Preface

I look at the floor and I see it needs sweeping.
– From "While My Guitar Gently Weeps" by George Harrison of The Beatles

What Part II is about

Think the time is right for a palace revolution.
– From "Street Fighting Man" by Mick Jagger and Keith Richards of The Rolling Stones

To help put the aim of this volume in perspective, we indulge in some relative volume comparison (with apologies to Bishop and Gromov). Let $g_{ij} \doteqdot$ '*The Ricci Flow: An Introduction*' by two of the authors (throughout this book we shall refer to this as '**Volume One**' [**142**]), let $\Gamma_{ij}^k \doteqdot$ '*Hamilton's Ricci Flow*' by three of the authors [**146**], let[1] $R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part I: Geometric Aspects*' by The Ricci Flowers[2] (throughout this book we shall refer to this as '**Part I of this volume**' [**135**]), let $\frac{\partial}{\partial t} R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part II: Analytic Aspects*' (i.e., this book) by The Ricci Flowers, and let $\Delta R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part III: Geometric-Analytic Aspects*' (forthcoming) by The Ricci Flowers (we shall refer to this as '**Part III of this volume**' [**136**]).[3] Both g_{ij} and Γ_{ij}^k are introductory books, whereas the latter more comprehensive monographs $R_{ijk\ell} \oplus \frac{\partial}{\partial t} R_{ijk\ell} \oplus \Delta R_{ijk\ell}$ are derived from the former. Finally, **Volume Two** refers to the collection of Parts I, II, and III.

This is Part II, the sequel to Part I of this volume. In $R_{ijk\ell}$ we discussed various geometric topics in Ricci flow in more detail, such as Ricci solitons, the Kähler–Ricci flow, the compactness theorem, Perelman's energy and entropy monotonicity and the application to no local collapsing, the reduced distance function and its application to the analysis of ancient solutions, and finally a primer on 3-manifold topology.

[1] We would like to thank Andrejs Treibergs and Junfang Li for suggesting the notation $R_{ijk\ell}$.

[2] 'The Ricci Flowers' is an abbreviation for the manifold/variety of authors of this volume.

[3] The originally intended two parts now comprise three parts.

Here, in $\frac{\partial}{\partial t}R_{ijk\ell}$, we discuss mostly analytic topics in Ricci flow including weak and strong maximum principles for scalar heat-type equations and systems on compact and noncompact manifolds, the classification by Böhm and Wilking of closed manifolds with 2-positive curvature operator, Bando's result that solutions to the Ricci flow are real analytic in the space variables, Shi's local derivative of curvature estimates and some variants, and differential Harnack estimates of Li–Yau-type including Hamilton's matrix estimate for the Ricci flow and Perelman's estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow. In the appendices we review aspects of Ricci flow and related geometric analysis and tensor calculus on the frame bundle. Various topics in this part also include the works of others as well as the authors.

In Part III of this volume, i.e., in $\Delta R_{ijk\ell}$, we shall discuss aspects of Perelman's theory of ancient κ-solutions, Perelman's pseudolocality theorem, Hamilton's classification of nonsingular solutions, numerical simulations of Ricci flow, stability of the Ricci flow, the linearized Ricci flow, and the space-time formulation of the Ricci flow. In the appendices, for the convenience of the reader, we review and discuss aspects of metric and Riemannian geometry, the reduced distance function and ancient solutions, and limited aspects of Ricci flat metrics on the K3 surface.

As in previous volumes and as is perhaps typical in geometric analysis, throughout this book we apply both the techniques of the 'weak maximum detail principle' and 'exposition by parts'. To wit, we endeavor to supply the reader with as much detail as possible and we also endeavor, for the most part, to make the chapters independent of each other.

As such, $\frac{\partial}{\partial t}R_{ijk\ell}$ (and $\Delta R_{ijk\ell}$ as well) may be used either for a topics course or for self-study, where the lecturer or reader may wish to select portions from this book $\frac{\partial}{\partial t}R_{ijk\ell}$, its predecessors g_{ij}, Γ_{ij}^k, and $R_{ijk\ell}$, and its successor $\Delta R_{ijk\ell}$.

