Contents of Part II of Volume Two

Councillor Hamann: Almost no one comes down here, unless, of course, there's a problem.

– From the movie "The Matrix Reloaded" by the Wachowski brothers.

Chapter 10. In this chapter we discuss the general formulation of the weak maximum principle for systems on closed manifolds, which applies to bilinear forms such as curvature tensors. The maximum principle for scalars may be considered as stating that solutions to a semilinear PDE are bounded by the solutions to the associated ODE obtained by dropping the Laplacian and any gradient terms. In particular for subsolutions/supersolutions to the heat equation, the maximum/minimum is nonincreasing/nondecreasing. This last statement has a generalization to symmetric 2-tensors, considered in Chapter 4 of Volume One, which gives general sufficient conditions to prove that the nonnegativity of tensor supersolutions to heat-type equations is preserved. We had previously applied this to the Ricci tensor and also obtained pinching estimates for the curvatures this way.

To obtain various estimates for the curvatures in Volume One, we found it convenient to employ a more general formulation of the weak maximum principle. We prove this version in this chapter. More precisely we consider sections of vector bundles which satisfy a semilinear heat-type equation. The maximum principle for systems states that if the initial section lies in a subset of the vector bundle which is convex in the fibers and invariant under parallel translation and if the associated ODE obtained by dropping the Laplacian preserves this subset, then the solution to the PDE stays inside this convex set.

The idea of the proof of this maximum principle is as follows. One can prove the maximum principle for functions by considering the spatial maximum function, which is Lipschitz in time, and showing that it is nonincreasing for subsolutions to the heat equation. In the case of the maximum principle for systems, one can look at the function of time which is the maximum distance of the solution to the PDE from the subset. Using the support functions to the convex fibers, one can show that this maximum distance function $s(t)$, which is again Lipschitz in time, satisfies an ODE of the form $ds/dt \leq Cs$. Since $s(0) = 0$, we conclude that $s(t) \equiv 0$ and the maximum principle for systems follows.

Refinements of the maximum principle include the case when the subsets with convex fibers are time-dependent and also when there is a so-called avoidance set for the solutions of the PDE. In terms of applications, one of the most important special cases is when the sections of the bundle are bilinear forms. We discuss this case and applications to the curvature tensors. We also discuss the Aleksandrov–Bakelman–Pucci maximum principle for elliptic equations.

Chapter 11. In this chapter we present Böhm and Wilking's solution to the conjecture of Rauch and Hamilton on the classification of closed Riemannian manifolds with positive curvature operator.[1] The flavor of this chapter is more algebraic with an essential component of the proof being the irreducible decomposition of (algebraic) curvature tensors. Generally, one of the ideas is to study when linear transformations of convex preserved sets (with respect to the ODE corresponding to the PDE satisfied by Rm) remain preserved. More specifically, via a 1-parameter family of linear transformations the cone of 2-nonnegative curvature operators is mapped into the cone of nonnegative curvature operators. This reduces the classification problem for Riemannian manifolds with 2-positive curvature operator to that for manifolds with positive curvature operator. To study the latter problem, one would hope that the cones of 2-positive curvature operators with arbitrary Ricci pinching are preserved. Unfortunately there does not seem to be any known way to prove this. Instead, one can prove that suitable linear transformations of the cones of 2-positive curvature operators with arbitrary Ricci pinching are preserved. This is sufficient to prove the Rauch–Hamilton conjecture.

Chapter 12. This chapter comprises two main topics: (1) weak maximum principles on noncompact manifolds and (2) the strong maximum principle, which is a local result. In both cases we consider scalar parabolic equations and systems of parabolic equations.

Since singularity models are often noncompact, it is important to be able to apply the weak maximum principle on complete, noncompact manifolds. We begin with the heat equation and present the weak maximum principle of Karp and Li which applies to solutions with growth slower than exponential quadratic in distance. Using barrier functions, we then give a weak maximum principle for bounded solutions of heat-type systems. As a special case, we show that complete solutions to the Ricci flow on noncompact manifolds, with nonnegative curvature operator initially and bounded curvature on space and time, have nonnegative curvature operator for all time.

We also discuss mollifiers on Riemannian manifolds with a lower bound on the injectivity radius. One technical issue we discuss, using the mollifiers obtained above, is the construction of distance-like functions with bounds on

[1]They actually obtain the stronger result of classifying closed manifolds with 2-positive curvature operator.

their gradients and upper bounds of their Hessians on complete Riemannian manifolds with bounded curvature. These distance-like functions are used to construct the barrier functions referred to in the previous paragraph.

Moreover, this construction carries over to the case of the Ricci flow. The aforementioned mollifiers are not only applicable to the proof of the compactness theorem in regards to the center of mass (discussed in Chapter 4 of Part I of this volume), but also to constructing a barrier function used in the proof of Hamilton's matrix Harnack estimate in Chapter 15 on complete noncompact Riemannian manifolds with bounded nonnegative curvature operator.

A fundamental property of the heat equation is that a solution which is initially nonnegative immediately becomes either positive or identically zero. This property is known as the strong maximum principle. For a solution to the Ricci flow, the curvature tensor satisfies a heat-type equation. The strong maximum principle for systems, due to Hamilton based on earlier work of Weinberger and others, tells us that for solutions to the Ricci flow with nonnegative curvature operator (such as singularity models in dimension 3) the curvature operator has a special form after the initial time. In particular, the image of the curvature operator is independent of time and invariant under parallel translation in space. Moreover, using the natural Lie algebra structure on the fibers Λ_x^2, $x \in \mathcal{M}$, of the bundle of 2-forms, the image of the curvature operator is a Lie subalgebra. It is useful to observe that the strong maximum principle is a local result and does not require the solution to be complete or to have bounded curvature. We also formulate the strong maximum principle in the more general setting of Chapter 10, i.e., for sections of a vector bundle which solve a PDE.

In dimension 3, Λ_x^2 is isomorphic to $\mathfrak{so}(3)$ and its only proper nontrivial Lie subalgebras are isomorphic to $\mathfrak{so}(2)$, which is 1-dimensional. Hence, after the initial time, a solution to the Ricci flow on a 3-manifold with nonnegative sectional curvature either is flat, has positive sectional curvature, or admits a global parallel 2-form. In the last case, by taking the dual of this 2-form, we have a parallel 1-form and the solution splits locally as the product of a surface solution and a line. This classification is useful in studying the singularities which arise in dimension 3.

Chapter 13. In this chapter we discuss the following two topics in Ricci flow.

(1) Curvature conditions that are *not* preserved. The weak positivity (i.e., nonnegativity) of the following curvatures are preserved: scalar curvature, Ricci curvature in dimension 3, Riemann curvature operator, isotropic curvature, complex sectional curvature, as well as the 2-nonnegativity of the curvature operator. On the other hand, in this chapter we discuss some nonnegativity conditions which are not preserved under the Ricci flow. In particular, we consider the conditions of nonnegative Ricci curvature and nonnegative sectional curvature in dimensions at least 4 for solutions of the Ricci flow on both closed and noncompact manifolds.

(2) Bando's result that solutions to the Ricci flow on closed manifolds are real analytic in the space variables for positive time. The proof of this is based on keeping track of the constants in the higher derivatives of curvature estimates and summing these estimates.

Chapter 14. In Chapter 7 of Volume One we encountered the global derivative of curvature estimates. The idea is to assume a curvature bound K for the Riemann curvature tensor and, by applying the weak maximum principle to the appropriate quantities, obtain bounds for the m-th derivatives of the curvatures of the form $|\nabla^m \operatorname{Rm}| \leq CKt^{-m/2}$. In this chapter, we present Shi's local derivative of curvature estimates. The idea of localizing the derivative estimates is simply to multiply the quantities considered by a cutoff function.

For the global first derivative of curvature estimate we previously considered $t\,|\nabla \operatorname{Rm}|^2 + C\,|\operatorname{Rm}|^2$. This quantity does not seem to adapt well to localization. Instead we consider $\eta t \left(16K^2 + |\operatorname{Rm}|^2\right) |\nabla \operatorname{Rm}|^2$, where η is a cutoff function. The local *first* derivative estimate we prove says that if a solution is defined on a ball of radius r and time interval $[0, \tau]$ and if it has curvatures bounded by K, then we have

$$|\nabla \operatorname{Rm}| \leq CK \left(\frac{1}{r^2} + \frac{1}{\tau} + K\right)^{1/2}$$

on the concentric ball of radius $r/2$ on the time interval $[\tau/2, \tau]$.

Local *higher* derivative estimates are proven using a similar idea. For example, to bound the second derivatives, one applies the weak maximum principle to the quantity $\left(4A^2 + t\,|\nabla \operatorname{Rm}|^2\right) t^2 \left|\nabla^2 \operatorname{Rm}\right|^2$, where the constant A is chosen appropriately. We also discuss a version of the local derivative estimates where bounds on the higher derivatives of the curvatures of the initial metric are assumed up to some order. In this case we obtain improved bounds for all higher derivatives of the curvatures. This result is useful in an approach toward constructing Perelman's standard solution.

For complete solutions with bounded nonnegative curvature operator, the local derivative estimates, when combined with Hamilton's trace Harnack estimate, yield instantaneous local derivative bounds. Previously, in Chapters 7 and 8 of Part I of this volume, we saw applications of the local derivative estimates to the study of the reduced distance and the reduced volume.

We also briefly discuss D. Yang's local Ricci flow. Here the velocity $-2\operatorname{Rc}$ of the metric is multiplied by a nonnegative weight function with compact support. The local Ricci flow provides another approach to Shi's short time existence theorem for the Ricci flow on noncompact manifolds.

Chapter 15. This chapter discusses differential Harnack inequalities of Li–Yau–Hamilton-type for the Ricci flow. These gradient-type estimates, which are directly motivated by considering quantities which vanish on the (gradient) Ricci solitons as discussed in Chapter 1 of Part I of this volume, provide useful bounds for the solution. The general form of the main estimate, which holds for complete solutions with bounded nonnegative curvature operator and which is known as Hamilton's matrix Harnack estimate, says that a certain tensor involving two and fewer derivatives of the curvature is nonnegative definite.

The trace Harnack estimate, as the name suggests, is obtained by tracing the matrix inequality. This trace inequality has several important consequences, including the fact that scalar curvature does not decrease too fast in the sense that for a fixed point, t times the scalar curvature is a nondecreasing function of time. More generally, the trace inequality yields a lower bound for the scalar curvature at a point and time, in terms of the scalar curvature at any other point and earlier time, and the distance between the points and the time difference.

We begin the proof of the Harnack estimate with the case of surfaces (dimension 2), in which case the evolution equation for the Harnack quadratic simplifies (in comparison to higher dimensions) and one can prove the trace inequality directly. This is unlike the situation in higher dimensions, where the trace inequality apparently can only be demonstrated by proving the matrix inequality and then tracing.

In all dimensions, we recall the terms in the matrix Harnack quadratic obtained in Chapter 1 (p. 9) of Part I of this volume by differentiating the expanding gradient Ricci soliton equation. The Harnack calculations simplify when one uses the formalism, given in Appendix F, of considering tensors as vector-valued functions on the frame bundle. Using the above formalism, we present the evolution of Harnack calculations which are long but relatively straightforward. The evolution of the Harnack quantity looks formally similar to the evolution of the Riemann curvature operator and as such is amenable to the application of the weak maximum principle when the solution has nonnegative curvature operator. When the manifold is noncompact, the techniques used to enable this application are reminiscent of the techniques used to prove the maximum principle for functions.

We also give a variant on Hamilton's proof of the matrix Harnack estimate, based on reducing the problem to showing that a symmetric 2-tensor is nonnegative definite.

Chapter 16. In this chapter we give a proof of Perelman's differential Harnack-type inequality for solutions of the adjoint heat equation coupled to the Ricci flow. We begin by considering entropy and differential Harnack estimates for the heat equation. Our approach for proving Perelman's differential Harnack-type inequality is to first prove gradient estimates for positive solutions of the (adjoint) heat equation. Using this and heat kernel estimates, we give a proof that for solutions to the adjoint heat equation coupled to the Ricci flow, Perelman's Harnack quantity (or, geometrically, the modified scalar curvature) is nonpositive:

$$\tau\left(R+2\Delta f-|\nabla f|^2\right)+f-n\leq 0.$$

Appendix D. In this appendix we review some basic results for the Ricci flow. In particular, we recall the results on the short time and long time existence and uniqueness of the Ricci flow on closed and noncompact manifolds, convergence results on closed manifolds assuming some sort of positivity of curvature, the rotationally symmetric neckpinch, curvature pinching estimates, strong maximum principle, derivative estimates, differential Harnack estimates, Perelman's energy and entropy monotonicity and no local collapsing, compactness theorems, and the existence of singularity models.

Appendix E. In this appendix we review some basic geometric analysis related to the Ricci flow with an emphasis on the heat equation. We recall Duhamel's principle and its application to basic results for the heat kernel. We discuss the Cheeger–Yau comparison theorem for the heat kernel, the Li–Yau differential Harnack estimate, and Hamilton's gradient estimates.

Appendix F. The material in this appendix is in preparation for Chapter 15 on Hamilton's matrix Harnack estimate. Given a Riemannian manifold, we describe a formalism for considering tensors as vector-valued functions on the (orthonormal) frame bundle. In the context of the Ricci flow, where we have a 1-parameter family of metrics, we add to this a modified time derivative which may be considered as a version of Uhlenbeck's trick. We discuss tensor calculus in this setting, including commutator formulas for the heat operator and covariant derivatives. These calculations are used for the Harnack calculations in Chapter 15.

CHAPTER 11

Closed Manifolds with Positive Curvature

Yes, there are two paths you can go by.

– From "Stairway to Heaven" by Jimmy Page and Robert Plant of Led Zeppelin

One of the basic problems in Riemannian geometry is to relate curvature and topology. As we saw in Volume One, the Ricci flow may be used to topologically classify closed 3-manifolds with positive Ricci curvature as spherical space forms. It is natural to ask if there is a generalization of this result to all dimensions. Remarkably, the answer is yes. In this chapter we give a presentation of Böhm and Wilking's remarkable proof that n-dimensional closed Riemannian manifolds with 2-positive curvature operators are diffeomorphic to spherical space forms, i.e., they admit metrics with constant positive sectional curvature.[1]

Let $\{\mu_\alpha(\mathrm{Rm})\}_{\alpha=1}^N$, $N \doteqdot \frac{n(n-1)}{2}$, denote the eigenvalues of the curvature operator Rm.[2] Recall that a Riemannian manifold $(\mathcal{M}^n, g)$ has **positive curvature operator** (**PCO**) if $\mu_\alpha(\mathrm{Rm}) > 0$ for all α.[3]

Definition 11.1 (2-positive curvature operator). A Riemannian manifold $(\mathcal{M}^n, g)$ has **2-positive curvature operator** (**2-PCO**) if

$$\mu_\alpha(\mathrm{Rm}) + \mu_\beta(\mathrm{Rm}) > 0 \quad \text{for all } \alpha \neq \beta. \tag{11.1}$$

That is, the sum of the smallest two eigenvalues of the Riemann curvature operator is positive. If the $>$ in (11.1) is replaced by $\geq$, we say that $(\mathcal{M}, g)$ has **2-nonnegative curvature operator**.

When $n = 3$, this condition is the same as positive Ricci curvature: $\mathrm{Rc} > 0$ (see (6.33) on p. 189 of Volume One). For $n = 4$, this curvature condition was studied in the context of the Ricci flow by Haiwen Chen [**112**], with the earlier seminal work of Hamilton [**245**] on 4-manifolds with positive curvature operator. Clearly, in all dimensions, if a metric g has positive curvature operator, then g has 2-positive curvature operator.

[1] In even dimensions $n = 2m$, spherical space forms are either S^{2m} or $\mathbb{R}P^{2m}$. In odd dimensions, there are many finite subgroups of $\mathrm{O}(n+1)$ acting freely, linearly, and isometrically on S^n (see Wolf's book [**502**] for what is known about the classification of these group actions).

[2] We would like to thank Nolan Wallach for explaining to us many aspects of representation theory related to curvature operators.

[3] We say $(\mathcal{M}^n, g)$ has **nonnegative curvature operator** if $\mu_\alpha(\mathrm{Rm}) \geq 0$ for all α.

Generalizing Hamilton's classification of closed 3-manifolds with positive Ricci curvature and H. Chen's classification of closed 4-manifolds with 2-positive curvature operator, we have the following solution of the spherical space form conjecture of Rauch and Hamilton.

THEOREM 11.2 (Böhm and Wilking). *If* $(\mathcal{M}^n, g_0)$ *is a closed Riemannian manifold with* 2*-positive curvature operator, then there exists a unique solution* $(\mathcal{M}^n, g(t))$, $t \in [0, \infty)$, *to the initial-value problem for the volume normalized Ricci flow*

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc} + \frac{2}{n} r g,$$
$$g(0) = g_0,$$

where $r(t) \doteqdot \left(\int R d\mu / \int d\mu \right)(t)$. *The solution converges exponentially fast in each* C^k*-norm as* $t \to \infty$ *to a constant positive sectional curvature metric.*

In particular, a closed Riemannian manifold with 2-positive curvature operator admits a metric with constant positive sectional curvature.

From the work of Hamilton [**244**] it follows that a sufficient condition for the convergence of solutions of the Ricci flow on closed manifolds with positive scalar curvature is the following '*effective* pinching improves estimate':

$$\left| \widetilde{\operatorname{Rm}} \right| \leq C R^{1-\delta}, \tag{11.2}$$

where $C < \infty$ and $\delta > 0$ (see also Theorem 3.1 and §4 in Huisken [**280**], Definition 5.1 and Convergence Criterion 5.2 in Hamilton [**245**], and Definition 11.59 below). In [**245**], the maximum principle for systems is introduced and applied to the study of (11.27) below, i.e.,

$$\frac{\partial}{\partial t} \operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^{\#},$$

by relating it to the study of the associated ODE (11.31), i.e.,

$$\frac{d}{dt} \mathbf{R} = \mathbf{R}^2 + \mathbf{R}^{\#}.$$

In particular, in dimension 3 under the positive Ricci curvature condition and in dimension 4 under the positive curvature operator condition, Hamilton proves the estimate (11.2).

One of the main ideas of Böhm and Wilking's proof is the following, quoted from their paper [**43**].

> "By analyzing how the differential equation changes under linear equivariant transformations, we provide a general method for constructing new invariant curvature conditions from known ones."

The differential equation to which Böhm and Wilking are referring is the ODE (11.31) associated to the PDE (11.27) satisfied by the Riemann curvature operator. They exhibit some miraculous properties of this ODE. In particular, the operator $\mathbf{R} \mapsto Q(\mathbf{R}) \doteqdot \mathbf{R}^2 + \mathbf{R}^{\#}$ on the space of algebraic curvature operators exhibits special behavior under linear transformations of this space.

In Section 1 we discuss some multilinear algebra related to the study of algebraic curvature operators. In Section 2 we investigate the properties of algebraic curvature operators and the operator $Q(\mathbf{R})$. In Section 3 we define a family of linear transformations $\ell_{a,b}$ and state crucial properties of $Q(\mathbf{R})$ under these transformations. In Section 4 we prove these properties of $Q(\mathbf{R})$. In Section 5 we discuss the convex cone of 2-nonnegative algebraic curvature operators and both the action of $\ell_{a,b}$ and the behavior of $Q(\mathbf{R})$ on this cone. In Section 6 we construct pinching families of algebraic curvature operators. In Section 7 we demonstrate the existence of a generalized pinching set (in lieu of a set defined by (11.2)) from these pinching families. This enables us to prove the main result of this chapter, i.e., Theorem 11.2, based on the maximum principle for systems applied to the evolution of Rm. In Section 8 we give a summary of the proof of Theorem 11.2. Those readers who would like to first acquaint themselves with the main ideas of the proof of Theorem 11.2 may wish to first look at this section. In the notes and commentary we give some historical remarks concerning work related to Theorem 11.2.

1. Multilinear algebra related to the curvature operator

In this section we recall some basic facts about tensor spaces and their representations which are needed for our later study of the space of algebraic curvature operators. Eventually all of these constructions on $\mathbb{E}^n$ will be transplanted to the tangent space $T_x\mathcal{M}$ and the tangent bundle $T\mathcal{M}$ of a Riemannian manifold $(\mathcal{M}^n, g)$ through the identification of $T_x\mathcal{M}$ with $\mathbb{E}^n$ via an orthonormal frame. The intent of our discussion is to serve as a quick review rather than as a comprehensive introduction.

1.1. Tensor spaces. Let $\mathbb{E}^n$ denote Euclidean n-space with the standard metric g_{can} and let $(\mathbb{E}^n)^*$ denote the dual vector space. In this section we shall discuss some real tensor spaces formed out of $\mathbb{E}^n$; their equivalent dual forms will be identified using the metric g_{can}. Later in this chapter we will freely use any one of these equivalent forms. Below, $\{e_i\}_{i=1}^n$ denotes the standard basis for $\mathbb{E}^n$ and $\{e_i^*\}_{i=1}^n$ denotes the dual standard basis for $(\mathbb{E}^n)^*$.

1.1.1. *Tensor spaces* $\bigotimes^2 \mathbb{E}^n$ *and* $S^2\mathbb{E}^n$. The tensor space $\bigotimes^2 \mathbb{E}^n \doteqdot \mathbb{E}^n \otimes \mathbb{E}^n$ is equipped with the inner product

$$\langle x \otimes y, u \otimes v \rangle \doteqdot \langle x, u \rangle \cdot \langle y, v \rangle .$$

The set $\{e_i \otimes e_j\}_{i,j=1}^n$ forms an orthonormal basis. Moreover, $\bigotimes^2 \mathbb{E}^n$ can be identified with the vector space $\mathfrak{gl}(n, \mathbb{R})$ of linear transformations of $\mathbb{E}^n$, where for any $x \otimes y \in \bigotimes^2 \mathbb{E}^n$ and $z \in \mathbb{E}^n$,

$$(x \otimes y)(z) \doteqdot \langle y, z \rangle x$$

defines the corresponding element of $\mathfrak{gl}(n,\mathbb{R})$. This is equivalent to identifying the second $\mathbb{E}^n$ in $\mathbb{E}^n \otimes \mathbb{E}^n$ with its dual $(\mathbb{E}^n)^*$ using the metric $g_{\text{can}} = \langle\ ,\ \rangle$. Under this identification, the space of symmetric 2-tensors $S^2\mathbb{E}^n = \mathbb{E}^n \otimes_S \mathbb{E}^n$ corresponds to the space of self-adjoint transformations $S^2(\mathbb{E}^n) \subset \mathfrak{gl}(n,\mathbb{R})$.

