GL(U) \text{ and } b \in SO^*(W,\mathbf{s}|_{W\times W}) \right\}. \tag{2.5.11}$$

Let $X = \mathbb{H}^{r\times s} \cong \mathrm{Hom}_{\mathbb{F}}(W,U)$. Once again, computation with (2.5.11) shows that

$$SO^*(V,\mathbf{s}) = X \rtimes (GL(U) \times SO^*(W,\mathbf{s}|_{W\times W})), \text{ semidirect product for the action } \left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right) : x \mapsto axb^{-1} \text{ of } GL(U)\times SO^*(W,\mathbf{s}|_{W\times W}) \text{ on } X \tag{2.5.12}$$

2.6. Automorphism Groups of Algebras

Let $\mathcal{A}$ be a finite dimensional algebra over $\mathbb{R}$ or $\mathbb{C}$, not necessarily associative. In other words, the multiplication on $\mathcal{A}$ could be any bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Then one has the **automorphism group** of $\mathcal{A}$,

$$\mathrm{Aut}(\mathcal{A}) = \{g \in GL(\mathcal{A}) \mid g(xy) = g(x)g(y)\ \forall\ x,y \in \mathcal{A}\}. \tag{2.6.1}$$

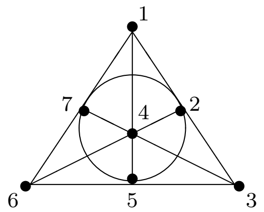
It is a closed subgroup of $GL(\mathcal{A})$ defined by some quadratic equations, so it is a topological group.

For example, if $\mathcal{A}$ is the associative[1] algebra $M_n(\mathbb{F})$ of $n \times n$ matrices over $\mathbb{F}$, then every automorphism is inner, so $\mathrm{Aut}(\mathcal{A})$ is the projective general linear group $PGL(\mathcal{A}) = GL(\mathcal{A})/\mathbb{F}^*$. If $\mathcal{A}$ is a semisimple Lie algebra then $\mathrm{Aut}(\mathcal{A})$ has identity component that is the adjoint group of $\mathcal{A}$.

Let $\mathbb{O}$ denote the algebra of octonions (Cayley division algebra) over $\mathbb{R}$. It can be defined as the algebra with basis $\{e_0,\dots,e_7\}$ over $\mathbb{R}$ with multiplication defined by (a) $e_0e_j = e_j = e_je_0$ for $1 \leqq j \leqq 7$, (b) $e_j^2 = -e_0$ for $1 \leqq j \leqq 7$, (c) $e_je_k + e_ke_j = 0$ for $1 \leqq j,k \leqq 7$ with $j \neq k$, (d) $e_1e_2 = e_3$, $e_3e_5 = e_6$, $e_6e_7 = e_1$, $e_1e_4 = e_5$, $e_3e_4 = -e_7$, $e_6e_4 = e_2$ and $e_2e_5 = e_7$, and (e) each equation in (d) remains true when the subscripts involved in it are cyclically permuted. This

[1] The algebra $\mathcal{A}$ is called **associative** if its multiplication is associative, i.e. if $(ab)c = a(bc)$ in $\mathcal{A}$.

multiplication table is summarized in the diagram
(2.6.2)



(1, 2, 3)
(3, 5, 6)
(6, 7, 1)
(1, 4, 5)
(3, 4, 7)
(6, 4, 2)
(2, 5, 7)

Then $\mathrm{Aut}(\mathbb{O})$ is the compact 14–dimensional simple Lie group G_2. Elements of $\mathrm{Im}\ \mathbb{O} = \mathrm{Span}_{\mathbb{R}}\{e_1, \dots, e_7\}$ are called *pure imaginary*, and $\mathrm{Im}\ \mathbb{O}$ has a positive definite inner product given by: $u \cdot v$ is the e_0–component of $-uv$. An element $g \in GL(\mathbb{O}; \mathbb{R})$ is an automorphism just when (say with $g(e_j) = e'_j$) $e_0 = e'_0$, e'_1 and e'_2 are orthonormal in $\mathrm{Im}\ \mathbb{O}$ and $e'_3 = e'_1 e'_2$, e'_4 is a unit vector in $\mathrm{Im}\ \mathbb{O}$ that is orthogonal to e'_1, e'_2 and e'_3. Then e'_5, e'_6 and e'_7 are given by $e'_1 e'_4 = e'_5$, $e'_2 e'_4 = -e'_6$ and $e'_3 e'_4 = e'_7$. Thus elements of $\mathrm{Aut}(\mathbb{O})$ are specified by the choice of pure imaginary unit element e'_1, pure imaginary unit element e'_2 orthogonal to e'_1, and pure imaginary unit element e'_4 orthogonal to e'_1, e'_2 and $e'_1 e'_2$. This is how one understands the group $G_2 = \mathrm{Aut}(\mathbb{O})$.

A **Jordan algebra** over a commutative field $\mathbb{F}$ of characteristic $\neq 2$ is a commutative algebra $\mathcal{J}$ over $\mathbb{F}$ such that $(a^2 b)a = a^2(ba)$ for $a, b \in \mathcal{J}$ (the Jordan identity). A standard construction: let $\mathcal{A}$ be an associative algebra over $\mathbb{F}$, let $\mathcal{J}$ be a subspace of $\mathcal{A}$ that is stable under the composition $a \bullet b = \frac{1}{2}(ab + ba)$, and let $\mathcal{J}$ be viewed as an algebra with that composition. A Jordan algebra $\mathcal{J}$ is called **special** if it can be obtained in this way, **exceptional** if it cannot. In addition to the automorphism group, a Jordan algebra $\mathcal{J}$ has two larger groups of transformations, the **structure group** $\mathrm{Str}(\mathcal{J})$ and the **reduced structure group** $\mathrm{Str}_0(J)$. They are defined as follows. If $x \in \mathcal{J}$ then $L(x) : \mathcal{J} \to \mathcal{J}$ denotes the left translation, $L(x)y = xy$. It defines the trace form $\tau(x, y) = \mathrm{trace}\, L(xy)$ on $\mathcal{J}$. One says that $\mathcal{J}$ is semisimple if τ is nondegenerate as a bilinear form. This is equivalent to $\mathcal{J}$ being the algebra direct sum of simple ideals. Suppose that $\mathcal{J}$ is semisimple. If g is a linear transformation of $\mathcal{J}$ let g' denote its transpose relative to τ. The "quadratic representation" of $\mathcal{J}$ is $x \mapsto P(x) := 2L(x)^2 - L(x^2)$. By definition the structure group $\mathrm{Str}(\mathcal{J}) = \{g \in GL(\mathcal{J}) \mid P(gx) = gP(x)g' \text{ for } x \in \mathcal{J}\}$. Note that $\mathrm{Str}(\mathcal{J})$ contains the scalar dilations $s_t : x \mapsto tx$, $t \in \mathbb{F}^\times$, as a normal subgroup. The reduced structure group $\mathrm{Str}_0(\mathcal{J}) = \mathrm{Str}(\mathcal{J}) \cap SL(\mathcal{J})$ is a finite cover of the quotient of $\mathrm{Str}(\mathcal{J})$ by the scalar dilations.

Fact: $\mathrm{Aut}(\mathcal{J}) = \{g \in \mathrm{Str}(\mathcal{J}) \mid g(I) = I\}$.

A Jordan algebra $\mathcal{J}$ over $\mathbb{R}$ is called **formally real** or **euclidean** if it has the property $x^2 + y^2 = 0 \Rightarrow x = 0 = y$. These algebras are important in analysis on homogeneous cones and in the geometry of complex bounded symmetric domains. They have the property that $\mathrm{Aut}(\mathcal{J})$ is a maximal compact subgroup of $\mathrm{Str}_0(\mathcal{J})$. They are given as follows.

First, $\mathcal{J}$ could be the special real simple Jordan algebra of $n \times n$ hermitian matrices over $\mathbb{F}$ ($= \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) with composition $x \bullet y = \frac{1}{2}(xy + yx)$. Then $\mathrm{Str}(\mathcal{J}) \cong GL(\mathcal{J})/\{\pm I\}$ and $\mathrm{Aut}(\mathcal{J}) \cong U(n; \mathbb{F})/\{\pm I\}$, each acting on $\mathcal{J}$ by $\pm g : a \to gag^*$.

Second, $\mathcal{J}$ could be the real simple Jordan algebra $V \oplus \mathbb{R}$, where V is a real vector space of dimension $n < \infty$ with a positive definite symmetric bilinear form f. $\mathcal{J}$ has composition $(u,a) \bullet (v,b) = (av + bu, ab + f(u,v)$. Here $\mathrm{Str}_0(\mathcal{J})$ is the orthogonal group of signature $(n,1)$ associated to the form $\tilde{f}((u,a),(v,b)) = f(u,v) - ab$ and $\mathrm{Aut}(\mathcal{J})$ is the orthogonal group of (V,f). It is not obvious, but in this case $\mathcal{J}$ is special.

Finally, third, we have the exceptional simple real Jordan algebra $\mathcal{J}$ consisting of 3×3 hermitian matrices over the octonion division algebra $\mathbb{O}$. It has composition $x \bullet y = \frac{1}{2}(xy + yx)$. Then $\mathrm{Str}_0(\mathcal{J})$ is the 78–dimensional simple Lie group E_{6,F_4} whose maximal compact subgroup is the 52–dimensional compact simple Lie group F_4, and that F_4 is $\mathrm{Aut}(\mathcal{J})$. We will see these groups later in connection with projective planes. (One can try this construction for the $n \times n$ hermitian matrices over $\mathbb{O}$, but that does not result in a Jordan algebra when $n > 3$.)

Summarizing, the simple formally real Jordan algebras and their automorphism and structure groups are

(2.6.3)

$\mathcal{J}$	$\mathrm{Str}(\mathcal{J})$	$\mathrm{Aut}(\mathcal{J})$
$\mathrm{Sym}\,\mathbb{R}^{n\times n}$	$GL(n;\mathbb{R})$	$SO(n)$
$\mathrm{Re}\,\mathbb{C}^{n\times n}$	$GL(n;\mathbb{C})$	$SU(n)$
$\mathrm{Re}\,\mathbb{H}^{n\times n}$	$GL(n;\mathbb{H})$	$Sp(n)$
$\mathbb{R}^n \oplus \mathbb{R}$	$O(n,1)$	$O(n)$
$\mathrm{Re}\,\mathbb{O}^{3\times 3}$	$E_{6,F_4} \times \mathbb{R}^\times$	F_4

where the hermitian and skew–hermitian parts of a square matrix m are given by

$$\mathrm{Re}\, m := \tfrac{1}{2}(m + m^*) \quad \text{and} \quad \mathrm{Im}\, m := \tfrac{1}{2}(m - m^*). \tag{2.6.4}$$

(These are real and imaginary parts only in a rather generalized sense.) See [**F-K**] for complete proofs.

2.7. Spheres, Projective Spaces and Grassmannians

The underlying set of the **projective space** $P(V)$ of a vector space V over $\mathbb{F}$ is the set of all 1–dimensional linear subspaces of V. Thus it can be described in terms of compact groups as a homogeneous space of a unitary group, or in terms of noncompact groups as a homogeneous space of a general linear group. If $n = \dim_{\mathbb{F}} V$ then

$$P(V) \cong U(n;\mathbb{F})/U(1;\mathbb{F}) \times U(n-1;\mathbb{F}), \tag{2.7.1}$$

which can also be described as

$$GL(n;\mathbb{F})/\left\{\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \mid a \in GL(1,\mathbb{F}),\ b \in GL(n-1;\mathbb{F}) \text{ and } x \in \mathbb{F}^{1\times(n-1)}\right\}.$$

For $U(n;\mathbb{F})$ is transitive on $P(V)$ and the stabilizer of $e_1\mathbb{F}$ is $U(1;\mathbb{F}) \times U(n-1;\mathbb{F})$, while $GL(n;\mathbb{F})$ is of course also transitive and the isotropy subgroup of $GL(n;\mathbb{F})$ at $e_1\mathbb{F}$ is the subgroup (call it $Q_n(\mathbb{F})$) described just above. Thus we have real, complex and quaternion n–space given by

$$P^n(\mathbb{R}) = O(n+1)/O(1)\times O(n) = GL(n+1;\mathbb{R})/Q_{n+1}(\mathbb{R}) \tag{2.7.2}$$

$$P^n(\mathbb{C}) = U(n+1)/U(1)\times U(n) = GL(n+1;\mathbb{C})/Q_{n+1}(\mathbb{C}) \tag{2.7.3}$$

$$P^n(\mathbb{H}) = Sp(n+1)/Sp(1)\times Sp(n) = GL(n+1;\mathbb{H})/Q_{n+1}(\mathbb{H}) \tag{2.7.4}$$

For $n > 1$ the universal cover of $P^n(\mathbb{R})$ is the sphere S^n with the covering given by

$$SO(n+1)/SO(n) \cong O(n+1)/O(n) \to O(n+1)/O(1) \times O(n).$$

The complex and quaternion projective spaces are simply connected. There is another locally compact projective space in the background, the octonion (or Cayley) projective plane,

$$P^2(\mathbb{O}) = F_4/Spin(9) = E_{6,F_4}/Q_3(\mathbb{O}). \tag{2.7.5}$$

Here E_{6,F_4} and F_4 are the groups mentioned above in connection with the exceptional simple Jordan algebra $\mathcal{J}$, which is used to construct[2] $P^2(\mathbb{O})$, and $Q_3(\mathbb{O})$ and $Spin(9)$ are certain subgroups. $P^2(\mathbb{O})$ is simply connected. The projective lines in $P^n(\mathbb{R})$ are circles S^1, in $P^n(\mathbb{C})$ are Riemann spheres S^2, in $P^n(\mathbb{H})$ are 4–spheres S^4, and in $P^2(\mathbb{O})$ are 8–spheres S^8.

More generally, the **Grassmann manifold** of k–dimensional linear subspaces of V is given by

$$\begin{aligned} G_k(V) &\cong U(n;\mathbb{F})/U(k;\mathbb{F}) \times U(n-k;\mathbb{F}) \\ &\cong GL(n;\mathbb{F})/\left\{ \left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \mid a \in GL(k,\mathbb{F}),\ b \in GL(n-k;\mathbb{F}) \text{ and } x \in \mathbb{F}^{k\times(n-k)} \right\} \end{aligned} \tag{2.7.6}$$

For again $U(n;\mathbb{F})$ is transitive on $P(V)$ and the stabilizer of $e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_k\mathbb{F}$ is $U(k;\mathbb{F}) \times U(n-k;\mathbb{F})$, while $GL(n;\mathbb{F})$ is of course also transitive and the isotropy subgroup of $GL(n;\mathbb{F})$ at $e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_k\mathbb{F}$ is the subgroup (call it $Q_{k,n}(\mathbb{F})$) described just above. Thus we have real, complex and quaternion Grassmann manifolds given by

$$G_{k,n}(\mathbb{R}) = O(n)/O(k) \times O(n-k) = GL(n+1;\mathbb{R})/Q_{n+1}(\mathbb{R}) \tag{2.7.7}$$

$$G_{k,n}(\mathbb{C}) = U(n)/U(k) \times U(n-k) = GL(n+1;\mathbb{C})/Q_{n+1}(\mathbb{C}) \tag{2.7.8}$$

$$G_{k,n}(\mathbb{H}) = Sp(n)/Sp(k) \times Sp(n-k) = GL(n+1;\mathbb{H})/Q_{n+1}(\mathbb{H}) \tag{2.7.9}$$

For $k(n-k) > 1$ the universal cover of $G_{k,n}(\mathbb{R})$ is the 2–sheeted cover by the grassmannian $\widetilde{G}_{k,n}(\mathbb{R})$ of oriented k–dimensional linear subspaces of an oriented $\mathbb{R}^n$, and that covering is given by

$$SO(n)/SO(k) \times SO(n-k) \to SO(n)/\{g \in O(k) \times O(n-k) \mid \det(g) = 1\}.$$

The complex and quaternion Grassmann manifolds are simply connected.

There is an important natural generalization of the real, complex and quaternion Grassman manifolds, the compact riemannian symmetric spaces. See Section 11.2 for a quick description and Cartan's classification. In differential geometry, in a certain sense "most" of the compact riemannian symmetric spaces are grassmannians. By this I mean that if a theorem can be verified, or a phenomenon can be understood, for the Grassmann manifolds mentioned above, then this often is enough of an indication to carry it through for compact riemannian symmetric spaces.

[2]Here is a uniform construction of all four types of projective spaces $P^{n-1}(\mathbb{F})$. Let $\mathcal{J}$ denote the Jordan algebra of $n \times n$ hermitian matrices over $\mathbb{F}$, with $n = 3$ in the case $\mathbb{F} = \mathbb{O}$. An element $e \in \mathcal{J}$ is **idempotent** if $e^2 = e$. An idempotent $e \in \mathcal{J}$ is **primitive** if $e \neq 0$ and e cannot be expressed in the form $e_1 + e_2$ where the e_i are nonzero idempotents that are orthogonal in the sense $e_1e_2 = 0$. Then the set $\mathcal{I}(\mathcal{J})$ of primitive idempotents is a compact submanifold of $\mathcal{J}$ that is $\mathrm{Str}_0(\mathcal{J})$–equivariantly diffeomorphic to $P^{n-1}(\mathbb{F})$. See [**F-K**, Exercises 4 & 5, Chapter IV] and [**F-K**, Exercise 5, Chapter V].