Although the intent of this series of books on the Ricci flow is expository, there is no substitute for reading the original source material in Ricci flow. In particular, the papers of Hamilton and Perelman contain a wealth of original and deep ideas. We encourage interested readers to consult these papers. We also encourage the reader to consult other sources for Ricci flow including Cao and Zhu [**78**], Chen and Zhu [**110**], Ding [**172**], Kleiner and Lott [**303**], Morgan and Tian [**363**], Müller [**371**], Tao [**461**], Topping [**475**], two of the authors [**142**], three of the authors [**146**], The Ricci Flowers [**135**] and [**136**] (Parts I and III of this volume), and sources for geometric evolution equations (e.g., the mean curvature flow) such as Chou and Zhu [**127**], Ecker [**179**], Zhu [**522**], and two of the authors [**139**]. Certain material, originally intended to appear in a successor volume to [**146**], has now been incorporated in Parts I, II, and III of this volume.

Highlights and interdependencies of Part II

Half my life is in books' written pages.
– From "Dream On" by Steven Tyler of Aerosmith

0.1. Highlights. In this book we consider the following mostly analytic topics, described in more detail in the section "Contents of Part II of Volume Two" below.

(1) Proofs are given of the weak maximum principles for scalars and systems on both compact and complete noncompact manifolds. In the noncompact case we have strived to present complete proofs of general results which are readily applicable. A proof is given of the strong maximum principle for systems with an emphasis on the evolution of the curvature operator under the Ricci flow. The application of the maximum principle to the Ricci flow was pioneered by Hamilton.

(2) We present the solution of Böhm and Wilking of the conjecture of Rauch and Hamilton that closed manifolds, in any dimension, with positive curvature operator are diffeomorphic to spherical space forms. Böhm and Wilking prove that the normalized Ricci flow evolves Riemannian metrics on closed manifolds with 2-positive curvature operator to constant positive sectional curvature metrics.[4]

(3) We discuss the following two topics: (i) nonnegative curvature conditions which are not preserved under the Ricci flow and (ii) Bando's result that solutions of the Ricci flow on closed manifolds are real analytic.

(4) We present Shi's local derivative of curvature estimates (including all higher derivatives) based on the Bernstein technique. We also present a refinement, due to one of the authors, where bounds on some higher derivatives of the initial metric are assumed and consequently improved bounds of all higher derivatives are obtained in space and time.

(5) We discuss the differential Harnack estimates of Li–Yau–Hamilton-type—giving a detailed proof of Hamilton's matrix estimate for complete solutions of the Ricci flow with bounded nonnegative curvature operator (this includes the noncompact case). An application is Hamilton's result that eternal solutions are steady gradient Ricci solitons. We also discuss a variant on Hamilton's proof of the matrix Harnack estimate.

[4] Partly based on Böhm and Wilking's work is the recent result of Brendle and Schoen [**48**] proving that positively $\frac{1}{4}$-pinched closed manifolds are diffeomorphic to spherical space forms. Related to this, Brendle and Schoen [**48**] and Nguyen [**376**] independently proved that the condition of positive isotropic curvature is preserved in all dimensions (a result previously known only in dimension 4 by the work of Hamilton).

(6) We present Perelman's differential Harnack estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow. This result, of which we give a detailed proof, will be used in the proof of Perelman's pseudolocality discussed in Part III.
(7) We review tensor calculus on the frame bundle—a framework for proving Hamilton's matrix Harnack estimate for the Ricci flow.

The appendices are intended to make this book more self-contained.

0.2. Interdependencies. The chapters are for the most part independent. However, many of the results discussed in this book rely on various forms of the maximum principle.[5] For example we have the following reliances.

(1) Chapter 11 on manifolds with positive curvature operator, Section 1 of Chapter 13 on curvature conditions that are not preserved, and Chapter 15 on the matrix Harnack estimate all require the (time-independent) maximum principles for tensors and systems.
(2) The maximum principles on noncompact manifolds in Chapter 12 require some familiarity with the maximum principles on compact manifolds in Chapter 10.
(3) Chapter 14 on local derivative estimates, Section 2 of Chapter 13 on the real analyticity of solutions, and Chapter 16 on Perelman's differential Harnack estimate all require the maximum principle for scalars.

[5]Perhaps we may say, 'Geometric analysis is simple; just apply the maximum principle!'