1.1.2. *The space $\Lambda^2(\mathbb{E}^n)^*$ of 2-forms and its Lie bracket.* We equip the wedge product space $\Lambda^2\mathbb{E}^n$ with the standard inner product

$$\langle x \wedge y, u \wedge v \rangle \doteqdot \det \begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{pmatrix}.$$

The set $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis for $\Lambda^2\mathbb{E}^n$. Through the identification of $\mathbb{E}^n$ with $(\mathbb{E}^n)^*$, this induces an inner product on the space of 2-forms $\Lambda^2(\mathbb{E}^n)^*$ and $e_i^* \wedge e_j^* \doteqdot e_i^* \otimes e_j^* - e_j^* \otimes e_i^*$ is an orthonormal basis for $\Lambda^2(\mathbb{E}^n)^*$. The pairing of $\Lambda^2(\mathbb{E}^n)^*$ with $\Lambda^2\mathbb{E}^n$ via linear functions is given by

$$e_i^* \wedge e_j^* (e_k \wedge e_l) \doteqdot \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}.$$

The space $\Lambda^2(\mathbb{E}^n)^*$ can be identified with the Lie algebra $\mathfrak{so}(n)$, where $e_i^* \wedge e_j^*$ is identified with the matrix E_{ij} defined to be 1 at the (i,j)-th position, -1 at the (j,i)-th position, and 0 elsewhere. Note that, with respect to the identification of $\mathfrak{so}(n)$ with $\Lambda^2(\mathbb{E}^n)^*$, the inner product on $\mathfrak{so}(n)$ is given by

$$\langle A, B\rangle = \frac{1}{2}\operatorname{tr}(A^t B) = -\frac{1}{2}\operatorname{tr}(AB).$$

Indeed, with respect to this inner product, $\langle E_{ij}, E_{k\ell}\rangle = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

The action of $\Lambda^2(\mathbb{E}^n)^*$ as $\mathfrak{so}(n)$ on $\mathbb{E}^n$ is given by

$$\begin{aligned}(e_i^* \wedge e_j^*)(e_k) &= (e_i^* \otimes e_j^* - e_j^* \otimes e_i^*)(e_k) \\ &= (\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell})e_\ell.\end{aligned} \tag{11.3}$$

An element of $\Lambda^2(\mathbb{E}^n)^*$ can be considered as a linear function on $\mathbb{E}^n \times \mathbb{E}^n$:

$$e_i^* \wedge e_j^*(e_k, e_\ell) \doteqdot \frac{1}{2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}). \tag{11.4}$$

This induces the pairing of $\Lambda^2(\mathbb{E}^n)^*$ with $\Lambda^2\mathbb{E}^n$ mentioned above.

The action of $\bigotimes^2 \mathbb{E}^n$ on $\mathbb{E}^n$ induces the following identity:

$$(e_i \otimes e_i) \wedge (e_j \otimes e_j) = \frac{1}{2}(e_i \wedge e_j) \otimes (e_i \wedge e_j). \tag{11.5}$$

Indeed,

$$\begin{aligned}&((e_i \otimes e_i) \wedge (e_j \otimes e_j))(e_k \wedge e_\ell) \\ &= \frac{1}{2}((e_i \otimes e_i)(e_k) \wedge (e_j \otimes e_j)(e_\ell) + (e_j \otimes e_j)(e_k) \wedge (e_i \otimes e_i)(e_\ell)) \\ &= \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell})(e_k \wedge e_\ell)\end{aligned}$$

and

$$\left(\frac{1}{2}(e_i \wedge e_j) \otimes (e_i \wedge e_j)\right)(e_k \wedge e_\ell) = \frac{1}{2}(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})(e_i \wedge e_j)$$

are equal.

As usual, for any $\varphi \in \Lambda^2(\mathbb{E}^n)^*$, we define $\varphi_{ij} \doteqdot \varphi(e_i, e_j)$. From (11.4) we may write

$$\varphi = \sum_{i,j} \varphi_{ij} e_i^* \wedge e_j^* = 2 \sum_{i<j} \varphi_{ij} e_i^* \wedge e_j^*.$$

The identification of $\Lambda^2(\mathbb{E}^n)^*$ with the Lie algebra $\mathfrak{so}(n)$ equips $\Lambda^2(\mathbb{E}^n)^*$ with a Lie algebra structure, where the Lie bracket is defined by

$$[\phi, \psi]_{ij} \doteqdot \phi_{ik}\psi_{kj} - \psi_{ik}\phi_{kj} \tag{11.6}$$

for $\phi, \psi \in \Lambda^2(\mathbb{E}^n)^*$. From this we have by identifying $e_i^* \wedge e_j^*$ with the matrix E_{ij}

$$\begin{aligned} [e_i^* \wedge e_j^*, e_k^* \wedge e_\ell^*] &= \left(e_i^* \wedge e_j^*\right) \cdot \left(e_k^* \wedge e_\ell^*\right) - \left(e_k^* \wedge e_\ell^*\right) \cdot \left(e_i^* \wedge e_j^*\right) \\ &= -\delta_{j\ell} e_i^* \wedge e_k^* - \delta_{ik} e_j^* \wedge e_\ell^* + \delta_{jk} e_i^* \wedge e_\ell^* + \delta_{i\ell} e_j^* \wedge e_k^*, \end{aligned} \tag{11.7}$$

where $\cdot$ denotes the matrix product. We also have

$$\begin{aligned} &\langle [e_k \wedge e_\ell, e_s \wedge e_t], e_a \wedge e_b \rangle \\ &= \langle -\delta_{\ell t} e_k \wedge e_s - \delta_{ks} e_\ell \wedge e_t + \delta_{\ell s} e_k \wedge e_t + \delta_{kt} e_\ell \wedge e_s, e_a \wedge e_b \rangle \\ &= -\delta_{\ell t}\left(\delta_{ka}\delta_{sb} - \delta_{kb}\delta_{sa}\right) - \delta_{ks}\left(\delta_{\ell a}\delta_{tb} - \delta_{\ell b}\delta_{ta}\right) \\ &\quad + \delta_{\ell s}\left(\delta_{ka}\delta_{tb} - \delta_{kb}\delta_{ta}\right) + \delta_{kt}\left(\delta_{\ell a}\delta_{sb} - \delta_{\ell b}\delta_{sa}\right) \end{aligned}$$

and hence

$$\begin{aligned} &\langle [e_k \wedge e_\ell, e_s \wedge e_t], e_a \wedge e_b \rangle \langle [e_i \wedge e_j, e_s \wedge e_t], e_c \wedge e_d \rangle \\ &= 2\left(\delta_{\ell s}\left(\delta_{ka}\delta_{tb} - \delta_{kb}\delta_{ta}\right) + \delta_{kt}\left(\delta_{\ell a}\delta_{sb} - \delta_{\ell b}\delta_{sa}\right)\right) \\ &\quad \times \begin{pmatrix} -\delta_{jt}\left(\delta_{ic}\delta_{sd} - \delta_{id}\delta_{sc}\right) - \delta_{is}\left(\delta_{jc}\delta_{td} - \delta_{jd}\delta_{tc}\right) \\ +\delta_{js}\left(\delta_{ic}\delta_{td} - \delta_{id}\delta_{tc}\right) + \delta_{it}\left(\delta_{jc}\delta_{sd} - \delta_{jd}\delta_{sc}\right) \end{pmatrix}. \end{aligned} \tag{11.8}$$

Hamilton [**245**] observed that the Lie algebra structure of $\Lambda^2(\mathbb{E}^n)^*$ is of fundamental importance in the study of the evolution of the curvature operator under the Ricci flow (see Section 2 below). We now discuss this Lie algebra in preparation for the study of Rm. For any orthonormal basis $\{\varphi^\alpha\}_{\alpha=1}^N$ of $\Lambda^2(\mathbb{E}^n)^*$, the **structure constants** for the Lie bracket are given by

$$[\varphi^\alpha, \varphi^\beta] \doteqdot \sum_\gamma c_\gamma^{\alpha\beta} \varphi^\gamma. \tag{11.9}$$

By definition, $c_\gamma^{\alpha\beta}$ is anti-symmetric in α and β. Since the 2-forms φ^α are orthonormal, $c_\gamma^{\alpha\beta} = \left\langle [\varphi^\alpha, \varphi^\beta], \varphi^\gamma \right\rangle$. We claim

$$\langle [\phi, \psi], \omega \rangle = -\langle [\omega, \psi], \phi \rangle.$$

Indeed, by relabelling indices and using the anti-symmetry of 2-forms, we calculate with respect to the orthonormal basis $\left\{e_i^* \wedge e_j^*\right\}_{i<j}$ that

$$\left(\phi_{ik}\psi_{kj} - \psi_{ik}\phi_{kj}\right)\omega_{ij} = -\left(\omega_{ik}\psi_{kj} - \psi_{ik}\omega_{kj}\right)\phi_{ij}.$$

This implies the structure constants $c_\gamma^{\alpha\beta}$ are anti-symmetric in all three components.

Let $\{\sigma_\alpha\}_{\alpha=1}^N$ be the orthonormal basis for $\Lambda^2\mathbb{E}^n$ dual to $\{\varphi^\alpha\}_{\alpha=1}^N$. We may also define the structure constants for the dual Lie algebra $\Lambda^2\mathbb{E}^n$ by

$$[\sigma_\alpha, \sigma_\beta] \doteqdot \sum_\gamma c_{\alpha\beta}^\gamma \sigma_\gamma. \tag{11.10}$$

From the identification of $\Lambda^2\mathbb{E}^n$ with $\Lambda^2(\mathbb{E}^n)^*$, we have

$$c_{\alpha\beta}^\gamma = c_\gamma^{\alpha\beta}. \tag{11.11}$$

1.1.3. *The tensor space* $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$. By duality we may write $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^* \doteqdot S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$. The vector space $S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$ has the inner product $\langle A, B\rangle \doteqdot \operatorname{tr}\left(A^t B\right) = \operatorname{tr}(AB)$. This space $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$ may also be identified with $S^2(\mathfrak{so}(n))$. An element of $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$ is both a symmetric bilinear form on $\Lambda^2\mathbb{E}^n$ and a self-adjoint endomorphism of $\Lambda^2\mathbb{E}^n$. We are interested in the tensor space $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$ because the Riemann curvature operator Rm_x is an element of $\left(\Lambda^2 T_x^*\mathcal{M}\right) \otimes_S \left(\Lambda^2 T_x^*\mathcal{M}\right)$ (which also satisfies the **first Bianchi identity**[4]). In the next subsection we will discuss operations related to $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$ in detail.

1.2. Algebraic operations on $\Lambda^2(\mathbb{E}^n)^* \otimes_S \Lambda^2(\mathbb{E}^n)^*$. We will discuss three algebraic operations in this subsection.

1.2.1. *The sharp operator.* Given $A \in S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$, let $A_{\alpha\beta} \doteqdot A(\sigma_\alpha, \sigma_\beta)$, where $\{\sigma_\alpha\}_{\alpha=1}^N$ is the dual (orthonormal) basis for $\Lambda^2\mathbb{E}^n$ introduced above. Given $A, B \in S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$, using the notation in (11.10), we define the **sharp product** by

$$(A\#B)_{\alpha\beta} = (A\#B)(\sigma_\alpha, \sigma_\beta) \doteqdot \frac{1}{2} c_\alpha^{\gamma\eta} c_\beta^{\delta\theta} A_{\gamma\delta} B_{\eta\theta}. \tag{11.12}$$

We denote $A\#A \doteqdot A^\#$. It is easy to see that $(A\#B)_{\alpha\beta}$ is symmetric in α and β. Hence

$$A\#B \in S^2\left(\Lambda^2(\mathbb{E}^n)^*\right).$$

We call the bilinear operator

$$\# : S^2\left(\Lambda^2(\mathbb{E}^n)^*\right) \times S^2\left(\Lambda^2(\mathbb{E}^n)^*\right) \to S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$$

the **sharp operator**.

From the anti-symmetry of $c_\gamma^{\alpha\beta}$,

$$A\#B = B\#A.$$

Let $\mathbf{I}$ and $\mathbf{0}$ be the identity and zero element in $S^2\left(\Lambda^2(\mathbb{E}^n)^*\right)$, respectively. Note that $A\#\mathbf{0} = \mathbf{0}$; however $A\#\mathbf{I}$ is not necessary A.

[4]The first Bianchi identity says that $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$.

A computation also shows that for $\phi, \psi \in \Lambda^2\mathbb{E}^n$,

$$\langle (A\#B)(\phi), \psi\rangle = \frac{1}{2}\sum_{\alpha,\beta}\langle [A(\omega_\alpha), B(\omega_\beta)], \phi\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \psi\rangle \tag{11.13}$$

for any orthonormal basis $\{\omega_\alpha\}_{\alpha=1}^N$ of $\Lambda^2\mathbb{E}^n$.

EXERCISE 11.3 (Formula for $A\#B$). Prove (11.13).

SOLUTION TO EXERCISE 11.3. Let $c_\gamma^{\alpha\beta}$ ($c_{\alpha\beta}^\gamma$) denote the (dual) structure constants corresponding to the orthonormal basis $\{\omega_\alpha\}$. We compute using identity (11.11)

$$\begin{aligned}\langle (A\#B)(\phi), \psi\rangle &= \frac{1}{2}c_\eta^{\gamma\delta}c_\theta^{\alpha\beta}A_{\gamma\alpha}B_{\delta\beta}\phi^\eta\psi^\theta \\ &= \frac{1}{2}\sum_{\alpha,\beta}\langle [A(\omega_\alpha), B(\omega_\beta)], \omega_\eta\rangle \cdot \langle \phi, \omega_\eta\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \omega_\theta\rangle \cdot \langle \psi, \omega_\theta\rangle\,,\end{aligned}$$

from which (11.13) follows.

EXERCISE 11.4. Prove that $\# : S^2(\mathfrak{so}(n)) \times S^2(\mathfrak{so}(n)) \to S^2(\mathfrak{so}(n))$ is bilinear and $\mathrm{O}(n)$-equivariant.

1.2.2. *Operator $Q(A)$ and a trilinear form.* Motivated by the consideration of the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$ (see Section 2 below), we define the operator

$$Q : S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right) \to S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right),$$

$$Q(A) = A \circ A + A\#A = A^2 + A^\#. \tag{11.14}$$

Related to this operator is the bilinear operator defined by

$$Q(A,B) \doteqdot \frac{1}{2}(AB + BA) + A\#B, \tag{11.15}$$

for $A, B \in S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$. It is clear that $Q(A,A) = Q(A)$ and that $Q(A,\mathbf{I}) = A + A\#\mathbf{I}$.

There is a trilinear form tri on $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$ introduced by Huisken and defined by

$$\operatorname{tri}(A,B,C) \doteqdot \operatorname{tr}((AB + BA + 2A\#B) \circ C). \tag{11.16}$$

Note that (11.13) implies that $\operatorname{tr}((A\#B)\cdot C)$ is symmetric in A, B, C since

$$\begin{aligned}\operatorname{tr}((A\#B)\cdot C) &= \sum_\gamma \langle (A\#B)\cdot C(\omega_\gamma), \omega_\gamma\rangle \\ &= \frac{1}{2}\sum_{\alpha,\beta,\gamma}\langle [A(\omega_\alpha), B(\omega_\beta)], C(\omega_\gamma)\rangle\langle [\omega_\alpha, \omega_\beta], \omega_\gamma\rangle.\end{aligned}$$

Hence tri is symmetric in all three variables. We also observe that

$$\operatorname{tri}(A,B,C) = 2\langle Q(A,B), C\rangle\,,$$

where $Q(A,B)$ is defined by (11.15) and $\langle A, B\rangle \doteqdot \operatorname{tr}(AB)$.

The ODE $\frac{d}{dt}A = A^2 + A^{\#}$ on $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$ (compare with (11.31) below) is the *gradient flow* for the functional

$$P(A) \doteqdot \frac{1}{3}\operatorname{tr}(A^3 + A^{\#}\circ A) = \frac{1}{6}\operatorname{tri}(A, A, A). \tag{11.17}$$

To see this, we take a variation $\frac{\partial}{\partial s}A = B$ and compute

$$\begin{aligned}\frac{\partial}{\partial s}P(A) &= \frac{1}{3}\operatorname{tr}(BA^2 + ABA + A^2B + (A\#B + B\#A)\circ A + A^{\#}\circ B)\\ &= \operatorname{tr}(\left(A^2 + A^{\#}\right)\cdot B) = \left\langle A^2 + A^{\#}, B\right\rangle,\end{aligned}$$

using the fact that $\operatorname{tr}((A\#B)\cdot C)$ is totally symmetric and using the standard inner product on $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$.

1.2.3. *The wedge product of two elements in* $S^2(\mathbb{E}^n)$. The sharp operator enables us to construct a new element in $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$ from two given elements. There is another method to construct an element of $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$ from two elements in $S^2(\mathbb{E}^n)$. Namely we define the **wedge product**

$$A \wedge B : \Lambda^2\mathbb{E}^n \to \Lambda^2\mathbb{E}^n$$

of two elements $A, B \in \mathfrak{gl}(n, \mathbb{R})$ by

$$(A\wedge B)(v\wedge w) \doteqdot \frac{1}{2}\left(A(v)\wedge B(w) + B(v)\wedge A(w)\right). \tag{11.18}$$

(Note of course that this 'wedge product' is not the usual wedge product defined for differential forms.)

We have the following.

LEMMA 11.5. *If* $A, B \in S^2(\mathbb{E}^n)$, *then*

(1) $A\wedge B = B\wedge A$,
(2) $\mathrm{id}_{\mathbb{E}^n}\wedge\mathrm{id}_{\mathbb{E}^n} = \mathrm{id}_{\Lambda^2\mathbb{E}^n}$,
(3) $A\wedge B \in S^2(\mathfrak{so}(n)) = S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$.

PROOF. It is easy to see (1) and (2). As for (3) we compute

$$\begin{aligned}\langle (A\wedge B)(v\wedge w), t\wedge u\rangle &= \frac{1}{2}\left\langle \left(A(v)\wedge B(w) + B(v)\wedge A(w)\right), t\wedge u\right\rangle\\ &= \frac{1}{4}\left(\langle A(v), t\rangle\langle B(w), u\rangle - \langle A(v), u\rangle\langle B(w), t\rangle\right)\\ &\quad + \frac{1}{4}\left(\langle B(v), t\rangle\langle A(w), u\rangle - \langle B(v), u\rangle\langle A(w), t\rangle\right)\\ &= \frac{1}{4}\left(\langle A(t), v\rangle\langle B(u), w\rangle - \langle A(u), v\rangle\langle B(t), w\rangle\right)\\ &\quad + \frac{1}{4}\left(\langle B(t), v\rangle\langle A(u), w\rangle - \langle B(u), v\rangle\langle A(t), w\rangle\right)\\ &= \frac{1}{2}\left\langle \left(A(t)\wedge B(u) + B(t)\wedge A(u)\right), v\wedge w\right\rangle\\ &= \langle (A\wedge B)(t\wedge u), v\wedge w\rangle.\end{aligned}$$

□

Note that using the wedge product, we can define a natural map from $S^2(\mathbb{E}^n)$ into $S^2(\mathfrak{so}(n))$ by

$$A \to A \wedge \mathrm{id}\,.$$

The wedge product $\wedge$, restricted to the space of symmetric 2-tensors, which can be identified with $S^2(\mathbb{E}^n)$, is equal to $-1/2$ times the **Kulkarni–Nomizu product** $\odot$ (see p. 176 of Volume One), i.e.,

$$\begin{aligned}(A \wedge B)(x, y, z, t) &= (A \wedge B)(x \wedge y, z \wedge t) \\ &= \frac{1}{2}\left(A(x,z)B(y,t) + A(y,t)B(x,z)\right) \\ &\quad - \frac{1}{2}\left(A(x,t)B(y,z) + A(y,z)B(x,t)\right) \\ &= -\frac{1}{2}(A \odot B)(x,y,z,t).\end{aligned} \tag{11.19}$$

Alternatively, in terms of an orthonormal frame,

$$\begin{aligned}\langle (A \wedge B)(e_i \wedge e_j), e_k \wedge e_\ell \rangle &= \frac{1}{2}\langle A_{ip}B_{jq}e_p \wedge e_q + B_{ip}A_{jq}e_p \wedge e_q, e_k \wedge e_\ell \rangle \\ &= \frac{1}{2}\left(A_{ik}B_{j\ell} + B_{ik}A_{j\ell} - A_{i\ell}B_{jk} - B_{i\ell}A_{jk}\right) \\ &= -\frac{1}{2}(A \odot B)_{ijk\ell}.\end{aligned}$$

1.3. Representations and their tensor products. Here we collect some relevant algebraic representation theory facts concerning the spaces in which the curvature operator and Ricci curvature belong. We refer the reader to Fulton and Harris [**198**] and Goodman and Wallach [**209**] for the background material for these results.[5]

1.3.1. *A primer on basic notions in representation theory.* Let V be a finite-dimensional real vector space and let $\mathrm{GL}(V)$ be the Lie group of invertible linear transformations of V.[6] A **representation** of a Lie group G on V is a Lie group morphism $\rho : G \to \mathrm{GL}(V)$. When it is clear how the Lie group morphism ρ is defined, we just say that V **is a representation of** G.[7] There are several important operations on representations which we now define.

If $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are representations, then the **direct sum representation** $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ is given by

$$(\rho \oplus \sigma)(A)(v \oplus w) \doteqdot Av \oplus Aw.$$

The **tensor product representation** $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W)$ is

$$(\rho \otimes \sigma)(A)(v \otimes w) \doteqdot \rho(A)(v) \otimes \sigma(A)(w)$$

[5] For example, related material is given in Lectures 6 and 15 and §26.3 of [**198**].

[6] Also called the group of automorphisms and denoted $\mathrm{Aut}(V)$.

[7] One also says that V is a G-**module**.

for $A \in G$, $v \in V$, and $w \in W$. More succinctly, we usually suppress the morphisms ρ and σ and write this as

$$A\,(v \otimes w) = Av \otimes Aw.$$

When $V = W$, we can define the **wedge product representation** $\rho \wedge \sigma : G \to \mathrm{GL}\left(\Lambda^2 V\right)$ by

$$(\rho \wedge \sigma)\,(A)\,(v \wedge w) \doteqdot \frac{1}{2}\left\{\rho\,(A)\,(v) \wedge \sigma\,(A)\,(w) + \sigma\,(A)\,(v) \wedge \rho\,(A)\,(w)\right\}$$

for $A \in G$ and $v, w \in V$.

Given a representation $\rho : G \to \mathrm{GL}\,(V)$, the tensor and wedge product representations induce representations $\bigotimes^r V$, $S^r V$ (or $\mathrm{Sym}^r V$), and $\Lambda^r V$ of G, which are the r-th tensor product, symmetric tensor product, and alternating tensor product of V, respectively.

EXAMPLE 11.6. Given the Euclidean vector space $\mathbb{E}^n$, the Lie group

$$\mathrm{O}(n) \doteqdot \{A \in \mathrm{GL}\,(\mathbb{E}^n) : \langle Av, Aw\rangle = \langle v, w\rangle \text{ for all } v, w \in \mathbb{E}^n\}$$

of orthogonal transformations acts on $\mathbb{E}^n$ by multiplication on the left. That is, the inclusion map $\iota : \mathrm{O}(n) \hookrightarrow \mathrm{GL}\,(\mathbb{E}^n)$ is a representation of $\mathrm{O}(n)$ on $\mathbb{E}^n$. Hence we have induced representations of $\mathrm{O}(n)$ on $\bigotimes^r \mathbb{E}^n$, $S^r\mathbb{E}^n$, and $\Lambda^r\mathbb{E}^n$.