For noncompact symmetric spaces there is something similar. Let **h** be the hermitian form on $\mathbb{F}^n$ with matrix $\begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$ Then $U(k, n-k; \mathbb{F})$ is its unitary group. The orbits of $U(k, n-k; \mathbb{F})$ on $G_{k,n}(\mathbb{F})$ are the various subsets on which **h** has equivalent restriction. The possibilities for that restriction are given by triples (p, q, u) where p is the number of +'s, q is the number of -'s, and u is the dimension of the kernel, of the restriction of **h** to our k–dimensional subspace. Thus $p \leqq k, q \leqq n-k, u \leq \min\{k, n-k\}$, and $p+q+u=k$. The most important of these orbits, for riemannian geometry, is

$$(2.7.10) \qquad U(k, n-k; \mathbb{F})([e_1 \wedge \cdots \wedge e_k]) \cong U(k, n-k; \mathbb{F})/U(k; \mathbb{F}) \times U(n-k; \mathbb{F}),$$

where $[e_1 \wedge \cdots \wedge e_k] = \mathrm{Span}_{\mathbb{F}}\{e_1, e_2, \ldots, e_k\}$. The manifold (2.7.10) can be realized as the bounded domain of $k \times (n-k)$ matrices Z over $\mathbb{F}$ such that $I - ZZ^* \gg 0$. The other orbits, at least in the case $\mathbb{F} = \mathbb{C}$, are useful in complex function theory and in the unitary representation theory of the group $U(k, n-k; \mathbb{F})$.

2.8. Complexification of Real Groups

Certain of our groups are obtained from others by a process that could be called complexification. For example, $GL(n; \mathbb{C})$ is the complexification of $GL(n; \mathbb{R})$ and $O(n; \mathbb{C})$ is the complexification of $O(n; \mathbb{R})$. This notion, however, goes well beyond those obvious cases, and it has a number of geometric and analytic consequences. We will formalize it from the viewpoint of linear groups.

Definition 2.8.1. Let G be a topological group. We say that another topological group $G_{\mathbb{C}}$ is a **complexification** of G if there exist $n > 0$ and homomorphisms $\phi : G \to GL(n; \mathbb{R})$ and $\psi : G_{\mathbb{C}} \to GL(n; \mathbb{C})$ such that

- ϕ is a homeomorphism of G onto $\phi(G)$ and ψ is a homeomorphism of $G_{\mathbb{C}}$ onto $\psi(G_{\mathbb{C}})$,
- $\phi(G) = \{g \in \mathbb{R}^{n \times n} \mid F(g) = 0 \text{ for all } F \in \mathcal{I}\}$ for some set $\mathcal{I}$ of real polynomial functions on $\mathbb{R}^{n \times n}$, and
- $\psi(G_{\mathbb{C}}) = \{g \in \mathbb{C}^{n \times n} \mid F(g) = 0 \text{ for all } F \in \mathcal{I}\}$ when we view the elements of $\mathcal{I}$ as polynomial functions on $\mathbb{C}^{n \times n}$.

If $G_{\mathbb{C}}$ is a complexification of G then we also say that G is a **real form** of $G_{\mathbb{C}}$. ♢

From now on, when we write $G_{\mathbb{C}}$ and G, it is understood that $G_{\mathbb{C}}$ is a complexification of G.

We now make sure that the definition of complexification is what we want, by verifying

$$(2.8.2) \qquad GL(n; \mathbb{R})_{\mathbb{C}} = GL(n; \mathbb{C}) \text{ and } SL(n; \mathbb{R})_{\mathbb{C}} = SL(n; \mathbb{C}).$$

For this, let $\phi(g) = \psi(g) = \begin{pmatrix} g & 0 \\ 0 & 1/\det(g) \end{pmatrix}$. Then $\phi(GL(n; \mathbb{R}))$ consists of all real $(n+1) \times (n+1)$ matrices m that satisfy the equations $m_{n+1,j} = 0$ for $1 \leqq j \leqq n$, $m_{i,n+1} = 0$ for $1 \leqq i \leqq n$, and $\det(m) = 1$. Note that $\psi(GL(n; \mathbb{C}))$ consists of all complex $(n+1) \times (n+1)$ matrices that satisfy the same system of equations. That proves (2.8.2) for GL. The same argument, with $\phi(g) = \psi(g) = g$, proves it for SL.

Now let's run through some cases that we met earlier.

$$GL(n;\mathbb{H})_{\mathbb{C}} = GL(2n;\mathbb{C}) \text{ and } SL(n;\mathbb{H})_{\mathbb{C}} = SL(2n;\mathbb{C}). \tag{2.8.3}$$

When we view $\mathbb{H}^n$ as a $4n$–dimensional real vector space $\mathbb{R}^{4n}$, we obtain a map $\phi : GL(n;\mathbb{H}) \to GL(4n;\mathbb{R})$, and the scalar multiplications by the pure imaginary unit quaternions $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are transformed to anticommuting linear transformations $\mathbf{I}, \mathbf{J}, \mathbf{K} \in GL(4n;\mathbb{R})$ of square $-I$. Note that

$$\phi(GL(n;\mathbb{H})) = \{g \in GL(4n;\mathbb{R}) \mid g\mathbf{I} = \mathbf{I}g, g\mathbf{J} = \mathbf{J}g \text{ and } g\mathbf{K} = \mathbf{K}g\}.$$

When we view $\mathbb{C}^{2n}$ as $\mathbb{R}^{4n}$ we obtain a map $\psi : GL(2n;\mathbb{C}) \to GL(4n;\mathbb{R})$, such that scalar multiplication by $\mathbf{i}$ is transformed to the same linear transformation $\mathbf{I} \in GL(4n;\mathbb{R})$, and $\psi(GL(2n;\mathbb{C})) = \{g \in GL(4n;\mathbb{R}) \mid g\mathbf{I} = \mathbf{I}g\}$. Now we view the space of $4n \times 4n$ real matrices that commute with $\mathbf{I}$ as a complex vector space V, with complex conjugation given by $x \mapsto \mathbf{J}x\mathbf{J}^{-1}$. So $U = \{x \in V \mid x\mathbf{J} = \mathbf{J}x\}$ is a real vector space with complexification V. Making use of the map that realizes (2.8.2), we obtain the first statement of (2.8.3). The second statement follows.

$$O(p,q)_{\mathbb{C}} = O(p+q;\mathbb{C}) \text{ and } SO(p,q)_{\mathbb{C}} = SO(p+q;\mathbb{C}). \tag{2.8.4}$$

For the full orthogonal groups we use the system of equations on the matrix entries given by ${}^t g I_{p,q} g - I_{p,q} = 0$, and for the special orthogonal groups we use the additional equations that come from $\det(g) - 1 = 0$. Then (2.8.4) is immediate.

$$Sp(n;\mathbb{R})_{\mathbb{C}} = Sp(n;\mathbb{C}). \tag{2.8.5}$$

The proof of (2.8.5) is the same as the proof of (2.8.4), with J in place of $I_{p,q}$.

$$U(p,q)_{\mathbb{C}} = GL(p+q;\mathbb{C}) \text{ and } SU(p,q)_{\mathbb{C}} = SL(p+q;\mathbb{C}). \tag{2.8.6}$$

Write $z \in U(p,q)$ as $z = x + iy$ with x, y real, and define $\phi(z) = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$. Then $\phi(U(p,q)) = \{g \in O(2p,2q) \mid gJ = Jg\}$ where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Thus $\phi(U(p,q))$ is the set of all $2n \times 2n$ real matrices, $n = p+q$, that satisfy certain real polynomial equations. As affine variety or as differentiable manifold it has real dimension n^2, so its complexification has real dimension $2n^2$. That complexification still consists of matrices that commute with J, in other words of matrices in $\phi(GL(n;\mathbb{C}))$, which has real dimension $2n^2$. That proves (2.8.6).

$$Sp(p,q)_{\mathbb{C}} = Sp(p+q;\mathbb{C}). \tag{2.8.7}$$

The proof of (2.8.7) is the same as the proof of (2.8.6).

$$SO^*(2n)_{\mathbb{C}} = SO(2n;\mathbb{C}). \tag{2.8.8}$$

The proof of (2.8.8) is the same as the proof of (2.8.6).

$$GL(n;\mathbb{C})_{\mathbb{C}} = GL(n;\mathbb{C}) \times GL(n;\mathbb{C}) \text{ and } SL(n;\mathbb{C})_{\mathbb{C}} = SL(n;\mathbb{C}) \times SL(n;\mathbb{C}). \tag{2.8.9}$$

Define $\widetilde{g} = \left(\begin{smallmatrix} g & 0 \\ 0 & 1/\det(g) \end{smallmatrix}\right) \in SL(n+1;\mathbb{C})$ for $g \in GL(n;\mathbb{C})$. Define $\phi(g) = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ where $\widetilde{g} = x + iy$ with x, y real. Then $\phi(GL(n;\mathbb{C}))$ complexifies to the group of all $\left(\begin{smallmatrix} z & w \\ -w & z \end{smallmatrix}\right)$ where z and w are of the form $\left(\begin{smallmatrix} g & 0 \\ 0 & 1/\det(g) \end{smallmatrix}\right)$ with $g \in GL(n;\mathbb{C})$. Thus $GL(n;\mathbb{C})_{\mathbb{C}} = GL(n;\mathbb{C}) \times GL(n;\mathbb{C})$. The statement on SL follows.

(2.8.10) $\quad O(n;\mathbb{C})_{\mathbb{C}} = O(n;\mathbb{C}) \times O(n;\mathbb{C})$ and $SO(n;\mathbb{C})_{\mathbb{C}} = SO(n;\mathbb{C}) \times SO(n;\mathbb{C})$.

The proof of (2.8.10) is essentially the same as the proof of (2.8.9).

(2.8.11) $$Sp(n;\mathbb{C})_{\mathbb{C}} = Sp(n;\mathbb{C}) \times Sp(n;\mathbb{C}).$$

The proof of (2.8.11) is essentially the same as the proof of (2.8.9).

Lemma 2.8.12. *If $\mathcal{A}$ is an algebra over $\mathbb{R}$ then* $\mathrm{Aut}(\mathcal{A})_{\mathbb{C}} = \mathrm{Aut}(\mathcal{A}_{\mathbb{C}})$.

Proof. Choose a basis of $\mathcal{A}$ over $\mathbb{R}$; then $\mathrm{Aut}(\mathcal{A})$ consists of all invertible real linear transformations of $\mathcal{A}$ that preserve the multiplication table (relative to that basis). Now both $\mathrm{Aut}(\mathcal{A})_{\mathbb{C}}$ and $\mathrm{Aut}(\mathcal{A}_{\mathbb{C}})$ consist of all invertible complex linear transformations of $\mathcal{A}_{\mathbb{C}}$ that preserve the multiplication table. □

2.9. $\mathfrak{p}$–adic Groups

The p–adic and adèle groups are related more with number theory than with harmonic or geometric analysis, but we mention them because they are important examples of locally compact topological groups.

A topological field is a field (in the sense of algebra) in which the field operations are continuous. When a topological field is locally compact one obtains some locally compact topological groups that are of importance in number theory. Formally,

Definition 2.9.1. Let $\mathbb{F} = (\mathbb{F}, +, \times)$ be both a field and a topological space, then $\mathbb{F}$ is a **topological field** if

the additive structure $(\mathbb{F}, +)$ and the topology form a commutative topological group,

the multiplicative structure $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ is a topological group, and

the map $\mathbb{F}^\times \times \mathbb{F} \to \mathbb{F}$, given by $(a, x) \mapsto ax$, is continuous. ♢

Of course $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are locally compact topological fields. Any field with the discrete topology is a locally compact topological field, but somehow that is not very interesting. There are, however, a number of other interesting ones, for example the p–adic number fields $\mathbb{Q}_p$.

We recall the definition of $\mathbb{Q}_p$. Let p be a prime number. The p–adic valuation on the rational number field $\mathbb{Q}$ is given by: $|0|_p = 0$, and if $0 \neq x \in \mathbb{Q}$ then $|x|_p = p^{-n}$ where $x = p^n u/v$ in such a way that n, u and v are integers (note: n can be negative) with u and v not divisible by p. That gives the metric $d_p(x, y) = |x-y|_p$ on $\mathbb{Q}$. Its completion (as a field and as a metric space) is the locally compact field

$\mathbb{Q}_p$ of p–adic numbers. Sometimes it is convenient to denote $\mathbb{R}$ as $\mathbb{Q}_\infty$. The point of this, for us in the present setting, is

PROPOSITION 2.9.2. *Let $\mathbb{F}$ be a nondiscrete locally compact field of characteristic zero. Then $\mathbb{F}$ is a division algebra of finite dimension over a p–adic number fields $\mathbb{Q}_p$ or over $\mathbb{R} = \mathbb{Q}_\infty$. Conversely every such division algebra is a nondiscrete locally compact field.*

Nondiscrete locally compact field of finite characteristic have an analogous description. Let $\mathbb{F}_p$ denote the pure transcendental extension of degree 1 of the field of p elements. Then the nondiscrete locally compact fields of characteristic p are just the division algebras of finite dimension over $\mathbb{F}_p$.

We omit the proof of Proposition 2.9.2. That, and related results described below in Section 3.8, are found in André Weil's book "Basic Number Theory."

Let $\mathbb{F}$ be a nondiscrete locally compact field of characteristic zero. Then $GL(n;\mathbb{F})$ consists of the invertible elements of $\mathbb{F}^{n\times n}$. Any closed subgroup defined by polynomial equations on the matrix entries is a locally compact group. Our orthogonal groups, symplectic groups, and other linear Lie groups, were defined by equations with integer coefficients, so they make sense over any field, in particular over $\mathbb{F}$. This gives a rich supply of locally compact groups that are extremely interesting in number theory. In Section 3.8, using the notions of Haar measure and the module of an automorphism, we will see how the integer matrices give maximal compact subgroups for linear Lie groups over p–adic fields, and how the various p–adic groups fit together to form the adèle groups.

2.10. Heisenberg Groups

The "Heisenberg groups" are ubiquitous in mathematics, playing important rôles in number theory, harmonic analysis, and the theory of homogeneous riemannian manifolds. In Part 2 we will use them to illustrate an important construction of Mackey for unitary representations. They are in the background in Part 3 when we discuss uncertainty principles on commutative spaces. In Part 4 we will see how they enter into several of the basic constructions of commutative spaces. For applications in those considerations of group structure and harmonic analysis we give the definitions in greater than usual generality.

Let $\mathbb{F}$ denote a real division algebra $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) or $\mathbb{O}$ (octonions). Then $\mathbb{F}$ has a standard decomposition $\mathbb{F} = \mathbb{R} + \operatorname{Im}\mathbb{F}$ where $\operatorname{Im}\mathbb{F}$, the space of pure imaginary elements of $\mathbb{F}$, has real dimension 0, 1, 3 or 7. This is consistent with (2.6.4), and we view Re and Im as the projection $\mathbb{F} \to \mathbb{R}$ and $\mathbb{F} \to \operatorname{Im}\mathbb{F}$.

We view the space $\mathbb{F}^n$ of n–tuples over $\mathbb{F}$ as a right vector space, so scalars act on the right and linear transformations act on the left. When p and q are non–negative integers, $p + q = n$, we have the hermitian vector space

$$\mathbb{F}^{p,q} : \ \mathbb{F}^n \text{ with hermitian form } h(x,y) = \sum\nolimits_1^p x^\ell \overline{y}^\ell - \sum\nolimits_1^q x^{p+\ell}\overline{y}^{p+\ell}$$

and its unitary group $U(p,q;\mathbb{F})$. In the octonion case one has to be careful: $U(p,q;\mathbb{O}) = SO(p,q) \times Spin(7)$. We now have a group

$$\text{(2.10.1)} \qquad \begin{aligned} &H_{p,q;\mathbb{F}} : \text{ real vector space } \operatorname{Im}\mathbb{F} + \mathbb{F}^{p,q} \text{ with group composition} \\ &\qquad\qquad (z,w)(z',w') = (z + z' + \operatorname{Im} h(w,w'), w + w'). \end{aligned}$$

Groups $H_{p,q;\mathbb{F}} \cong H_{p',q';\mathbb{F}'}$ if and only if (i) $\mathbb{F} = \mathbb{F}'$ and (ii) $p+q = p'+q'$. Finally, $g(z,w) = (z, g(w))$ defines an action of the unitary group $U(p,q;\mathbb{F})$ by automorphisms on $H_{p,q;\mathbb{F}}$. The semidirect product group $G_{p,q;\mathbb{F}} = H_{p,q;\mathbb{F}} \rtimes U(p,q;\mathbb{F})$ has group composition

$$\text{(2.10.2)} \qquad (z,w,g)(z',w',g') = (z + z' + \operatorname{Im} h(w, g(w')), w + g(w'), gg').$$

We will mostly be concerned with the case $q = 0$, leading to the groups

$$\text{(2.10.3)} \qquad H_{n;\mathbb{F}} := H_{n,0;\mathbb{F}} \text{ and } G_{n;\mathbb{F}} := G_{n;0;\mathbb{F}}.$$

The usual Heisenberg group is $H_n := H_{n;\mathbb{C}}$. So the groups $H_{p,q;\mathbb{F}}$, and even the groups $H_{n;\mathbb{F}}$, form a slight generalization. Of course the $H_{p,q;\mathbb{R}}$ are just a real vector groups. The others are 2–step nilpotent.

Occasionally one wants more of a generalization of the usual Heisenberg group.

