Preface

The theory of classical matrix Lie algebras can be viewed from at least two related but different perspectives. On the one hand, the special linear, orthogonal and symplectic Lie algebras form four infinite series, A_n, B_n, C_n, D_n, which together with five exceptional Lie algebras, E_6, E_7, E_8, F_4, G_2, comprise a complete list of the simple Lie algebras over the field of complex numbers. The structure of these Lie algebras is uniformly described in terms of certain finite sets of vectors in a Euclidean space called the root systems. The symmetries of the root systems play a key role in the representation theory of all simple Lie algebras providing the dimension and character formulas for the representations. On the other hand, the matrix realizations of the classical Lie algebras allow some specific tools to be used for their study which are not always available for the exceptional Lie algebras. The theory of *Yangians* and *twisted Yangians* which we develop in this book is one of such tools bringing in new symmetries and shedding new light on this classical subject.

The Yangians and twisted Yangians are associative algebras whose defining relations are written in a specific matrix form. We describe the structure of these algebras and classify their finite-dimensional irreducible representations. The results exhibit many analogies with the representation theory of the classical Lie algebras themselves, including the triangular decompositions of the (twisted) Yangians and the parametrization of the representations by their highest weights. In the simplest cases explicit constructions of the irreducible representations are also given. Then we apply the Yangian symmetries to the classical Lie algebras. The applications include constructions of several families of Casimir elements, derivations of the characteristic identities and Capelli identities, and explicit constructions of all finite-dimensional irreducible representations of the classical Lie algebras via weight bases of Gelfand–Tsetlin type.

Let us discuss the relationship between the classical Lie algebras and the (twisted) Yangians in more detail. The term *Yangian* was introduced by V. G. Drinfeld (in honor of C. N. Yang) in his fundamental paper (1985). In that paper, Drinfeld also defined the *quantized Kac–Moody algebras*, which together with the work of M. Jimbo (1985), who introduced these algebras independently, marked the beginning of the era of *quantum groups*. The Yangians form a remarkable family of quantum groups related to rational solutions of the classical Yang–Baxter equation. For each simple finite-dimensional Lie algebra $\mathfrak{a}$ over the field $\mathbb{C}$ of complex numbers, the corresponding Yangian is defined as a canonical deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[z])$ for the polynomial current Lie algebra $\mathfrak{a}[z]$. Importantly, the deformation is considered in the class of Hopf algebras, which guarantees its uniqueness under some natural homogeneity conditions. Another presentation of the Yangian for $\mathfrak{a}$ was given later by Drinfeld (1988).

A few years earlier, the algebra, which is now called the *Yangian for the general linear Lie algebra* $\mathfrak{gl}_N$ and denoted by $\mathrm{Y}(\mathfrak{gl}_N)$, was considered in the work of L. D. Faddeev and the St. Petersburg (Leningrad) school. The defining relations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ can be written in the form of a single *ternary* (or *RTT*) relation on the matrix of generators. This relation has a rich and extensive background. It originates from the *quantum inverse scattering method*; see for instance L. A. Takhtajan and Faddeev (1979), P. P. Kulish and E. K. Sklyanin (1982), and Faddeev (1984). The Yangians were primarily regarded as a vehicle for producing rational solutions of the Yang–Baxter equation; cf. Drinfeld (1985). Conversely, the ternary relation is a powerful tool for studying quantum groups themselves; see e.g. N. Yu. Reshetikhin, Takhtajan and Faddeev (1990). The Hopf algebra structure of $\mathrm{Y}(\mathfrak{gl}_N)$ can also be conveniently described in a matrix form.

From the algebraic point of view, the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ and the closely related Yangian $\mathrm{Y}(\mathfrak{sl}_N)$ for the special linear Lie algebra $\mathfrak{sl}_N$ are exceptional in the following sense. For any simple Lie algebra $\mathfrak{a}$, the corresponding Yangian contains the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ as a subalgebra. However, only in the case $\mathfrak{a} = \mathfrak{sl}_N$ does there exist a homomorphism from the Yangian to $\mathrm{U}(\mathfrak{a})$ (the *evaluation homomorphism*) which is identical on the subalgebra $\mathrm{U}(\mathfrak{a})$ (Drinfeld, 1985). This property plays a key role in the applications of the Yangians to the conventional representation theory. In this book we concentrate on these distinguished algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}(\mathfrak{sl}_N)$.