If a subspace W of V is invariant under G, i.e., $Aw \in W$ for all $w \in W$ and $A \in G$, then we call W a **subrepresentation** of V. (We also say that W is **G-invariant**.) We say that W is a **proper subrepresentation** if $W \neq \{0\}$ and $W \neq V$. A standard fact is that a representation V is **irreducible**, i.e., V does not contain any proper subrepresentations if and only if V is not a nontrivial direct sum of representations (see Proposition 1.5 on p. 6 of [**198**]). Hence, any representation V is the direct sum of *irreducible* subrepresentations:

$$V = V_1 \oplus \cdots \oplus V_k.$$

1.3.2. *Irreducible decompositions of tensor product representations.* For the standard representation of $\mathrm{O}(n)$ on $\mathbb{E}^n$, we are interested in the tensor product representations of $\mathrm{O}(n)$ and their irreducible subrepresentations. First we recall the following $\mathrm{O}(n)$-invariant decomposition of tensor space $\bigotimes^2\mathbb{E}^n$.

PROPOSITION 11.7 (Irreducible decomposition of $\bigotimes^2 \mathbb{E}^n$).

$$\bigotimes{}^2\mathbb{E}^n = \Lambda^2\mathbb{E}^n \oplus S_0^2\mathbb{E}^n \oplus \mathbb{R}g_{\mathrm{can}} \tag{11.20}$$

is an $\mathrm{O}(n)$*-invariant irreducible orthogonal decomposition of the space of* 2*-tensors. Here* $S_0^2\mathbb{E}^n$ *denotes the space of trace-free symmetric* 2*-tensors and* $g_{\mathrm{can}} = \mathrm{id}$ *is the Euclidean metric on* $\mathbb{E}^n$*.*

Using the identification of $\bigotimes^2\mathbb{E}^n$ with $\mathfrak{gl}(n, \mathbb{R})$, one can also write this as

$$\bigotimes{}^2\mathbb{E}^n = \Lambda^2\mathbb{E}^n \oplus S_0^2(\mathbb{E}^n) \oplus \mathbb{R}\,\mathrm{id}\,.$$

Here $S_0^2(\mathbb{E}^n)$ is the space of trace-free symmetric transformations of $\mathbb{E}^n$. The nontrivial aspect of the proposition is the *irreducibility* of the decomposition. In particular, it is easy to see that each of the factors $\Lambda^2\mathbb{E}^n$, $S_0^2\mathbb{E}^n$, and $\mathbb{R}g_{\mathrm{can}}$ is $\mathrm{O}(n)$-invariant. For example,

$$A\,(v \otimes w - w \otimes v) = Av \otimes Aw - Aw \otimes Av \in \Lambda^2\mathbb{E}^n$$

for $A \in \mathrm{O}(n)$ and $v, w \in \mathbb{E}^n$. We leave it as an exercise to show that $A \in \mathrm{O}(n)$ if and only if $A\,(g_{\mathrm{can}}) = g_{\mathrm{can}}$.

We also have the following $\mathrm{O}(n)$-invariant irreducible decomposition of the tensor space $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$, where the notation shall be defined below (in particular, see (11.45)). The following proposition is discussed further in subsection 2.2 below.

PROPOSITION 11.8 (Irreducible decomposition of $S^2(\mathfrak{so}(n))$).

$$S^2(\mathfrak{so}(n)) = \langle \mathbf{I} \rangle \oplus \langle \mathrm{Rc}_0 \rangle \oplus \langle \mathbf{W} \rangle \oplus \Lambda^4(\mathbb{E}^n) \tag{11.21}$$

is an $\mathrm{O}(n)$*-invariant irreducible orthogonal decomposition.*

2. Algebraic curvature operators and Rm

In this section first we motivate our discussion of algebraic curvature operators by reviewing the evolution equation for the Riemann curvature operator under Ricci flow. We then define the space of algebraic curvature operators and its $\mathrm{O}(n)$-invariant irreducible decomposition. We end this section with a formula for the Ricci tensor (i.e., trace) of the operator $Q\,(\mathbf{R}) \doteqdot \mathbf{R}^2 + \mathbf{R}^{\#}$.

2.1. Motivation for algebraic curvature operator and operator $Q\,(\mathbf{R})$. Let $(\mathcal{M}^n, g)$ be a Riemannian manifold with Levi-Civita connection ∇.[8] Recall that the Riemann curvature tensor is defined by

$$\begin{aligned} R(X,Y)Z &\doteqdot \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ R(X,Y,Z,W) &\doteqdot \langle R(X,Y)Z, W \rangle \end{aligned} \tag{11.22}$$

for tangent vectors X, Y, Z, W. The curvature operator, denoted by Rm, is the symmetric bilinear form on $\Lambda^2 T\mathcal{M}$ (or self-adjoint transformation of $\Lambda^2 T\mathcal{M}$) defined by[9]

$$\mathrm{Rm}(X \wedge Y, Z \wedge W) = \langle \mathrm{Rm}(X \wedge Y), Z \wedge W \rangle \doteqdot 2\langle R(X,Y)W, Z \rangle. \tag{11.23}$$

Note that we have $\mathrm{Rm} = 2\mathbf{I}$ for the standard n-sphere of radius 1.

[8]The reader familiar with §§1–3 of Chapter 6 of Volume One may skip this subsection (referring to it only to confirm notation and conventions).

[9]This definition agrees with the definition of Rm on p. 183 of Volume One. The factor 2 on the RHS of (11.23) is introduced to make our notation (e.g., the definition of Rm) consistent with Volume One.

Let $x \in \mathcal{M}$; if $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_x\mathcal{M}$,[10] then $\{e_i \wedge e_j\}_{i<j}$ forms an orthonormal basis of $\Lambda^2 T_x\mathcal{M}$ with its induced inner product.[11] So $\mathrm{Rm} \in S^2\left(\Lambda^2 T_x^*\mathcal{M}\right) \doteqdot \left(\Lambda^2 T_x^*\mathcal{M}\right) \otimes_S \left(\Lambda^2 T_x^*\mathcal{M}\right)$ can also be viewed as an $N \times N$ symmetric matrix, where $N \doteqdot \frac{n(n-1)}{2}$. Using $\{e_i\}_{i=1}^n$, we can identify $T_x\mathcal{M}$ with $\mathbb{E}^n$ and identify $S^2\left(\Lambda^2 T_x^*\mathcal{M}\right)$ with either $S^2\left(\Lambda^2\left(\mathbb{E}^n\right)^*\right)$ or $S^2(\mathfrak{so}(n))$.

The sharp operation defined in (11.12) can be used to define the sharp operator

$$\#: S^2\left(\Lambda^2 T_x^*\mathcal{M}\right) \times S^2\left(\Lambda^2 T_x^*\mathcal{M}\right) \to S^2\left(\Lambda^2 T_x^*\mathcal{M}\right). \tag{11.24}$$

Because of its $\mathrm{O}(n)$-invariance, the sharp operator is well defined (independent of the choice of $\{e_i\}_{i=1}^n$ used in identifying $T_x\mathcal{M}$ with $\mathbb{E}^n$). Note that this identification enables us to use the properties of # and the other algebraic operations we discussed in the previous section; in particular $\Lambda^2 T_x^*\mathcal{M}$ has the Lie algebra structure of $\mathfrak{so}(n)$.

Now we consider the Ricci flow $\frac{\partial}{\partial t} g = -2\operatorname{Ric}(g)$. After pulling back Rm to a bundle with a fixed metric on the fibers (i.e., using Uhlenbeck's trick as in Section 2 of Chapter 6 in Volume One), we have the following evolution equation for the curvature tensor (see (6.21) of Volume One):

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{abcd} = 2\left(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}\right), \tag{11.25}$$

where[12]

$$B_{abcd} \doteqdot -R_{apbq}R_{cpdq}.$$

Let $\{e_a\}_{a=1}^n$ be an orthonormal basis of local sections of $(\mathcal{V}, h)$, a vector bundle isometric to $(T\mathcal{M}, g(t))$ via a bundle isometry $\iota(t)$. From (11.22) and (11.23), we have

$$R_{abcd} \doteqdot \langle R(\iota_* e_a, \iota_* e_b)\iota_* e_c, \iota_* e_d\rangle = \frac{1}{2}\langle(\iota^* \mathrm{Rm})(e_a \wedge e_b), e_d \wedge e_c\rangle.$$

Abusing notation, we shall often write

$$R_{abcd} = \langle R(e_a, e_b)e_c, e_d\rangle = \frac{1}{2}\langle \mathrm{Rm}(e_a \wedge e_b), e_d \wedge e_c\rangle.$$

We also define

$$\mathrm{Rm}_{abcd} \doteqdot \langle \mathrm{Rm}(e_a \wedge e_b), e_c \wedge e_d\rangle = -2R_{abcd}.$$

Note that

$$\mathrm{Rm}(e_a \wedge e_b) = 2\sum_{c<d} R_{abcd}\, e_d \wedge e_c, \tag{11.26}$$

[10] Our notation, i.e., the use of e_i, is redundant with the notation for the standard basis for $\mathbb{E}^n$.

[11] We adopt the convention that for $X, Y \in T_x\mathcal{M}$, $X \wedge Y = X \otimes Y - Y \otimes X$.

[12] In Volume One on p. 182, the third line from the bottom, a minus sign is missing in the formula for B_{abcd}.

so that for any section $\psi \in \Gamma\left(\Lambda^2 T\mathcal{M}\right)$

$$\operatorname{Rm}(\psi) = 2\sum_{a<b} \psi_{ab} \operatorname{Rm}(e_a \wedge e_b) = 2\sum_{\substack{a<b,\\ c<d}} R_{abcd}\psi_{ab} e_d \wedge e_c$$

and

$$\operatorname{Rm}(\psi)_{cd} = -2\sum_{a<b} R_{abcd}\psi_{ab}.$$

In terms of the squaring operator and the sharp operator # in (11.24), the evolution of Rm is given as follows by (6.27) of Volume One; for the convenience of the reader, we recall its derivation.

PROPOSITION 11.9 (Evolution of Rm).

$$\left(\frac{\partial}{\partial t} - \Delta\right)\operatorname{Rm} = \operatorname{Rm}^2 + \operatorname{Rm}^{\#}, \tag{11.27}$$

where $\operatorname{Rm}^2 \doteqdot \operatorname{Rm}\circ\operatorname{Rm}$ *and* $\operatorname{Rm}^{\#} \doteqdot \operatorname{Rm} \# \operatorname{Rm}$.

PROOF. Thus the LHS of (11.25) is equal to

$$-\frac{1}{2}\left(\frac{\partial}{\partial t} - \Delta\right)\operatorname{Rm}_{abcd}.$$

Since $\operatorname{Rm}^2 = \operatorname{Rm}\circ\operatorname{Rm}$, we have

$$\begin{aligned}\left(\operatorname{Rm}^2\right)_{abcd} &\doteqdot \langle \operatorname{Rm}^2(e_a \wedge e_b), e_c \wedge e_d\rangle \\ &= \left\langle \operatorname{Rm}\left(2\sum_{e<f} R_{abef} e_f \wedge e_e\right), e_c \wedge e_d\right\rangle \\ &= 4\sum_{e<f} R_{abef}R_{cdef} = 2\sum_{e,f} R_{abef}R_{cdef}.\end{aligned}$$

On the other hand, by the first Bianchi identity,

$$\begin{aligned}R_{aebf}R_{cdef} &= \frac{1}{2}\left(R_{aebf}R_{cdef} + R_{afbe}R_{cdfe}\right) \\ &= \frac{1}{2}\left(R_{aebf}R_{cdef} + R_{ebaf}R_{cdef}\right) \\ &= -\frac{1}{2}R_{baef}R_{cdef} = \frac{1}{2}R_{abef}R_{cdef} \qquad (11.28)\end{aligned}$$

and

$$\begin{aligned}B_{abcd} - B_{abdc} &= -R_{aebf}R_{cedf} + R_{aebf}R_{decf} \\ &= R_{aebf}\left(R_{ecdf} + R_{decf}\right) \\ &= -R_{aebf}R_{cdef}.\end{aligned}$$

This shows that the first two terms on the RHS of (11.25) may be expressed as

$$2\left(B_{abcd} - B_{abdc}\right) = -R_{abef}R_{cdef} = -\frac{1}{2}\left(\operatorname{Rm}^2\right)_{abcd}.$$

By definition and (11.13),

$$
\begin{aligned}
&(\mathrm{Rm}^{\#})_{abcd} \\
&= \langle (\mathrm{Rm} \# \mathrm{Rm})(e_a \wedge e_b), e_c \wedge e_d \rangle \\
&= \frac{1}{2} \sum_{\alpha,\beta} \langle [\mathrm{Rm}(\omega_\alpha), \mathrm{Rm}(\omega_\beta)], e_a \wedge e_b \rangle \cdot \langle [\omega_\alpha, \omega_\beta], e_c \wedge e_d \rangle \\
&= \frac{1}{2} \sum_{e<f,s<t} \langle [\mathrm{Rm}(e_e \wedge e_f), \mathrm{Rm}(e_s \wedge e_t)], e_a \wedge e_b \rangle \, \langle [e_e \wedge e_f, e_s \wedge e_t], e_c \wedge e_d \rangle \\
&= \frac{1}{8} \sum R_{efk\ell} R_{stpq} \langle [e_\ell \wedge e_k, e_q \wedge e_p], e_a \wedge e_b \rangle \, \langle [e_e \wedge e_f, e_s \wedge e_t], e_c \wedge e_d \rangle ;
\end{aligned}
\tag{11.29}
$$

here we have used (11.26).

Applying formula (11.7) for the Lie bracket to (11.29), we obtain

$$
(\mathrm{Rm}^{\#})_{abcd} = 4\left(-R_{apdt} R_{bpct} + R_{apct} R_{bpdt}\right) = 4\left(B_{adbc} - B_{acbd}\right), \tag{11.30}
$$

which identifies the last two terms on the RHS of (11.25) with $\mathrm{Rm}^{\#}$:

$$
2\left(B_{acbd} - B_{adbc}\right) = -\frac{1}{2} (\mathrm{Rm}^{\#})_{abcd}.
$$

We have proved (11.27). □

REMARK 11.10. Equation (11.30) can also be proved as follows:[13]

$$
\begin{aligned}
&4\left(R_{apdt} R_{bpct} - R_{apct} R_{bpdt}\right) \\
&= \mathrm{Rm}_{\alpha\beta} \, \varphi^\alpha_{ap} \varphi^\beta_{td} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\gamma_{bp} \varphi^\delta_{tc} - \mathrm{Rm}_{\alpha\beta} \, \varphi^\alpha_{ap} \varphi^\beta_{tc} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\gamma_{bp} \varphi^\delta_{td} \\
&= \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\alpha_{ap} \varphi^\gamma_{bp} \left(\varphi^\beta_{td} \varphi^\delta_{tc} - \varphi^\beta_{tc} \varphi^\delta_{td} \right) \\
&= \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\alpha_{ap} \varphi^\gamma_{bp} [\varphi^\beta, \varphi^\delta]_{cd} \\
&= \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\alpha_{ap} \varphi^\gamma_{bp} c^{\beta\delta}_\eta \varphi^\eta_{cd} \\
&= \frac{1}{2} \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\alpha_{ap} \varphi^\gamma_{bp} c^{\beta\delta}_\eta \varphi^\eta_{cd} - \frac{1}{2} \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, \varphi^\gamma_{ap} \varphi^\alpha_{bp} c^{\beta\delta}_\eta \varphi^\eta_{cd} \\
&= -\frac{1}{2} \mathrm{Rm}_{\alpha\beta} \, \mathrm{Rm}_{\gamma\delta} \, c^{\beta\delta}_\eta \varphi^\eta_{cd} [\varphi^\alpha, \varphi^\gamma]_{ab} \\
&= -(\mathrm{Rm}^{\#})_{abcd},
\end{aligned}
$$

where in the third line we used $[\phi^\beta, \phi^\delta]_{cd} = \phi^\beta_{ct} \phi^\delta_{td} - \phi^\beta_{dt} \phi^\delta_{tc}$ from (11.6).

In view of the Weinberger–Hamilton maximum principle for systems, we study the ODE:

$$
\frac{d}{dt} \mathbf{R} = \mathbf{R}^2 + \mathbf{R}^{\#} \doteqdot Q(\mathbf{R}) \tag{11.31}
$$

[13] The identification of $T_x\mathcal{M}$ with $\mathbb{E}^n$ enables us to use the orthonormal basis $\{\varphi^\alpha\}_{\alpha=1}^N$ of $\Lambda^2 T_x^* \mathcal{M}$ with $\varphi^\alpha_{ab} \doteqdot \varphi^\alpha(e_a, e_b)$.

on $S^2\left(\Lambda^2 T_x^*\mathcal{M}\right) \cong S^2\left(\Lambda^2\mathbb{E}^n\right)$. In particular, this is the ODE corresponding to the PDE for Rm:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{Rm} = \mathrm{Rm}^2 + \mathrm{Rm}^{\#}.$$

We want to find $\mathrm{O}(n)$-invariant convex closed sets preserved by the ODE. We shall study this ODE in the subspace of algebraic curvature operators $S_B^2(\mathfrak{so}(n)) \subset S^2(\mathfrak{so}(n)) = S^2\left(\Lambda^2 T_x^*\mathcal{M}\right)$.[14] Note that if $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$, then $\mathbf{R}^2 + \mathbf{R}^{\#} \in S_B^2(\mathfrak{so}(n))$ (see (11.54) below). Now we turn to the discussion of algebraic curvature operators.

2.2. Algebraic curvature operators. Since the Riemannian curvature operator satisfies the *first Bianchi identity*, we shall define the space of algebraic curvature operators to be the subspace of $S^2(\mathfrak{so}(n))$ consisting of elements which satisfy the first Bianchi identity.

In the remainder of this section we shall use notation from Section 1 of this chapter.

2.2.1. *The Bianchi map and the space of algebraic curvature operators.* We define the **Bianchi map** $b : \bigotimes^4 \mathbb{E}^n \to \bigotimes^4 \mathbb{E}^n$ by

$$b(R)(x,y,z,t) \doteqdot \frac{1}{3}\left(R(x,y,z,t) + R(y,z,x,t) + R(z,x,y,t)\right) \tag{11.32}$$

for $R \in \bigotimes^4 \mathbb{E}^n$. Given a Riemannian manifold $(\mathcal{M}^n, g)$, we may analogously define the **Bianchi map** $b : \bigotimes^4 T\mathcal{M} \to \bigotimes^4 T\mathcal{M}$. The first Bianchi identity then says that $b(\mathrm{Rm}) = 0$.

LEMMA 11.11. *The Bianchi map has the following properties.*

(i) *The subspace $S^2(\Lambda^2\mathbb{E}^n) \subset \bigotimes^4 \mathbb{E}^n$ is invariant under the action of b, i.e.,*

$$b(S^2(\Lambda^2\mathbb{E}^n)) \subset S^2(\Lambda^2\mathbb{E}^n). \tag{11.33}$$

(ii) *For $R \in S^2(\Lambda^2\mathbb{E}^n)$,*

$$b(R)(x,z,y,t) = -b(R)(x,y,z,t). \tag{11.34}$$

Hence $b(R) \in \Lambda^4\mathbb{E}^n$, i.e.,

$$b|_{S^2(\Lambda^2\mathbb{E}^n)} : S^2(\Lambda^2\mathbb{E}^n) \to \Lambda^4\mathbb{E}^n.$$

For any $R \in \Lambda^4\mathbb{E}^n$, we have $b(R) = R$, so that the above map is onto.

(iii) *The map b is self-adjoint, i.e., for any $R, S \in \bigotimes^4 \mathbb{E}^n$,*

$$\langle b(R), S\rangle = \langle R, b(S)\rangle. \tag{11.35}$$

---

[14] See (11.36) for the definition of $S_B^2(\mathfrak{so}(n))$.

PROOF. (i) If $R \in S^2(\Lambda^2 \mathbb{E}^n)$, then

$$\begin{aligned} b(R)(y,x,z,t) &= \frac{1}{3}\left(R(y,x,z,t)+R(x,z,y,t)+R(z,y,x,t)\right) \\ &= \frac{1}{3}\left(-R(x,y,z,t)-R(z,x,y,t)-R(y,z,x,t)\right) \\ &= -b(R)(x,y,z,t). \end{aligned}$$

Similarly one can show $b(R)(x,y,t,z) = -b(R)(x,y,z,t)$. Finally,

$$\begin{aligned} b(R)(z,t,x,y) &= \frac{1}{3}\left(R(z,t,x,y)+R(t,x,z,y)+R(x,z,t,y)\right) \\ &= \frac{1}{3}\left(R(x,y,z,t)+R(z,y,t,x)+R(z,x,y,t)\right) \\ &= b(R)(x,y,z,t). \end{aligned}$$

(ii) We compute

$$\begin{aligned} b(R)(x,z,y,t) &= \frac{1}{3}\left(R(x,z,y,t)+R(z,y,x,t)+R(y,x,z,t)\right) \\ &= \frac{1}{3}\left(R(y,x,z,t)+R(z,y,x,t)+R(x,z,y,t)\right) \\ &= -b(R)(x,y,z,t). \end{aligned}$$

(iii) We also compute

$$\begin{aligned} &\langle b(R), S\rangle \\ &= \sum_{i,j,k,\ell} b(R)(e_i,e_j,e_k,e_\ell)\cdot S(e_i,e_j,e_k,e_\ell) \\ &= \frac{1}{3}\sum \left(R(e_i,e_j,e_k,e_\ell)+R(e_j,e_k,e_i,e_\ell)+R(e_k,e_i,e_j,e_\ell)\right)\cdot S(e_i,e_j,e_k,e_\ell) \\ &= \frac{1}{3}\sum R(e_i,e_j,e_k,e_\ell)\cdot\left(S(e_i,e_j,e_k,e_\ell)+S(e_k,e_i,e_j,e_\ell)+S(e_j,e_k,e_i,e_\ell)\right) \\ &= \langle R, b(S)\rangle . \end{aligned}$$

□

Note that $S^2(\Lambda^2\mathbb{E}^n) = S^2(\mathfrak{so}(n))$; we define

$$S^2_B(\mathfrak{so}(n)) \doteqdot \ker\left(b|_{S^2(\mathfrak{so}(n))}\right). \tag{11.36}$$

We call any element $\mathbf{R} \in S^2_B(\mathfrak{so}(n))$ an **algebraic curvature operator**. It is clear that the curvature operator of any Riemannian manifold lies inside the space $S^2_B\left(\Lambda^2 T^*\mathcal{M}\right)$.

CAVEAT. The sign convention for the notation $\mathbf{R}_{ijk\ell} \doteqdot \langle \mathbf{R}(e_i\wedge e_j), e_k \wedge e_\ell\rangle$ for the components of the algebraic curvature operator $\mathbf{R}$ (compare with (11.23)) differs from that for the Riemann curvature tensor

$$R_{ijk\ell} \doteqdot \langle R(e_i,e_j)e_\ell, e_k\rangle = \frac{1}{2}\langle \mathrm{Rm}(e_i\wedge e_j), e_\ell\wedge e_k\rangle$$

(see (11.22)). That is, we have the following relation between the components of the Riemann curvature *tensor* and the components of the Riemann curvature *operator*

$$R_{ijk\ell} = -\frac{1}{2} \operatorname{Rm}_{ijk\ell} .$$

REMARK 11.12. Another way to define the Bianchi map is by

$$\frac{1}{2} b(\omega_\alpha \otimes \omega_\beta + \omega_\beta \otimes \omega_\alpha) \doteqdot \omega_\alpha \wedge \omega_\beta.$$

We have, using (11.5), that

$$\begin{aligned} b\left((e_i \otimes e_i) \wedge (e_j \otimes e_j)\right) &= \frac{1}{2} b\left((e_i \wedge e_j) \otimes (e_i \wedge e_j)\right) \\ &= \frac{1}{2}(e_i \wedge e_j) \wedge (e_i \wedge e_j) = 0. \end{aligned}$$

That is, $(e_i \otimes e_i) \wedge (e_j \otimes e_j) \in S^2_B(\mathfrak{so}(n))$.