Given an integer $s \geqq 1$ we decompose $\mathbb{F}^{s\times s} = \operatorname{Im}\mathbb{F}^{s\times s} + \operatorname{Re}\mathbb{F}^{s\times s}$, direct sum of real vector spaces, where the projections are given by

$$\operatorname{Im} z = \tfrac{1}{2}(z - z^*) \text{ and } \operatorname{Re} z = \tfrac{1}{2}(z + z^*) \text{ where } z^* \text{ is conjugate transpose.}$$

Given integers $t, u \geqq 0$, we have a hermitian map

$$\mathcal{H} : \mathbb{F}^{s\times(t+u)} \times \mathbb{F}^{s\times(t+u)} \to \mathbb{F}^{s\times s} \text{ given by } \mathcal{H}((x_1,x_2),(y_1,y_2)) = x_1y_1^* - x_2y_2^*$$

where $x_1, y_1 \in \mathbb{F}^{s\times t}$ and $x_2, y_2 \in \mathbb{F}^{s\times u}$. We write $\mathbb{F}^{s\times(t,u)}$ for $\mathbb{F}^{s\times(t+u)}$ with the hermitian map $\mathcal{H}$. Putting these together we have a (very) generalized Heisenberg group

$$\text{(2.10.4)} \qquad \begin{aligned} &H_{s,t,u;\mathbb{F}} : \text{ real vector space } \operatorname{Im}\mathbb{F}^{s\times s} + \mathbb{F}^{s\times(t,u)} \text{ with group composition} \\ &\qquad\qquad (z,w)(z',w') = (z + z' + \operatorname{Im}\mathcal{H}(w,w'), w + w'). \end{aligned}$$

Since the (t,u) pertains to rows in $F^{s\times(t,u)}$ the automorphism action of $U(t,u;\mathbb{F})$ on $H_{s,t,u;\mathbb{F}}$ is $g(z,w) = (z, wg^*)$. The semidirect product group $G_{s,t,u;\mathbb{F}} = H_{s,t,u;\mathbb{F}} \rtimes U(t,u;\mathbb{F})$ has group composition

$$\text{(2.10.5)} \qquad (z,w,g)(z',w',g') = (z + z' + \operatorname{Im}\mathcal{H}(w, w'g^*), w + w'g^*, gg').$$

In particular $G_{s,t,u;\mathbb{F}}$ has center $\operatorname{Im}\mathbb{F}^{s\times s}$, $H_{1,t,u;\mathbb{F}} \cong H_{t,u;\mathbb{F}}$ and $G_{1,t,u;\mathbb{F}} \cong G_{t,u;\mathbb{F}}$, and $H_{s,t,u;\mathbb{F}}$ is commutative just when either $s = 1$ with $\mathbb{F} = \mathbb{R}$, or $t = u = 0$.

The groups $H_{s,t,u;\mathbb{F}}$ and $G_{s,t,u;\mathbb{F}}$ appear in the study of maximal parabolic subgroups of unitary groups $U(p,q;\mathbb{F})$. See [**W9**], [**W10**] and [**W11**]. We will meet a few of them in Chapter 13 when we look at the classification of *commutative nilmanifolds*, and in Chapter 14 we will see that harmonic analysis on those groups, based on a theory of *square integrable representations*, is particularly elegant.

CHAPTER 8

Basic Theory of Commutative Spaces

In this chapter we introduce the basic ideas of commutative harmonic analysis, starting with the definition of Gelfand pair and the notion of spherical function. We discuss spherical functions from the viewpoints of algebra homomorphism, functional equation, and (in the Lie group case) invariant differential operators. We then consider the notion of positive definite spherical function, preparing for development of the spherical transform and spherical inversion theorems. Finally we consider the important case of principal series representations of semisimple Lie groups and we give an example that is as far as possible from the Lie group case.

8.1. Preliminaries

Fix a locally compact group G and a compact subgroup K. We always assume the normalization $\int_K d\mu_K(k) = 1$. The projection

$$p : G \to K\backslash G/K \text{ by } p(g) = KgK$$

identifies $C_c(K\backslash G/K)$ with

$$\{f \in C_c(G) \mid f(k_1 g k_2) = f(g) \text{ for } k_i \in K, g \in G\}.$$

Of course we also identify $C(K\backslash G/K)$, $\mathcal{C}_\infty(K\backslash G/K)$, and $L^p(K\backslash G/K)$ with the bi–K–invariant functions of the same class on G. Here $\mathcal{C}_\infty$ means the commutative Banach algebra of continuous functions that vanish at ∞ with pointwise multiplication and sup norm. In other words, $\mathcal{C}_\infty(G)$ consists of the continuous functions $f : G \to \mathbb{C}$ such that, if $\varepsilon > 0$ then there is a compact set $C \subset G$ such that $|f(x)| < \varepsilon$ for all $x \in G \setminus C$.

We now have the projections $C(G) \to C(K\backslash G/K)$, $C_c(G) \to C_c(K\backslash G/K)$, $\mathcal{C}_\infty(G) \to \mathcal{C}_\infty(K\backslash G/K)$, and $L^p(G) \to L^p(K\backslash G/K)$, all denoted $f \to f^\sharp$, given by

$$f^\sharp(g) = \int_K \int_K f(k_1 g k_2)\, d\mu_K(k_1) d\mu_K(k_2)\,. \tag{8.1.1}$$

Some immediate properties are

$$\text{if } f \in C(K\backslash G/K) \text{ and } h \in C(G) \text{ then } (f\,h)^\sharp = f\,h^\sharp$$

for the pointwise product, and

$$\text{if } f \in C_c(K\backslash G/K) \text{ and } h \in C_c(G) \text{ then } (h * f)^\sharp = h^\sharp * f \text{ and } (f * h)^\sharp = f * h^\sharp$$

for the convolution product. In particular $C_c(K\backslash G/K)$ is a subalgebra of the convolution algebra $C_c(G)$ and $L^1(K\backslash G/K)$ is a subalgebra of $L^1(G)$.

Definition 8.1.2. We say that (G, K) is a **Gelfand pair** if the convolution algebra $L^1(K\backslash G/K)$ is commutative. If (G, K) is a Gelfand pair then G/K is a

commutative space relative to G, and we also say that (G, K) is a **commutative pair**. Since $C_c(K\backslash G/K)$ is dense in $L^1(K\backslash G/K)$ it is equivalent to require that $C_c(K\backslash G/K)$ be commutative. ♢

There is a special class of Gelfand pairs that is easy to identify:

PROPOSITION 8.1.3. **(Gelfand)** *If G has an automorphism θ such that $\theta^2 = 1$ and $\theta(g^{-1}) \in KgK$ for all $g \in G$, then (G, K) is a Gelfand pair. In particular, if there is an involutive automorphism θ of G such that $\theta(K) = K$ and $G = KP$, where $P = \{p \in G \mid \theta(p) = p^{-1}\}$, then (G, K) is a Gelfand pair.*

PROOF. For $f \in C_c(G)$ set $f^\theta(g) = f(\theta(g))$. The automorphism θ of G has module $||\theta|| = 1$ because $||\theta|| > 0$ and $||\theta||^2 = ||\theta^2|| = 1$. Now $f^\theta * h^\theta = (f * h)^\theta$. If $f \in C_c(K\backslash G/K)$ then $f^\theta(g^{-1}) = f(\theta(g^{-1})) = f(k_1 g k_2) = f(g)$, so $\int_G f(g^{-1})\, d\mu_G(g) = \int_G f^\theta(g^{-1})\, d\mu_G(g) = \int_G f(g)\, d\mu_G(g)$. Thus G is unimodular. If we denote $f^\diamond(g) = f(g^{-1})$ then $(f * h)^\diamond = h^\diamond * f^\diamond$. But K–bi–invariant functions satisfy $f^\theta = f^\diamond$. If $f, h \in C_c(G)$ now

$$(f * h)^\theta = f^\theta * h^\theta = f^\diamond * h^\diamond = (h * f)^\diamond = (h * f)^\theta$$

so $f * h = h * f$. In other words (G, K) is a Gelfand pair. That proves the first statement.

For the second statement let $g = kp$ and compute $\theta(g^{-1}) = \theta(p^{-1})\theta(k^{-1}) = p\theta(k^{-1}) \in pK \subset KpK = KgK$. □

COROLLARY 8.1.4. *If $X = G/K$ is a riemannian symmetric space, and G is the largest connected group of isometries of X, then (G, K) is a Gelfand pair.*

Indication of proof. The symmetry at $x_0 = 1K \in X$ induces an involutive automorphism θ of G such that $\theta(k) = k$ for every $k \in K$. Covering X by geodesic rays from x_0 gives a subset $P \subset G$ such that $G = KP$ and $\theta(p) = p^{-1}$ for all $p \in P$. Thus (G, K) is a Gelfand pair by the second statement of Proposition 8.1.3. □

At this point we should note

LEMMA 8.1.5. *Let (G, K) be a Gelfand pair. Then G is unimodular.*

PROOF. Left–invariant Haar measure μ_G is right K–invariant, because K is compact, so we need only check that $\int_G f(g)\, d\mu_G(g) = \int_G f(g^{-1})\, d\mu_G(g)$ for every $f \in C_c(K\backslash G/K)$. Given f let $h \in C_c(K\backslash G/K)$ such that $f(g) \neq 0$ implies $h(g) = h(g^{-1}) = 1$. Then $\int_G f(g)\, d\mu_G(g) = (f*h)(1) = (h*f)(1) = \int_G f(g^{-1})\, d\mu_G(g)$. □

The group $K \times K$ acts on G by $T(k_1, k_2) : g \mapsto k_1 g k_2^{-1}$. The orbits are the double coset spaces KgK, and $K\backslash G/K$ is the orbit space. Each orbit KgK has a unique $K \times K$–invariant probability measure μ_{KgK}, which can also be viewed as a measure on G:

$$\int_G f(x)\, d\mu_{KgK}(x) = \int_{KgK} f(x)\, d\mu_{KgK}(x) : \quad \text{defined to be } \int_K \int_K f(k_1 g k_2^{-1})\, d\mu_K(k_1)\, d\mu_K(k_2). \tag{8.1.6}$$

These characterize Gelfand pairs and commutative spaces as follows.

PROPOSITION 8.1.7. *Let K be a compact subgroup in a locally compact group G. Then the following conditions are equivalent.*

(i) *(G, K) is a Gelfand pair, i.e. $L^1(K\backslash G/K)$ is commutative.*

(ii) *If $g, g' \in G$ then $(KgK)(Kg'K) = (Kg'K)(KgK)$* (multiplication of sets).

(iii) *If $g, g' \in G$ then $\mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$* (convolution of measures as in Definition 3.7.2)

PROOF. Fix $x, y \in G$. Let $\{\phi_\alpha\}_{\alpha \in A} \subset C_c(G)$ be an approximate identity in $L^1(G)$, let $f_\alpha = \lambda(x)\phi_\alpha$, and let $h_\alpha = \lambda(y)\phi_\alpha$. Then $\{\phi_\alpha \mu_G\}$ converges to the point mass measure at 1, $\{f_\alpha \mu_G\}$ converges to the point mass at x, and $\{h_\alpha \mu_G\}$ converges to the point mass at y. If $q \in C_c(G)$ then

$$\lim_{\alpha \in A} \int_G q(z)(f_\alpha * h_\alpha)(z)\, d\mu_G(z) = \lim_{\alpha \in A} \int_G q(uv) f_\alpha(u) h_\alpha(v)\, d\mu_G(u)\, d\mu_G(v) = q(xy).$$

If (G, K) is a Gelfand pair, that gives us

$$\begin{aligned}\int_G q(z)\, d(\mu_{KxK} * \mu_{KyK})(z) &= \int_G \int_G q(uv)\, d\mu_{KxK}(u)\, d\mu_{KyK}(v) \\ &= \lim_{\alpha \in A} \int_G q(z)(f^\sharp_\alpha * h^\sharp_\alpha)(z)\, d\mu_G(z) \\ &= \lim_{\alpha \in A} \int_G q(z)(h^\sharp_\alpha * f^\sharp_\alpha)(z)\, d\mu_G(z) = \int_G q(z)\, d(\mu_{KyK} * \mu_{KxK})(z).\end{aligned}$$

Thus (i) implies (iii). Conversely, (iii) implies commutativity of $M(K\backslash G/K)$, which in turn implies commutativity of $L^1(K\backslash G/K)$.

Note that $(KxK)(KyK)$ is the support of the convolution $\mu_{KxK} * \mu_{KyK}$, so (iii) implies (ii). Conversely

$$\int_G q(z)\, d(\mu_{KxK} * \mu_{KyK})(z) = \iiint q(k_1 x k_2 y k_3)\, d\mu_K(k_1)\, d\mu_K(k_2)\, d\mu_K(k_3),$$

so (ii) implies (iii). □

The following consequence of Theorem 8.1.7 plays an important rôle in Part 4.

COROLLARY 8.1.8. *Let (G, K) be a Gelfand pair and N a closed normal subgroup of G. Then*

(i) *NK is a closed subgroup of G, thus is a locally compact group;*

(ii) *(NK, K) is a Gelfand pair;*

(iii) *the centralizer $Z_K(N)$ of N in K is a closed normal subgroup of NK, and furthermore $(NK/Z_K(N), K/Z_K(N))$ is a Gelfand pair.*

PROOF. Assertion (i) is immediate. For Assertion (ii), let $n_1, n_2 \in N$. Since the $n_i \in G$, Theorem 8.1.7 says $Kn_1Kn_2K = Kn_2Kn_1K$, and now, again by Theorem 8.1.7, (NK, K) is a Gelfand pair. To see (iii) let $z \in Z_K(N)$, $n \in N$, and $k \in K$. Compute $(kzk^{-1})n(kz^{-1}k^{-1}) = k(z(k^{-1}nk)z^{-1})k^{-1} = k(k^{-1}nk)k^{-1} = n$ to see that $Z_K(N)$ is normal in K. It is obviously closed in NK. Now $Z_K(N)$ is a closed normal subgroup in NK, and $NK \to NK/Z_K(N)$ maps the relation $Kn_1Kn_2K = Kn_2Kn_1K$ in NK to the relation $(K/Z_K(N))m_1(K/Z_K(N))m_2(K/Z_K(N)) = (K/Z_K(N))m_2(K/Z_K(N))m_1(K/Z_K(N))$ in $NK/Z_K(N)$. Thus (iii) follows as a consequence of (ii). □

In Part 4 we will also have use for

Corollary 8.1.9. *Let G be a locally compact group and K a compact subgroup. Suppose that $G = G^0K$, in other words that the identity component G^0 is transitive on G/K. If (G^0, K^0) is a Gelfand pair then $(G^0, K \cap G^0)$ is a Gelfand pair. If $(G^0, K \cap G^0)$ is a Gelfand pair then (G, K) is a Gelfand pair.*

Proof. Let (G^0, K^0) be a Gelfand pair. Suppose that $g_0, g_0', g_0'' \in G^0$ with $g_0'' \in g_0(K \cap G^0)$. Then we have $g_0K^0g_0' \subset K^0g_0'K^0g_0K^0$. Take the union as g_0 runs over $g_0''(K \cap G^0)$. Then

$$g_0(K \cap G^0)g_0' \subset K^0g_0'K^0g_0(K \cap G^0) \subset (K \cap G^0)g_0'(K \cap G^0)g_0(K \cap G^0),$$

so $(G^0, K \cap G^0)$ is a Gelfand pair. That is the first assertion.

Let $(G^0, K \cap G^0)$ be a Gelfand pair and express $G = \bigcup G^0k_i$ where the $k_i \in K$. Then $K = \bigcup (K \cap G^0)k_i$. Let $g, g' \in G$. Express $g = k_ig_0$ and $g' = g_0'k_j$ where $g_0, g_0' \in G^0$. Now $g_0(K \cap G^0)g_0' \subset (K \cap G^0)g_0'(K \cap G^0)g_0(K \cap G^0)$ gives us

$$g(K \cap G^0)g' \subset k_i(K \cap G^0)g'k_j^{-1}(K \cap G^0)k_i^{-1}g(K \cap G^0)k_j \subset Kg'KgK.$$

Here we can replace g by and gk, so now $gKg' \subset Kg'KgK$, and (G, K) is a Gelfand pair. That is the second assertion. □

8.2. Spherical Measures and Spherical Functions

In this section we introduce spherical functions. They are the Gelfand pair analog of characters on locally compact abelian groups. As before, G is a locally compact group and K is a compact subgroup. At first we do not require (G, K) to be a Gelfand pair.

Definition 8.2.1. A **spherical measure** for (G, K) is a nonzero Radon measure m on G such that

(*i*) m is K–bi–invariant : $m(k_1Ek_2^{-1}) = m(E)$ for every Borel set $E \subset G$, and

(*ii*) $f \mapsto m(f) = \int_G f(g)\, dm(g)$ is an algebra homomorphism $C_c(K\backslash G/K) \to \mathbb{C}$, i.e. $m(f_1 * f_2) = m(f_1)m(f_2)$.

In other words, m is a **multiplicative linear functional** on $C_c(K\backslash G/K)$. ♢

Example 8.2.2. Here is the example that guides Definition 8.2.1. Suppose that $\omega : G \to \mathbb{C}^*$ is a continuous homomorphism (quasi–character). Define

$$m(f) = \int_G f(g)\omega(g^{-1})\, d\mu_G(g).$$

If $f_1, f_2 \in C_c(K\backslash G/K)$ then

$$m(f_1 * f_2) = \int_G \int_G f_1(x) f_2(x^{-1}y)\omega(y^{-1})\, d\mu_G(x)\, d\mu_G(y)$$

$$= \int_G \int_G f_1(x) f_2(y)\omega(x^{-1})\omega(y^{-1})\, d\mu_G(x)\, d\mu_G(y) = m(f_1)m(f_2),$$

so $m : C_c(K\backslash G/K) \to \mathbb{C}$ either is identically zero or is a spherical measure. Let $F = KFK$ be a compact subset of G that is the closure of a nonempty open set, let 1_F be the function that has value 1 on F and 0 on $G\backslash F$, and define $h(x) = \overline{\omega^\sharp(x^{-1})}$. Express h as the uniform limit $\lim f_n$ of functions $f_n \in C_c(K\backslash G/K)$. Then

$$\lim m(f_n) = \lim \int_G f_n(x)\omega^\sharp(x^{-1})\, d\mu_G(x) = \int_F |\omega^\sharp(x^{-1})|^2 d\mu_G(x) > 0,$$

so $m : C_c(K\backslash G/K) \to \mathbb{C}$ is not identically zero. Thus $m : C_c(K\backslash G/K) \to \mathbb{C}$ is a spherical measure. ♢

Example 8.2.3. Suppose that the compact subgroup K is open in G. This is the case, for example, if G is a p–adic linear algebraic group $G_{\mathbb{Q}_p}$ and K is the subgroup $G_{\mathbb{Z}_p}$ consisting of matrices in $G_{\mathbb{Q}_p}$ whose entries are p–adic integers, as in Section 3.8. Then every nonzero associative algebra homomorphism $\phi : C_c(K\backslash G/K) \to \mathbb{C}$ is a spherical measure. To see this we just have to check that $\widetilde{\phi} : C_c(G) \to \mathbb{C}$, given by $\widetilde{\phi}(f) = \phi(f^\sharp)$, is a Radon measure. In other words, if $L \subset G$ is compact we have to check that $\widetilde{\phi}$ is continuous on the space $'C(L) = \{f \in C(G) \mid \mathrm{Supp}\,(f) \subset L\}$. But $'C(L)^\sharp \subset C_c(K\backslash G/K) \cap C(KLK)$, and each KgK is compact and open, so KLK is a finite union of double cosets. It follows that $\dim C_c(K\backslash G/K) \cap {}'C(KLK) < \infty$, so $\widetilde{\phi}$ is continuous on $'C(L)$. ♢

Theorem 8.2.4. *If m is a spherical measure for (G, K), then m is absolutely continuous with respect to Haar measure on G,*

$$\int_G f(x)\, dm(x) = \int_G f(x)\omega(x^{-1})\, d\mu_G(x) \text{ for } f \in C_c(K\backslash G/K),$$

and here we may choose $\omega \in C(K\backslash G/K)$ with $\omega(1) = 1$.