We will use the symbol $\mathfrak{g}_N$ to denote either the orthogonal Lie algebra $\mathfrak{o}_N$ or symplectic Lie algebra $\mathfrak{sp}_N$, assuming $N = 2n$ even for the latter. For each of these Lie algebras G. Olshanski (1992) introduced another algebra which he called the *twisted Yangian*. We will denote it by $\mathrm{Y}(\mathfrak{g}_N)$. When $\mathfrak{a} = \mathfrak{g}_N$, the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ should not be confused with the Yangian for $\mathfrak{g}_N$ defined by Drinfeld. The latter Yangian will not be considered in the main exposition of the present book; see, however, Examples 2.16.2 and 4.6.1.

The classical Lie algebra $\mathfrak{g}_N$ can be regarded as a fixed point subalgebra of an appropriate involution σ of the Lie algebra $\mathfrak{gl}_N$. Then the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ can be defined as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$. The algebra $\mathrm{Y}(\mathfrak{g}_N)$ is a deformation of the universal enveloping algebra for the twisted polynomial current Lie algebra

$$\mathfrak{gl}_N[z]^\sigma = \{\, A(z) \in \mathfrak{gl}_N[z] \mid \sigma\big(A(z)\big) = A(-z) \,\}.$$

This is *not* a Hopf algebra deformation. However, the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ as a subalgebra, and there exists a homomorphism $\mathrm{Y}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$ identical on the subalgebra $\mathrm{U}(\mathfrak{g}_N)$. It is called the *evaluation homomorphism* by analogy with the $\mathfrak{gl}_N$ case. Moreover, the twisted Yangian turns out to be a (left) coideal of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

Similar to the Yangian for $\mathfrak{gl}_N$, the twisted Yangians can be equivalently presented by generators and defining relations which can be written as a *quaternary* (or *reflection*) equation for the matrix of generators, together with a *symmetry* relation. Relations of this type appeared for the first time in the papers by I. V. Cherednik (1984) and Sklyanin (1988), where integrable systems with boundary conditions were studied.

This matrix form of the defining relations for the Yangian and the twisted Yangians allows special algebraic techniques (the so-called *R-matrix formalism*) to be used to describe the structure and to study representations of these algebras. On the other hand, the defining relations can also be observed inside the enveloping algebras. To be more precise, consider the general linear Lie algebra $\mathfrak{gl}_N$ with its standard basis E_{ij}, $i,j=1,\dots,N$. The commutation relations are given by

$$[E_{ij},E_{kl}]=\delta_{kj}E_{il}-\delta_{il}E_{kj}, \tag{0.1}$$

where δ_{ij} is the Kronecker delta. Introduce the $N\times N$ matrix E whose ij-th entry is E_{ij}. The matrix elements of the powers of the matrix E are known to satisfy the relations

$$[E_{ij},(E^s)_{kl}]=\delta_{kj}(E^s)_{il}-\delta_{il}(E^s)_{kj}. \tag{0.2}$$

These, in particular, imply that the traces of powers of E are central elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ known as the *Gelfand invariants*. The following generalization of (0.2) which can be verified by induction, appears to be less known:

$$[(E^{r+1})_{ij},(E^s)_{kl}]-[(E^r)_{ij},(E^{s+1})_{kl}]=(E^r)_{kj}(E^s)_{il}-(E^s)_{kj}(E^r)_{il}, \tag{0.3}$$

where $r,s\geqslant 0$ and $E^0=1$ is the identity matrix. The definition of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ can be motivated by these relations: replacing $(E^r)_{ij}$ by an abstract generator $t_{ij}^{(r)}$ we obtain the Yangian defining relations; see (1.1) below. Introducing the generating series

$$e_{ij}(u)=\delta_{ij}+\sum_{r=1}^{\infty}(E^r)_{ij}u^{-r},$$

where u is a formal (complex) parameter, we can rewrite (0.3) in the form

$$(u-v)\,[e_{ij}(u),e_{kl}(v)]=e_{kj}(u)\,e_{il}(v)-e_{kj}(v)\,e_{il}(u) \tag{0.4}$$

which is equivalent to the *RTT* relation; see (1.19) below.