By Lemma 11.11(i) and (ii),

$$\begin{aligned} S^2(\mathfrak{so}(n)) &= \ker(b|_{S^2(\Lambda^2\mathbb{E}^n)}) \oplus \operatorname{image}(b|_{S^2(\Lambda^2\mathbb{E}^n)}) \\ &= S^2_B(\mathfrak{so}(n)) \oplus \Lambda^4\mathbb{E}^n. \end{aligned} \tag{11.37}$$

The following gives examples of algebraic curvature operators.

LEMMA 11.13. *For any $A \in S^2(\mathbb{E}^n)$ we have the following.*

(i) $A \wedge \mathrm{id} \in S^2_B(\mathfrak{so}(n))$. *Hence we may define the map*

$$\begin{aligned} \mathrm{id}_\wedge : S^2(\mathbb{E}^n) &\to S^2_B(\mathfrak{so}(n)), \\ \mathrm{id}_\wedge (A) &\doteqdot \mathrm{id} \wedge A. \end{aligned}$$

(ii) $A \wedge A \in S^2_B(\mathfrak{so}(n))$.

PROOF. (i) We compute using (11.19)

$$\begin{aligned} &b(A \wedge \mathrm{id})(x,y,z,t) \\ &= \frac{1}{3}\left((A \wedge \mathrm{id})(x,y,z,t) + (A \wedge \mathrm{id})(y,z,x,t) + (A \wedge \mathrm{id})(z,x,y,t)\right) \\ &= \frac{1}{6}\begin{pmatrix} A(x,z)\langle y,t\rangle + A(y,t)\langle x,z\rangle - A(x,t)\langle y,z\rangle - A(y,z)\langle x,t\rangle \\ +A(y,x)\langle z,t\rangle + A(z,t)\langle y,x\rangle - A(y,t)\langle x,z\rangle - A(x,z)\langle y,t\rangle \\ +A(z,y)\langle x,t\rangle + A(x,t)\langle z,y\rangle - A(z,t)\langle x,y\rangle - A(x,y)\langle z,t\rangle \end{pmatrix} \\ &= 0. \end{aligned}$$

(ii) We compute

$$
\begin{aligned}
&b(A\wedge A)(x,y,z,t)\\
&=\frac{1}{3}\left((A\wedge A)(x,y,z,t)+(A\wedge A)(y,z,x,t)+(A\wedge A)(z,x,y,t)\right)\\
&=\frac{1}{6}\begin{pmatrix} A(x,z)A(y,t)+A(y,t)A(x,z)-A(x,t)A(y,z)-A(y,z)A(x,t)\\ +A(y,x)A(z,t)+A(z,t)A(y,x)-A(y,t)A(x,z)-A(x,z)A(y,t)\\ +A(z,y)A(x,t)+A(x,t)A(z,y)-A(z,t)A(x,y)-A(x,y)A(z,t)\end{pmatrix}\\
&=0.
\end{aligned}
$$

□

We may define the **Ricci tensor** $\operatorname{Rc}(\mathbf{R})\in S^2(\mathbb{E}^n)$ corresponding to an algebraic curvature operator $\mathbf{R}\in S_B^2(\Lambda^2\mathbb{E}^n)$ by[15]

$$
\langle \operatorname{Rc}(\mathbf{R})(e_i),e_j\rangle=\sum_{k=1}^{n}\langle \mathbf{R}(e_i\wedge e_k),e_j\wedge e_k\rangle=\mathbf{R}_{ikjk}. \tag{11.38}
$$

That is, the Ricci tensor corresponding to an algebraic curvature operator is the same trace as is used to obtain the usual Ricci tensor from the Riemann curvature operator:

$$
\langle \operatorname{Rc}(\operatorname{Rm})(e_i),e_j\rangle=\sum_{k=1}^{n}\langle \operatorname{Rm}(e_i\wedge e_k),e_j\wedge e_k\rangle=2\sum_{k=1}^{n}R(e_i,e_k,e_k,e_j)=2R_{ij}.
$$

Caveat. Note that for a Riemannian metric the Ricci tensor of its curvature operator Rm is twice the Ricci curvature tensor of the metric.

We define the **scalar curvature** associated to an algebraic curvature operator $\mathbf{R}$ by[16]

$$
\operatorname{Scal}(\mathbf{R})\doteqdot\operatorname{tr}(\operatorname{Rc}(\mathbf{R}))=\mathbf{R}_{ikik}. \tag{11.39}
$$

Hence, for a Riemannian metric, the scalar curvature of the curvature operator, i.e., Scal(Rm), is equal to twice the scalar curvature of the metric.

2.2.2. *Decomposition of* $S_B^2(\mathfrak{so}(n))$. When studying the curvature operator on a Riemannian n-manifold, it is important to decompose the Riemann curvature tensor into the sum of the Weyl tensor, $\frac{2}{n-2}\operatorname{Rc}_0\wedge\operatorname{id}$ and $\frac{S}{n(n-1)}\mathbf{I}$, where Rc_0 and S denote the trace-free Ricci tensor and the scalar curvature, respectively. Algebraic curvature operators admit the same decomposition, which we now discuss after some preparation.

The following identities regarding traces (i.e., the Rc operator) are an immediate consequence of the definition of $\operatorname{Rc}(\mathbf{R})$ and (11.19).

---

[15]This definition also applies to any $R\in S^2(\Lambda^2\mathbb{E}^n)=S^2(\mathfrak{so}(n))$.

[16]Note that

$$
\operatorname{trace}(\mathbf{R})=\sum_{\alpha=1}^{N}\langle \mathbf{R}(\omega_\alpha),\omega_\alpha\rangle=\frac{1}{2}\operatorname{Scal}(\mathbf{R}),
$$

where $\omega_\alpha\doteqdot e_i\wedge e_j$.

LEMMA 11.14 (Trace of the wedge product). *If* $A, B \in S^2(\mathbb{E}^n)$, *then* $A \wedge B \in S^2(\mathfrak{so}(n))$ *satisfies*

$$\operatorname{Rc}(A \wedge B) = \frac{1}{2}\left(A(\operatorname{tr}(B)\operatorname{id} - B)\right) + \frac{1}{2}\left(B(\operatorname{tr}(A)\operatorname{id} - A)\right). \tag{11.40}$$

In particular,

$$\operatorname{Rc}(A \wedge A) = \operatorname{tr}(A)A - A^2 \tag{11.41}$$

and

$$\operatorname{Rc}(A \wedge \operatorname{id}) = \frac{n-2}{2}A + \frac{\operatorname{tr}(A)}{2}\operatorname{id}. \tag{11.42}$$

Hence, if $\operatorname{tr}(A) = 0$, *then*

$$\operatorname{Rc}(A \wedge A) = -A^2 \quad \textit{and} \quad \operatorname{Rc}(A \wedge \operatorname{id}) = \frac{n-2}{2}A. \tag{11.43}$$

PROOF. We compute

$$\begin{aligned}
\langle \operatorname{Rc}(A \wedge B)(e_i), e_j \rangle &= \sum_k \langle (A \wedge B)(e_i \wedge e_k), e_j \wedge e_k \rangle \\
&= \frac{1}{2}\left(A_{ij}B_{kk} + B_{ij}A_{kk} - A_{ik}B_{kj} - B_{ik}A_{kj}\right) \\
&= \frac{1}{2}\left(A_{ij}\operatorname{tr}(B) - A_{ik}B_{kj} + B_{ij}\operatorname{tr}(A) - B_{ik}A_{kj}\right).
\end{aligned}$$

□

An important algebraic fact is given by the following.

LEMMA 11.15.

(1) *If* $n > 2$, *then the map*

$$\operatorname{id}_\wedge : S^2(\mathbb{E}^n) \to S_B^2(\Lambda^2 \mathbb{E}^n),$$

defined by

$$\operatorname{id}_\wedge(A) \doteqdot \operatorname{id} \wedge A,$$

is injective.

(2) *The operator* $\operatorname{id}_\wedge$ *is the adjoint of* $\operatorname{Rc} : S_B^2(\Lambda^2 \mathbb{E}^n) \to S^2(\mathbb{E}^n)$.

PROOF. From tracing (11.42), we have

$$\operatorname{Scal}(A \wedge \operatorname{id}) = (n-1)\operatorname{tr}(A).$$

Hence if $A \wedge \mathrm{id} = 0$, we then have that $\mathrm{tr}(A) = 0$ and thus $A = 0$. We calculate using (11.19)

$$\begin{aligned}\langle \mathrm{id} \wedge A, \mathbf{R} \rangle &= \sum_{i<j} \langle (\mathrm{id} \wedge A)(e_i \wedge e_j), \mathbf{R}(e_i \wedge e_j) \rangle \\ &= \sum_{i<j,\, k<\ell} \langle (\mathrm{id} \wedge A)(e_i \wedge e_j), e_k \wedge e_\ell \rangle \langle \mathbf{R}(e_i \wedge e_j), e_k \wedge e_\ell \rangle \\ &= \frac{1}{8} \sum_{i,j,k,\ell} \left(\delta_{ik} A_{j\ell} + A_{ik} \delta_{j\ell} - \delta_{i\ell} A_{jk} - A_{i\ell} \delta_{jk} \right) \mathbf{R}_{ijk\ell} \\ &= \frac{1}{8} \sum \left(\mathbf{R}_{iji\ell} A_{j\ell} + \mathbf{R}_{ijkj} A_{ik} - \mathbf{R}_{ijki} A_{jk} - \mathbf{R}_{ijj\ell} A_{i\ell} \right) \\ &= \frac{1}{2} \mathrm{Rc}(\mathbf{R})_{j\ell} A_{j\ell} = \langle \mathrm{Rc}(\mathbf{R}), A \rangle.\end{aligned}$$

□

With the help of the lemma above, we can decompose $S_B^2(\Lambda^2 \mathbb{E}^n)$ as

$$\begin{aligned}S_B^2(\Lambda^2 \mathbb{E}^n) &= \mathrm{image}(\mathrm{id}_\wedge) + \ker(\mathrm{Rc}) \\ &= \mathrm{id}_\wedge \left(S^2(\mathbb{E}^n) \right) \oplus \ker(\mathrm{Rc}).\end{aligned}$$

Since $S^2(\mathbb{E}^n) = S_0^2(\mathbb{E}^n) \oplus \mathbb{R}\,\mathrm{id}$, we finally arrive at the decomposition

$$\begin{aligned}S_B^2(\Lambda^2 \mathbb{E}^n) &= \mathrm{id}_\wedge(\mathbb{R}\,\mathrm{id}) \oplus \mathrm{id}_\wedge S_0^2(\mathbb{E}^n) \oplus \ker(\mathrm{Rc}) \\ &= \langle \mathbf{I} \rangle \oplus \langle \mathrm{Rc}_0 \rangle \oplus \langle \mathbf{W} \rangle, \qquad (11.44)\end{aligned}$$

where

$$\langle \mathbf{I} \rangle \doteqdot \mathbb{R}\,\mathrm{id} \wedge \mathrm{id}\,, \quad \langle \mathrm{Rc}_0 \rangle \doteqdot \mathrm{id}_\wedge S_0^2(\mathbb{E}^n), \quad \text{and} \quad \langle \mathbf{W} \rangle \doteqdot \ker(\mathrm{Rc}) \qquad (11.45)$$

(recall that $\mathbf{I} \in S_B^2(\mathfrak{so}(n))$ is the identity map of $\mathfrak{so}(n)$, which is equal to $\mathrm{id} \wedge \mathrm{id}$). Summarizing this discussion, we have the following.

LEMMA 11.16 (Irreducible decomposition of $S_B^2(\mathfrak{so}(n))$). *The irreducible decomposition of $S_B^2(\mathfrak{so}(n))$ is given by*

$$S_B^2(\mathfrak{so}(n)) = \langle \mathbf{I} \rangle \oplus \langle \mathrm{Rc}_0 \rangle \oplus \langle \mathbf{W} \rangle. \qquad (11.46)$$

Combining this lemma with (11.37), we have the decomposition for $S^2(\mathfrak{so}(n))$ claimed before in Proposition 11.8. Again the nontrivial aspect of the statement of Proposition 11.8 is that this decomposition is irreducible; however, we do not need this fact for our discussion later.

2.2.3. *Algebraic curvature operator decomposition.* Now for any algebraic curvature operator $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$ we denote by $\mathbf{R_I}$, $\mathbf{R}_{\mathrm{Rc}_0}$, and $\mathbf{R_W}$ its components with respect to the irreducible decomposition (11.46) so that

$$\mathbf{R} = \mathbf{R_I} + \mathbf{R}_{\mathrm{Rc}_0} + \mathbf{R_W}.$$

One can calculate these components as follows. Let

$$\mathbf{R_I} \doteqdot c\,\mathrm{id} \wedge \mathrm{id} \quad \text{and} \quad \mathbf{R}_{\mathrm{Rc}_0} \doteqdot \mathrm{id} \wedge A$$

for some $A \in S_0^2(\mathbb{E}^n)$ and $c \in \mathbb{R}$. From Lemma 11.14 we have for $A \in S_0^2(\mathbb{E}^n)$

$$\begin{aligned} \operatorname{Rc}(\mathbf{I}) &= (n-1)\,\mathrm{id}, \quad & \operatorname{Rc}(\mathrm{id}\wedge A) &= \frac{n-2}{2}A, \\ \operatorname{Scal}(\mathbf{I}) &= n(n-1), & \operatorname{Scal}(\mathrm{id}\wedge A) &= 0. \end{aligned}$$

Let

(11.47)
$$\bar{\lambda} \doteqdot \frac{1}{n}\operatorname{Scal}(\mathbf{R}), \quad \operatorname{Rc}_0(\mathbf{R}) \doteqdot \operatorname{Rc}(\mathbf{R}) - \bar{\lambda}\,\mathrm{id}\,, \quad \text{and} \quad \sigma \doteqdot \frac{1}{n}\|\operatorname{Rc}_0(\mathbf{R})\|^2.$$

Then $c = \frac{\bar{\lambda}}{n-1}$, which then implies that $A = \frac{2}{n-2}\operatorname{Rc}_0(\mathbf{R})$. Hence we have for any algebraic curvature operator

$$\mathbf{R} = \frac{\bar{\lambda}}{n-1}\mathbf{I} + \frac{2}{n-2}\,\mathrm{id}\wedge\operatorname{Rc}_0(\mathbf{R}) + \mathbf{R}_{\mathbf{W}} \tag{11.48}$$

$$= \frac{\operatorname{Scal}(\mathbf{R})}{n\,(n-1)}\mathbf{I} + \frac{2}{n-2}\,\mathrm{id}\wedge\left(\operatorname{Rc}(\mathbf{R}) - \frac{\operatorname{Scal}(\mathbf{R})}{n}\,\mathrm{id}\right) + \mathbf{R}_{\mathbf{W}} \tag{11.49}$$

$$= -\frac{\operatorname{Scal}(\mathbf{R})}{(n-1)\,(n-2)}\mathbf{I} + \frac{2}{n-2}\,\mathrm{id}\wedge\operatorname{Rc}(\mathbf{R}) + \mathbf{R}_{\mathbf{W}}. \tag{11.50}$$

As a consequence of (11.46), we also have the following, which we shall later apply to $A = \operatorname{Rc}_0$.

COROLLARY 11.17 (Decomposition of $A \wedge A$). *For any $A \in S^2(\mathbb{E}^n)$ with $\operatorname{tr}(A) = 0$, we have*

$$A \wedge A = -\frac{\operatorname{tr}\left(A^2\right)}{n\,(n-1)}\mathbf{I} - \frac{2}{n-2}(A^2)_0 \wedge \mathrm{id} + (A\wedge A)_{\mathbf{W}}, \tag{11.51}$$

where $(A^2)_0$ is the trace-free part of $A^2 \in S^2(\mathbb{E}^n)$.

PROOF. Recall that $A \wedge A \in S_B^2(\mathfrak{so}(n))$ by Lemma 11.13. By (11.41) and (11.49), we have for any $A \in S^2(\mathbb{E}^n)$

$$\begin{aligned} A \wedge A = {} & \frac{2}{n-2}\,\mathrm{id}\wedge\left(\operatorname{tr}(A)A - A^2 - \frac{\operatorname{tr}\left(\operatorname{tr}(A)A - A^2\right)}{n}\,\mathrm{id}\right) \\ & + \frac{\operatorname{tr}\left(\operatorname{tr}(A)A - A^2\right)}{n\,(n-1)}\mathbf{I} + (A\wedge A)_{\mathbf{W}}. \end{aligned}$$

The result now follows from the assumption $\operatorname{tr}(A) = 0$. □

2.3. The quadratic $\mathbf{R}^2 + \mathbf{R}^\#$. Recall that $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#$ for any algebraic curvature operator $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$. The calculation in the proof of Proposition 11.9 only uses the algebraic properties and identities satisfied by Rm. Hence the quadratic formulas for the curvature operator Rm we derived still hold for algebraic curvature operators in general. In particular, for any algebraic curvature operator $\mathbf{R}$ we have[17]

(11.52)
$$(\mathbf{R}^2+\mathbf{R}^{\#})_{ijk\ell} = \mathbf{R}_{ipjq}\mathbf{R}_{kp\ell q} - \mathbf{R}_{ipjq}\mathbf{R}_{\ell pkq} + \mathbf{R}_{ipkq}\mathbf{R}_{jp\ell q} - \mathbf{R}_{ip\ell q}\mathbf{R}_{jpkq}$$

(11.53)
$$= -\mathbf{B}_{ijk\ell} + \mathbf{B}_{ij\ell k} - \mathbf{B}_{ikj\ell} + \mathbf{B}_{i\ell jk},$$

where $\mathbf{B}_{ijk\ell} \doteqdot -\mathbf{R}_{ipjq}\mathbf{R}_{kp\ell q}$. Also, from (11.28) we have

$$\mathbf{R}_{aebf}\mathbf{R}_{cdef} = \frac{1}{2}\mathbf{R}_{abef}\mathbf{R}_{cdef}.$$

It follows from (11.52) that for any $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$

$$Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^{\#} \in S_B^2(\mathfrak{so}(n)). \tag{11.54}$$

Indeed, one readily checks that the right-hand side satisfies the first Bianchi identity.

Viewing $Q(\mathbf{R})$ as an algebraic curvature operator, we can compute its Ricci tensor and scalar curvature.

Corollary 11.18 (Trace of $\mathbf{R}^2+\mathbf{R}^{\#}$). *For any algebraic curvature operator* $\mathbf{R}$,

$$\operatorname{Rc}(\mathbf{R}^2+\mathbf{R}^{\#})_{ij} = \sum_{k,\ell} \operatorname{Rc}(\mathbf{R})_{k\ell}\mathbf{R}_{ikj\ell}, \tag{11.55}$$

$$\operatorname{Scal}\left(\mathbf{R}^2+\mathbf{R}^{\#}\right) = \sum_{k,\ell}\left(\operatorname{Rc}(\mathbf{R})_{k\ell}\right)^2. \tag{11.56}$$

In particular if $\mathbf{R} \in \langle \mathbf{W} \rangle$, *then* $\mathbf{R}^2+\mathbf{R}^{\#} \in \langle \mathbf{W}\rangle$. *That is, if* $\operatorname{Rc}(\mathbf{R}) = 0$, *then* $\operatorname{Rc}(\mathbf{R}^2+\mathbf{R}^{\#}) = 0$ *and* $\operatorname{Scal}\left(\mathbf{R}^2+\mathbf{R}^{\#}\right) = 0$.

Remark 11.19. When $\mathbf{R} = \operatorname{Rm}$, this can be seen from standard formulas for the Ricci flow since the evolution for Rc is given by (6.7) and the trace of (6.27) both in Volume One.

Proof. From (11.52) we have

$$\operatorname{Rc}(\mathbf{R}^2+\mathbf{R}^{\#})_{ik} = (\mathbf{R}^2+\mathbf{R}^{\#})_{ijkj} = \mathbf{R}_{ipjq}\mathbf{R}_{kpjq} - 2\mathbf{R}_{ipjq}\mathbf{R}_{jpkq} + \mathbf{R}_{ipkq}\mathbf{R}_{jpjq}.$$

Using the identity $\mathbf{R}_{aebf}\mathbf{R}_{cdef} = \frac{1}{2}\mathbf{R}_{abef}\mathbf{R}_{cdef}$, we have

$$\mathbf{R}_{ipjq}\mathbf{R}_{jpkq} = \mathbf{R}_{ipqj}\mathbf{R}_{kqpj} = \frac{1}{2}\mathbf{R}_{iqpj}\mathbf{R}_{kqpj} = \frac{1}{2}\mathbf{R}_{ipjq}\mathbf{R}_{kpjq}.$$

Hence $\operatorname{Rc}(\mathbf{R}^2+\mathbf{R}^{\#})_{ik} = \mathbf{R}_{ipkq}\mathbf{R}_{jpjq} = \mathbf{R}_{ipkq}\operatorname{Rc}(\mathbf{R})_{pq}$, which proves (11.55). The lemma follows easily. □

With respect to the curvature decomposition we have the following properties of the bilinear form Q and the trilinear form tri defined in (11.15) and (11.16), respectively.

Lemma 11.20. *If* $\mathbf{R} \in \langle \operatorname{Rc}_0 \rangle$ *and* $\mathbf{S}, \mathbf{W} \in \langle \mathbf{W} \rangle$, *then*

[17] Again note our sign convention.

(i)

$$\operatorname{tri}(\mathbf{S}, \mathbf{R}, \mathbf{W}) = \operatorname{tr}((\mathbf{SR} + \mathbf{RS} + 2\mathbf{R}\#\mathbf{S})\mathbf{W}) = 0,$$

(ii)

$$\operatorname{tri}(\mathbf{S}, \mathbf{R}, \mathbf{I}) = \operatorname{tr}((\mathbf{SR} + \mathbf{RS} + 2\mathbf{R}\#\mathbf{S})\mathbf{I}) = 0,$$

(iii)

$$Q(\mathbf{S}, \mathbf{R}) = \mathbf{SR} + \mathbf{RS} + 2\mathbf{R}\#\mathbf{S} \in \langle \mathrm{Rc}_0 \rangle. \tag{11.57}$$

PROOF. To prove (i), we first recall that $\operatorname{tri}(\mathbf{S}, \mathbf{R}, \mathbf{W}) = \operatorname{tri}(\mathbf{W}, \mathbf{S}, \mathbf{R})$. On the other hand, by Corollary 11.18,

$$(\mathbf{W} + \mathbf{S})^2 + (\mathbf{W} + \mathbf{S})^\# \in \langle \mathbf{W} \rangle$$

for any $\mathbf{W}, \mathbf{S} \in \langle \mathbf{W} \rangle$, and hence $\mathbf{WS} + \mathbf{SW} + 2\mathbf{W}\#\mathbf{S} \in \langle \mathbf{W} \rangle$. Therefore, since $\mathbf{R} \in \langle \mathrm{Rc}_0 \rangle$,

$$0 = \langle \mathbf{WS} + \mathbf{SW} + 2\mathbf{W}\#\mathbf{S}, \mathbf{R} \rangle = \operatorname{tr}((\mathbf{WS} + \mathbf{SW} + 2\mathbf{W}\#\mathbf{S})\mathbf{R}) = \operatorname{tri}(\mathbf{W}, \mathbf{S}, \mathbf{R}).$$

To prove (ii), using the symmetry of tri, we have

$$\operatorname{tri}(\mathbf{S}, \mathbf{R}, \mathbf{I}) = \operatorname{tri}(\mathbf{S}, \mathbf{I}, \mathbf{R}) = \operatorname{tr}((2\mathbf{S} + 2\mathbf{S}\#\mathbf{I})\mathbf{R}) = 0;$$

the reason for why the last equality holds is that (11.78) below implies $\mathbf{S} + \mathbf{S}\#\mathbf{I} = 0$.