Proof. Choose $f_0 \in C_c(K\backslash G/K)$ with $m(f) \neq 0$. If $f \in C_c(K\backslash G/K)$ then

$$m(f)m(f_0) = m(f * f_0) = \int_G \int_G f(x) f_0(x^{-1}y)\, d\mu_G(x)\, dm(y)$$

so

$$m(f) = \int_G f(x)\omega(x^{-1})\, d\mu_G(x) \text{ where } \omega(x^{-1}) = \tfrac{1}{m(f_0)} \int_G f_0(x^{-1}y)\, dm(y).$$

Here ω is continuous because $f_0 \in C_c(K\backslash G/K)$, and ω is K–bi–invariant because f_0 and m have that property. Finally, $\omega(1) = (1/m(f_0)) \int_G f_0(y)\, dm(y) = 1$. □

Definition 8.2.5. A **spherical function** for (G, K) is a continuous function $\omega : G \to \mathbb{C}$ such that the measure $m(f) = \int_G f(x)\omega(x^{-1})\, d\mu_G(x)$ is spherical for (G, K). Then it is automatic that ω is K–bi–invariant and that $\omega(1) = 1$. ♢

Theorem 8.2.6. *Let G be a locally compact group and $K \subset G$ a compact subgroup. The following conditions are equivalent.*

1. *$\omega : G \to \mathbb{C}$ is a spherical function for (G, K).*

2. $\omega : G \to \mathbb{C}$ *is a continuous* K*–bi–invariant function with* $\omega(1) = 1$ *and such that:*

$$\text{if } f \in C_c(K\backslash G/K) \text{ there exists } \lambda_f \in \mathbb{C} \text{ such that } f * \omega = \lambda_f\, \omega.$$

3. (Functional Equation) $\omega : G \to \mathbb{C}$ *is a continuous function not identically zero; and if* $g_1, g_2 \in G$ *then* $\omega(g_1)\omega(g_2) = \int_K \omega(g_1 k g_2)\, d\mu_K(k)$.

Proof. We check that (1) implies (2). Let m be the spherical measure with $dm(x) = \omega(x^{-1}) d\mu_G(x)$. If $f, f' \in C_c(K\backslash G/K)$ with $m(f') \neq 0$ then

$$\begin{aligned}\int_G f(x) m(f')\omega(x^{-1})\, d\mu_G(x) &= m(f)m(f') = m(f * f') = [(f * f') * \omega](1) \\ &= [f * (f' * \omega)](1) = \int_G f(x)(f' * \omega)(x^{-1})\, d\mu_G(x)\end{aligned}$$

so

$$m(f')\omega(x^{-1}) = (f' * \omega)(x^{-1}) \text{ a.e. } K\backslash G/K.$$

Since both $m(f')\omega(x^{-1})$ and $(f' * \omega)(x^{-1})$ are continuous and K–bi–invariant, that says

$$m(f')\omega(x^{-1}) = (f' * \omega)(x^{-1}) \text{ for all } x \in G, \text{ i.e. } f' * \omega = \lambda_{f'}\, \omega \text{ where } \lambda_{f'} = m(f'),$$

which is the assertion of (2).

Next we check that (2) implies (3). Compute

$$\int_G f(x)\omega(x^{-1})\, d\mu_G(x) = (f * \omega)(1) = \lambda_f\, \omega(1) = \lambda_f\ .$$

Similarly

$$\lambda_f\, \omega(y) = (f * \omega)(y) = \int_G f(x)\omega(x^{-1}y)\, d\mu_G(x).$$

Change x to $k^{-1}x$ and integrate over K:

$$\begin{aligned}\lambda_f\, \omega(y) &= \int_K \left(\int_G f(x)\omega(x^{-1}ky)\, d\mu_G(x) \right) d\mu_K(k) \quad (\text{since } f(k^{-1}x) = f(x)) \\ &= \int_K f(x)\omega_y(x^{-1})\, d\mu_G(x) \quad (\text{where } \omega_y(x^{-1}) = \int_K \omega(x^{-1}ky)\, d\mu_K(k))\end{aligned}$$

Now

$$\lambda_f\, \omega(y) = \int_G f(x)\omega_y(x^{-1})\, d\mu_G(x) \text{ while } \lambda_f\, \omega(y) = \big(\int_G f(x)\omega(x^{-1})\, d\mu_G(x)\big)\omega(y).$$

This holds for every $f \in C_c(K\backslash G/K)$. Thus $\omega_y(x^{-1}) = \omega(x^{-1})\omega(y)$ a.e. By continuity that holds for all $x, y \in G$. Change x^{-1} to x and this becomes $\omega(x)\omega(y) = \omega_y(x) = \int_K \omega(xky)\, d\mu_K(k)$, which is the functional equation of (3).

Finally we check that (3) implies (1). Define $m(f) = \int_G f^\sharp(x)\omega(x^{-1})\, d\mu_G(x)$ for $f \in C_c(G)$. Compute

$$\begin{aligned} m(f_1 * f_2) &= \int_G \left(\int_G f_1(x) f_2(x^{-1}y)\, d\mu_G(x) \right) \omega(y^{-1})\, d\mu_G(y) \\ &= \int_G f_1(x) \left(\int_G f_2(x^{-1}y)\omega(y^{-1})\, d\mu_G(y) \right) d\mu_G(x) \\ &= \int_G f_1(x) \left(\int_G f_2(z)\omega(z^{-1}x^{-1})\, d\mu_G(z) \right) d\mu_G(x). \end{aligned}$$

Change z to kz, integrate over K, and use $f_2(kz) = f_2(z)$:

$$m(f_1 * f_2) = \int_G f_1(x) \left[\int_G f_2(z) \left(\int_K \omega(z^{-1}k^{-1}x^{-1})\, d\mu_K(k) \right) d\mu_G(z) \right] d\mu_G(x).$$

Now apply (3) to see

$$m(f_1 * f_2) = \int_G \int_G f_1(x) f_2(z)\omega(z^{-1})\omega(x^{-1})\, d\mu_G(z)\, d\mu_G(x) = m(f_1)m(f_2).$$

Thus $m : C_c(K\backslash G/K) \to \mathbb{C}$ is an associative algebra homomorphism. Continuity of ω says that m is a Radon measure. The functional equation (3) implies K–bi–invariance of ω, so the Radon measure m is K–bi–invariant. Also, ω is not identically zero on $C_c(k\backslash G/K)$, so the same follows for m. Thus m is a spherical measure, and ω is a spherical function, for (G, K). That completes the proof of Theorem 8.2.6. □

The associative algebra homomorphisms $m : C_c(K\backslash G/K) \to \mathbb{C}$ are not always continuous. If we view $C_c(K\backslash G/K)$ as a subalgebra of the normed algebra (under convolution) $L^1(K\backslash G/K)$ with the subspace topology, then continuity is given by

THEOREM 8.2.7. *The continuous homomorphisms $C_c(K\backslash G/K) \to \mathbb{C}$ (and $L^1(K\backslash G/K) \to \mathbb{C}$) are the maps $f \mapsto \int_G f(x)\omega(x^{-1})d\mu_G(x)$ where ω is a bounded (G, K)–spherical function on G.*

PROOF. Let ω be a bounded (G, K)–spherical function on G. Then the map $f \mapsto \int_G f(x)\omega(x^{-1})d\mu_G(x)$ sends products to products because ω is spherical, and is continuous because ω is bounded.

Let $T : L^1(K\backslash G/K) \to \mathbb{C})$ be a continuous homomorphism. Then we have $\omega \in L^\infty(K\backslash G/K)$ such that $T(f) = \int_G f(x)\omega(x^{-1})d\mu_G(x)$ for all $f \in L^1(K\backslash G/K)$. As f ranges over $C_c(K\backslash G/K)$, and approximating ω from $C_c(K\backslash G/K)$, we can use the argument that (2) implies (3), in the proof of Theorem 8.2.6. This shows that $\int_K \omega(xkz)d\mu_K(k) = \omega(x)\omega(z)$ a.e. $(x, z) \in G \times G$. Now choose $h \in C_c(G)$ with $\int_G \omega(z)h(z)d\mu_G(z) \neq 0$ and compute

$$\begin{aligned} \omega(x) \int_G \omega(z)h(z)d\mu_G(z) &= \int_G h(z) \left(\int_K \omega(xkz)d\mu_K(k) \right) d\mu_G(z) \\ &= \int_K \left(\int_G h(k^{-1}x^{-1}y)\omega(y)d\mu_G(y) \right) d\mu_K(k) \\ &= \int_G \left(\int_K h(k^{-1}x^{-1}y)d\mu_K(k) \right) \omega(y)d\mu_G(y), \end{aligned}$$

which is continuous in x. $\square$

8.3. Alternate Formulation in the Differentiable Setting

When G is a Lie group, the notion of commutative space G/K can be formulated in terms of the algebra of G–invariant differential operators on G/K, and the notion of spherical function can be formulated as joint eigenfunction of that algebra. We discuss that now, making free use of material mentioned in the first three sections of Chapter 6.

For the rest of this section, except for Remark 8.3.17 at the end, G is a connected Lie group and K is a compact subgroup. Thus K is a Lie subgroup and G/K has a natural (and unique) G–invariant structure of C^ω manifold. A differential operator D on G/K is G–**invariant** if if commutes with the action of G on $C^\infty(G/K)$, in other words if $\lambda(g)(Df) = D(\lambda(g)f)$ for every $g \in G$ and every C^∞ function f on G/K. The set of all G–invariant differential operators on G/K forms an associative algebra $\mathcal{D}(G,K)$ over $\mathbb{C}$. We write $\mathcal{D}(G)$ for $\mathcal{D}(G,\{1\})$. Then $\mathcal{D}(G)$ is the associative algebra generated by the left invariant vector fields (the Lie algebra $\mathfrak{g}$) on G, in other words the universal enveloping algebra, and $\mathcal{D}(G,K)$ consists of the right K–invariant elements of $\mathcal{D}(G)$.

In this section we will prove the following three theorems and illustrate their use in constructing spherical functions.

Theorem 8.3.1. (**Thomas** [**T**]) *Let G be a connected Lie group and K a compact subgroup. Then (G,K) is a Gelfand pair if and only if $\mathcal{D}(G,K)$ is commutative.*

Now it is reasonable to speak of joint $\mathcal{D}(G,K)$–eigenfunctions when G is a connected Lie group and (G,K) is a Gelfand pair.

Remark 8.3.2. Along the lines of Corollary 8.1.9 one can sometimes weaken the connectedness requirement in Theorem 8.3.1. We will need this in Section 11.3. Let G be a Lie group and K a compact subgroup. Suppose that $G = G^0K$, i.e. that the identity component G^0 is transitive on $M = G/K$. If $\mathcal{D}(G^0,K^0)$ is commutative then its subalgebra $\mathcal{D}(G^0, K\cap G^0)$ is of course commutative. If $\mathcal{D}(G^0, K\cap G^0)$ is commutative, in other words the algebra of G^0–invariant differential operators on M is commutative, then its subalgebra consisting of the G–invariant differential operators on M is commutative, in other words $\mathcal{D}(G,K)$ is commutative. $\diamondsuit$

Theorem 8.3.3. (**Gelfand** [**Ge1**], **Godement** [**Go**], **Helgason** [**H1**]) *Let G be a connected Lie group and K a compact subgroup such that (G,K) is a Gelfand pair. Then a K–bi–invariant C^∞ function $f : G \to \mathbb{C}$ is spherical for (G,K) if and only if* (i) *$f(1) = 1$ and* (ii) *if $D \in \mathcal{D}(G,K)$ then $Df = \chi(D)f$ for some number $\chi(D) \in \mathbb{C}$.* (Note that the joint eigenvalue $\chi : \mathcal{D}(G,K) \to \mathbb{C}$ is an associative algebra homomorphism.)

The joint eigenvalue in Theorem 8.3.3 determines the spherical function:

Theorem 8.3.4. *Let G be a connected Lie group and K a compact subgroup such that (G,K) is a Gelfand pair. Let f_1 and f_2 be spherical functions for (G,K) with the same $\mathcal{D}(G,K)$–eigenvalue: $Df_1 = \chi(D)f_2$ and $Df_2 = \chi(D)f_2$ for all $D \in \mathcal{D}(G,K)$. Then $f_1 = f_2$.*

The proof of Theorem 8.3.1 depends on the basic facts in the theory of distributions on manifolds. There are a number of short good references, for example Appendix 2 (pages 479–482) of Warner's book [**War**]. We write $C^{-\infty}(G)$, $C^{-\infty}(G/K)$ and $C^{-\infty}(K\backslash G/K)$ for the spaces of distributions on G, G/K and $K\backslash G/K$. We write $C_c^{-\infty}(G)$, $C_c^{-\infty}(G/K)$ and $C_c^{-\infty}(K\backslash G/K)$ for the respective subspaces of compactly supported distributions. Now we run through Thomas' proof.

Lemma 8.3.5. *If $S \in C_c^{-\infty}(K\backslash G/K)$ then right convolution $A_S : T \mapsto T * S$ is a continuous linear operator on $C^{-\infty}(G/K)$ that commutes with left translation by elements of G. Conversely every continuous linear operator on $C^{-\infty}(G/K)$ that commutes with left translation by elements of G is of the form A_S for a unique $S \in C_c^{-\infty}(K\backslash G/K)$. Also, $A_{S_1 * S_2} = A_{S_2} A_{S_1}$.*

Proof. The first statement is straightforward. Here $S = A_S(\delta_K)$ where $\delta_K(f) = \int_K f(x) d\mu_K(x)$. For the second statement let A be the operator and set $S = A(\delta_K)$. Then $S \in C^{-\infty}(G/K)$. Since S is left K–invariant this implies $S \in C^{-\infty}(K\backslash G/K)$.

We check that S has compact support. If $\{g_n\} \to \infty$ in G and $\{c_n\} \subset \mathbb{C}$ then $\{c_n \lambda(g_n)(\delta_K)\} \to 0$ in $C^{-\infty}(G)$ because $[\lambda(g_n)(\delta_K)](f) = 0$ as soon as n is big enough that the supports of $\lambda(g_n)(\delta_K)$ and f are disjoint. Thus $\{c_n \lambda(g_n)(S)\} = \{A(c_n \lambda(g_n)(\delta_K)\} \to 0$. If $\{g_n\} \to \infty$ in Supp (S) then $1 \in$ Supp $(\lambda(g_n)(S))$ for all n, so if C is a compact neighborhood of 1 then there exist functions $f_n \in C_C^\infty(G)$ (support in C) such that $(\lambda(g_n)(S))(f_n) = 1$ for all n. Let $c_n = n p_n(f_n)$ where $\{p_n\}$ is a fundamental sequence of continuous seminorms on $C_C^\infty(G)$. As $\{c_n \lambda(g_n)(S)\} \to 0$ the Banach–Steinhaus Theorem says that its restriction to $C_C^\infty(G)$ is equicontinuous, so there exist $m, M > 0$ such that $|(c_n \lambda(g_n)(S))(f)| \leqq M p_m(f)$ for all $f \in C_C^\infty(G)$ and all n. If $f = f_n$ this says $n \leqq M$ for all n, which is absurd. Thus one cannot have $\{g_n\} \to \infty$ in Supp (S), and so S has compact support.

We check that $A(T) = T * S$ for $T \in C^{-\infty}(G/K)$. This is clear for $T = \delta_g * \delta_K$, and thus for $T = \nu * \delta_K$ where ν is a finite linear combination of point masses. Those linear combinations are dense in $C^{-\infty}(G/K)$, so the formula is verified for all $T \in C^{-\infty}(G/K)$. That completes the proof of the second statement.

The third statement is associativity of convolution, which is defined here because the S_i have compact support. □

Lemma 8.3.6. *If one of the convolution algebras $C_c^\infty(K\backslash G/K)$, $C_c(K\backslash G/K)$, $L^1(K\backslash G/K)$ and $C_c^{-\infty}(K\backslash G/K)$ is commutative, so are all the others.*

This is a straightforward approximation. Compare it with Proposition 8.1.7.

Lemma 8.3.7. *(G, K) is a Gelfand pair if and only if the algebra of continuous linear operators on $C^{-\infty}(G/K)$, that commute with the action of G, is commutative.*

Combine Lemmas 8.3.5 and 8.3.6. That gives Lemma 8.3.7. Note that $\mathcal{D}(G, K)$ is a subalgebra of the algebra $C_c^{-\infty}(K\backslash G/K)$ of Lemma 8.3.7. If (G, K) is a Gelfand pair now $\mathcal{D}(G, K)$ is commutative by Lemma 8.3.7. For the converse we have to describe $\mathcal{D}(G, K)$ explicitly as a subalgebra of $C_c^{-\infty}(K\backslash G/K)$. Define $\mathcal{A}(K\backslash G/K) = \{\delta_K * E * \delta_K \mid E \in C^{-\infty}(G) \text{ with support Supp}\,(E) = \{1\}\}$.