Alternatively, the generators of the Yangian can be realized as the *Capelli minors*. Keeping the notation E for the matrix of the basis elements of $\mathfrak{gl}_N$, we introduce the *Capelli determinant*

$$\det(1+Eu^{-1})=\sum_{p\in\mathfrak{S}_N}\operatorname{sgn}p\cdot(1+Eu^{-1})_{p(1),1}\cdots(1+E(u-N+1)^{-1})_{p(N),N}. \tag{0.5}$$

When multiplied by $u(u-1)\cdots(u-N+1)$ this determinant becomes a polynomial in u whose coefficients (with the exception of the leading coefficient 1) constitute a family of algebraically independent generators of the center of $\mathrm{U}(\mathfrak{gl}_N)$. The value of this polynomial at $u=N-1$ is the distinguished central element whose image in a natural representation of $\mathfrak{gl}_N$ by differential operators is given by the celebrated *Capelli identity*. For a positive integer $M\leqslant N$ introduce the subsets of indices $\mathcal{B}_i=\{i,M+1,M+2,\dots,N\}$ and for any $1\leqslant i,j\leqslant M$ consider the Capelli minor

$$\det(1+Eu^{-1})_{\mathcal{B}_i\mathcal{B}_j}=\delta_{ij}+c_{ij}^{(1)}u^{-1}+c_{ij}^{(2)}u^{-2}+\cdots$$

defined as in (0.5), whose rows and columns are respectively enumerated by $\mathcal{B}_i$ and $\mathcal{B}_j$. These minors turn out to satisfy the Yangian defining relations; i.e., there is an algebra homomorphism

$$\mathrm{Y}(\mathfrak{gl}_M)\to\mathrm{U}(\mathfrak{gl}_N),\qquad t_{ij}^{(r)}\mapsto c_{ij}^{(r)}.$$

These two interpretations of the Yangian defining relations (which will reappear in Sections 1.4 and 1.12) indicate a close relationship between the representation theory of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ and the representation theory of the general linear Lie algebra.

Similar calculations applied to the orthogonal and symplectic Lie algebras lead to the defining relations for the corresponding *twisted Yangians*. For instance, consider the orthogonal Lie algebra $\mathfrak{o}_N$ as the subalgebra of $\mathfrak{gl}_N$ spanned by the skew-symmetric matrices. The elements $F_{ij} = E_{ij} - E_{ji}$ with $i < j$ form a basis of $\mathfrak{o}_N$. Introduce the $N \times N$ matrix F whose ij-th entry is F_{ij}. The matrix elements of the powers of the matrix F are known to satisfy the relations

$$[F_{ij}, (F^s)_{kl}] = \delta_{kj}(F^s)_{il} - \delta_{il}(F^s)_{kj} - \delta_{ik}(F^s)_{jl} + \delta_{lj}(F^s)_{ki}. \tag{0.6}$$

A counterpart of (0.3) for the elements $(F^r)_{ij}$ can be explicitly written down. Introduce the generating series

$$f_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} (F^r)_{ij} \Big(u + \frac{N-1}{2} \Big)^{-r}.$$

Then we have the relations for these series analogous to (0.4):

$$\begin{aligned}(u^2 - v^2)\,[f_{ij}(u), f_{kl}(v)] = {} & (u+v)\,\big(f_{kj}(u)\,f_{il}(v) - f_{kj}(v)\,f_{il}(u)\big) \\ & - (u-v)\,\big(f_{ik}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{lj}(u)\big) \\ & + f_{ki}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{jl}(u).\end{aligned} \tag{0.7}$$

This motivates the defining relations of the twisted Yangians; see (2.6) below. They also include a symmetry relation which reflects the fact that the matrix F is skew-symmetric.

We now describe the contents of the book in more detail. Chapters 1 and 2 contain a detailed exposition of the algebraic properties of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ and the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$. We develop the R-matrix techniques as a powerful instrument to investigate the structure of these algebras. The key results there are the constructions of special formal power series called the *quantum determinant* and the *Sklyanin determinant* originating from the works of A. G. Izergin and V. E. Korepin (1981), Kulish and Sklyanin (1982), and Olshanski (1992) (more detailed discussions of the origins of these constructions and other results contained in this book can be found in the bibliographical notes at the end of each chapter). The coefficients of these power series generate the centers of the Yangian and twisted Yangian, respectively. The *quantum Liouville formula*, which is originally due to M. L. Nazarov (1991), explicit formulas for the quantum determinant and the Sklyanin determinant, as well as factorizations of these determinants will be important for the applications to the corresponding classical Lie algebras.