Part (iii) follows directly from (i) and (ii). □

3. A family of linear transformations and their effect on $\mathbf{R}^2 + \mathbf{R}^\#$

As noted earlier, we are looking for families of $\mathrm{O}(n)$-invariant closed convex sets in $S^2_B(\mathfrak{so}(n))$ preserved by the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$. The main idea is to consider invertible $\mathrm{O}(n)$-invariant linear transformations of known $\mathrm{O}(n)$-invariant closed convex sets in $S^2_B(\mathfrak{so}(n))$ preserved by the ODE. In this section we discuss such linear transformations, their properties, and the conditions under which the transformed convex sets will be preserved.

3.1. $\mathrm{O}(n)$**-invariant linear transformations of** $S^2_B(\mathfrak{so}(n))$**.** We first discuss some preliminaries. Let $L : S^2_B(\mathfrak{so}(n)) \to S^2_B(\mathfrak{so}(n))$ be a self-adjoint $\mathrm{O}(n)$-invariant linear transformation, so that

$$L(g(\mathbf{R})) = g(L(\mathbf{R}))$$

for all $g \in \mathrm{O}(n)$ and $\mathbf{R} \in S^2_B(\mathfrak{so}(n))$. Then L preserves the irreducible decomposition of $S^2_B(\mathfrak{so}(n))$, i.e.,

$$L(\langle \mathbf{I} \rangle) \subset \langle \mathbf{I} \rangle, \qquad L(\langle \mathrm{Rc}_0 \rangle) \subset \langle \mathrm{Rc}_0 \rangle, \qquad L(\langle \mathbf{W} \rangle) \subset \langle \mathbf{W} \rangle,$$

and furthermore, L is a multiple of the identity on each component. A way to see this is as follows. Since L is self-adjoint, it has real eigenvalues $\lambda_1 < \cdots < \lambda_k$ with corresponding eigenspaces $E_1, \ldots, E_k$ giving an orthogonal decomposition of $S^2_B(\mathfrak{so}(n))$:

$$S^2_B(\mathfrak{so}(n)) = E_1 \oplus \cdots \oplus E_k.$$

Each eigenspace is $\mathrm{O}(n)$-invariant and hence contains one of the irreducible components $\langle \mathbf{I} \rangle$, $\langle \mathrm{Rc}_0 \rangle$, or $\langle \mathbf{W} \rangle$ of $S_B^2(\mathfrak{so}(n))$. Therefore L is a multiple of the identity on each of the components $\langle \mathbf{I} \rangle$, $\langle \mathrm{Rc}_0 \rangle$, and $\langle \mathbf{W} \rangle$.

We shall be interested in those invertible linear transformations which *preserve the Weyl parts* of algebraic curvature operators. Note that every $\mathrm{O}(n)$-invariant linear transformation L is a constant multiple of a linear transformation which preserves the Weyl part.

EXERCISE 11.21. Show that if $L : S_B^2(\mathfrak{so}(n)) \to S_B^2(\mathfrak{so}(n))$ is an $\mathrm{O}(n)$-invariant linear transformation and if C is an $\mathrm{O}\,(n)$-invariant closed convex cone, then $L\,(C)$ is an $\mathrm{O}\,(n)$-invariant closed convex cone.

It is natural to define a 2-parameter family of $\mathrm{O}(n)$-invariant linear transformations on the space of algebraic curvature operators as follows.

DEFINITION 11.22. Given $a, b \in \mathbb{R}$, define the $\mathrm{O}(n)$-invariant linear transformation

$$\ell_{a,b} : S_B^2(\mathfrak{so}(n)) \to S_B^2(\mathfrak{so}(n))$$

by

$$\ell_{a,b}\,(\mathbf{R}) \doteqdot \mathbf{R} + 2(n-1)a\mathbf{R}_{\mathbf{I}} + (n-2)b\mathbf{R}_{\mathrm{Rc}_0}. \tag{11.58}$$

Using the decomposition (11.48), we can rewrite this as

$$\begin{aligned}\ell_{a,b}(\mathbf{R}) &= \mathbf{R} + 2a\bar{\lambda}\mathbf{I} + 2b\,\mathrm{id} \wedge \mathrm{Rc}_0(\mathbf{R}) \\ &= (1 + 2(n-1)a)\mathbf{R}_{\mathbf{I}} + (1 + (n-2)b)\mathbf{R}_{\mathrm{Rc}_0} + \mathbf{R}_{\mathbf{W}}.\end{aligned}$$

It is easy to see that $\ell_{0,0}(\mathbf{R}) = \mathbf{R}$, $\ell_{a,b}(S_B^2(\mathfrak{so}(n))) \subset S_B^2(\mathfrak{so}(n))$ and $\ell_{a,b}$ is invertible provided both $a \neq -\frac{1}{2(n-1)}$ and $b \neq -\frac{1}{n-2}$; we shall always make these assumptions on a and b. Each transformation $\ell_{a,b}$ preserves the Weyl part and is a multiple of the identity on each of the other two irreducible components. (In particular, $\ell_{a,b}(\mathbf{R}) = \mathbf{R}$ if $\mathbf{R} \in \langle \mathbf{W} \rangle$.) Hence the 2-parameter family $\{\ell_{a,b}\}$ consists of *all* $\mathrm{O}\,(n)$-invariant linear transformations of $S_B^2(\mathfrak{so}(n))$ which preserve the Weyl parts of algebraic curvature operators.

REMARK 11.23. For the linear maps $\ell_{a,b}$, as $a > 0$ gets large, the identity component $(\ell_{a,b}(\mathbf{R}))_{\mathbf{I}} = (1 + 2\,(n-1)\,a)\,\mathbf{R}_{\mathbf{I}}$ also gets large, which increases the 'likelihood' of an algebraic curvature operator $\mathbf{R}$ with $\mathrm{Scal}\,(\mathbf{R}) > 0$ having positive curvature operator. In fact, as $a \to \infty$ (while $\frac{b}{a} \to 0$), we have for $\mathbf{R}$ with $\mathrm{Scal}\,(\mathbf{R}) > 0$

$$\frac{\ell_{a,b}(\mathbf{R})}{|\ell_{a,b}(\mathbf{R})|} \to \frac{\mathbf{I}}{|\mathbf{I}|}.$$

3.2. Conjugating $\mathbf{R}^2 + \mathbf{R}^{\#}$ by $\ell_{a,b}$ and the operator $D_{a,b}(\mathbf{R})$. Using $\ell_{a,b}$, we define two maps $X_{a,b}$ and $D_{a,b}$ of the space of algebraic curvature operators to itself. The analysis of $X_{a,b}$ and $D_{a,b}$ determines when, for a convex cone C, the transformed cone $\ell_{a,b}\,(C)$ is preserved by the ODE. The main result in this section is Theorem 11.27, which gives a decomposition for the algebraic curvature operator $D_{a,b}(\mathbf{R})$ and shows that $D_{a,b}(\mathbf{R})$ has the crucial property that it is independent of the Weyl curvature part $\mathbf{R}_{\mathbf{W}}$ of $\mathbf{R}$.

DEFINITION 11.24. We define maps

$$X_{a,b} : S_B^2(\mathfrak{so}(n)) \to S_B^2(\mathfrak{so}(n)),$$
$$D_{a,b} : S_B^2(\mathfrak{so}(n)) \to S_B^2(\mathfrak{so}(n))$$

by

$$X_{a,b}(\mathbf{R}) \doteqdot \ell_{a,b}^{-1}\left((\ell_{a,b}(\mathbf{R}))^2 + (\ell_{a,b}(\mathbf{R}))^\#\right) = \left(\ell_{a,b}^{-1} \circ Q \circ \ell_{a,b}\right)(\mathbf{R}) \tag{11.59}$$

and

$$D_{a,b}(\mathbf{R}) \doteqdot X_{a,b}(\mathbf{R}) - \mathbf{R}^2 - \mathbf{R}^\# = \left(\ell_{a,b}^{-1} \circ Q \circ \ell_{a,b}\right)(\mathbf{R}) - Q(\mathbf{R}) \tag{11.60}$$

where $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#$.

The operator $D_{a,b}$ measures how the map Q (which we think of as a vector field on $S_B^2(\mathfrak{so}(n))$) changes under conjugation by $\ell_{a,b}$.

We give an example of calculating $D_{a,b}(\mathbf{R})$.

EXAMPLE 11.25. Recall from Corollary 11.18 that if $\mathbf{S} \in \langle \mathbf{W} \rangle$, then $\mathbf{S}^2 + \mathbf{S}^\# \in \langle \mathbf{W} \rangle$, which together with $\ell_{a,b}(\mathbf{S}) = \mathbf{S}$ implies

$$D_{a,b}(\mathbf{S}) = \ell_{a,b}^{-1}\left(\mathbf{S}^2 + \mathbf{S}^\#\right) - \left(\mathbf{S}^2 + \mathbf{S}^\#\right) = 0. \tag{11.61}$$

The following elementary result gives a criterion for when the linear transformation of a cone is preserved by the ODE.

LEMMA 11.26 (A criterion for when $\ell_{a,b}(C)$ is preserved by the ODE). *An* $\mathrm{O}(n)$*-invariant closed convex cone* $\ell_{a,b}(C)$ *is preserved by the* ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#$ *if and only if* C *is preserved by the* ODE $\frac{d}{dt}\mathbf{R} = X_{a,b}(\mathbf{R})$.

PROOF. We have that $\mathbf{R}(t)$ is a solution of the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ if and only if $\ell_{a,b}^{-1}(\mathbf{R}(t))$ is a solution of the ODE $\frac{d}{dt}\mathbf{R} = X_{a,b}(\mathbf{R})$. Indeed,

$$\begin{aligned} \frac{d}{dt}\ell_{a,b}^{-1}(\mathbf{R}(t)) &= \ell_{a,b}^{-1}(Q(\mathbf{R}(t))) \\ &= \left(\ell_{a,b}^{-1} \circ Q \circ \ell_{a,b}\right)\left(\ell_{a,b}^{-1}(\mathbf{R}(t))\right) \\ &= X_{a,b}(\ell_{a,b}^{-1}(\mathbf{R}(t))). \end{aligned}$$

We also have $\mathbf{R} \in C$ if and only if $\ell_{a,b}(\mathbf{R}) \in \ell_{a,b}(C)$. Hence a solution $\mathbf{R}(t)$ of $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ stays in $\ell_{a,b}(C)$ if and only if the solution $\ell_{a,b}^{-1}(\mathbf{R}(t))$ of $\frac{d}{dt}\mathbf{R} = X_{a,b}(\mathbf{R})$ stays in C. The lemma follows. □

In view of the lemma, we want to show that $X_{a,b}(\mathbf{R}) = Q(\mathbf{R}) + D_{a,b}(\mathbf{R})$ lies in the tangent cone of C at $\mathbf{R} \in \partial C$. Assuming that C itself is preserved by the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ and that the tangent cone of C is invariant under the addition of nonnegative operators (as is the case when C is the cone of 2-nonnegative curvature operators), it suffices to show that $D_{a,b}(\mathbf{R})$ is nonnegative for $\mathbf{R} \in \partial C$. (More generally, since the tangent cones of C are invariant under addition by Exercise 10.42, it suffices to show that $D_{a,b}(\mathbf{R})$ lies in the tangent cone of C at $\mathbf{R}$.) The following gives a fundamental formula for $D_{a,b}(\mathbf{R})$ (see Theorem 2 in [**43**]).

3.3. The main formula for $D_{a,b}(\mathbf{R})$. We have the following magical formula for $D_{a,b}(\mathbf{R})$.

THEOREM 11.27 (Main formula for $D_{a,b}$). *For any $a, b \in \mathbb{R}$ such that $a \neq -\frac{1}{2(n-1)}$ and $b \neq -\frac{1}{n-2}$,*

(11.62)

$$D_{a,b}(\mathbf{R}) = \left((n-2)b^2 - 2(a-b)\right) \operatorname{Rc}_0 \wedge \operatorname{Rc}_0 + 2a \operatorname{Rc} \wedge \operatorname{Rc} + 2b^2 \operatorname{Rc}_0^2 \wedge \operatorname{id} + \frac{\operatorname{tr}(\operatorname{Rc}_0^2)}{n + 2n(n-1)a} \left(nb^2(1-2b) - 2(a-b)(1-2b+nb^2)\right) \mathbf{I},$$

where Rc_0 *denotes the trace-free part of* $\operatorname{Rc}(\mathbf{R})$. *We also have*

$$\operatorname{Rc}(D_{a,b}(\mathbf{R})) = -2b \operatorname{Rc}_0^2 + 2(n-2)a\bar{\lambda} \operatorname{Rc}_0 + 2(n-1)a\bar{\lambda}^2 \operatorname{id} + \frac{2(n-1)b + (n-2)^2 b^2 - 2(n-1)a(1-2b)}{1 + 2(n-1)a} \sigma \operatorname{id}. \quad (11.63)$$

We will give a proof of Theorem 11.27 in the next section.

The consideration of the quantity $D_{a,b}(\mathbf{R}) = \ell_{a,b}^{-1}(Q(\ell_{a,b}(\mathbf{R}))) - Q(\mathbf{R})$ is natural for the following reasons. Suppose we know that a set $C \subset S_B^2(\mathfrak{so}(n))$ is preserved by the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ and we want to construct a new set preserved by the ODE of the form $\ell_{a,b}(C)$. We then need to check that the vector $Q(\ell_{a,b}(\mathbf{R}))$ lies inside the tangent cone of $\ell_{a,b}(C)$. This is equivalent to seeing if $\ell_{a,b}^{-1}(Q(\ell_{a,b}(\mathbf{R})))$ lies inside the tangent cone of C. The advantage of expression (11.62) for $D_{a,b}(\mathbf{R})$ is that it is independent of the Weyl curvature part $\mathbf{R_W}$ of $\mathbf{R}$. Furthermore, $D_{a,b}(\mathbf{R})$ may be put into diagonal form and as we shall see in Corollary 11.29 below, we can compute the eigenvalues of $D_{a,b}(\mathbf{R}) : \mathfrak{so}(n) \to \mathfrak{so}(n)$ in terms of n, a, b, and the eigenvalues of $\operatorname{Rc}(\mathbf{R})$. This greatly simplifies the task of understanding how $\mathbf{R}^2 + \mathbf{R}^\#$ changes under the linear map $\ell = \ell_{a,b}$. The above properties of $D_{a,b}$ are key to obtaining new $\mathrm{O}(n)$-invariant cones and sets in $S_B^2(\mathfrak{so}(n))$ preserved by the ODE from known ones.

Now we give a simplified expression for $D_{a,b}$ for certain choices of a, b. By (11.62), if we choose a and b such that

$$(n-2)b^2 = 2(a-b), \quad (11.64)$$

then we have

$$D_{a,b}(\mathbf{R}) = 2a \operatorname{Rc} \wedge \operatorname{Rc} + 2b^2 \operatorname{Rc}_0^2 \wedge \operatorname{id} + \frac{\sigma b^2}{1+2(n-1)a}\left(n(1-2b) - (n-2)(1-2b+nb^2)\right)\mathbf{I} \tag{11.65}$$

$$= 2a \operatorname{Rc} \wedge \operatorname{Rc} + 2b^2 \operatorname{Rc}_0^2 \wedge \operatorname{id} + \frac{\sigma b^2}{1+2(n-1)a}\left(-n(n-2)b^2 - 4b + 2\right)\mathbf{I}. \tag{11.66}$$

REMARK 11.28. Note that the roots of the quadratic equation

$$-n(n-2)b^2 - 4b + 2 = 0$$

are

$$b = \frac{-2 \mp \sqrt{2n(n-2)+4}}{n(n-2)}.$$

In particular, $-n(n-2)b^2 - 4b + 2 \geq 0$ for $b \in \left(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\right]$.

We end this subsection with a consequence of Theorem 11.27 which gives the formulas for the eigenvalues of $D_{a,b}(\mathbf{R})$ and $\operatorname{Rc}(D_{a,b}(\mathbf{R}))$ in terms of the eigenvalues of $\operatorname{Rc}(\mathbf{R})$.

COROLLARY 11.29 (Eigenvalues of $D_{a,b}$ and $\operatorname{Rc}(D_{a,b})$). *Let $\{e_i\}_{i=1}^n$ be an orthonormal basis consisting of the eigenvectors of $\operatorname{Rc}_0(\mathbf{R})$ with corresponding eigenvalues λ_i. Then $e_i \wedge e_j$, where $i < j$, is an eigenvector of $D_{a,b}(\mathbf{R})$ with eigenvalue*

$$d_{ij} = \left((n-2)b^2 - 2(a-b)\right)\lambda_i\lambda_j + 2a(\lambda_i + \bar{\lambda})(\lambda_j + \bar{\lambda}) + b^2(\lambda_i^2 + \lambda_j^2) + \frac{\sigma}{1+2(n-1)a}\left(nb^2(1-2b) - 2(a-b)(1-2b+nb^2)\right). \tag{11.67}$$

Furthermore, e_i is an eigenvector of $\operatorname{Rc}(D_{a,b}(\mathbf{D}))$ with eigenvalue

$$r_i = -2b\lambda_i^2 + 2a(n-2)\bar{\lambda}\lambda_i + 2a(n-1)\bar{\lambda}^2 + \frac{\sigma}{1+2(n-1)a}\left(n^2b^2 - 2(n-1)(a-b)(1-2b)\right). \tag{11.68}$$

PROOF OF THE COROLLARY. Let $\omega_\alpha \doteqdot e_i \wedge e_j$. By Remark 11.12, $\omega_\alpha \otimes \omega_\alpha$ is an algebraic curvature operator. From (11.5) we have

$$\operatorname{Rc}_0^2 \wedge \operatorname{id} = \frac{1}{2}\sum_{i<j}\left(\lambda_i^2 + \lambda_j^2\right)(e_i \otimes e_i) \wedge (e_j \otimes e_j)$$
$$= \frac{1}{2}\sum_{i<j}\left(\lambda_i^2 + \lambda_j^2\right)\omega_\alpha \otimes \omega_\alpha.$$

Since $\mathrm{Rc} = \mathrm{Rc}_0 + \bar{\lambda}\,\mathrm{id}$, we have

$$\begin{aligned}\mathrm{Rc} \wedge \mathrm{Rc} &= \sum_{i,j} (\mathrm{Rc}_{ii} e_i \otimes e_i) \wedge (\mathrm{Rc}_{jj} e_j \otimes e_j) = \sum_{i<j} \mathrm{Rc}_{ii} \mathrm{Rc}_{jj} \omega_\alpha \otimes \omega_\alpha \\ &= \sum_{i<j} \left(\lambda_i + \bar{\lambda}\right)\left(\lambda_j + \bar{\lambda}\right) \omega_\alpha \otimes \omega_\alpha .\end{aligned}$$

Similarly we have

$$\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 = \sum_{i<j} \lambda_i \lambda_j \omega_\alpha \otimes \omega_\alpha .$$

Note also that $\mathbf{I} = \sum_{i<j} \omega_\alpha \otimes \omega_\alpha$.

Hence, by (11.62) we have, for $i < j$, that $\omega_\alpha = e_i \wedge e_j$ is an eigenvector of $D_{a,b}(\mathbf{D})$ with eigenvalue d_{ij} given by (11.67).

From (11.63) we have

$$\begin{aligned} r_i = &-2b\lambda_i^2 + 2(n-2)a\bar{\lambda}\lambda_i + 2(n-1)a\bar{\lambda}^2 \\ &+ \frac{2(n-1)b + (n-2)^2 b^2 - 2(n-1)a(1-2b)}{1+2(n-1)a}\sigma ,\end{aligned}$$

which simplifies to (11.68). □

Remark 11.30. A motivation for considering the relation (11.64), i.e., $2a = 2b + (n-2)b^2$ is that it eliminates a term in (11.67).

4. Proof of the main formula for $D_{a,b}(\mathbf{R})$

In this section we give a proof of Theorem 11.27. Since the proof is long, first we give a proof of the theorem modulo the proof of two formulas used in the proof and then we give a proof of these two formulas. For the sake of simplifying notation, let $D \doteqdot D_{a,b}$ and let $\ell \doteqdot \ell_{a,b}$.

4.1. Proof of Theorem 11.27 assuming formulas (11.78) and (11.81). We divide the proof of the theorem into three steps. In Step 1 we use (11.78) to reduce the theorem to the case where $\mathbf{R}_{\mathbf{W}} = 0$. In Step 2, assuming that $\mathbf{R}_{\mathbf{W}} = 0$, we use a consequence of (11.81) to show that both sides of formula (11.62) have the same projection to $\langle \mathbf{W} \rangle$. In Step 3, again assuming $\mathbf{R}_{\mathbf{W}} = 0$, we use another consequence of (11.81) to show that both sides of (11.62) have the same Rc. Formulas (11.78) and (11.81) will be proved in the next subsection.

Step 1. Reduction of proof of Theorem 11.27 to the case where $\mathbf{R}_{\mathbf{W}} = 0$. We shall prove the following.

Proposition 11.31. *The algebraic curvature operator $D(\mathbf{R})$ is independent of the Weyl part $\mathbf{R}_{\mathbf{W}}$ of $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$. Equivalently,*

$$D(\mathbf{R} + \mathbf{S}) = D(\mathbf{R})$$

for all $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$ and $\mathbf{S} \in \langle \mathbf{W} \rangle$.

Note that by replacing $\mathbf{R}$ by $\mathbf{R}-\mathbf{S}$ and $\mathbf{S}$ by $2\mathbf{S}$, proving the proposition is equivalent to showing that

$$B(\mathbf{R},\mathbf{S}) \doteqdot \frac{1}{4}\left(D(\mathbf{R}+\mathbf{S}) - D(\mathbf{R}-\mathbf{S})\right) \tag{11.69}$$

vanishes for all $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$ and $\mathbf{S} \in \langle \mathbf{W} \rangle$.

The following exercise, which gives a formula for $D(\mathbf{R}+\mathbf{S}) - D(\mathbf{R})$, will be used in the proof of the proposition.