LEMMA 8.3.8. *Let* $S \in C_c^{-\infty}(K\backslash G/K)$. *Then* $A_S \in \mathcal{D}(G,K)$ *if and only if* $S \in \mathcal{A}(K\backslash G/K)$.

The point is that the left–invariant differential operators on G are in one–one correspondence with distributions supported at 1.

LEMMA 8.3.9. $C^\omega(G)$ *is dense in* $C^\infty(G)$.

PROOF. The short proof follows the idea of Gårding's proof [**Gar**] of Nelson's theorem on density of analytic vectors in a representation of G, as follows. If $u(x,t)$ is the solution to the heat equation with initial data $f \in C_c^\infty(G)$ as in [**Gar**], then Gårding's arguments combine with the Sobolev Inequalities to show that $\lim_{t\downarrow 0} u(\cdot,t) = f$ in $C^\infty(G)$. As $u(x,t)$ is C^ω in x and $C_c^\infty(G)$ is dense in $C^\infty(G)$ now $C^\omega(G)$ is dense in $C^\infty(G)$.

The less technical proof uses Nelson's theorem to construct a sort of approximate identity consisting of analytic functions. Let $\mathcal{B}$ denote the Banach space $L^2(G)\cap\mathcal{C}_\infty(G)$ with norm $||f||_{\mathcal{B}} = ||f||_2+||f||_\infty$. The dense space of C^ω vectors in $\mathcal{B}$, for the left action of G, is a space of C^ω functions. Take a sequence $\{\varphi_n\} \subset C_c(G)$ such that $\lim_{n\to\infty} ||\varphi_n||_2 = 1$ and $\lim_{n\to\infty} \int_{G\backslash U} |\varphi_n(x)|^2 d\mu_G(x) = 0$ for every neighborhood U of 1. Choose analytic vectors $\{f_n\} \subset \mathcal{B}$ with $||f_n-\varphi_n||_{\mathcal{B}} \leqq 1/n$. Then for appropriate constants $r_n > 0$ the $h_n = r_n|f_n|^2$ are positive analytic functions with the properties $\lim_{n\to\infty} \int_G h_n(x)d\mu_G(x) = 1$ and $\lim_{n\to\infty} \int_{G\backslash U} h_n(x)d\mu_G(x) = 0$ for every neighborhood U of 1. If D is a left invariant differential operator on G and $\varphi \in C_c^\infty(G)$ now $D(h_n * \varphi) = h_n * D\varphi$ converges uniformly to $D\varphi$. Thus $h_n * \varphi$ converges to φ in $C^\infty(G)$. As the $h_n * \varphi$ are C^ω that proves $C^\omega(G)$ dense in $C^\infty(G)$. □

LEMMA 8.3.10. *If* $f \in C^\omega(G)$ *and* $T \in C_c^{-\infty}(G)$ *then* $f * T$, $T * f \in C^\omega(G)$.

LEMMA 8.3.11. *If* $T \in C_c^{-\infty}(G)$ *there exist compactly supported Radon measures* $\mu_1, \ldots, \mu_n$ *and left invariant differential operators* $D_1, \ldots, D_n$ *on* G *such that* $T = \sum_{1\leqq i\leqq n} D_i\mu_i$.

Here Lemma 8.3.11 reduces the proof of Lemma 8.3.10 to the case where T is a compactly supported measure, thus a finite linear combination of positive measures μ with compact support contained in the domain of a C^ω coordinate chart on G. For such a measure μ, the function $x \mapsto \int f(xy^{-1})d\mu(y)$ is seen C^∞ from term by term integration of the power series of f, so $f*T \in C^\omega(G)$, and the same argument give $T * f \in C^\omega(G)$. For Lemma 8.3.11, the topology of $C^\infty(G)$ is defined by the seminorms $N_{D,C}(f) = \sum_{x\in C} |Df(x)| = ||Df||_{\infty,C}$ where D is a differential operator and $C \subset G$ is compact. It suffices to consider left invariant differential operators, say $\Xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\ldots\xi_m^{\alpha_m}$ where $\{\xi_1,\ldots,\xi_m\}$ is a basis of $\mathfrak{g}$. If C is a compact neighborhood of $\mathrm{Supp}\,(T)$ and D_i of the form Ξ^α on C, and $M > 0$ such that $|T(f)| \leqq M \sum_{1\leqq i\leqq n} ||D_i(f)||_{\infty,C}$ for all $f \in C^\infty(G)$, then Hahn–Banach Theorem gives the measures μ_i with support in C such that $T(f) = \sum \mu_I(D_i(f))$, and $T = \sum_{1\leqq i\leqq n} {}^tD_i\mu_i$, which is Lemma 8.3.11.

We give $C_c^{-\infty}(G)$ the weakest topology for which the maps $T \mapsto T(f)$, where $f \in C^\omega(G)$, are continuous. Item (5.) shows that this topology is Hausdorff. The next item is an observation of Godement [**Go**].

LEMMA 8.3.12. *$\{E \in C^{-\infty}(G) \mid \mathrm{Supp}\,(E) = \{1\}\}$ is dense in $C_c^{-\infty}(G)$.*

This comes right out of the Hahn–Banach Theorem. Suppose $f \in C^{\omega}(G)$ such that $E(f) = 0$ whenever $E \in C^{-\infty}(G)$ with $\mathrm{Supp}\,(E) = \{1\}$. Then $Df(1) = 0$ for all left–invariant differential operators D on G, so $f = 0$.

LEMMA 8.3.13. *Let $S \in C^{-\infty}(G)$. Then the operators $T \mapsto T*S$ and $T \mapsto S*T$ on $C_c^{-\infty}(G)$ are continuous.*

For example $(T * S)(f) = T(f * S')$ for some $S' \in C^{-\infty}(G)$, and $f * S'$ is C^{ω} by Lemma 8.3.10.

PROOF OF THEOREM 8.3.1 By Lemma 8.3.13, $T \mapsto \delta_K * T * \delta_K$ is continuous on $C_c^{-\infty}(G)$. By Lemma 8.3.8 its image is the subalgebra $\mathcal{A}(K\backslash G/K)$ of $C_c^{-\infty}(K\backslash G/K)$ corresponding to $\mathcal{D}(G,K)$. Thus by Lemma 8.3.12 $\mathcal{A}(K\backslash G/K)$ is dense in $C_c^{-\infty}(K\backslash G/K)$. If $\mathcal{D}(G,K)$ is commutative now $C_c^{-\infty}(K\backslash G/K)$ is commutative, and then Lemma 8.3.6 says that (G,K) is a Gelfand pair. $\square$

The proof of Theorem 8.3.3 combines results of Gelfand [**Ge1**], Godement [**Go**] and Helgason [**H1**]. Gelfand found the differential equations for the spherical functions and Godement developed their properties and related them to work of Harish–Chandra [**Ha1**]. Helgason characterized the solutions to these differential equations by a functional equation

$$0 \neq \varphi \in C(G) \text{ and } \varphi(x)\varphi(y) = \int_K \varphi(xky)\, d\mu_K(k) \text{ for } x, y \in G, \tag{8.3.14}$$

which is that of Theorem 8.2.6 for spherical functions. This characterization proves Theorem 8.3.3.

PROPOSITION 8.3.15. *Suppose that $\mathcal{D}(G,K)$ is commutative and $0 \neq \varphi \in C(G)$. Then φ satisfies* (8.3.14) *if and only if* (i) *$\varphi \in C^{\infty}(G)$,* (ii) *$\varphi(1) = 1$, and* (iii) *$\varphi$ is a joint eigenfunction of $\mathcal{D}(G,K)$.*

PROOF. We follow Helgason's argument [**H1**, Chapter X, Proposition 3.2]. Suppose first that φ is C^{∞} and is a joint eigenfunction of $\mathcal{D}(G,K)$, and that $\varphi(1) = 1$. The manifold G/K is C^{ω}, as is any G–invariant riemannian metric, so the Laplace–Beltrami operator Δ for any such metric is C^{ω}. As $\Delta \in \mathcal{D}(G,K)$ and Δ is elliptic, now φ is C^{ω} by elliptic regularity.

Fix $x \in G$ and define $h(y) = \int_K \varphi(xky) d\mu_K(k)$. Then $h \in C^{\omega}(K\backslash G/K)$. If $D \in \mathcal{D}(G,K)$, say $d\varphi = \chi(D)\varphi$, then $(Dh)(y) = \int_K (D\varphi)(xky)\, d\mu_K(k) = \chi(D)h(y)$. Thus $[D(\varphi(1)h - h(1)\varphi)](1) = 0$. The map $\widetilde{D} \mapsto \int_K (\widetilde{D} \cdot r(k)) d\mu_K(k)$ sends the algebra $\mathcal{D}(G)$ of left–invariant differential operators on G, onto $\mathcal{D}(G,K)$. Now $[\widetilde{D}(\varphi(1)h - h(1)\varphi)](1) = 0$ for every $\widetilde{D} \in \mathcal{D}(G)$, where we have pulled the functions φ and h back to G. Thus all derivatives of $\varphi(1)h - h(1)\varphi$ vanish at 1. Since φ is C^{ω}, so is the function $\varphi(1)h - h(1)\varphi$, while its Taylor series expansion at 1 is identically zero. Thus $\varphi(1)h(y) = h(1)\varphi(y)$ for all $y \in G$. Since $\varphi(1) = 1$ and $h(1) = \varphi(x)$ now $\int_K \varphi(xky) d\mu_K(k) = \varphi(x)\varphi(y)$. Thus φ satisfies (8.3.14).

Conversely suppose that φ satisfies (8.3.14). Choose $x_0 \in G$ with $\varphi(x_0) \neq 0$. Then (8.3.14) says $\varphi(xk)\varphi(x_0) = \varphi(x_0)\varphi(x) = \varphi(x_0)\varphi(kx)$ for $x \in G$ and $k \in K$. Now ϕ is K–bi–invariant. Set $y = 1$ in (8.3.14) to see $\varphi(1) = 1$. Choose $u \in C_c^\infty(G)$ such that $\int_G u(g)\varphi(g)d\mu_G(g) \neq 0$. Compute

$$\begin{aligned}\varphi(x)\int_G u(g)\varphi(g)d\mu_G(g) &= \int_G u(g)\left(\int_K \varphi(xkg)d\mu_K(k)\right)d\mu_G(g)\\ &= \int_K\left(\int_G u(g)\varphi(xkg)d\mu_G(g)\right)d\mu_K(k)\\ &= \int_K\left(\int_G u(k^{-1}x^{-1}z)\varphi(z)d\mu_G(z)\right)d\mu_K(k)\\ &= \int_G\left(\int_K u(kx^{-1}z)d\mu_K(k)\right)\varphi(z)d\mu_G(z).\end{aligned}$$

That transfers differentiation in x from φ to the C^∞ function u, proving $\phi \in C^\infty(G)$. Now look again at (8.3.14). Fix $x \in G$ and let $D \in \mathcal{D}(G,K)$. Then $\varphi(x)(D\varphi)(y) = \int_K (D\varphi)(xky)d\mu_K(k)$. With $y = 1$ that becomes $(D\varphi)(x) = (D\varphi)(1)\varphi(x)$, so φ is a joint eigenfunction of $\mathcal{D}(G,K)$. That completes the proof of the converse. Proposition 8.3.15 is proved. □

Proof of Theorem 8.3.3 According to Theorem 8.2.6, a continuous function $\varphi : G \to \mathbb{C}$ is spherical if and only if it satisfies (8.3.14). According to Proposition 8.3.15, φ satisfies (8.3.14) if and only if (i) $\varphi \in C^\infty(G)$, (ii) $\varphi(1) = 1$, and (iii) φ is a joint eigenfunction of $\mathcal{D}(G,K)$. □

Lemma 8.3.16. *Identify the space $\mathcal{D}(G)$ of left G–invariant differential operators on G with the universal enveloping algebra $\mathfrak{G}$, so $\mathcal{D}(G,K)$ is identified with the algebra of all $D|_{C^\infty(G/K)}$ as D ranges over the fixed point set of $\mathrm{Ad}_G(K)$ on $\mathfrak{G}$. Let $\pi : \mathcal{D}(G) \to \mathcal{D}(G,K)$ denote the projection $D \mapsto (\int_K (\mathrm{Ad}(k)D)d\mu_K(k))|_{C^\infty(G/K)}$. Then π is surjective, and if $D \in \mathcal{D}(G)$ and $\varphi \in C^\infty(K\backslash G/K)$ then $(D\varphi)(1) = (\pi(D)\varphi)(1)$.*

Proof. If $\Xi \in \mathfrak{G}$ then $\mathrm{Ad}_G(K)\Xi$ lies in a finite dimensional subspace of $\mathfrak{G}$, because K is compact. Thus $\mathfrak{G}$ is the algebraic direct sum of the convolutions $\psi * \mathfrak{G}$ as ψ runs over the normalized characters of irreducible representations of K. In particular π is surjective.

If $f \in C^\infty(K\backslash G/K)$, $\zeta_i \in \mathfrak{g}$ for $1 \leqq i \leqq \ell$, and $k \in K$, then

$$\begin{aligned}((\mathrm{Ad}(k)(\zeta_1\dots\zeta_\ell))f)(1) &= \tfrac{d}{dt_1}\Big|_{t_1=0}\cdots\tfrac{d}{dt_\ell}\Big|_{t_\ell=0} f(k\exp(t_1\zeta_1)\dots\exp(t_\ell\zeta_\ell)k^{-1})\\ &= \tfrac{d}{dt_1}\Big|_{t_1=0}\cdots\tfrac{d}{dt_\ell}\Big|_{t_\ell=0} f(\exp(t_1\zeta_1)\dots\exp(t_\ell\zeta_\ell))\\ &= (\zeta_1\dots\zeta_\ell)(f)(1)\end{aligned}$$

If $D \in \mathcal{D}(G)$ now $((\mathrm{Ad}(k)D)f)(1) = (Df)(1)$. Taking the integral over K we conclude that $(\pi(D)f)(1) = (Df)(1)$. □

Proof of Theorem 8.3.4 At the end of the proof of Proposition 8.3.15 we saw that any spherical function φ satisfies $(D\varphi)(x) = (D\varphi)(1)\varphi(x)$ for every

$D \in \mathcal{D}(G, K)$. Thus the joint eigenvalue $\chi : \mathcal{D}(G, K) \to \mathbb{C}$ for φ is given by $\chi(D) = (D\varphi)(1)$. In view of Lemma 8.3.16 we have $(D\varphi)(1) = (\pi(D)\varphi)(1)$ for every $D \in \mathcal{D}(G)$. Now if f_1 and f_2 are (G, K)–spherical functions with the same joint eigenvalue for $\mathcal{D}(G, K)$, $(Df_1)(1) = (\pi(D)f_1)(1) = (\pi(D)f_2)(1) = Df_2(1)$ for every $D \in \mathcal{D}(G)$. They are real analytic, so $f_1 = f_2$. □

Remark 8.3.17. Theorem 8.3.1 can fail for disconnected G. For example consider the case $G = \mathbb{R}^n \times F$ and $K = \{(0, 1)\}$ where F is any group with the discrete topology. Since K consists of just the identity element, the Gelfand pair condition $KgKg'K = Kg'KgK$ becomes $gg' = g'g$, which fails whenever F is not commutative. On the other hand, here $\mathcal{D}(G, K)$ consists of the constant (in the $\mathbb{R}^n$ variable) coefficient differential operators on $\mathbb{R}^n \times \{1\}$, transported to the other components of G/K be the action of F, and is commutative.

One can also look at the important intermediate situation where G need not be connected but G/K is connected. If G/K is connected and $(G^0, K \cap G^0)$ is a Gelfand pair, then $\mathcal{D}(G^0, K \cap G^0)$ is commutative by Theorem 8.3.1, and $\mathcal{D}(G, K)$ is a subalgebra of $\mathcal{D}(G^0, K \cap G^0)$, so $\mathcal{D}(G, K)$ is commutative. Also, it is straightforward to check that (G, K) inherits the Gelfand pair condition $KgKg'K = Kg'KgK$ from $(G^0, K \cap G^0)$. It would be worthwhile to clarify this situation of possibly–disconnected groups G. ♢

8.4. Positive Definite Functions

In this section G is locally compact and we consider positive definite spherical functions.

Definition 8.4.1. Let G be a topological group. A function $\phi : G \to \mathbb{C}$ is **positive definite** if $\sum_1^n \phi(g_j^{-1} g_i)\overline{c_j} c_i \geqq 0$ whenever $n \geqq 0$, $\{g_1, \ldots, g_n\} \subset G$, and $\{c_1, \ldots, c_n\} \subset \mathbb{C}$. ♢

It is immediate from the definition that, if H is a closed subgroup of G and $\phi : G \to \mathbb{C}$ is positive definite, then $\phi|_H$ is positive definite. Also, if $\pi : G \to L$ is projection to a quotient group, then the positive definite functions on G that are constant on left cosets of L, are just the $\phi = \psi \cdot \pi$ where $\psi : L \to \mathbb{C}$ is positive definite.

Proposition 8.4.2. *Let $\phi : G \to \mathbb{C}$ be positive definite. Then* (i) $\phi(1) \geqq |\phi(g)|$ *for every $g \in G$,* (ii) $\phi(g^{-1}) = \overline{\phi(g)}$ *for $g \in G$,* (iii) *a non–negative linear combination of positive definite functions is positive definite, and* (iv) *ϕ is continuous and $\int\int \phi(h^{-1}g)\overline{f(h)}f(g)\, d\mu_G(h)\, d\mu_G(g) \geqq 0$ for $f \in L^1(G)$.*

Proof. If $n = 1$ Definition 8.4.1 says that $\phi(1)$ is real and non–negative. If $g \in G$ and $a, b \in \mathbb{C}$ then $\phi(1)(a^2 + b^2) + a\bar{b}\phi(g) + \bar{a}b\phi(g^{-1}) \geqq 0$; (i) and (ii) follow with $a = b = 1$. Assertion (iii) is obvious.