In Chapters 3 and 4 we prove classification theorems for the irreducible finite-dimensional representations of the algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}(\mathfrak{g}_N)$, respectively. For the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ the classification results are a part of the Drinfeld theorem (1988). Our approach employs the *RTT* presentation of the Yangian, and it is based on the original work of V. O. Tarasov (1985, 1986). Note that an alternative exposition of the Yangian representation theory was given in the book by V. Chari and A. Pressley (1994, Chapter 12), whose methods rely on the *Drinfeld presentation* of the Yangians. In Chapter 3 we give a proof of the isomorphism theorem between the two presentations of $\mathrm{Y}(\mathfrak{gl}_N)$ following J. Brundan and A. Kleshchev (2005). For both the Yangian and twisted Yangians we give a complete description of the irreducible finite-dimensional representations in terms of their *highest weights* and *Drinfeld polynomials*. In the simplest case $N = 2$ explicit constructions of all such representations as tensor products of the evaluation modules are also given.

Chapters 5 and 6 are devoted to explicit constructions of finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. For a wide class of such representations it is possible to construct a basis and produce explicit formulas for the matrix elements of the generators of the Yangian in this basis (Chapter 5). The crucial observation here is the fact that the *lowering operators* for the reduction $\mathrm{Y}(\mathfrak{gl}_N) \downarrow \mathrm{Y}(\mathfrak{gl}_{N-1})$ can be written in terms of quantum minors. This basis may be regarded as a generalization of the original basis of I. M. Gelfand and M. L. Tsetlin (1950) for the representations of the Lie algebra $\mathfrak{gl}_N$. The techniques of lowering operators are developed further in Chapter 6, where we prove an irreducibility criterion for tensor products of the Yangian evaluation modules. An essential ingredient here is the *fusion procedure* for the symmetric group due to Cherednik (1986), which we also discuss in detail. It allows us to apply Weyl's tensor approach to the Yangian evaluation modules and realize them as submodules in the tensor products of the vector representations of the Lie algebra $\mathfrak{gl}_N$. This makes it possible to establish an important *binary property* for the tensor product modules due to Nazarov and Tarasov (2002).

In the remaining part of the book (Chapters 7–9) we consider various applications of the Yangian theory to the classical Lie algebras. Taking the images of the central elements of the (twisted) Yangian onto the corresponding classical enveloping algebra with respect to the evaluation homomorphism, we get Casimir elements for the corresponding Lie algebra. Several families of Casimir elements are discussed in Chapter 7. Many of them are well known; some have appeared quite recently. The Yangian perspective provides a unifying picture of all these families and relations between them. We also consider the images of the Casimir elements under the natural representations of the Lie algebras in the differential operators, thus providing various generalizations of the classical Capelli identity. These include the *higher Capelli identities* originally discovered by A. Okounkov (1996), which are related to a distinguished linear basis of the center of $\mathrm{U}(\mathfrak{gl}_N)$ formed by the *quantum immanants*.

Chapter 8 contains an account of the *centralizer construction*, which was the original motivation for Olshanski to discover the twisted Yangians. In order to explain the main ideas of the construction and its applications to the weight bases in Chapter 9, consider a complex reductive Lie algebra $\mathfrak{g}$ and let $\mathfrak{a} \subset \mathfrak{g}$ be a reductive subalgebra. Suppose that V is a finite-dimensional irreducible $\mathfrak{g}$-module and consider its restriction to the subalgebra $\mathfrak{a}$. This restriction is isomorphic to a direct sum of irreducible finite-dimensional $\mathfrak{a}$-modules W_μ which occur with certain multiplicities m_μ,

$$V|_{\mathfrak{a}} \cong \bigoplus_{\mu} m_\mu W_\mu.$$

If the decomposition is multiplicity-free (i.e., $m_\mu \leqslant 1$ for all μ) and each W_μ is provided with a basis, then it can be used to get a basis of V as the union of the bases of the spaces W_μ which occur in the decomposition. This was a key observation for the constructions of the bases for the representations of the general linear and orthogonal Lie algebras given by Gelfand and Tsetlin (1950). Alternatively, if the decomposition is not necessarily multiplicity-free, we can interpret it as the vector space isomorphism