Exercise 11.32. Show that for all $\mathbf{R},\mathbf{S} \in S_B^2(\mathfrak{so}(n))$,

$$\begin{aligned} &D(\mathbf{R}+\mathbf{S}) - D(\mathbf{R}) \\ &= \ell^{-1}\left(\ell(\mathbf{S})^2 + \ell(\mathbf{R})\ell(\mathbf{S}) + \ell(\mathbf{S})\ell(\mathbf{R}) + \ell(\mathbf{S})^{\#} + 2\ell(\mathbf{R})\#\ell(\mathbf{S})\right) \\ &\quad - \mathbf{S}^2 - \mathbf{RS} - \mathbf{SR} - \mathbf{S}^{\#} - 2\mathbf{R}\#\mathbf{S}. \end{aligned} \tag{11.70}$$

Show also that if $\mathbf{S} \in \langle \mathbf{W} \rangle$, then the above expression may be rewritten as

$$\begin{aligned} &D(\mathbf{R}+\mathbf{S}) - D(\mathbf{R}) \\ &= \ell^{-1}\left(\ell(\mathbf{R})\mathbf{S} + \mathbf{S}\ell(\mathbf{R}) + 2\ell(\mathbf{R})\#\mathbf{S}\right) - \mathbf{RS} - \mathbf{SR} - 2\mathbf{R}\#\mathbf{S} \\ &= 2B(\mathbf{R},\mathbf{S}). \end{aligned}$$

Hint: By Corollary 11.18, if $\mathbf{S} \in \langle \mathbf{W} \rangle$, then $\mathbf{S}^2 + \mathbf{S}^{\#} \in \langle \mathbf{W} \rangle$.

From the exercise it is clear that $B(\mathbf{R},\mathbf{S})$ is bilinear in $\mathbf{R}$ and $\mathbf{S}$.

Proof of Proposition 11.31. Suppose $\mathbf{S} \in \langle \mathbf{W} \rangle$; it suffices to show that $B(\mathbf{R},\mathbf{S}) = 0$ for each $\mathbf{R}$ in any one of the $\mathrm{O}(n)$-irreducible components of $S_B^2(\mathfrak{so}(n))$.

(1) $B(\mathbf{R},\mathbf{S}) = 0$ if $\mathbf{R} \in \langle \mathbf{W} \rangle$. Since $\mathbf{R}+\mathbf{S}$, $\mathbf{R}-\mathbf{S} \in \langle \mathbf{W} \rangle$, it follows from (11.61) that

$$B(\mathbf{R},\mathbf{S}) = \frac{1}{4}\left(D(\mathbf{R}+\mathbf{S}) - D(\mathbf{R}-\mathbf{S})\right) = 0.$$

(2) $B(\mathbf{I},\mathbf{S}) = 0$. By definition (11.58),

$$\ell(\mathbf{I}) = (1 + 2(n-1)a)\,\mathbf{I} \doteqdot \left(1 + a'\right)\mathbf{I}$$

and by (11.78) and $\mathbf{S} \in \langle \mathbf{W} \rangle$,

$$\mathbf{S} + \mathbf{S}\#\mathbf{I} = (n-1)\mathbf{S}_{\mathbf{I}} + \frac{n-2}{2}\mathbf{S}_{\mathrm{Rc}_0} = 0, \tag{11.71}$$

so that

$$\begin{aligned} B(\mathbf{I},\mathbf{S}) &= \ell^{-1}\left(\frac{1}{2}(\ell(\mathbf{I})\mathbf{S} + \mathbf{S}\ell(\mathbf{I})) + \ell(\mathbf{I})\#\mathbf{S}\right) - \frac{1}{2}(\mathbf{IS} + \mathbf{SI}) - \mathbf{I}\#\mathbf{S} \\ &= \ell^{-1}\left(\frac{1}{2}\left((1+a')\mathbf{IS} + \mathbf{S}(1+a')\mathbf{I}\right) + (1+a')\mathbf{I}\#\mathbf{S}\right) - \mathbf{S} - \mathbf{I}\#\mathbf{S} \\ &= (1+a')\ell^{-1}\left(\mathbf{S} + \mathbf{S}\#\mathbf{I}\right) - \left(\mathbf{S} + \mathbf{S}\#\mathbf{I}\right) = 0. \end{aligned}$$

(3) $B(\mathbf{R},\mathbf{S})=0$ for $\mathbf{R}\in\langle\mathrm{Rc}_0\rangle$. Since

$$B(\mathbf{R},\mathbf{S})=\ell^{-1}\left(\frac{1}{2}(\ell(\mathbf{R})\mathbf{S}+\mathbf{S}\ell(\mathbf{R}))+\ell(\mathbf{R})\#\mathbf{S}\right)-\frac{1}{2}(\mathbf{RS}+\mathbf{SR})-\mathbf{R}\#\mathbf{S},$$

it suffices to show that

$$\ell(\mathbf{RS}+\mathbf{SR}+2\mathbf{R}\#\mathbf{S})=\ell(\mathbf{R})\mathbf{S}+\mathbf{S}\ell(\mathbf{R})+2\ell(\mathbf{R})\#\mathbf{S} \tag{11.72}$$

for all $\mathbf{R}\in\langle\mathrm{Rc}_0\rangle$. This follows from (11.57)[18] and the fact that

$$\ell(\tilde{\mathbf{R}})=(1+(n-2)b)\,\tilde{\mathbf{R}}\doteqdot(1+b')\tilde{\mathbf{R}}$$

for any $\tilde{\mathbf{R}}\in\langle\mathrm{Rc}_0\rangle$. The proposition is proved. □

This completes Step 1.

Since $D(\mathbf{R})$ is independent of the Weyl part $\mathbf{R_W}$, we only need to prove formula (11.62) for $\mathbf{R}=\frac{\bar\lambda}{n-1}\mathbf{I}+\mathbf{R}_0$, where $\mathbf{R}_0\doteqdot\mathbf{R}_{\mathrm{Rc}_0}\in\langle\mathrm{Rc}_0\rangle$. Moreover, as for any algebraic curvature operator, it suffices to check that both sides of (11.62) have the same Rc and the same projection to $\langle\mathbf{W}\rangle$.

Step 2. Both sides of (11.62) have the same projection to $\langle\mathbf{W}\rangle$ for $\mathbf{R}=\frac{\bar\lambda}{n-1}\mathbf{I}+\mathbf{R}_0$. First we note that the Weyl part of the LHS of (11.62) is

$$\begin{aligned}(D(\mathbf{R}))_{\mathbf{W}}&=\left(\ell^{-1}\left(\ell(\mathbf{R})^2+\ell(\mathbf{R})^{\#}\right)\right)_{\mathbf{W}}-(\mathbf{R}^2+\mathbf{R}^{\#})_{\mathbf{W}}\\&=\left(\ell(\mathbf{R})^2+\ell(\mathbf{R})^{\#}\right)_{\mathbf{W}}-(\mathbf{R}^2+\mathbf{R}^{\#})_{\mathbf{W}}\end{aligned}$$

since for any $\mathbf{S}\in S^2_B(\mathfrak{so}(n))$ we have $\left(\ell^{-1}(\mathbf{S})\right)_{\mathbf{W}}=\mathbf{S_W}$. Since $\mathbf{R_W}=0$, by definition (11.58),

$$\ell(\mathbf{R})=(1+2(n-1)a)\frac{\bar\lambda}{n-1}\mathbf{I}+(1+(n-2)b)\frac{2}{n-2}\,\mathrm{Rc}_0\wedge\mathrm{id}, \tag{11.73}$$

so that

$$\mathrm{Rc}(\ell(\mathbf{R}))=(1+2(n-1)a)\bar\lambda\,\mathrm{id}+(1+(n-2)b)\,\mathrm{Rc}_0.$$

Therefore

$$\mathrm{Rc}_0\,(\ell(\mathbf{R}))=(1+(n-2)b)\,\mathrm{Rc}_0.$$

[18]Note that (11.57) is also a consequence of (11.78).

Since we have $\mathbf{R}_{\mathbf{W}} = 0$ and $\ell(\mathbf{R})_{\mathbf{W}} = 0$, we can apply (11.36) to conclude that

$$\begin{aligned}(D(\mathbf{R}))_{\mathbf{W}} &= \left(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}\right)_{\mathbf{W}} - (\mathbf{R}^2 + \mathbf{R}^{\#})_{\mathbf{W}} \\ &= \frac{1}{n-2}\left(\mathrm{Rc}_0\left(\ell(\mathbf{R})\right) \wedge \mathrm{Rc}_0\left(\ell(\mathbf{R})\right)\right)_{\mathbf{W}} \\ &\quad - \frac{1}{n-2}\left(\mathrm{Rc}_0(\mathbf{R}) \wedge \mathrm{Rc}_0(\mathbf{R})\right)_{\mathbf{W}} \\ &= \frac{1}{n-2}\left((1+(n-2)b)^2 - 1\right)(\mathrm{Rc}_0 \wedge \mathrm{Rc}_0)_{\mathbf{W}} \\ &= ((n-2)b^2 + 2b)(\mathrm{Rc}_0 \wedge \mathrm{Rc}_0)_{\mathbf{W}}.\end{aligned}$$

On the other hand, since $(\mathrm{Rc} \wedge \mathrm{Rc})_{\mathbf{W}} = (\mathrm{Rc}_0 \wedge \mathrm{Rc}_0)_{\mathbf{W}}$ (the operator $\mathrm{id} \wedge A$ is orthogonal to $\langle \mathbf{W} \rangle$) and $\mathbf{I}_{\mathbf{W}} = 0$, the Weyl part of the right-hand side of equation (11.62) is

$$\begin{aligned}&\left(\left((n-2)b^2 - 2(a-b)\right)\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 + 2a\,\mathrm{Rc} \wedge \mathrm{Rc}\right)_{\mathbf{W}} \\ &= \left(\left((n-2)b^2 + 2b\right)\mathrm{Rc}_0 \wedge \mathrm{Rc}_0\right)_{\mathbf{W}} = D_{\mathbf{W}}.\end{aligned}$$

Thus the Weyl part of equation (11.62) holds.

Step 3. Both sides of (11.62) have the same Rc **for** $\mathbf{R} = \frac{\bar{\lambda}}{n-1}\mathbf{I} + \mathbf{R}_0$**.** For the Rc part of (11.62), we notice that

$$\bar{\lambda}(\ell(\mathbf{R})) = (1+2(n-1)a)\bar{\lambda} \quad \text{and} \quad \sigma(\ell(\mathbf{R})) = (1+(n-2)b)^2\sigma,$$

where $\bar{\lambda}(\mathbf{R})$ and $\sigma(\mathbf{R})$ are defined in (11.47). Hence, by (11.83), we have

$$\begin{aligned}\mathrm{Rc}(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}) &= -\frac{2}{n-2}(1+(n-2)b)^2(\mathrm{Rc}_0^2)_0 \\ &\quad + \frac{n-2}{n-1}(1+2(n-1)a)\bar{\lambda}(1+(n-2)b)\,\mathrm{Rc}_0 \\ &\quad + \left(((1+2(n-1)a)\bar{\lambda})^2 + (1+(n-2)b)^2\sigma\right)\mathrm{id}\end{aligned} \tag{11.74}$$

since

$$\mathrm{Rc}_0(\ell(\mathbf{R})) = (1+(n-2)b)\,\mathrm{Rc}_0(\mathbf{R})$$

and

$$\left(\mathrm{Rc}_0^2\right)_0(\ell(\mathbf{R})) = (1+(n-2)b)^2\left(\mathrm{Rc}_0^2\right)_0(\mathbf{R}).$$

In particular,

$$\begin{aligned}\mathrm{Rc}_0(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}) &= -\frac{2}{n-2}(1+(n-2)b)^2(\mathrm{Rc}_0^2)_0 \\ &\quad + \frac{n-2}{n-1}(1+2(n-1)a)\bar{\lambda}(1+(n-2)b)\,\mathrm{Rc}_0.\end{aligned}$$

For any $\tilde{\mathbf{R}} \in S_B^2(\mathfrak{so}(n))$,

$$\ell^{-1}(\tilde{\mathbf{R}}) = \tilde{\mathbf{R}}_{\mathbf{W}} + \frac{1}{1+2(n-1)a}\tilde{\mathbf{R}}_{\mathbf{I}} + \frac{1}{1+(n-2)b}\tilde{\mathbf{R}}_{\mathrm{Rc}_0},$$

which implies

$$\operatorname{Rc}(\ell^{-1}(\tilde{\mathbf{R}})) = \frac{\bar{\lambda}(\tilde{\mathbf{R}})}{1+2(n-1)a}\operatorname{id} + \frac{1}{1+(n-2)b}\operatorname{Rc}_0(\tilde{\mathbf{R}})$$

since $\operatorname{Rc}\left(\tilde{\mathbf{R}}_{\mathbf{I}}\right) = \bar{\lambda}(\tilde{\mathbf{R}})\operatorname{id}$ and $\operatorname{Rc}\left(\tilde{\mathbf{R}}_{\operatorname{Rc}_0}\right) = \operatorname{Rc}_0(\tilde{\mathbf{R}})$. Applying this to $\tilde{\mathbf{R}} = \ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}$, we have

$$\begin{aligned} \operatorname{Rc}(\ell^{-1}(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#})) = &-\frac{2}{n-2}(1+(n-2)b)\left(\operatorname{Rc}_0^2\right)_0 \\ &+ \frac{n-2}{n-1}(1+2(n-1)a)\bar{\lambda}\operatorname{Rc}_0 \\ &+ \frac{(1+2(n-1)a)^2\bar{\lambda}^2 + (1+(n-2)b)^2\sigma}{1+2(n-1)a}\operatorname{id} \end{aligned} \tag{11.75}$$

since (11.74) implies

$$\bar{\lambda}(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}) = \left((1+2(n-1)a)\bar{\lambda}\right)^2 + (1+(n-2)b)^2\sigma.$$

Combining (11.83) and (11.75), we have

$$\begin{aligned} \operatorname{Rc}(D(\mathbf{R})) &= \operatorname{Rc}\left(\ell^{-1}\left(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}\right)\right) - \operatorname{Rc}\left(\mathbf{R}^2 + \mathbf{R}^{\#}\right) \\ &= -2b\left(\operatorname{Rc}_0^2\right)_0 + 2(n-2)a\bar{\lambda}\operatorname{Rc}_0 - \left(\bar{\lambda}^2 + \sigma\right)\operatorname{id} \\ &\quad + \frac{(1+2(n-1)a)^2\bar{\lambda}^2 + (1+(n-2)b)^2\sigma}{1+2(n-1)a}\operatorname{id} \\ &= -2b\left(\operatorname{Rc}_0^2\right)_0 + 2(n-2)a\bar{\lambda}\operatorname{Rc}_0 + 2(n-1)a\bar{\lambda}^2\operatorname{id} \\ &\quad + \frac{2(n-2)b + (n-2)^2b^2 - 2(n-1)a}{1+2(n-1)a}\sigma\operatorname{id}. \end{aligned}$$

Converting $\left(\operatorname{Rc}_0^2\right)_0$ to Rc_0^2 in the first term and simplifying, formula (11.63) then follows.

Let E denote the right-hand side of (11.62). We shall show that $\operatorname{Rc}(E) = \operatorname{Rc}(D(\mathbf{R}))$, from which Step 3 follows. Indeed, we compute using (11.41), (11.42), and $\operatorname{Rc} = \operatorname{Rc}_0 + \bar{\lambda}\operatorname{id}$,

$$\begin{aligned} \operatorname{Rc}(E) = &-((n-2)b^2 - 2(a-b))\operatorname{Rc}_0^2 + 2an\bar{\lambda}\left(\operatorname{Rc}_0 + \bar{\lambda}\operatorname{id}\right) \\ &- 2a(\operatorname{Rc}_0^2 + 2\bar{\lambda}\operatorname{Rc}_0 + \bar{\lambda}^2\operatorname{id}) + b^2(n-2)\operatorname{Rc}_0^2 + b^2n\sigma\operatorname{id} \\ &+ \sigma\frac{nb^2(1-2b) - 2(a-b)(1-2b+nb^2)}{1+2(n-1)a}(n-1)\operatorname{id}. \end{aligned} \tag{11.76}$$

By combining terms in (11.76), we obtain

$$\begin{aligned} \operatorname{Rc}(E) = &-((n-2)b^2 - 2(a-b) + 2a - b^2(n-2))\operatorname{Rc}_0^2 \\ &+ 2a\left(n\bar{\lambda} - 2\bar{\lambda}\right)\operatorname{Rc}_0 + \left(2an\bar{\lambda}^2 - 2a\bar{\lambda}^2 + b^2n\sigma\right)\operatorname{id} \\ &+ \sigma\frac{nb^2(1-2b) - 2(a-b)(1-2b+nb^2)}{1+2(n-1)a}(n-1)\operatorname{id}, \end{aligned}$$

which is the same as (11.63), i.e., $\mathrm{Rc}\,(E) = \mathrm{Rc}\,(D\,(\mathbf{R}))$. Step 3 and hence Theorem 11.27 are now proved.

4.2. Two formulas for the decomposition of some algebraic curvature operators. Recall that if $A \in S^2(\mathbb{E}^n)$, then $A \wedge \mathrm{id} \in S_B^2(\mathfrak{so}(n))$; if $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$, then $\mathrm{Rc}(\mathbf{R}) \in S^2(\mathbb{E}^n)$. As we have seen, the following identity for $Q\,(\mathbf{R},\mathbf{I}) = \mathbf{R} + \mathbf{R}\#\mathbf{I}$ holds a key to the proof of Theorem 11.27.

LEMMA 11.33 (Formula for $\mathbf{R} + \mathbf{R}\#\mathbf{I}$). *If* $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$, *then*

$$\mathbf{R} + \mathbf{R}\#\mathbf{I} = \mathrm{Rc}(\mathbf{R}) \wedge \mathrm{id}\,. \tag{11.77}$$

Hence,

$$\mathbf{R} + \mathbf{R}\#\mathbf{I} = (n-1)\mathbf{R}_{\mathbf{I}} + \frac{n-2}{2}\mathbf{R}_{\mathrm{Rc}_0}. \tag{11.78}$$

REMARK 11.34. Note that by definition (11.15), $Q\,(\mathbf{R},\mathbf{I}) = \mathbf{R} + \mathbf{R}\#\mathbf{I}$. The proof we give of the lemma is via a direct calculation. For the original proof by Böhm and Wilking, see the proof of Lemma 2.1 in [**43**].

PROOF. For the sake of simplicity, we shall denote $\mathrm{Rc} \doteqdot \mathrm{Rc}(\mathbf{R})$ and $\mathrm{Rc}_0 \doteqdot \mathrm{Rc}_0(\mathbf{R})$. By the identities $\mathrm{Rc} \wedge \mathrm{id} = \left(\mathrm{Rc}_0 + \bar{\lambda}\,\mathrm{id}\right) \wedge \mathrm{id}$, $\mathbf{R}_{\mathbf{I}} = \frac{\bar{\lambda}}{n-1}\mathbf{I}$, and $\mathbf{R}_{\mathrm{Rc}_0} = \frac{2}{n-2}\,\mathrm{id} \wedge \mathrm{Rc}_0$, the second stated identity in the lemma follows from the first one. Given an orthonormal basis $\{e_a\}$, denote $r_{ab} \doteqdot \mathrm{Rc}(\mathbf{R})_{ab} = \mathrm{Rc}\,(e_a, e_b)$. We have by (11.18),

$$\begin{aligned}
(\mathrm{Rc} \wedge \mathrm{id})_{abcd} &= \langle (\mathrm{Rc} \wedge \mathrm{id})(e_a \wedge e_b), e_c \wedge e_d \rangle \\
&= \frac{1}{2}\left\langle \mathrm{Rc}\,(e_a) \wedge e_b + e_a \wedge \mathrm{Rc}\,(e_b)\,, e_c \wedge e_d \right\rangle \\
&= \frac{1}{2}\left(r_{ac}\delta_{bd} - r_{ad}\delta_{bc} + r_{bd}\delta_{ac} - r_{bc}\delta_{ad}\right). \qquad (11.79)
\end{aligned}$$

On the other hand, by (11.13) we have

$$\begin{aligned}
(\mathbf{R}\#\mathbf{I})_{abcd} &= \langle (\mathbf{R}\#\mathbf{I})\,(e_a \wedge e_b)\,, e_c \wedge e_d \rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [\mathbf{R}(\omega_\alpha), \omega_\beta], e_a \wedge e_b \rangle \langle [\omega_\alpha, \omega_\beta], e_c \wedge e_d \rangle \\
&= \frac{1}{8}\sum_{i,j,s,t} \langle [\mathbf{R}(e_i \wedge e_j), e_s \wedge e_t], e_a \wedge e_b \rangle \langle [e_i \wedge e_j, e_s \wedge e_t], e_c \wedge e_d \rangle \\
&= \frac{1}{16}\sum \mathbf{R}_{ijk\ell} \langle [e_k \wedge e_\ell, e_s \wedge e_t], e_a \wedge e_b \rangle \langle [e_i \wedge e_j, e_s \wedge e_t], e_c \wedge e_d \rangle \\
&= \frac{1}{2}\left(r_{ac}\delta_{bd} - r_{ad}\delta_{ac} + r_{bd}\delta_{ac} - r_{bc}\delta_{ad}\right) - \mathbf{R}_{abcd}. \qquad (11.80)
\end{aligned}$$

We now justify the last equality. From (11.8), we have

$$\begin{aligned}
&\mathbf{R}_{ijk\ell}\langle[e_k\wedge e_\ell, e_s\wedge e_t], e_a\wedge e_b\rangle\langle[e_i\wedge e_j, e_s\wedge e_t], e_c\wedge e_d\rangle\\
&= 4\mathbf{R}_{ijk\ell}\left(\delta_{\ell s}\left(\delta_{ka}\delta_{tb}-\delta_{kb}\delta_{ta}\right)+\delta_{kt}\left(\delta_{\ell a}\delta_{sb}-\delta_{\ell b}\delta_{sa}\right)\right)\\
&\quad\times\left(-\delta_{jt}\left(\delta_{ic}\delta_{sd}-\delta_{id}\delta_{sc}\right)+\delta_{js}\left(\delta_{ic}\delta_{td}-\delta_{id}\delta_{tc}\right)\right)\\
&= 8\mathbf{R}_{ijk\ell}\left(\delta_{\ell s}\left(\delta_{ka}\delta_{tb}-\delta_{kb}\delta_{ta}\right)-\delta_{\ell t}\left(\delta_{ka}\delta_{sb}-\delta_{kb}\delta_{sa}\right)\right)\\
&\quad\times\delta_{js}\left(\delta_{ic}\delta_{td}-\delta_{id}\delta_{tc}\right)\\
&= 8\mathbf{R}_{ijk\ell}\begin{pmatrix}\delta_{\ell j}\left(\delta_{ka}\delta_{db}-\delta_{kb}\delta_{da}\right)\delta_{ic}-\delta_{\ell d}\left(\delta_{ka}\delta_{jb}-\delta_{kb}\delta_{ja}\right)\delta_{ic}\\ -\delta_{\ell j}\left(\delta_{ka}\delta_{cb}-\delta_{kb}\delta_{ca}\right)\delta_{id}+\delta_{\ell c}\left(\delta_{ka}\delta_{jb}-\delta_{kb}\delta_{ja}\right)\delta_{id}\end{pmatrix}\\
&= 8\begin{pmatrix}\delta_{db}r_{ca}-\delta_{da}r_{cb}-\left(\mathbf{R}_{cbad}-\mathbf{R}_{cabd}\right)\\ -\delta_{cb}r_{da}+\delta_{ca}r_{db}+\left(\mathbf{R}_{dbac}-\mathbf{R}_{dabc}\right)\end{pmatrix}\\
&= 8\left(\delta_{db}r_{ca}-\delta_{da}r_{cb}-\delta_{cb}r_{da}+\delta_{ca}r_{db}+2\mathbf{R}_{bacd}\right),
\end{aligned}$$

where we used the first Bianchi identity to obtain the last equality. From this we obtain (11.80). The lemma follows from combining formulas (11.79) and (11.80). □

Note that from Step 1 of the proof of Theorem 11.27, we know that $D(\mathbf{R})$ is independent of $\mathbf{R}_{\mathbf{W}}$. We now focus on understanding how the operator $\mathbf{R}\mapsto\mathbf{R}^2+\mathbf{R}^\#$ acts on the orthogonal complement of $\langle\mathbf{W}\rangle$.