For assertion (iv) we can assume $f \in C_c(G)$. If $S \subset G$ is finite then

$$\sum\nolimits_{g,h \in S} \phi(h^{-1}g)\overline{f(h)}f(g) \geqq 0.$$

These sums approximate $\int\int \phi(h^{-1}g)\overline{f(h)}f(g)\, d\mu_G(h)\, d\mu_G(g)$. In other words the integral of (iv) is real and non–negative. □

REMARK 8.4.3. If $\phi : G \to \mathbb{C}$ is positive definite and is a spherical function, then it is a bounded spherical function, so Theorem 8.2.7 tells us that $f \mapsto \int_G f(x)\phi(x^{-1})d\mu_G(x)$ defines continuous homomorphisms $C_c(K\backslash G/K) \to \mathbb{C}$ and $L^1(K\backslash G/K) \to \mathbb{C}$. ♢

EXAMPLE 8.4.4. Let $f \in L^1(G)$. Then $f * f^*$ is a continuous positive definite function on G. For continuity see the proof of Theorem 3.5.7. Compute

$$\begin{aligned}\sum_{a,b=1}^{n} (f * f^*)(g_b^{-1}g_a)\overline{c_b}c_a &= \sum_{a,b=1}^{n} \overline{c_b}c_a \int_G f(h)f^*(h^{-1}g_b^{-1}g_a)\, d\mu_G(h) \\ &= \sum_{a,b=1}^{n} \overline{c_b}c_a \int_G f(g_b^{-1}h)f^*(h^{-1}g_a)\, d\mu_G(h) \\ &= \sum_{a,b=1}^{n} \overline{c_b}c_a \int_G f(g_b^{-1}h)\overline{f(g_a^{-1}h)}\, d\mu_G(h) \\ &= \int_G \left|\sum_{b=1}^{n} \overline{c_b} f(g_b^{-1}h)\right|^2 d\mu_G(h) \geqq 0.\end{aligned}$$

♢

Note that positive definite for $\phi(g) = \sum_{i=1}^{s} r_i\langle \pi_i(g)v_i, v_i\rangle$ is the same as positive definite for $\overline{\phi}$ where $\overline{\phi}(g) = \overline{\phi(g)} = \sum_{i=1}^{s} r_i\langle v_i, \pi_i(g)v_i\rangle$. Here of course each $r_i \geqq 0$.

EXAMPLE 8.4.5. Consider the case where G is a commutative locally compact group and $K = \{1\}$, and let $\omega : G \to \mathbb{C}$. Proposition 10.1.1 says that ω is spherical for $(G, \{1\})$ if and only if it is a quasi–character (continuous homomorphism to $\mathbb{C}^\times$), and that if ω is spherical for $(G, \{1\})$ then it is positive definite if and only if it is a unitary character on G. ♢

PROPOSITION 8.4.6. *Let $\phi : G \to \mathbb{C}$ be a continuous positive definite function with $\phi(1) = 1$. Then there exist a unitary representation π of G and a cyclic*[1] *unit vector $u \in H_\pi$ such that $\phi(g) = \langle u, \pi(g)u\rangle$ for all $g \in G$. If (π', u') is another such pair then there is a unitary equivalence $A \in I(\pi, \pi')$ such that $A(u) = u'$.*

PROOF. Let V be the vector space of all functions $f : G \to \mathbb{C}$ of **finite** support, with inner product $\langle f, h\rangle = \sum_{x,y\in G} \phi(x^{-1}y)f(x)\overline{h(y)}$. This inner product is positive semidefinite and satisfies $|\langle f, h\rangle| \leqq |\langle f, f\rangle|^{1/2}|\langle h, h\rangle|^{1/2}$. The group G acts on V by $[\widetilde{\pi}(g)f](x) = f(g^{-1}x)$. This preserves the inner product:

$$\langle \widetilde{\pi}(g)f, \widetilde{\pi}(g)h\rangle = \sum_{x,y} \phi(x^{-1}y)f(g^{-1}x)\overline{h(g^{-1}y)} = \sum_{x',y'} \phi(x'^{-1}y')f(x')\overline{h(y')} = \langle f, h\rangle.$$

Let N be the kernel of the inner product, $N = V \cap V^\perp = \{f \in V \mid \langle f, f\rangle = 0\}$, and let H denote the Hilbert space completion of V/N. Each $\widetilde{\pi}(g)$ preserves the inner product in V, hence preserves N and defines a unitary transformation $\pi(g)$ of H. Evidently $\pi : G \to \mathcal{U}(H)$ is a homomorphism. We must show that π is continuous.

[1] A vector $v \in H_\pi$ is called **cyclic** if $\pi(G)v$ is not contained in a proper closed subspace of H_π. The representation π is called **cyclic** if there is a cyclic vector $v \in H_\pi$.

Let $u = \delta_1 + N \in H$. Then $\langle u, \pi(g)u\rangle = \sum_{x,y} \phi(x^{-1}y)\delta_1(x)\delta_1(g^{-1}y)$. Note $\delta_1(x)\delta_1(g^{-1}y) = 0$ unless $x = 1$ and $y = g$, in which case $\delta_1(x)\delta_1(g^{-1}y) = 1$ and $x^{-1}y = g$. Thus $\langle u, \pi(g)u\rangle = \phi(g)$. In particular it is continuous in g. If $s, t \in G$ now $\langle \pi(s)u, \pi(g)\pi(t)u\rangle = \phi(s^{-1}gt)$, which is continuous in g. Taking finite linear combinations now $g \mapsto \langle v, \pi(g)w\rangle$ is continuous for $v, w \in V$ taken modulo N. In other words, $g \mapsto \langle v, \pi(g)w\rangle$ is continuous for v, w in a dense subspace of H, and thus for all $v, w \in H$. This constructs our unit vector $u \in H = H_\pi$ and proves weak continuity of π. But weak continuity is the same as strong continuity for unitary representations.

Finally let π' be another unitary representation of G, with cyclic unit vector u' such that $\langle u', \pi'(g)u'\rangle = \phi(g)$. Let L be the dense subspace of H_π spanned by the $\pi(g)u$ and L' the dense subspace of $H_{\pi'}$ spanned by the $\pi'(g)u'$. Define $A_0 : L \to L'$ by $A_0(\sum r_i\pi(g_i)u) = \sum r_i\pi'(g_i)u'$. This is well defined and isometric because

$$\begin{aligned} ||A_0(\sum_i r_i\pi(g_i)u)||^2 &= ||\sum_i r_i\pi'(g_i)u'||^2 \\ &= \sum_{i,j}\langle r_i\pi'(g_i)u', r_j\pi'(g_j)u'\rangle = \sum_{i,j}\phi(g_i^{-1}g_j)r_i\overline{r_j} \end{aligned}$$

while

$$||\sum_i r_i\pi(g_i)u||^2 = \sum_{i,j}\langle r_i\pi(g_i)u, r_j\pi(g_j)u\rangle = \sum_{i,j}\phi(g_i^{-1}g_j)r_i\overline{r_j}.$$

Now A_0 extends by continuity to a unitary transformation $A \in I(\pi, \pi')$ that carries u to u'. □

Definition 8.4.7. Fix a locally compact group G and a compact subgroup K. By **positive definite spherical function** for (G, K) we mean a positive definite function ϕ on G that is a spherical function for (G, K). ♢

As noted in Remark 8.4.3, Theorem 8.2.7 says in particular that if ϕ is a bounded spherical function for (G, K) then $f \mapsto \int_G f(x)\phi(x^{-1})d\mu_G(x)$ defines continuous homomorphisms $C_c(K\backslash G/K) \to \mathbb{C}$ and $L^1(K\backslash G/K) \to \mathbb{C}$.

Theorem 8.4.8. *The relation between positive definite spherical functions for (G, K) and irreducible unitary representation of G with a K–fixed vector is given as follows.*

1. *Let ϕ be a positive definite spherical function for (G, K). Write (π, u) for the corresponding unitary representation and cyclic unit vector such that $\phi(g) = \langle u, \pi(g)u\rangle$ as in* Proposition 8.4.6. *Then π is irreducible, $\pi(k)u = u$ for all $k \in K$, and u spans the space H_π^K of K–fixed vectors in H_π.*

2. *Let π be an irreducible unitary representation of G such that H_π^K is spanned by a single unit vector u. Then $\phi(g) = \langle u, \pi(g)u\rangle$ is a positive definite spherical function for (G, K), and (π, u) corresponds to ϕ up to unitary equivalence as in* Proposition 8.4.6.

Proof. We first prove (1). Compute $\langle u, \pi(k)u\rangle = \phi(k) = 1$ to see $\pi(k)u = u$ for all $k \in K$. Now let $f \in C_c(K\backslash G/K)$. Recall $f * \phi = \lambda_f\,\phi$. Then $\langle \dot\pi(f)u, \pi(g)u\rangle = \int_G f(x)\langle \pi(x)u, \pi(g)u\rangle\, d\mu_G(x) = \int_G f(x)\phi(x^{-1}g)\, d\mu_G(x) = (f * \phi)(g) = \lambda_f\,\phi(g) = \lambda_f\,\langle u, \pi(g)u\rangle$ shows that $\dot\pi(f)u = \lambda_f\, u$.

We now prove that u spans H_π^K. Let $v \in H_\pi^K$ with $v \perp u$. We must show $v = 0$, and for that it suffices to show $v \perp \dot\pi(f)u$ for every $f \in C_c(G)$. Let W denote the kernel of the projection $f \mapsto f^\sharp$ of $C_c(G)$ onto $C_c(K\backslash G/K)$, so $C_c(G) = C_c(K\backslash G/K) \oplus W$. Now $f = f_1 + f_2$ with $f_1 \in C_c(K\backslash G/K)$ and $f_2 \in W$. We just saw that $\dot\pi(f_1)u = \lambda_{f_1} u$, so $\dot\pi(f_1)u \perp v$. Now

$$\begin{aligned}\langle \dot\pi(f)u, v\rangle = \langle \dot\pi(f_2)u, v\rangle &= \int_G f_2(x)\langle \pi(x)u, v\rangle\, d\mu_G(x)\\ &= \int_K\int_K \left(\int_G f_2(x)\langle \pi(x)\pi(k_2)u, \pi(k_1^{-1})v\rangle\, d\mu_G(x)\right) d\mu_K(k_1)\, d\mu_K(k_2)\\ &= \int_G f_2^\sharp(x)\langle \pi(x)u, v\rangle\, d\mu_G(x) = 0,\end{aligned}$$

as required. Thus u spans H_π^K.

Let $H_1 \subset H_\pi$ be a closed $\pi(G)$–invariant subspace. The orthogonal projection $p : H_\pi \to H_1$ is G–equivariant, hence K–equivariant, so it keeps u in H_π^K. Thus $p(u) = \lambda u$ for some number λ. But $\lambda u = p(u) = p(p(u)) = \lambda^2 u$, so λ is 0 or 1. If $\lambda = 0$ then $\pi(G)u \subset H_1^\perp$, so $H_1 = 0$. If $\lambda = 1$ then $\pi(G)u \subset H_1$ so $H_1 = H_\pi$. That proves irreducibility of π. Now the proof of (1) is complete.

Conversely let (π, u) be given and define $\phi(g) = \langle u, \pi(g)u\rangle$. Then ϕ is continuous and positive definite with $\phi(1) = 1$. If $f \in C_c(K\backslash G/K)$ now $\pi(k)\dot\pi(f)u = \int_G f(x)\pi(kx)u\, d\mu_G(x) = (\text{by } x \mapsto k^{-1}x) \int_G f(x)\pi(x)u\, d\mu_G(x) = \dot\pi(f)u$. As u spans H_π^K now $\dot\pi(f)u$ is some multiple $\lambda_f\, u$ of u. More or less as we computed before, that gives $(f * \phi)(g) = \int_G f(x)\phi(x^{-1}g)\, d\mu_G(x) = \int_G f(x)\langle \pi(x)u, \pi(g)u\rangle\, d\mu_G(x) = \langle \dot\pi(f)u, \pi(g)u\rangle = \lambda_f \langle u, \pi(g)u\rangle = \lambda_f\, \phi(g)$. In view of Theorem 8.2.6 now ϕ is a spherical function for (G, K).

At this point the given (π, u) and the pair (π', u') derived from ϕ as in Proposition 8.4.6 are related by $\langle u, \pi(g)u\rangle = \phi(g) = \langle u', \pi'(g)u'\rangle$, so Proposition 8.4.6 provides a unitary $A \in I(\pi, \pi')$ with $A(u) = u'$. That completes the proof of (2), and thus of the theorem. □

8.5. Induced Spherical Functions

Fix a locally compact group G, a compact subgroup $K \subset G$, and a closed subgroup $Q \subset G$ such that K is transitive on G/Q, i.e. $G = KQ$, i.e. $G = QK$, i.e. Q is transitive on G/K. Let $\zeta : Q \to \mathbb{C}$ be spherical for $(Q, Q \cap K)$. We are going to construct spherical functions $\mathrm{Ind}_Q^{G,p}(\zeta) : G \to \mathbb{C}$ for $1 \leqq p \leqq \infty$ in a way that will mirror the construction of induced representations.

The most interesting case of this induced spherical function construction occurs when G is a semisimple Lie group, K is a maximal compact subgroup, Q is a "parabolic subgroup" of G, and the spherical function for $(Q, Q \cap K)$ is a unitary character on Q as described in Example 8.2.2. We will discuss that case more precisely in Section 8.6. It results in the "spherical principal series" of unitary representations of G.

DEFINITION 8.5.1. Fix G, K, Q, ζ and p as above. Then the L^p **induced spherical function** $\mathrm{Ind}_Q^{G,p}(\zeta) : G \to \mathbb{C}$ is given by

$$[\mathrm{Ind}_Q^{G,p}(\zeta)](g) = \int_K \widetilde{\zeta}(gk)\, d\mu_K(k) \text{ where } \widetilde{\zeta}(kq) = \zeta(q)\Delta_{G/Q}(q)^{-1/p} \tag{8.5.2}$$

We write $\mathrm{Ind}_Q^G(\zeta)$ for $\mathrm{Ind}_Q^{G,2}(\zeta)$. Here $\widetilde{\zeta}$ extends ζ from Q to $KQ = G$ and inserts a modular function term whose significance will appear in the context of induced representations. $\widetilde{\zeta}$ is well–defined because $\zeta(k'q) = \zeta(q)$ for $k' \in Q \cap K$ and $q \mapsto \Delta_{G/Q}(q)^{-1/p}$ is a homomorphism whose kernel contains the compact subgroup $Q \cap K$. Note that $\widetilde{\zeta}$ is left K–invariant by construction. The integral forces right K–invariance. ♢

PROPOSITION 8.5.3. $\mathrm{Ind}_Q^{G,p}(\zeta) : G \to \mathbb{C}$ *is a spherical function for* (G, K).

PROOF. Denote $\omega = \mathrm{Ind}_Q^{G,p}(\zeta)$. Then ω is K–bi–invariant by construction and $\omega(1) = \int_K \zeta(1)\, d\mu_K(k) = \int_K d\mu_K(k) = 1$. Further, ω is continuous because ζ is continuous, so $\widetilde{\zeta}$ is continuous, and K is compact.

Let $f \in C_c(K\backslash G/K)$. We use $\Delta_{G/Q}(qk_1q_1) = \Delta_{G/Q}(q)\Delta_{G/Q}(q_1)$ and $g = kq$ to compute

$$\begin{aligned}\int_{Q\cap K} \widetilde{\zeta}(gk_1q_1)\, d\mu_{Q\cap K}(k_1) &= \int_{Q\cap K} \widetilde{\zeta}(qk_1q_1)\, d\mu_{Q\cap K}(k_1)\\ &= \Delta_{G/Q}(q)^{-1/p}\Delta_{G/Q}(q_1)^{-1/p} \int_{Q\cap K} \zeta(qk_1q_1)\, d\mu_{Q\cap K}(k_1)\\ &= \Delta_{G/Q}(q)^{-1/p}\Delta_{G/Q}(q_1)^{-1/p}\zeta(q)\zeta(q_1) = \widetilde{\zeta}(g)\widetilde{\zeta}(q_1).\end{aligned}$$

Since $g = kq$ this says

$$\begin{aligned}(f * \widetilde{\zeta})(g) &= \int_G f(h)\widetilde{\zeta}(h^{-1}kq)\, d\mu_G(h)\\ &= \int_G f(kh)\widetilde{\zeta}(h^{-1}q)\, d\mu_G(h)\\ &= \int_G f(h)\widetilde{\zeta}(h^{-1}q)\, d\mu_G(h)\\ &= \int_G f(h)\left(\int_{Q\cap K} \widetilde{\zeta}(h^{-1}k_1q) d\mu_{Q\cap K}(k_1)\right) d\mu_G(h)\\ &= \int_G f(h)\widetilde{\zeta}(h^{-1})\, d\mu_G(h)\cdot\widetilde{\zeta}(q) = \lambda_f\,\widetilde{\zeta}(q) = \lambda_f\,\widetilde{\zeta}(g)\end{aligned}$$

where $\lambda_f = (f * \widetilde{\zeta})(1)$, and where we note that $\widetilde{\zeta}(q) = \widetilde{\zeta}(g)$. Now

$$\begin{aligned}\lambda_f\,\omega(g) &= \int_K \lambda_f\,\widetilde{\zeta}(gk)\, d\mu_K(k) = \int_K (f * \widetilde{\zeta})(gk)\, d\mu_K(k)\\ &= \int_K\int_G f(g_1)\widetilde{\zeta}(g_1^{-1}gk)\, d\mu_G(g_1)\, d\mu_K(k)\\ &= \int_G f(g_1)\omega(g_1^{-1}g)\, d\mu_G(g_1) = (f * \omega)(g).\end{aligned}$$

Thus, finally, $f * \omega = \lambda_f \,\omega$. Now ω is a spherical function for (G, K) by Theorem 8.2.6. □

We now explain the connection between L^2 induced spherical functions and unitary induction of representations.