$$V \cong \bigoplus_{\mu} U_\mu \otimes W_\mu, \tag{0.8}$$

where

$$U_\mu = \mathrm{Hom}_{\mathfrak{a}}(W_\mu, V), \qquad \dim U_\mu = m_\mu.$$

It is well known that the vector space U_μ is an irreducible module over the algebra $\mathrm{C}(\mathfrak{g},\mathfrak{a}) = \mathrm{U}(\mathfrak{g})^{\mathfrak{a}}$, the centralizer of $\mathfrak{a}$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. Now, if some bases of the spaces U_μ and W_μ are given, then the decomposition (0.8) yields the natural tensor product basis of V. The general difficulty of this approach is the complicated structure of the algebra $\mathrm{C}(\mathfrak{g},\mathfrak{a})$. As was observed by Olshanski, for each pair of the classical Lie algebras

$$(\mathfrak{g},\mathfrak{a}) \quad = \quad (\mathfrak{gl}_N, \mathfrak{gl}_M), \qquad (\mathfrak{g}_N, \mathfrak{g}_M),$$

there exist algebra homomorphisms

$$\mathrm{Y}(\mathfrak{gl}_{N-M}) \to \mathrm{C}(\mathfrak{gl}_N, \mathfrak{gl}_M), \qquad \mathrm{Y}(\mathfrak{g}_{N-M}) \to \mathrm{C}(\mathfrak{g}_N, \mathfrak{g}_M). \tag{0.9}$$

These homomorphisms turn out to be consistent for different values of M and N provided the difference $N - M$ is fixed, which allows one to embed the (twisted) Yangian into a projective limit of the centralizers. The structure of the "limit" algebras is much simpler than that of the corresponding centralizers $\mathrm{C}(\mathfrak{g},\mathfrak{a})$: the (twisted) Yangian can be presented by quadratic and linear defining relations. Furthermore, the $\mathrm{C}(\mathfrak{g},\mathfrak{a})$-module U_μ in (0.8) can be equipped with the structure of a representation of the Yangian or twisted Yangian, respectively, via the homomorphisms (0.9). Luckily, in the case $N - M = 2$ the $\mathrm{Y}(\mathfrak{g}_2)$-module U_μ admits an extension to a module over the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$. This fact plays a key role in the construction: using the embedding

$$\mathrm{Y}(\mathfrak{gl}_1) \subset \mathrm{Y}(\mathfrak{gl}_2)$$

we can get a natural basis of Gelfand–Tsetlin type for the $\mathrm{Y}(\mathfrak{gl}_2)$-module U_μ and then, by induction, a weight basis of the representation V of the orthogonal and symplectic Lie algebra. Moreover, the matrix elements of the generators of the Lie algebras can be written down in an explicit form.

On the other hand, the (twisted) Yangian modules U_μ emerging from the centralizer construction give rise to a natural class of *skew representations*. These are described in Chapter 8 for the Yangian case, where we calculate their highest weights, Drinfeld polynomials and the Gelfand–Tsetlin characters. The skew representations of the twisted Yangians are considered in Chapter 9 in connection with the weight bases for representations of the orthogonal and symplectic Lie algebras.

The basis vectors of these representations are expressed explicitly in terms of the *lowering operators*. We describe these operators in the context of the *Mickelsson algebra theory* developed by D. P. Zhelobenko (1990, 1994). A brief account of this theory is given in the beginning of Chapter 9. The lowering operators were first used by J. G. Nagel and M. Moshinsky (1964) to construct the Gelfand–Tsetlin bases for the representations of $\mathfrak{gl}_N$. A similar construction of the bases for the orthogonal case was produced by S. C. Pang and K. T. Hecht (1967) and M. K. F. Wong (1967). J. Mickelsson (1972) gave some formulas for basis vectors of the representations of the symplectic Lie algebra as ordered products of the lowering operators. However, the action of the Lie algebra generators in such a basis does not seem to be computable. The reason is the fact that, unlike the cases of $\mathfrak{gl}_N$ and $\mathfrak{o}_N$, the lowering operators do not commute so that the basis depends on the chosen ordering. A "hidden symmetry" has been needed to make a natural choice of the appropriate combination of the lowering operators. That symmetry was provided by the action of the twisted Yangian $\mathrm{Y}(\mathfrak{sp}_2)$ on the homomorphism space U_μ in (0.8), and this action can be written in terms of the lowering operators. The basis of the representation of $\mathfrak{sp}_{2n}$ is then obtained by induction with the use of the chain of subalgebras

$$\mathfrak{sp}_2 \subset \mathfrak{sp}_4 \subset \cdots \subset \mathfrak{sp}_{2n}.$$