LEMMA 11.35 ($\mathbf{R}^2+\mathbf{R}^\#$ when $\mathbf{R}_{\mathbf{W}}=0$). *Let* $\mathbf{R}$ *be an algebraic curvature operator such that* $\mathbf{R}_{\mathbf{W}}=0$. *Then*

$$\mathbf{R}^2+\mathbf{R}^\# = \frac{1}{n-2}\operatorname{Rc}_0\wedge\operatorname{Rc}_0+\frac{2\bar\lambda}{n-1}\operatorname{Rc}_0\wedge\operatorname{id} - \frac{2}{(n-2)^2}\left(\operatorname{Rc}_0^2\right)_0\wedge\operatorname{id}+\left(\frac{\bar\lambda^2}{n-1}+\frac{\sigma}{n-2}\right)\mathbf{I}, \tag{11.81}$$

where $\bar\lambda=\frac{1}{n}\operatorname{Scal}(\mathbf{R})$ *and* $\sigma=\frac{1}{n}\left|\operatorname{Rc}_0\right|^2$.

As a consequence, we have the following.

COROLLARY 11.36 (Weyl component and trace of $\mathbf{R}^2+\mathbf{R}^\#$ when $\mathbf{R}_{\mathbf{W}}=0$). *If* $\mathbf{R}_{\mathbf{W}}=0$, *then*

$$\left(\mathbf{R}^2+\mathbf{R}^\#\right)_{\mathbf{W}} = \frac{1}{n-2}\left(\operatorname{Rc}_0\wedge\operatorname{Rc}_0\right)_{\mathbf{W}}, \tag{11.82}$$

$$\operatorname{Rc}(\mathbf{R}^2+\mathbf{R}^\#) = -\frac{2}{n-2}\left(\operatorname{Rc}_0^2\right)_0+\frac{n-2}{n-1}\bar\lambda\operatorname{Rc}_0+\left(\bar\lambda^2+\sigma\right)\operatorname{id}, \tag{11.83}$$

where $\operatorname{Rc}_0^2\doteqdot(\operatorname{Rc}_0)^2$.

PROOF OF THE COROLLARY. Since $\langle\mathbf{W}\rangle$ is orthogonal to $\operatorname{id}_\wedge S^2(\mathbb{E}^n)$ (see (11.44)), the first formula follows directly from (11.81).

For the second formula, from (11.81) we can compute $\mathrm{Rc}(\mathbf{R}^2+\mathbf{R}^{\#})$ using (11.41) and (11.42) as follows:

$$\begin{aligned}
&\mathrm{Rc}(\mathbf{R}^2+\mathbf{R}^{\#})\\
&=\frac{1}{n-2}\,\mathrm{Rc}\left(\mathrm{Rc}_0\wedge\mathrm{Rc}_0\right)+\frac{2\bar\lambda}{n-1}\,\mathrm{Rc}\left(\mathrm{Rc}_0\wedge\mathrm{id}\right)\\
&\quad-\frac{2}{(n-2)^2}\,\mathrm{Rc}\left(\left(\mathrm{Rc}_0^2\right)_0\wedge\mathrm{id}\right)+\frac{\bar\lambda^2}{n-1}\,\mathrm{Rc}\left(\mathbf{I}\right)+\frac{\sigma}{n-2}\,\mathrm{Rc}\left(\mathbf{I}\right)\\
&=-\frac{1}{n-2}\,\mathrm{Rc}_0^2+\frac{n-2}{n-1}\bar\lambda\,\mathrm{Rc}_0-\frac{1}{n-2}\left(\mathrm{Rc}_0^2\right)_0+\left(\bar\lambda^2+\frac{n-1}{n-2}\sigma\right)\mathrm{id}\\
&=-\frac{2}{n-2}\left(\mathrm{Rc}_0^2\right)_0-\frac{\sigma}{n-2}\,\mathrm{id}+\frac{(n-2)}{n-1}\bar\lambda\,\mathrm{Rc}_0+\left(\bar\lambda^2+\frac{n-1}{n-2}\sigma\right)\mathrm{id}
\end{aligned}$$

since $\mathrm{Rc}_0^2=\left(\mathrm{Rc}_0^2\right)_0+\sigma\,\mathrm{id}$, which implies (11.83). □

We return to the

Proof of Lemma 11.35. Let $\mathbf{R}_0\doteqdot\mathbf{R}_{\mathrm{Rc}_0}=\frac{2}{n-2}\,\mathrm{id}\wedge\mathrm{Rc}_0$. Since $\mathbf{R}_{\mathbf{W}}=0$, we then have

$$\mathbf{R}=\frac{\bar\lambda}{n-1}\mathbf{I}+\mathbf{R}_0 \tag{11.84}$$

and

$$\begin{aligned}
&\mathbf{R}^2+\mathbf{R}^{\#}\\
&=\mathbf{R}_0^2+\frac{2\bar\lambda}{n-1}\mathbf{R}_0+\left(\frac{\bar\lambda}{n-1}\right)^2\mathbf{I}+\mathbf{R}_0^{\#}+\frac{2\bar\lambda}{n-1}\mathbf{R}_0\#\mathbf{I}+\left(\frac{\bar\lambda}{n-1}\right)^2\mathbf{I}\#\mathbf{I}\\
&=\mathbf{R}_0^2+\mathbf{R}_0^{\#}+\frac{2\bar\lambda}{n-1}\left(\mathbf{R}_0+\mathbf{R}_0\#\mathbf{I}\right)+\left(\frac{\bar\lambda}{n-1}\right)^2\left(\mathbf{I}+\mathbf{I}\#\mathbf{I}\right).
\end{aligned}$$

Observe that by (11.42),

$$\mathrm{Rc}(\mathbf{R}_0)=\frac{2}{n-2}\,\mathrm{Rc}\left(\mathrm{Rc}_0\wedge\mathrm{id}\right)=\mathrm{Rc}_0$$

and

$$\mathrm{Rc}(\mathbf{I})=(n-1)\,\mathrm{id},$$

so that by (11.77) we have

$$\mathbf{R}^2+\mathbf{R}^{\#}=\mathbf{R}_0^2+\mathbf{R}_0^{\#}+\frac{2\bar\lambda}{n-1}\,\mathrm{Rc}_0\wedge\mathrm{id}+\frac{\bar\lambda^2}{n-1}\,\mathrm{id}\wedge\mathrm{id}\,.$$

Comparing this with (11.81), we see that to prove the lemma it suffices to show that

$$\mathbf{R}_0^2+\mathbf{R}_0^{\#}=\frac{1}{n-2}\,\mathrm{Rc}_0\wedge\mathrm{Rc}_0-\frac{2}{(n-2)^2}\left(\mathrm{Rc}_0^2\right)_0\wedge\mathrm{id}+\frac{\sigma}{n-2}\mathbf{I}. \tag{11.85}$$

The rest of the proof is devoted to establishing this equation by a direct calculation.

Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of eigenvectors of Rc_0 with corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$. It is easy to see that

$$\left(\frac{1}{n-2}\,\mathrm{Rc}_0 \wedge \mathrm{Rc}_0\right)(e_i \wedge e_j) = \frac{\lambda_i \lambda_j}{n-2} e_i \wedge e_j$$

and

$$\left(\mathrm{Rc}_0^2\right)(e_i) = \lambda_i^2 e_i,$$
$$\left(\mathrm{Rc}_0^2\right)_0 (e_i) = \left(\lambda_i^2 - \sigma\right) e_i,$$

so that

$$-\frac{2}{(n-2)^2}\left(\left(\mathrm{Rc}_0^2\right)_0 \wedge \mathrm{id}\right)(e_i \wedge e_j) = -\frac{\lambda_i^2 + \lambda_j^2 - 2\sigma}{(n-2)^2} e_i \wedge e_j.$$

Combining these formulas, we have

$$\left(\frac{1}{n-2}\,\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 - \frac{2}{(n-2)^2}\left(\mathrm{Rc}_0^2\right)_0 \wedge \mathrm{id} + \frac{\sigma}{n-2}\mathbf{I}\right)(e_i \wedge e_j)$$
$$= \left(\frac{\lambda_i \lambda_j}{n-2} + \frac{n\sigma - \lambda_i^2 - \lambda_j^2}{(n-2)^2}\right) e_i \wedge e_j.$$

Thus, to prove (11.85), we need to show that

$$\left(\mathbf{R}_0^2 + \mathbf{R}_0^\#\right)(e_i \wedge e_j) = \left(\frac{\lambda_i \lambda_j}{n-2} + \frac{n\sigma - \lambda_i^2 - \lambda_j^2}{(n-2)^2}\right) e_i \wedge e_j. \tag{11.86}$$

First,

$$\mathbf{R}_0(e_i \wedge e_j) = \frac{2}{n-2}\left(\mathrm{id} \wedge \mathrm{Rc}_0\right)(e_i \wedge e_j) = \frac{\lambda_i + \lambda_j}{n-2} e_i \wedge e_j,$$

so that

$$\mathbf{R}_0^2(e_i \wedge e_j) = \left(\frac{\lambda_i + \lambda_j}{n-2}\right)^2 e_i \wedge e_j. \tag{11.87}$$

As in the proof of Lemma 11.33, using equations (11.13), (11.7), and (11.8), we compute

$$\begin{aligned}
&\langle (\mathbf{R}_0 \# \mathbf{R}_0)(e_a \wedge e_b), e_c \wedge e_d \rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [\mathbf{R}_0(\omega_\alpha), \mathbf{R}_0(\omega_\beta)], e_a \wedge e_b \rangle \langle [\omega_\alpha, \omega_\beta], e_c \wedge e_d \rangle \\
&= \frac{1}{8}\sum_{i \neq j,\ p \neq q} \frac{\lambda_i + \lambda_j}{n-2}\frac{\lambda_p + \lambda_q}{n-2} \langle [e_i \wedge e_j, e_p \wedge e_q], e_a \wedge e_b \rangle \\
&\quad \times \langle [e_i \wedge e_j, e_p \wedge e_q], e_c \wedge e_d \rangle \\
&= \sum_{j \neq a,b} \frac{\lambda_a + \lambda_j}{n-2}\frac{\lambda_j + \lambda_b}{n-2} \langle e_a \wedge e_b, e_c \wedge e_d \rangle.
\end{aligned}$$

We then have

$$(\mathbf{R}_0 \# \mathbf{R}_0)(e_a \wedge e_b) = \sum_{j \neq a,b} \frac{\lambda_a + \lambda_j}{n-2} \frac{\lambda_j + \lambda_b}{n-2} e_a \wedge e_b. \tag{11.88}$$

Combining (11.87) and (11.88), we have

$$\begin{aligned}\left(\mathbf{R}_0^2 + \mathbf{R}_0^{\#}\right)(e_i \wedge e_j) &= \left(\left(\frac{\lambda_i + \lambda_j}{n-2} \right)^2 + \sum_{k \neq i,j} \frac{\lambda_i + \lambda_k}{n-2} \frac{\lambda_k + \lambda_j}{n-2} \right) e_i \wedge e_j \\ &= \left(\frac{\lambda_i \lambda_j}{n-2} + \frac{n\sigma - \lambda_i^2 - \lambda_j^2}{(n-2)^2} \right) e_i \wedge e_j,\end{aligned}$$

which proves (11.86) and the equivalent (11.85); to obtain the second equality above, we used the fact that

$$\sum_{k \neq i,j} \lambda_k = -\lambda_i - \lambda_j \quad \text{and} \quad \sum_{k \neq i,j} \lambda_k^2 = n\sigma - \lambda_i^2 - \lambda_j^2$$

(since $\sum_{k=1}^n \lambda_k = 0$) implies

$$\begin{aligned}&\left(\frac{\lambda_i + \lambda_j}{n-2} \right)^2 + \sum_{k \neq i,j} \frac{\lambda_i + \lambda_k}{n-2} \frac{\lambda_k + \lambda_j}{n-2} \\ &= \frac{1}{(n-2)^2} \left((\lambda_i + \lambda_j)^2 + (n-2)\lambda_i \lambda_j + \lambda_k (\lambda_i + \lambda_j) + \lambda_k^2 \right) \\ &= \frac{1}{(n-2)^2} \left((n-2)\lambda_i \lambda_j + n\sigma - \lambda_i^2 - \lambda_j^2 \right).\end{aligned}$$

The lemma is proved. □

4.3. Theorem 11.27 revisited. Equation (11.63) in Theorem 11.27 is rather remarkable and the result hinges on Lemma 11.33. The following *alternate proof* shows how to determine the expression on the right-hand side of (11.63). Recall that $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^{\#}$; for simplicity we let $\alpha \doteqdot 2(n-1)a$ and $\beta \doteqdot (n-2)b$, so that by (11.73),

$$\ell(\mathbf{R}) = (1+\alpha) \frac{\bar{\lambda}}{n-1} \mathbf{I} + (1+\beta) \mathbf{R}_0.$$

By a direct calculation and Lemma 11.33,

(11.89)
$$\begin{aligned}
D(\mathbf{R}) &= \ell^{-1}\left(\left((1+\alpha)\frac{\bar{\lambda}}{n-1}\mathbf{I}+(1+\beta)\mathbf{R}_0\right)^2\right) \\
&\quad+\ell^{-1}\left(\left((1+\alpha)\frac{\bar{\lambda}}{n-1}\mathbf{I}+(1+\beta)\mathbf{R}_0\right)^{\#}\right)-Q(\mathbf{R}) \\
&=\ell^{-1}\left(\left((1+\alpha)\frac{\bar{\lambda}}{n-1}\right)^2\left(\mathbf{I}^2+\mathbf{I}\#\mathbf{I}\right)+(1+\beta)^2\left(\mathbf{R}_0^2+\mathbf{R}_0^{\#}\right)\right) \\
&\quad+\ell^{-1}\left(2(1+\alpha)(1+\beta)\frac{\bar{\lambda}}{n-1}\left(\mathbf{R}_0+\mathbf{I}\#\mathbf{R}_0\right)\right)-Q(\mathbf{R}) \\
&=\ell^{-1}\left(\frac{(1+\alpha)^2\bar{\lambda}^2}{n-1}\mathbf{I}+(1+\beta)^2Q(\mathbf{R}_0)+\frac{2(1+\alpha)(1+\beta)\bar{\lambda}}{n-1}\operatorname{Rc}_0\wedge\operatorname{id}\right) \\
&\quad-Q(\mathbf{R})
\end{aligned}$$

since $\mathbf{I}^2+\mathbf{I}\#\mathbf{I}=(n-1)\mathbf{I}$ and $\mathbf{R}_0+\mathbf{I}\#\mathbf{R}_0=\frac{n-2}{2}\mathbf{R}_0=\operatorname{Rc}_0\wedge\operatorname{id}$.

On the other hand, recall from (11.85) that

$$\begin{aligned}
Q(\mathbf{R}_0) &= \mathbf{R}_0^2+\mathbf{R}_0^{\#} \\
&=\frac{1}{n-2}\operatorname{Rc}_0\wedge\operatorname{Rc}_0-\frac{2}{(n-2)^2}\left(\operatorname{Rc}_0^2\right)_0\wedge\operatorname{id}+\frac{\sigma}{n-2}\mathbf{I}.
\end{aligned}$$

By the curvature decomposition formula and (11.43), we have

$$\operatorname{Rc}_0\wedge\operatorname{Rc}_0=-\frac{\sigma}{n-1}\mathbf{I}-\frac{2}{n-2}\left(\operatorname{Rc}_0^2\right)_0\wedge\operatorname{id}+\left(\operatorname{Rc}_0\wedge\operatorname{Rc}_0\right)_{\mathbf{W}}, \tag{11.90}$$

so that $Q(\mathbf{R}_0)$ may be rewritten as

$$Q(\mathbf{R}_0)=\frac{\sigma}{n-1}\mathbf{I}-\frac{4}{(n-2)^2}\left(\operatorname{Rc}_0^2\right)_0\wedge\operatorname{id}+\frac{1}{n-2}\left(\operatorname{Rc}_0\wedge\operatorname{Rc}_0\right)_{\mathbf{W}}.$$

Thus, by (11.89) and

$$\ell^{-1}(\mathbf{R})=\frac{1}{1+\alpha}\frac{\bar{\lambda}}{n-1}\mathbf{I}+\frac{1}{1+\beta}\mathbf{R}_0,$$

we have

$$\begin{aligned}
D(\mathbf{R}) &= \frac{(1+\alpha)^2\bar{\lambda}^2}{n-1}\ell^{-1}(\mathbf{I}) + \frac{2(1+\alpha)(1+\beta)\bar{\lambda}}{n-1}\ell^{-1}(\mathrm{Rc}_0 \wedge \mathrm{id}) \\
&\quad - Q(\mathbf{R}) + (1+\beta)^2\ell^{-1}(Q(\mathbf{R}_0)) \\
&= \frac{(1+\alpha)\bar{\lambda}^2}{n-1}\mathbf{I} + \frac{2(1+\alpha)\bar{\lambda}}{n-1}\mathrm{Rc}_0 \wedge \mathrm{id} - \frac{1}{n-2}\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 \\
&\quad - \frac{2\bar{\lambda}}{n-1}\mathrm{Rc}_0 \wedge \mathrm{id} + \frac{2}{(n-2)^2}\left(\mathrm{Rc}_0^2\right)_0 \wedge \mathrm{id} - \left(\frac{\bar{\lambda}^2}{n-1} + \frac{\sigma}{n-2}\right)\mathbf{I} \\
&\quad + \frac{(1+\beta)^2}{1+\alpha}\frac{\sigma}{n-1}\mathbf{I} - \frac{4(1+\beta)}{(n-2)^2}\left(\mathrm{Rc}_0^2\right)_0 \wedge \mathrm{id} + \frac{(1+\beta)^2}{n-2}(\mathrm{Rc}_0 \wedge \mathrm{Rc}_0)_{\mathbf{W}}.
\end{aligned}$$

Simplifying this expression for $D(\mathbf{R})$ and applying (11.51) to get

$$(\mathrm{Rc}_0 \wedge \mathrm{Rc}_0)_{\mathbf{W}} = \mathrm{Rc}_0 \wedge \mathrm{Rc}_0 + \frac{\sigma}{n-1}\mathbf{I} + \frac{2}{n-2}(\mathrm{Rc}_0^2)_0 \wedge \mathrm{id},$$

we obtain

$$\begin{aligned}
D(\mathbf{R}) &= \frac{\alpha\bar{\lambda}^2}{n-1}\mathbf{I} + \frac{2\alpha\bar{\lambda}}{n-1}\mathrm{Rc}_0 \wedge \mathrm{id} + \left(\frac{(1+\beta)^2}{1+\alpha} - 1 + \frac{2\beta+\beta^2}{n-2}\right)\frac{\sigma}{n-1}\mathbf{I} \\
&\quad + \frac{2\beta+\beta^2}{n-2}\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 + \frac{2\beta^2}{(n-2)^2}(\mathrm{Rc}_0^2)_0 \wedge \mathrm{id}.
\end{aligned}$$

Noticing that the first two terms may be rewritten as

$$\frac{\alpha}{n-1}(\mathrm{Rc} \wedge \mathrm{Rc} - \mathrm{Rc}_0 \wedge \mathrm{Rc}_0),$$

we then have (while substituting a and b back into α and β)

$$\begin{aligned}
D(\mathbf{R}) &= 2a\,\mathrm{Rc} \wedge \mathrm{Rc} + \left((n-2)b^2 + 2b - 2a\right)\mathrm{Rc}_0 \wedge \mathrm{Rc}_0 + 2b^2(\mathrm{Rc}_0^2)_0 \wedge \mathrm{id} \\
&\quad + \left(\frac{(1+(n-2)b)^2}{1+2(n-1)a} - 1 + b(2+(n-2)b)\right)\frac{\sigma}{n-1}\mathbf{I}.
\end{aligned}$$

Noting that $(\mathrm{Rc}_0^2)_0 = \mathrm{Rc}_0^2 - \sigma\,\mathrm{id}$ and $\mathrm{tr}(\mathrm{Rc}_0^2) = n\sigma$, we obtain (11.62) in Theorem 11.27.

5. The convex cone of 2-nonnegative algebraic curvature operators

In this section we discuss various properties of the curvature operator condition of 2-nonnegativity. We show that the convex cone of 2-nonnegative algebraic curvature operators is preserved by the ODE. Then we show that a family of O(n)-invariant linear transformations of this cone is also preserved by the ODE. This is the main result of this section.

5.1. Definition and properties of 2-nonnegative curvature operator. Analogous to the definition of positive Riemann curvature operator, we say that an algebraic curvature operator $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$ is **nonnegative** (**positive**) if the eigenvalues of $\mathbf{R}$ are all nonnegative (positive). Next we consider the following notion, which was first studied in the context of Ricci flow by H. Chen [**112**] (compare with Definition 11.1).

DEFINITION 11.37 (2-nonnegative curvature operator). We say that an algebraic curvature operator $\mathbf{R} \in S_B^2(\mathfrak{so}(n))$ is **2-nonnegative** if

$$\mu_\alpha + \mu_\beta \geq 0$$

for any $\alpha \neq \beta$, where $\{\mu_\gamma\}_{\gamma=1}^N$, $N \doteqdot \frac{n(n-1)}{2}$, denote the eigenvalues of $\mathbf{R}$. We say that $\mathbf{R}$ is **2-positive** if the inequality above is strict.

Clearly this condition only makes sense when $n \geq 3$. When $n = 3$, the condition of 2-nonnegativity is equivalent to the Ricci curvature $\mathrm{Rc}(\mathbf{R})$ being nonnegative. The cone $\mathfrak{C}$ of 2-nonnegative algebraic curvature operators is an $\mathrm{O}(n)$-invariant convex cone.

Another curvature condition related to 2-nonnegativity is the following.

DEFINITION 11.38. We say that an algebraic curvature operator $\mathbf{R}$ is a **nonnegative curvature operator of rank** 1 if $\mathbf{R} = \omega \otimes \omega$ for some $\omega \in \Lambda^2\mathbb{E}^n$. An algebraic curvature operator is called **geometrically nonnegative** if it can be written as the linear combination, with positive coefficients, of nonnegative curvature operators of rank 1.

If $\mathbf{R}$ is geometrically nonnegative, then $\mathbf{R} = \sum_\alpha c_\alpha \eta_\alpha \otimes \eta_\alpha$, where $c_\alpha > 0$. Thus, for any 2-form ξ we have

$$\mathbf{R}(\xi, \xi) = \sum_\alpha c_\alpha (\eta_\alpha(\xi))^2 \geq 0,$$

so that $\mathbf{R}$ is a nonnegative curvature operator.