THEOREM 8.5.4. *Let η_0 be an irreducible unitary representation of Q with a $(Q \cap K)$–fixed cyclic unit vector v. Let ζ be the positive definite spherical function for $(Q, Q \cap K)$ corresponding to (η_0, v). Let π be the unitarily induced representation $\mathrm{Ind}_Q^G(\eta_0)$ of G and let φ be the L^2 induced spherical function $\mathrm{Ind}_Q^G(\zeta)$ for (G, K). Define $u : G \to E_{\eta_0}$ by $u(kq) = \eta_0(q)^{-1}\Delta_{G/Q}(q)^{-1/2}\, v$. Then u is a K–fixed unit vector in $H_\pi = L^2(G/Q, \eta_0 \otimes \Delta_{G/Q}^{1/2})$ and $\varphi(g) = \langle u, \pi(g)u\rangle$ for all $g \in G$. In particular φ is a positive definite spherical function for (G, K).*

The representation π need not be irreducible. But the cyclic subspace H'_π, the closed span of $\pi(G)u$, defines an irreducible subrepresentation π' of π. The content of the theorem is that φ is the positive definite spherical function for (G, K) associated to (π', u).

PROOF. Since the compact group K is transitive on G/Q, one can characterize the elements of $L^2(G/Q, \eta_0 \otimes \Delta_{G/Q}^{1/2}) = H_\pi$ by

$$f(kq) = \eta_0(q)^{-1}\Delta_{G/Q}(q)^{-1/2} f(k) \text{ and } f|_K \in L^2(K).$$

The inner product is given by $\langle f, f'\rangle = \int_K \langle f(k), f'(k)\rangle_{E_{\eta_0}} \, d\mu_K(k)$. Compute $\int_K \langle u(k), u(k)\rangle_{E_{\eta_0}} \, d\mu_K(k) = \int_K \langle v, v\rangle_{E_{\eta_0}} \, d\mu_K(k) = 1$. Thus u is a well defined unit vector in H_π. It is K–fixed because $[\pi(k)u](k_1q_1) = u(k^{-1}k_1q_1) = u(q_1) = u(k_1q_1)$.

Let ψ be the positive definite spherical function for (G, K) given by $\psi(g) = \langle u, \pi(g)u\rangle$. We must prove $\varphi = \psi$. Writing $qk = k_1q_1$, $H = H_\pi$ and $E = E_{\eta_0}$ we compute

$$\begin{aligned}
\psi(q) = \langle u, \pi(q)u\rangle_H = \langle \pi(q^{-1})u, u\rangle_H &= \int_K \langle u(qk), u(k)\rangle_E \, d\mu_K(k)\\
= \int_K \langle u(k_1q_1), u(k)\rangle_E \, d\mu_K(k) &= \int_K \langle \eta_0(q_1^{-1})\Delta_{G/Q}(q_1)^{-1/2} v, v\rangle_E \, d\mu_K(k)\\
= \int_K \langle v, \eta_0(q_1)v\rangle_E \, \Delta_{G/Q}(q_1)^{-1/2} \, d\mu_K(k) &= \int_K \zeta(q_1)\Delta_{G/Q}(q_1)^{-1/2} \, d\mu_K(k)\\
= \int_K \widetilde{\zeta}(k_1q_1) \, d\mu_K(k) &= \int_K \widetilde{\zeta}(qk) \, d\mu_K(k) = \varphi(q).
\end{aligned}$$

Now $\varphi(g) = \varphi(q) = \psi(q) = \psi(g)$ where $g = kq$. That completes our proof. □

8.6. Example: Spherical Principal Series Representations

In this section we indicate the structure theory of real parabolic subgroups and then show how "spherical principal series" representations fit into the framework of Theorem 8.5.4.

Now G is a connected semisimple Lie group with finite center. In other words it is a finite covering group of a semisimple linear Lie group G'. In Section 6.2 we saw the structure theory for parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ and parabolic subgroups of $G_{\mathbb{C}}$. Now we indicate the corresponding theory for $\mathfrak{g}$ and G.

We define Cartan involution for $\mathfrak{g}$ and G, extending Definition 2.2.1 slightly beyond the case of linear groups. G is a finite covering group of a linear group G'. G' has a Cartan involution θ'. The Lie algebra map $d\theta'$ is the differential of an involutive automorphism $\widetilde{\theta}$ of the universal covering group $\widetilde{G}$ of G and G'. The center of $\widetilde{G}$ is contained in the fixed point set of $\widetilde{\theta}$, so $\widetilde{\theta}$ pushes down to a well defined automorphism θ of G, and θ has order 2 because $d\theta' = d\widetilde{\theta} = d\theta$ has order 2. Note that θ is the unique lift of θ' to an automorphism of G. θ is called a **Cartan involution** of G. Since $G \to G'$ has finite kernel, and the fixed point set $(G')^{\theta'}$ is a maximal compact subgroup of G', we see that the fixed point set $K = G^{\theta}$ is a maximal compact subgroup of G.

Any two Cartan involutions of G are $\mathrm{Int}(\mathfrak{g})$–conjugate in the automorphism group of G. Thus any two maximal compact subgroups of G are conjugate.

Let θ be a Cartan involution of G as just described. Then every G–conjugacy class of Cartan subgroups $H \subset G$ contains a θ–stable Cartan subgroup. If H is a θ–stable Cartan subgroup of G then $H = T \times A$ where $T = H \cap K$ is its compact part and $A = \exp(\mathfrak{a}), \mathfrak{a} = \{\xi \in \mathfrak{h} \mid d\theta(\xi) = -\xi\}$, is the "split" or "vector" part of H. The Cartan subgroup H is called **maximally split** or **maximally noncompact** if $\dim A$ is maximal among θ–stable Cartan subgroups of G. H is called **fundamental** or **maximally compact** if $\dim T$ is maximal among θ–stable Cartan subgroups of G, in other words is T is a Cartan subgroup of K.

Fix a θ–stable maximally split Cartan subgroup $H = T \times A$ in G. It doesn't matter which one, because any two are $\mathrm{Ad}(K)$–conjugate. Let M denote the centralizer $Z_K(A)$, so T is a Cartan subgroup of M. Then $\mathrm{ad}(\xi)$ has all eigenvalues real, for $\xi \in \mathfrak{a}$, so $\mathfrak{g}$ decomposes under the adjoint action of $\mathfrak{a}$ as

$$\mathfrak{g} = (\mathfrak{m} + \mathfrak{a}) + \sum\nolimits_{\gamma \in \Sigma_{\mathfrak{a}}} \mathfrak{g}^{\gamma} \text{ where } \mathfrak{g}^{\gamma} \text{ is the } \gamma\text{–eigenspace and } \mathfrak{m} + \mathfrak{a} \text{ is the 0–eigenspace.} \tag{8.6.1}$$

Here $\Sigma_{\mathfrak{a}}$ consists of nonzero real linear functionals γ on $\mathfrak{a}$ such that $\mathfrak{g}^{\gamma} \neq 0$. $\Sigma_{\mathfrak{a}}$ is called the **restricted root system** of $\mathfrak{g}$ relative to $\mathfrak{h}$. Just as for ordinary root systems we have positive subsystems $\Sigma_{\mathfrak{a}}^{+}$ and simple root systems $\Psi_{\mathfrak{a}}$.

Fix a positive restricted root system $\Sigma_{\mathfrak{a}}^{+}$. Then $\mathfrak{n}_{\mathfrak{a}} = \sum_{\gamma \in \Sigma_{\mathfrak{a}}^{+}} \mathfrak{g}^{-\gamma}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}_{\mathfrak{a}}$ is a parabolic subalgebra of $\mathfrak{g}$ that is minimal among such subalgebras. The $\mathrm{Ad}(G)$–conjugates of $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}_{\mathfrak{a}}$ are called **minimal parabolic subalgebras** of $\mathfrak{g}$. The **Iwasawa decomposition** is the decomposition $G = KAN_{\mathfrak{a}}$ where $N_{\mathfrak{a}}$ is the nilpotent subgroup $\exp(\mathfrak{n}_{\mathfrak{a}})$ with Lie algebra $\mathfrak{n}_{\mathfrak{a}}$. Iwasawa's decomposition theorem says that the map $K \times A \times N_{\mathfrak{a}} \to G$, given by $(k, a, n) \mapsto kan$, is a real analytic diffeomorphism.

Let $\Psi_{\mathfrak{a}}$ denote the simple restricted root system corresponding to the positive restricted root system $\Sigma_{\mathfrak{a}}^{+}$. Any subset $\Phi_{\mathfrak{a}} \subset \Psi_{\mathfrak{a}}$ determines a parabolic subalgebra $\mathfrak{q}_{\Phi_{\mathfrak{a}}}$ as in the complex setting. Let $\langle \Phi_{\mathfrak{a}} \rangle$ denote the set of restricted roots that are linear combinations of the roots of $\Phi_{\mathfrak{a}}$. Then $\mathfrak{q}_{\Phi_{\mathfrak{a}}} = \mathfrak{q}^{n}_{\Phi_{\mathfrak{a}}} + \mathfrak{q}^{r}_{\Phi_{\mathfrak{a}}}$ where $\mathfrak{q}^{n}_{\Phi_{\mathfrak{a}}} = \Sigma_{\gamma \in (\Sigma_{\mathfrak{a}}^{+} \setminus \langle \Phi_{\mathfrak{a}} \rangle)} \mathfrak{g}^{-\gamma}$ is the nilpotent radical and $\mathfrak{q}^{r}_{\Phi_{\mathfrak{a}}} = \mathfrak{m} + \mathfrak{a} + \Sigma_{\gamma \in \langle \Phi_{\mathfrak{a}} \rangle} \mathfrak{g}^{\gamma}$ is a reductive algebra. Write $Q_{\Phi_{\mathfrak{a}}}$ for the parabolic subgroup of G with Lie algebra $\mathfrak{q}_{\Phi_{\mathfrak{a}}}$. Then we have decompositions

(8.6.2) $$Q_{\Phi_\mathfrak{a}} = Q^n_{\Phi_\mathfrak{a}} \rtimes Q^r_{\Phi_\mathfrak{a}} \text{ and } Q^r_{\Phi_\mathfrak{a}} = M_{\Phi_\mathfrak{a}} \times A_{\Phi_\mathfrak{a}}$$

where $Q^n_{\Phi_\mathfrak{a}}$ is the unipotent radical of $Q_{\Phi_\mathfrak{a}}$, $Q^r_{\Phi_\mathfrak{a}}$ is reductive, $A_{\Phi_\mathfrak{a}} = Q_{\Phi_\mathfrak{a}} \cap A$, and $M_{\Phi_\mathfrak{a}}$ is θ–stable. Every parabolic subgroup of $Q \subset G$ is conjugate to just one of the groups $Q_{\Phi_\mathfrak{a}}$. In particular some conjugate Q' of Q contains $AN_\mathfrak{a}$, and it follows from the Iwasawa decomposition that

(8.6.3) if Q is a parabolic subgroup of G then $G = KQ$,

so every parabolic subgroup $Q \subset G$ is transitive on G/K.

If $\Phi_\mathfrak{a}$ is empty then $\mathfrak{q}_{\Phi_\mathfrak{a}}$ is a minimal parabolic subalgebra of $\mathfrak{g}$ and $Q_{\Phi_\mathfrak{a}}$ is a minimal parabolic subgroup of G. If $\Phi_\mathfrak{a} = \Psi_\mathfrak{a}$ then $\mathfrak{q}_{\Phi_\mathfrak{a}} = \mathfrak{g}$ and $Q_{\Phi_\mathfrak{a}} = G$. The number of G–conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ is $2^{|\Psi_\mathfrak{a}|}$, given by the $\mathfrak{q}_{\Phi_\mathfrak{a}}$ as $\Phi_\mathfrak{a}$ ranges over the subsets of $\Psi_\mathfrak{a}$.

Positive root systems $\Sigma^+_\mathfrak{m}$ for $\mathfrak{m}_\mathbb{C}$ relative to $\mathfrak{t}_\mathbb{C}$ and $\Sigma^+_\mathfrak{a}$ for $\mathfrak{g}$ relative to $\mathfrak{a}$ define a positive root system $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha|_\mathfrak{a} \in \Sigma^+_\mathfrak{a} \text{ or } \alpha|_\mathfrak{a} = 0 \text{ and } \alpha|_{\mathfrak{t}_\mathbb{C}} \in \Sigma^+_\mathfrak{m}\}$ for $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{h}_\mathbb{C}$. Let Ψ and $\Psi_\mathfrak{a}$ denote the corresponding simple root systems for Σ and $\Sigma_\mathfrak{a}$. Then every $\gamma \in \Psi_\mathfrak{a}$ is the restriction of some $\alpha \in \Psi$. Suppose $\Phi_\mathfrak{a} \subset \Psi_\mathfrak{a}$ and define $\Phi = \{\alpha \in \Psi \mid \alpha|_\mathfrak{a} \in \Phi_\mathfrak{a}\}$. Then $\mathfrak{q}_{\Phi_\mathfrak{a}} = \mathfrak{q}_\Phi \cap \mathfrak{g}$, and if $G \in G_\mathbb{C}$ then $Q_{\Phi_\mathfrak{a}} = Q_\Phi \cap G$.

Theorem 8.6.4. *Let Q be a parabolic subgroup of G, say $Q = M_{\Phi_\mathfrak{a}} A_{\Phi_\mathfrak{a}} Q^n_{\Phi_\mathfrak{a}}$ as in* (8.6.2). *Then the spherical functions for $(Q, Q \cap K)$ are just the*

$\zeta(man) = \zeta_M(m)\zeta_A(a)$ where

ζ_M is spherical for $(M_{\Phi_\mathfrak{a}}, M_{\Phi_\mathfrak{a}} \cap K)$ and ζ_A is a quasi–character on $A_{\Phi_\mathfrak{a}}$.

Here ζ is positive definite if and only if ζ_M is positive definite and ζ_A is unitary.

Proof. Let $\zeta(man) = \zeta_M(m)\zeta_A(a)$ as above. In particular ζ_M is spherical for $(M_{\Phi_\mathfrak{a}}, M_{\Phi_\mathfrak{a}} \cap K)$ and ζ_A is a quasi–character on $A_{\Phi_\mathfrak{a}}$. We check that ζ is spherical for $(Q, Q \cap K)$. It is continuous and not identically zero, so we need only check the functional equation. Note that $Q \cap K = M_{\Phi_\mathfrak{a}} \cap K$. Let $q_i = m_i a_i n_i \in Q$ as in (8.6.2). If $k \in M_{\Phi_\mathfrak{a}} \cap K$ then, since $Q^n_{\Phi_\mathfrak{a}}$ is normal in Q, $n_1 k = k n_3$ and $n_3 m_2 a_2 = m_2 a_2 n_4$ for some $n_3, n_4 \in Q^n_{\Phi_\mathfrak{a}}$. Now compute $\zeta(q_1 k q_2) = \zeta(m_1 a_1 n_1 k m_2 a_2 n_2) = \zeta(m_1 a_1 k n_3 m_2 a_2 n_2) = \zeta(m_1 a_1 k m_2 a_2 n_4 n_2) = \zeta(m_1 k m_2 a_1 a_2)$, so

$$\begin{aligned}\int_{Q\cap K} \zeta(q_1 k q_2) d\mu_{(M_{\Phi_\mathfrak{a}} \cap K)}(k) &= \int_{M_{\Phi_\mathfrak{a}} \cap K} \zeta(m_1 k m_2 a_1 a_2) d\mu_{(M_{\Phi_\mathfrak{a}} \cap K)}(k) \\ &= \left(\int_{M_{\Phi_\mathfrak{a}} \cap K} \zeta_M(m_1 k m_2) d\mu_{(M_{\Phi_\mathfrak{a}} \cap K)}(k) \right) \zeta_A(a_1 a_2) \\ &= \zeta_M(m_1)\zeta_M(m_2)\zeta_A(a_1)\zeta_A(a_2) = \zeta(q_1)\zeta(q_2)\end{aligned}$$

as required.