The same method can be applied to the pairs of orthogonal Lie algebras where we use the "two step reductions" $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-2}$ instead of the restrictions of the representations of $\mathfrak{o}_N$ to the subalgebra $\mathfrak{o}_{N-1}$ used by Gelfand and Tsetlin. To compare the two constructions, note that the basis of Gelfand and Tsetlin lacks the *weight* property; i.e., the basis vectors are not eigenvectors for the Cartan subalgebra. The reason for that is the fact that the restrictions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ involve Lie algebras of different types (B and D) and the embeddings are not compatible with the root systems. In the new approach we use instead the chains

$$\mathfrak{o}_2 \subset \mathfrak{o}_4 \subset \cdots \subset \mathfrak{o}_{2n} \qquad \text{and} \qquad \mathfrak{o}_3 \subset \mathfrak{o}_5 \subset \cdots \subset \mathfrak{o}_{2n+1}.$$

The embeddings here "respect" the root systems so that the bases possess the weight property in both the symplectic and orthogonal cases. However, the new weight bases, in their turn, lack the *orthogonality* property of the Gelfand–Tsetlin bases: the latter are orthogonal with respect to the standard inner product in the representation space.

At the end of each chapter we give brief bibliographical comments pointing towards the original articles and give further references. This book was intended to be an introductory text on the Yangian theory and its applications, and so the list of topics covered here is by no means complete. In particular, we do not discuss in detail the *Bethe subalgebras* of the (twisted) Yangians (see Section 1.14 for the definition and some basic properties in the Yangian case). These are commutative subalgebras playing an important role in the theory of quantum integrable models in relation with the *Bethe ansatz*; see e.g. Takhtajan and Faddeev (1979), A. N. Kirillov and Reshetikhin (1986), Sklyanin (1992), and Nazarov and Olshanski (1996). The *Drinfeld functor* connecting the representation theory of the degenerate affine Hecke algebras with that of the Yangians is not considered either; see Drinfeld (1986), T. Arakawa and T. Suzuki (1998), Arakawa (1999), S. Khoroshkin and Nazarov (2006, 2007). An application of this functor leads to the character formulas for the finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_N)$ expressing the characters in terms of the Kazhdan–Lusztig polynomials, as computed by Arakawa (1999); see also Brundan and Kleshchev (2007).

The Yangians, as well as their super and q-analogues, have found numerous applications in different areas of physics, including the theory of integrable models in statistical mechanics, conformal field theory, and quantum gravity. We do not discuss them in the book, although we give some references in the Bibliography to indicate at least a few directions for such applications. Some versions of the theorems proved in the book hold for other types of algebras, in particular, for the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ and the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$. We briefly discuss these versions in the examples which follow each chapter, and we also formulate some open problems there.

We have tried to keep the exposition as self-contained as possible relying only on the basic facts of the Lie algebra representation theory; see e.g. J. E. Humphreys (1972), W. Fulton and J. Harris (1991), R. Goodman and N. R. Wallach (1998).

However, some applications in Chapters 6–9 use a few results from the representation theory of the symmetric group, theory of symmetric functions, and the Mickelsson algebra theory. Appropriate references are given in those chapters. The part of the exposition devoted to the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ can be read separately. The corresponding results are contained in Chapters 1, 3, 5, 6 and the respective parts of Chapters 7 and 8. Throughout the book we use the standard notation $\mathbb{Z}$ and $\mathbb{C}$ to indicate the sets of integers and complex numbers, respectively, while $\mathbb{Z}_+$ is used to denote the set of nonnegative integers. The vector spaces and algebras are considered over the field of complex numbers, except for a few examples, where some extensions of $\mathbb{C}$ are needed.

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