Conversely, let ζ be spherical for $(Q, Q \cap K)$. Define $\zeta_M : M_{\Phi_\mathfrak{a}} \to \mathbb{C}$ and $\zeta_A : A_{\Phi_\mathfrak{a}} \to \mathbb{C}$ as the restrictions. They are continuous and not identically zero. ζ_M is spherical for $(M_{\Phi_\mathfrak{a}}, M_{\Phi_\mathfrak{a}} \cap K)$ because $Q \cap K = M_{\Phi_\mathfrak{a}} \cap K$, and ζ_A is spherical for $A_{\Phi_{\mathfrak{a}}}$ because $M_{\Phi_{\mathfrak{a}}} \cap K$ centralizes $A_{\Phi_{\mathfrak{a}}}$. In particular ζ_A is a quasi–character on $A_{\Phi_{\mathfrak{a}}}$. Now let $a \in A_{\Phi_{\mathfrak{a}}}$ and $n \in Q^n_{\Phi_{\mathfrak{a}}}$, and compute

$$\begin{aligned}
\zeta(n) = \zeta(a)\zeta(n)\zeta(a^{-1}) &= \left(\int_{M_{\Phi_{\mathfrak{a}}} \cap K} \zeta(a\,k\,n)d\mu_{(M_{\Phi_{\mathfrak{a}}} \cap K)}(k)\right)\zeta(a^{-1}) \\
&= \left(\int_{M_{\Phi_{\mathfrak{a}}} \cap K} \zeta(kan)d\mu_{(M_{\Phi_{\mathfrak{a}}} \cap K)}(k)\right)\zeta(a^{-1}) \\
&= \left(\int_{M_{\Phi_{\mathfrak{a}}} \cap K} \zeta(an)d\mu_{(M_{\Phi_{\mathfrak{a}}} \cap K)}(k)\right)\zeta(a^{-1}) \\
&= \zeta(an)\zeta(a^{-1}) = \int_{M_{\Phi_{\mathfrak{a}}} \cap K} \zeta(an\,k\,a^{-1})d\mu_{(M_{\Phi_{\mathfrak{a}}} \cap K)}(k) \\
&= \int_{M_{\Phi_{\mathfrak{a}}} \cap K} \zeta(ana^{-1}k)d\mu_{(M_{\Phi_{\mathfrak{a}}} \cap K)}(k) = \zeta(ana^{-1}).
\end{aligned}$$

Thus ζ is constant on every $\mathrm{Ad}(A_{\Phi_{\mathfrak{a}}})$–orbit in $Q^n_{\Phi_{\mathfrak{a}}}$. Let $\beta = \zeta|_{Q^n_{\Phi_{\mathfrak{a}}}}$ and write $\widetilde{\beta}$ for its lift to the Lie algebra $\mathfrak{q}^n_{\Phi_{\mathfrak{a}}}$. Now $\widetilde{\beta}$ is constant on every $\mathrm{Ad}(A_{\Phi_{\mathfrak{a}}})$–orbit in $\mathfrak{q}^n_{\Phi_{\mathfrak{a}}}$. But $\mathrm{Ad}(A_{\Phi_{\mathfrak{a}}})$ is unbounded on each of its joint eigenspaces (restricted root spaces) in $\mathfrak{q}^n_{\Phi_{\mathfrak{a}}}$. As $\widetilde{\beta}$ is real analytic, all the components of its power series expansion, except for the constant term, must vanish. We conclude that ζ is constant on $Q^n_{\Phi_{\mathfrak{a}}}$. We have proved that $\zeta(man) = \zeta_M(m)\zeta_A(a)$ for $m \in M_{\Phi_{\mathfrak{a}}}$, $a \in A_{\Phi_{\mathfrak{a}}}$ and $n \in Q^n_{\Phi_{\mathfrak{a}}}$, with ζ_M spherical for $(M_{\Phi_{\mathfrak{a}}}, M_{\Phi_{\mathfrak{a}}} \cap K)$ and ζ_A a quasi–character on $A_{\Phi_{\mathfrak{a}}}$.

This completes the proof of the first statement of Theorem 8.6.4.

Now suppose that $\zeta(man) = \zeta_M(m)\zeta_A(a)$ as above, where ζ_M is a positive definite spherical function for $(M_{\Phi_{\mathfrak{a}}}, M_{\Phi_{\mathfrak{a}}} \cap K)$ and ζ_A is a unitary character on $A_{\Phi_{\mathfrak{a}}}$. We have proved that ζ is spherical for $(Q, Q \cap K)$. To prove that it is positive definite let $q_i = m_i a_i n_i$ and $c_i \in \mathbb{C}$, $1 \leqq i \leqq r$, for some $r \geqq 1$. Then $q_j^{-1}q_i = n_j^{-1}a_j^{-1}m_j^{-1}m_i a_i n_i = m_j^{-1}m_i\ a_j^{-1}a_i\ n_{i,j}$ for some $n_{i,j} \in Q^n_{\Phi_{\mathfrak{a}}}$, because $Q^n_{\Phi_{\mathfrak{a}}}$ is normal in Q and $M_{\Phi_{\mathfrak{a}}}$ centralizes $A_{\Phi_{\mathfrak{a}}}$. Thus $\zeta(q_j^{-1}q_i) = \zeta_M(m_j^{-1}m_i)\zeta_A(a_j^{-1}a_i)$. Now

$$\sum \zeta(q_j^{-1}q_i)\overline{c_j}c_i = \sum \zeta_M(m_j^{-1}m_i)\overline{c_j'}c_i' \geqq 0$$

where $c_i' = \zeta_A(a_i)c_i$, so ζ is positive definite.

Conversely suppose that ζ is a positive definite spherical function. It is immediate from the definition that the restriction of a positive definite function to a subgroup is positive definite. Thus the spherical functions $\zeta_M = \zeta|_{M_{\Phi_{\mathfrak{a}}}}$ and $\zeta_A = \zeta|_{A_{\Phi_{\mathfrak{a}}}}$ are positive definite. We know from Example 8.4.5 that, consequently, ζ_A is a unitary character on $A_{\Phi_{\mathfrak{a}}}$. That completes the proof of the second statement of Theorem 8.6.4. □

COROLLARY 8.6.5. *Fix a minimal parabolic subgroup $Q = MAN$ of G. Then the spherical functions for (Q, M) are just the quasi–characters ζ_ν, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, given by $\zeta_\nu(m\ \exp(\xi)\ n) = e^{i\nu(\xi)}$ on Q.* (Here $m \in M$, $\xi \in \mathfrak{a}$, and $n \in N$.) *The spherical function ζ_ν is positive definite if and only if ν belongs to the real dual space $\mathfrak{a}^*$.*

Definition 8.6.6. Fix a minimal parabolic subgroup $Q = MAN$ of G. Let η be an irreducible finite dimensional unitary representation of Q. Thus $\eta(N) = 1$. If $m \in M, a \in A$ and $n \in N$ then $\eta(man) = \tau(m)\exp(\sqrt{-1}\,\nu(\xi))$ for some $[\tau] \in \widehat{M}$ and some $\nu \in \mathfrak{a}^*$ where $\xi \in \mathfrak{a}$ with $\exp(\xi) = a$. The corresponding **principal series representation** is the unitarily induced representation

$$\pi_\eta = \pi_{\tau,\nu} = \mathrm{Ind}_Q^G(\eta) \tag{8.6.7}$$

of G. If $\tau = 1$ then π_η is called a **spherical principal series representation** of G. The **principal series** of G is the set of unitary equivalence classes of principal series representations. The **spherical principal series** of G is the set of unitary equivalence classes of spherical principal series representations. ♢

Corollary 8.6.5 combines with Theorem 8.5.4 to give us

Theorem 8.6.8. *Fix a minimal parabolic subgroup $Q = MAN$ of G. Then the spherical principal series of G consists of the unitary equivalence classes of representations defined by the induced spherical functions* $\mathrm{Ind}_Q^G(\zeta_\nu)$*, $\nu \in \mathfrak{a}^*$, as in* Theorem 8.5.4.

In Section 11.5 we will see a more explicit version of Theorem 8.6.8, based on some delicate analysis of Harish–Chandra.

The p–adic analog of Theorem 8.6.8, or more precisely of the spherical function formulae in Section 11.5, is rather technical and very arithmetic; see Satake [**Sa1**].

8.7. Example: Double Transitivity and Homogeneous Trees

We now look at some Gelfand pairs that (at least at first glance) have nothing to do with the theory of Lie groups. However, see [**Cart1**] for a complete description, [**Cart2**] and [**Cart3**] for introductions, to the connection with the representation theory of p–adic groups. Part of our treatment is based on Faraut [**Fa**, pp. 343–344 and 363–364], part on [**Cart3**] and [**Ser**].

8.7A. Doubly Transitive Groups. Let (V, dist) be a metric space, and let G be a group of isometries that is locally compact in the compact–open topology. One says that the action of G on V is **doubly transitive** provided that

$$\begin{aligned}&\text{if } x, y, x', y' \in V \text{ with } \mathrm{dist}(x, y) = \mathrm{dist}(x', y')\\ &\text{then } x' = g(x) \text{ and } y' = g(y) \text{ for some } g \in G.\end{aligned}$$

Examples include (i) the spheres, (ii) the real, complex and quaternionic projective spaces, (iii) the real, complex and quaternionic hyperbolic spaces, and (iv) the homogeneous trees to be discussed in a moment.

Proposition 8.7.1. *Let G be a locally compact group and K a compact subgroup such that $V = G/K$ is a metric space with a G–invariant distance function* $\mathrm{dist}(x, y)$*. If G is doubly transitive on* (V, dist) *then (G, K) is a Gelfand pair.*

Proof. Let v_0 be the base point $1K$ in $V = G/K$. If $g \in G$ then $\mathrm{dist}(v_0, gv_0) = \mathrm{dist}(g^{-1}v_0, v_0)$, so there is an element $g' \in G$ such that $g'v_0 = v_0$ and $g'gv_0 = g^{-1}v_0$.

Here $g' \in K$ from $g'v_0 = v_0$, so $g \in g'^{-1}g^{-1}K \subset Kg^{-1}K$. Now Proposition 8.1.3, with $\theta = 1$, says that (G, K) is a Gelfand pair. □

Let G be doubly transitive on $V = G/K$, as in Proposition 8.7.1. A K–bi–invariant function f on G can be viewed as a K–invariant function on M, and the double transitivity shows that the value $f(v)$ depends only on the distance $\text{dist}(v_0, v)$. Define a real valued function F_f by

$$F_f(\text{dist}(v_0, v)) = f(v), \text{ i.e., } F_f(r) = f(v) \text{ where } r = \text{dist}(v_0, v). \tag{8.7.2}$$

If f_1 and f_2 are K–bi–invariant functions on G, viewed as K–invariant functions on V, then one computes

$$F_{f_1 * f_2}(\text{dist}(x, y)) = \int_M F_{f_1}(\text{dist}(x, v))F_{f_2}(\text{dist}(v, y))dv \tag{8.7.3}$$

where dv is the G–invariant measure on V derived from Haar measure on G.

8.7B. Homogeneous Trees. A **tree** is a connected graph that does not contain a cycle. Fix a tree T, let $V(T)$ denote the set of all its vertices, and let $E(T)$ denote the set of all its edges. Then if $v, v' \in V(t)$ there is a unique sequence $\{v_0, \dots, v_n\}$ of distinct vertices such that $v_0 = v$, each $e_i = \overline{v_{i-1}, v_i}$ is an edge, and $v_n = v'$, with $n \geqq 0$. Thus $V(T)$ is a metric space with distance function $\text{dist}(v, v') = n$. It's topology is discrete, thus locally compact. We now impose the condition that each vertex meets only finitely many edges. Then the compact subsets of $V(T)$ are finite.

The distance function on $V(T)$ is invariant under the group $G = \text{Aut}(T)$ of graph automorphisms of T. In fact, a moments thought shows that $\text{Aut}(T)$ is the isometry group of $(V(T), \text{dist})$.

The **index** of a vertex is the number of edges that meet it. The tree T is **homogeneous of index** $m \geqq 1$ if each of its vertices is of index $m + 1$. Suppose that T is homogeneous of index $m + 1$ and let $G = \text{Aut}(T)$. It is known (check directly or see [**Du**]) that G is transitive on $V(T)$, and that the G–stabilizer of a vertex $v \in V(T)$ is transitive on the set of m edges that meet v. From this, it is straightforward to check that G is doubly transitive on $V(T)$.

Let T be a homogeneous tree and $G = \text{Aut}(T)$. We want to apply a result [**D-W**, Satz I] of Van Dantzig and van der Waerden to see that G is locally compact in the compact–open topology and the G–stabilizer (say K) of a vertex $v_0 \in V(T)$ is compact. That theorem, however, applies only to isometry groups of connected locally compact separable metric spaces, and $V(T)$ is discrete rather than connected. One can try to extend the metric to T be extending it linearly along edges, but if $E(T)$ is infinite one loses track of invariance. So instead we use the metric dist of $V(T)$ to define a distance function δ on G by

$$\delta(g, g') = \sum_{v \in V(T)} \text{dist}(gv, g'v)(m + 1)^{-2\,\text{dist}(v_0, v)}.$$

Then (G, δ) is a metric space. Its topology is locally compact, and K is a compact open subgroup of G. In view of the double transitivity, Proposition 8.7.1 says that the (G, K) is a Gelfand pair.

As generally in the doubly transitive setting, we view a K–bi–invariant function f on G can be viewed as a K–invariant function on $V(T)$ and the double transitivity shows that the value $f(v)$ depends only on $\mathrm{dist}(v_0, v)$. As in (8.7.2) we define $F_f(r) = f(v)$ where $r = \mathrm{dist}(v_0, v)$. If $f \in L^1(K\backslash G/K)$ we compute

$$\int_G f(x) d\mu_G(x) = F(0) + \sum_{r=1}^{\infty} (m+1)m^{r-1} F(r).$$

For each n define a K–bi–invariant function f_n on G by

(8.7.4)
$$\begin{aligned} &\text{if } n \neq 0: \ F_{f_n}(r) = 0 \text{ for } r \neq n, \ \ 1/(m+1)m^{n-1} \text{ for } r = n \\ &\text{if } n = 0: \ F_{f_0}(r) = 0 \text{ for } r \neq n, \ \ 1 \text{ for } r = n \end{aligned}$$

Then one computes

$$f_1 * f_n = \tfrac{1}{m+1} f_{n-1} + \tfrac{m}{m+1} f_{n+1}.$$

Thus, if χ is a multiplicative linear functional on $C_c(K\backslash G/K)$ then

$$\chi(f_1)\chi(f_n) = \tfrac{1}{m+1}\chi(f_{n-1}) + \tfrac{m}{m+1}\chi(f_{n+1}),$$

so the sequence $\{\chi(f_n)\}$ is determined by the number $z = \chi(f_1)$ and the fact $\chi(f_0) = 1$. Specifically, $\chi(f_n) = p_n(z)$ where $z = \chi(f_1)$ and $\{p_n\}$ is the sequence of polynomials recursively defined by

(8.7.5)
$$p_0(z) = 1, \ p_1(z) = z, \ p_{n+1}(z) = \tfrac{m+1}{m} z p_n(z) - \tfrac{1}{m} p_{n-1}(z).$$

Proposition 8.7.6. *The polynomials p_n, defined recursively in* (8.7.5) *determine the (G, K)–spherical functions and certain of their properties, as follows.*

1. *For every complex number z, define $\phi_z : G \to \mathbb{C}$ by $\phi_z(g) = p_r(z)$ where $r = \mathrm{dist}(v_0, gv_0)$. Then ϕ_z is (G, K)–spherical, and every (G, K)–spherical function is one of the ϕ_z.*

2. *If $m = 1$, so $T \cong \mathbb{Z}$ and G is generated by integer translations and the reflection $v \mapsto -v$, then the p_n are the Chebyschev polynomials, $p_n(\cos(\theta)) = \cos(n\theta)$.*

3. *Let $z = x + iy$ as usual. Suppose $m > 1$. Then the spherical function ϕ_z is bounded if and only if z is contained in the ellipse $x^2 + (\frac{m+1}{m-1})^2 y^2 \leqq 1$, and ϕ_z is is positive definite if and only if $y = 0$ and $-1 \leqq x \leqq +1$.*

See Arnaud [**Ar**] for Part 1 of Proposition 8.7.6. Part 2 follows by a glance at the defining recursion relations for Chebyschev polynomials. For Part 3 see Dunau [**Du**].

8.7C. A Special Case. Let $\mathbb{K}$ be a field that is complete with respect to a discrete valuation ν, for example one of the p–adic number fields $\mathbb{Q}_p$. Let $\mathfrak{O}$ denote the ring of integers in $\mathbb{K}$, in other words $\mathfrak{O} = \{x \in \mathbb{K} \mid \nu(x) \leqq 1\}$. Also, let $\mathfrak{p} = \langle \pi \rangle$ denote the nonzero prime ideal in $\mathfrak{O}$; so $\mathfrak{p} = \{x \in \mathbb{K} \mid \nu(x) < 1\}$. Assume that the residue field $\mathfrak{O}/\mathfrak{p}$ is finite, say with q elements. If $\mathbb{K} = \mathbb{Q}_p$ then $q = p$.

The general linear group $GL(2; \mathbb{K})$ and the projective general linear group $PGL(2; \mathbb{K}) = GL(2; \mathbb{K})/(center)$ are locally compact because $\mathbb{K}$ is locally compact, and the subgroups $GL(2; \mathfrak{O})$ and $PGL(2; \mathfrak{O})$ are compact because $\mathfrak{O}$ is compact.

A lattice in $\mathbb{K}^2$ is, by definition, an $\mathfrak{O}$–submodule generated by two linearly independent elements, in other words generated by the columns of some element of $GL(2;\mathbb{K})$. We say that lattices Λ and Λ' are equivalent if $\Lambda' = c\Lambda$ for some $c \in \mathbb{K}$. In effect these equivalence classes correspond to the "columns" of the elements of $PGL(2;\mathbb{K})$. The set $V(T)$ of vertices will be the set of equivalence classes of lattices in $\mathbb{K}^2$, and $PGL(2;\mathbb{K})$ acts transitively on $V(T)$. Vertices $v_1, v_2 \in V(T)$ determine an edge if they have representative lattices Λ_1 and Λ_2 such that one of them is a sublattice of index q in the other, $\Lambda_1 \subset \Lambda_2$ with $\Lambda_2/\Lambda_1 = q$ or $\Lambda_2 \subset \Lambda_1$ with $\Lambda_1/\Lambda_2 = q$. Evidently, $PGL(2;\mathbb{K})$ acts transitively on the set $E(T)$ of edges. We have constructed the tree T and have seen that it is homogeneous under the action of the group $PGL(2;\mathbb{K})$.

Let v_0 denote the vertex represented by the lattice $\Lambda_0 = \mathfrak{O}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \mathfrak{O}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. If v is another vertex then $\{v_0, v\}$ determines an edge if and only if v is represented either by $\mathfrak{O}\left(\begin{smallmatrix}\pi\\0\end{smallmatrix}\right) + \mathfrak{O}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ or by $\mathfrak{O}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \mathfrak{O}\left(\begin{smallmatrix}x\\\pi\end{smallmatrix}\right)$ with $x \in \mathfrak{O}$ modulo $\mathfrak{p}$. Thus the number of edges from each vertex is $q+1$. Finally, $PGL(2;\mathfrak{O})$ is the $PGL(2;\mathbb{K})$–stabilizer of v_0, so we have $V(T) = PGL(2;\mathbb{K})/PGL(2;\mathfrak{O})$.