Note that if $\omega \otimes \omega$ is an algebraic curvature operator, then the first Bianchi identity implies $\omega \wedge \omega = 0$ and hence there exist $v, w \in \mathbb{E}^n$ such that[19]

$$\omega = v \wedge w.$$

[19]Indeed, the first Bianchi identity applied to $\omega \wedge \omega$ implies

$$(\omega \wedge \omega)_{ijk\ell} = \frac{1}{3}(\omega_{ij}\omega_{k\ell} + \omega_{ki}\omega_{j\ell} + \omega_{jk}\omega_{i\ell}) = 0.$$

We may prove, by induction on n, that $\omega \wedge \omega = 0$ implies there exist $v, w \in \mathbb{R}^n$ such that $\omega = v \wedge w$. The result is clearly true for $n = 1, 2$. Now suppose the result is true for dimension $n-1$, where $n \geq 4$. Let $\{e_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$. We may write

$$\omega = e_1 \wedge x + \eta,$$

where $x = \sum_{i \geq 2} x^i e_i$ and $\eta = \sum_{i,j \geq 2} \eta^{ij} e_i \wedge e_j$. From $\omega \wedge \omega = 0$ we find that

$$e_1 \wedge x \wedge \eta = 0 \quad \text{and} \quad \eta \wedge \eta = 0.$$

By our induction assumption, $\eta = y \wedge z$, where $y = \sum_{i \geq 2} y^i e_i$ and $z = \sum_{i \geq 2} z^i e_i$. Now since $n \geq 3$, $e_1 \wedge x \wedge \eta = 0$ implies $x \wedge y \wedge z = 0$, so that the vectors x, y, z are linearly dependent. Without loss of generality, we may assume that $x = c_1 y + c_2 z$ for some

Note also that the Riemann curvature operator of $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ is a nonnegative curvature operator of rank 1.

We have the following properties of 2-nonnegative algebraic curvature operators $\mathbf{R}$.

LEMMA 11.39. *If an algebraic curvature operator* $\mathbf{R}$ *is* 2*-nonnegative, then we have the following.*

(i) *If we order the eigenvalues of* $\mathbf{R}$ *so that* $\mu_1 \leq \cdots \leq \mu_N$*, then the eigenvalues* μ_β *are nonnegative for* $\beta \geq 2$ *and*

$$\mu_1 \geq -\mu_2 \geq -\frac{\operatorname{trace}(\mathbf{R})}{N-2} = -\frac{2\operatorname{trace}(\mathbf{R})}{n(n-1)-4}. \tag{11.91}$$

(ii) *The scalar curvature* $\operatorname{Scal}(\mathbf{R})$ *is nonnegative, with equality if and only if* $\mathbf{R} = 0$.

(iii) *The Ricci curvature* $\operatorname{Rc}(\mathbf{R})$ *is nonnegative.*

Moreover, the cone $\mathfrak{C}$ *of* 2*-nonnegative curvature operators contains the cone of geometrically nonnegative operators.*

PROOF. (i) Let $\{\tilde{\omega}_\alpha\}_{\alpha=1}^N$ be an orthonormal basis for $\mathfrak{so}(n)$ which satisfies $\mathbf{R}(\tilde{\omega}_\alpha) = \mu_\alpha \tilde{\omega}_\alpha$. Since $\mathbf{R}$ is 2-nonnegative, $\mu_\alpha + \mu_\beta \geq 0$ for any $\alpha \neq \beta$. Hence $\mu_\alpha \geq 0$ for $\alpha \geq 2$ and the only possible negative eigenvalue is μ_1. Since

$$\operatorname{trace}(\mathbf{R}) = \sum_{\alpha=1}^N \mu_\alpha \geq \sum_{\alpha \geq 3} \mu_\alpha \geq (N-2)\mu_3,$$

formula (11.91) follows.

(ii) Let $\{\omega_\alpha\}_{\alpha=1}^N \doteqdot \{e_i \wedge e_j\}_{i<j}$ be an orthonormal basis for $\mathfrak{so}(n)$, where each α corresponds to a pair (i,j) with $i < j$. From definition (11.39) we have

$$\operatorname{Scal}(\mathbf{R}) = \sum_{i,j} \mathbf{R}_{ijij} = \sum_\alpha \mathbf{R}_{\alpha\alpha}.$$

On the other hand, since $\mathbf{R}$ is 2-nonnegative,

$$0 \leq \sum_{\alpha \neq \beta} (\mathbf{R}_{\alpha\alpha} + \mathbf{R}_{\beta\beta}) = (N-1)\operatorname{Scal}(\mathbf{R}).$$

Hence, if $\operatorname{Scal}(\mathbf{R}) = 0$, then $\mathbf{R}_{\alpha\alpha} + \mathbf{R}_{\beta\beta} = 0$ for all pairs (α, β) with $\alpha \neq \beta$, which implies that $\mathbf{R}_{\alpha\alpha} = 0$ for all α, i.e., $\mathbf{R} = 0$.

(iii) From definition (11.38) we have

$$\operatorname{Rc}(\mathbf{R})_{ii} = \sum_{k \neq i} \langle \mathbf{R}(e_i \wedge e_k), e_i \wedge e_k \rangle = \sum_{\beta \in \mathcal{B}} \langle \mathbf{R}(\omega_\beta), \omega_\beta \rangle,$$

$c_1, c_2 \in \mathbb{R}$. We conclude

$$\omega = (y + c_2 e_1) \wedge (z - c_1 e_1) \doteqdot v \wedge w.$$

where

$$\mathcal{B} \doteqdot \{(1,i),(2,i),\ldots,(i-1,i),(i,i+1),(i,i+2),\ldots,(i,n)\}.$$

Since $\mathbf{R}$ is 2-nonnegative, $\mathbf{R}_{\alpha\alpha} + \mathbf{R}_{\beta\beta} \geq 0$ for any $\alpha \neq \beta$ and $\mathbf{R}_{\gamma\gamma} < 0$ for at most one $\gamma \in \{1,\ldots,N\}$. This is enough to conclude that $\operatorname{Rc}(\mathbf{R})_{ii} \geq 0$ for any orthonormal basis $\{e_j\}_{j=1}^n$. In fact,

$$\operatorname{Rc}(\mathbf{R})_{ii} \geq \inf_{\{\eta_k\}} \sum_{k=1}^{n-1} \langle \mathbf{R}(\eta_k), \eta_k \rangle \geq \mu_1 + \mu_2 + \cdots + \mu_{n-1},$$

where the infimum on the right-hand side is taken over all orthonormal bases $\{\eta_k\}_{k=1}^{n-1}$ for $\operatorname{span}\{\omega_\beta : \beta \in \mathcal{B}\}$ and where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ are the ordered eigenvalues of $\mathbf{R}$. Hence we have

$$\operatorname{Rc}(\mathbf{R})_{ii} \geq (n-3)\mu_3 \geq (n-3)|\mu_1|, \tag{11.92}$$

which completes the proof of (iii).

Finally, the last statement in the lemma is clear since the cone of nonnegative curvature operators contains the cone of geometrically nonnegative operators. □

The next lemma gives a relation between the algebraic curvature operator $D_{a,b}(\mathbf{R})$ and the cone of geometrically nonnegative operators.

LEMMA 11.40.

(i) *Suppose that $A \in S^2(\mathbb{E}^n)$ is a nonnegative symmetric 2-tensor, let A_0 denote the trace-free part of A, and let $A_0^2 \doteqdot (A_0)^2$. Then the algebraic curvature operators $A \wedge A$, $A_0^2 \wedge \mathrm{id}$, and $\mathbf{I}$ are all contained in the cone of geometrically nonnegative operators.*

(ii) *Suppose that the algebraic curvature operator $\mathbf{R}$ has nonnegative Ricci curvature. If constants $a, b \in [0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}]$ satisfy the relation (11.64), then the algebraic curvature operator $D_{a,b}(\mathbf{R})$ lies in the cone of geometrically nonnegative operators.*

PROOF. (i) Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of A_0 and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{E}^n$ satisfying $A_0(e_i) = \lambda_i e_i$. Then $A(e_i) = \bar{\lambda} + \lambda_i$, where $\bar{\lambda} \doteqdot \frac{\operatorname{tr}(A)}{n}$. By Lemma 11.13, $A \wedge A$, $A \wedge \mathrm{id}$, and $A_0^2 \wedge \mathrm{id}$ belong to

$S_B^2(\mathfrak{so}(n))$. Letting $\omega_\alpha \doteqdot e_i \wedge e_j$, we compute using (11.5) that

$$A \wedge A = \sum_{i,j} \left(\left(\bar{\lambda} + \lambda_i \right) e_i \otimes e_i \right) \wedge \left(\left(\bar{\lambda} + \lambda_j \right) e_j \otimes e_j \right) \tag{11.93}$$
$$= \sum_{i<j} \left(\bar{\lambda} + \lambda_i \right) \left(\bar{\lambda} + \lambda_j \right) \omega_\alpha \otimes \omega_\alpha,$$

$$A \wedge \mathrm{id} = \frac{1}{2} \sum_{i<j} \left(\lambda_i + \lambda_j \right) \left(e_i \wedge e_j \right) \otimes \left(e_i \wedge e_j \right) \tag{11.94}$$
$$= \frac{1}{2} \sum_{i<j} \left(\lambda_i + \lambda_j \right) \omega_\alpha \otimes \omega_\alpha$$

and

$$A_0^2 \wedge \mathrm{id} = \frac{1}{2} \sum_{i<j} \left(\lambda_i^2 + \lambda_j^2 \right) \left(e_i \wedge e_j \right) \otimes \left(e_i \wedge e_j \right) = \frac{1}{2} \sum_{i<j} \left(\lambda_i^2 + \lambda_j^2 \right) \omega_\alpha \otimes \omega_\alpha .$$

Hence both $A \wedge A$ and $A_0^2 \wedge \mathrm{id}$ are contained in the cone of geometrically nonnegative operators since $\bar{\lambda} + \lambda_i \geq 0$. Since $\mathbf{I} = \sum_{i<j} (e_i \wedge e_j) \otimes (e_i \wedge e_j)$, we know that $\mathbf{I}$ also belongs to the cone of geometrically nonnegative operators.

(ii) This follows easily from (i), which shows that each of the terms in (11.66) is geometrically nonnegative. □

Remark 11.41. Note that as a special case of the lemma we have that

$$\mathrm{Rc} \wedge \mathrm{Rc} = \sum_{i<j} \mathrm{Rc}_{ii} \, \mathrm{Rc}_{jj} \omega_\alpha \otimes \omega_\alpha,$$
$$\mathrm{Rc}_0^2 \wedge \mathrm{id} = \frac{1}{2} \sum_{i<j} \left(\lambda_i^2 + \lambda_j^2 \right) \left(e_i \wedge e_j \right) \otimes \left(e_i \wedge e_j \right)$$

are contained in the cone of geometrically nonnegative operators provided $\mathrm{Rc} \geq 0$.

The $\mathrm{O}(n)$-invariant convex cone $\mathfrak{C}$ is preserved by the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$.

Proposition 11.42 (2-nonnegativity is preserved by the ODE). *The convex cone $\mathfrak{C}$ of 2-nonnegative algebraic curvature operators is preserved by the* ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$.[20]

Proof. We only need to show that $\mathbf{R}^2 + \mathbf{R}^\#$ lies inside the tangent cone of the convex cone $\mathfrak{C}$ for $\mathbf{R} \in \partial\mathfrak{C}$. Let $\{\omega_\alpha\}_{\alpha=1}^N$ be an orthonormal basis of eigenvectors of $\mathbf{R}$ in $\mathfrak{so}(n)$ with corresponding eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$. Given $\mathbf{S} \in S_B^2(\mathfrak{so}(n))$, let $\mathbf{S}_{\alpha\beta} \doteqdot \mathbf{S}(\omega_\alpha, \omega_\beta)$. If $\mathbf{R} \in \partial\mathfrak{C}$, then a vector $\mathbf{S}$ at the point $\mathbf{R}$ lies in the tangent cone of $\mathfrak{C}$ if $\mathbf{S}_{\alpha\alpha} + \mathbf{S}_{\beta\beta} \geq 0$ for all α, β with $\alpha < \beta$ such that $\mathbf{R}_{\alpha\alpha} + \mathbf{R}_{\beta\beta} = 0$.

[20]By the proof it is clear that the same result holds on $S^2(\mathfrak{so}(n))$ (i.e., if we do not assume that $\mathbf{R}$ satisfies the Bianchi identity).

If $\mathbf{R}$ has nonnegative curvature operator, then $\mathbf{R}^2+\mathbf{R}^\#$ has nonnegative curvature operator and we are done. Otherwise, since the eigenvalues of $\mathbf{R}$ are ordered, the set of all pairs (α,β) with $\alpha<\beta$ and $\mathbf{R}_{\alpha\alpha}+\mathbf{R}_{\beta\beta}=0$ is equal to $\{(1,2),\dots,(1,k)\}$ for some $k\geq 2$ and we have $-\mathbf{R}_{11}=\mathbf{R}_{22}=\cdots=\mathbf{R}_{kk}>0$. We then have for $2\leq\alpha\leq k$,

(11.95)

$$(\mathbf{R}^2+\mathbf{R}^\#)_{11}+(\mathbf{R}^2+\mathbf{R}^\#)_{\alpha\alpha}=\mu_1^2+\mu_\alpha^2+2\sum_{\beta<\gamma}\left(\left(c_1^{\beta\gamma}\right)^2+\left(c_\alpha^{\beta\gamma}\right)^2\right)\mu_\beta\mu_\gamma .$$

Since $\mu_1+\mu_2\geq 0$, we have $\mu_\gamma>0$ for all $\gamma\geq 2$. Hence for $2\leq\alpha\leq k$,

$$\begin{aligned}&\sum_{\beta<\gamma}\left(\left(c_1^{\beta\gamma}\right)^2+\left(c_\alpha^{\beta\gamma}\right)^2\right)\mu_\beta\mu_\gamma\\ &=\sum_{2\leq\beta<\gamma}\left(c_1^{\beta\gamma}\right)^2\mu_\beta\mu_\gamma+\sum_{1\leq\beta<\gamma}\left(c_\alpha^{\beta\gamma}\right)^2\mu_\beta\mu_\gamma\\ &\geq\sum_{\gamma>\alpha}\left(c_1^{\alpha\gamma}\right)^2\mu_\alpha\mu_\gamma+\sum_{2\leq\beta<\alpha}\left(c_1^{\beta\alpha}\right)^2\mu_\beta\mu_\alpha+\sum_{\gamma\geq 2}\left(c_\alpha^{1\gamma}\right)^2\mu_1\mu_\gamma\\ &=\sum_{\gamma\geq 2}\left(c_1^{\alpha\gamma}\right)^2(\mu_\alpha+\mu_1)\,\mu_\gamma\\ &\geq 0,\end{aligned}\tag{11.96}$$

where we used $c_1^{1\gamma}=c_1^{\alpha\alpha}=0$. Therefore $(\mathbf{R}^2+\mathbf{R}^\#)_{11}+(\mathbf{R}^2+\mathbf{R}^\#)_{\alpha\alpha}\geq 0$ for $2\leq\alpha\leq k$, i.e., $\mathbf{R}^2+\mathbf{R}^\#$ lies in the tangent cone of $\mathfrak{C}$; the proposition follows.[21] □

EXERCISE 11.43. Show that if $\mathbf{R}\in\partial\mathfrak{C}$, then the cone of geometrically nonnegative operators lies in the tangent cone of $\mathfrak{C}$ at $\mathbf{R}$.

HINT: See the proof of Proposition 11.42.

5.2. Ricci flow preserves 2-nonnegative curvature operator. Using Hamilton's weak and strong maximum principles for Rm, one can prove the following two results concerning the 2-nonnegativity of Rm (see H. Chen [**112**] for the consequence of the weak maximum principle on closed manifolds and see B. Wu and one of the authors [**391**] for the complete noncompact with bounded curvature case). The weak maximum principle for systems tells us the following.

Let $C_0\subset S_B^2(\mathfrak{so}(n))$ be an $\mathrm{O}(n)$-invariant closed convex set of algebraic curvature operators. Using the identification of $(T\mathcal{M}_x,g)$ with $\mathbb{E}^n$ for $x\in\mathcal{M}$, we can define a closed convex set $\mathcal{K}\subset\mathcal{V}\doteqdot\mathrm{Sym}^2\left(\Lambda^2T^*\mathcal{M}\right)$ (which is invariant under parallel translation and independent of time) by

$$C_0=\mathcal{K}_x\subset\mathrm{Sym}^2\left(\Lambda^2T_x^*\mathcal{M}\right)=S^2(\mathfrak{so}(n))$$

[21] In the proof we actually have $\mu_1+\mu_2=0$ and $\mu_\alpha+\mu_1=0$.

for all $x \in \mathcal{M}$. In the remainder of this chapter we shall call $\mathcal{K}$ the subset of $\mathcal{V}$ corresponding to C_0.

PROPOSITION 11.44 (Weak maximum principle and 2-nonnegativity). *If $(\mathcal{M}^n, g(t))$ is a complete solution to the Ricci flow with bounded curvature such that the curvature operator* $\operatorname{Rm}(g(0))$ *is 2-nonnegative, then for any $t \geq 0$ the curvature operator* $\operatorname{Rm}(g(t))$ *is 2-nonnegative.*

PROOF. Recall that by using Uhlenbeck's trick, the evolution equation for the curvature operator Rm is

$$\frac{\partial}{\partial t} \operatorname{Rm} - \Delta \operatorname{Rm} = \operatorname{Rm}^2 + \operatorname{Rm}^{\#} . \tag{11.97}$$

By Proposition 11.42 the convex cone $\mathfrak{C}$ of 2-nonnegative operators $\mathbf{R} \in S^2(\mathfrak{so}(n))$ is preserved by the ODE corresponding to the PDE (11.97). Let $\mathcal{K} \subset \mathcal{V} = \operatorname{Sym}^2\left(\Lambda^2 T^* \mathcal{M}\right)$ be the corresponding set of 2-nonnegative operators on $\mathcal{M}$ (which is invariant under parallel translation and independent of time). By the weak maximum principle for systems, i.e., Theorem 10.13 when $\mathcal{M}$ is closed and Theorem 12.34 in the next chapter when $\mathcal{M}$ is noncompact, we conclude that if $\operatorname{Rm}(g(0)) \subset \mathcal{K}$, then $\operatorname{Rm}(g(t)) \subset \mathcal{K}$, i.e., $\operatorname{Rm}(g(t))$ is 2-nonnegative for $t \geq 0$. □

The strong maximum principle for systems implies the following.

PROPOSITION 11.45 (Strong maximum principle and 2-nonnegativity). *Let $(\mathcal{M}^n, g(t))$ be a complete solution to the Ricci flow with bounded curvature such that* $\operatorname{Rm}(g(0))$ *is 2-nonnegative.*

(1) *Then for any $t > 0$ the curvature operator* $\operatorname{Rm}(g(t))$ *is either nonnegative or 2-positive.*
(2) *If in addition $g(0)$ has 2-positive curvature operator at some point in $\mathcal{M}$, then $g(t)$ has 2-positive curvature operator everywhere for $t > 0$.*

REMARK 11.46. In the proof below of part (2) we are essentially proving a special case of the strong maximum principle for systems (for a more general version, see Proposition 12.47 and Proposition 12.49 in Chapter 12).[22]

PROOF. First recall that by Proposition 11.44 we have

$$\mu_1 (\operatorname{Rm}(g(t))) + \mu_2 (\operatorname{Rm}(g(t))) \geq 0$$

for all $t \geq 0$. We prove the parts of the proposition in reverse order.

(2) Let $x_0 \in \mathcal{M}$ be a point such that $(\mu_1 + \mu_2) \operatorname{Rm}(x_0, 0) > 0$. Given any point $y \in \mathcal{M}$, let $\Omega \subset \mathcal{M}$ be a connected open set such that $\bar{\Omega}$ is a compact manifold with smooth boundary and Ω contains both x_0 and y. Let φ_1 to be a smooth nonnegative function satisfying

$$\varphi_1(x) \leq \frac{1}{2} (\mu_1 + \mu_2) \operatorname{Rm}(x, 0)$$

[22]The reader may wish to first read the proof of Proposition 12.47.

for all $x \in \bar{\Omega}$, $\varphi_1(x_0) \geq \frac{1}{4}(\mu_1+\mu_2)\operatorname{Rm}(x_0,0) > 0$, and $\varphi_1 = 0$ on $\partial\Omega$. Given a large constant $A > 0$ to be chosen later, there exists a solution $f : \bar{\Omega} \times [0,T) \to \mathbb{R}$ of

$$\frac{\partial f}{\partial t} = \Delta f - Af$$

with $f(x,0) = \varphi_1(x)$ and $f|_{\partial\Omega\times[0,T]} = 0$. By the strong maximum principle for scalars (Theorem 12.40), we have $f(x,t) > 0$ on $\Omega \times (0,T]$. By the weak maximum principle we have

$$f(x,t) \leq \frac{1}{2}\sup_{x\in\bar{\Omega}}(\mu_1+\mu_2)\operatorname{Rm}(x,0) \text{ on } \bar{\Omega}\times[0,T].$$

Define

$$\overline{\operatorname{Rm}}(x,t) \doteqdot \operatorname{Rm}(x,t) + \left(\varepsilon e^{At} - f(x,t)\right)\operatorname{id}_{\Lambda^2}(x),$$

where $\varepsilon > 0$. For A sufficiently large we have that for $\varepsilon \in (0, e^{-AT}]$,

$$\frac{\partial}{\partial t}\overline{\operatorname{Rm}} > \Delta\overline{\operatorname{Rm}} + \overline{\operatorname{Rm}}^2 + \overline{\operatorname{Rm}}^{\#}.$$

This implies that $\overline{\operatorname{Rm}}$ is 2-positive on $\bar{\Omega}\times[0,T]$ for any $\varepsilon \in (0,e^{-AT}]$ (we leave this as an exercise for the reader). Taking the limit as $\varepsilon \to 0$, we conclude that $\operatorname{Rm} - f\operatorname{id}_{\Lambda^2}$ is 2-nonnegative on $\bar{\Omega}\times[0,T]$. Since $f(y,t) > 0$, this implies that Rm is 2-positive at (y,t). This completes the proof of part (2).

(1) By part (2), if $g(t_1)$ is 2-nonnegative everywhere in $\mathcal{M}$ and 2-positive at a point in $\mathcal{M}$, then $g(t)$ is 2-positive everywhere for $t > t_1$. Hence, if for some $t_0 > 0$ we have $\mu_1(\operatorname{Rm}(g(t_0))) + \mu_2(\operatorname{Rm}(g(t_0))) = 0$ at some point (i.e., if $\operatorname{Rm}(g(t_0))$ is not 2-positive for some $t_0 > 0$), then

$$\mu_1(\operatorname{Rm}(g(t))) + \mu_2(\operatorname{Rm}(g(t))) = 0$$

everywhere for $t < t_0$ (which by continuity implies the same for $t = t_0$). We **claim** that

$$\mu_1(\operatorname{Rm}(g(t))) = \mu_2(\operatorname{Rm}(g(t))) = 0$$

everywhere for $t \leq t_0$. (The theorem then follows since this implies that $\operatorname{Rm} \geq 0$ on $\mathcal{M}\times[0,t_0]$, which in turn implies $\operatorname{Rm}(g(t)) \geq 0$ for all $t \geq 0$ since $\operatorname{Rm} \geq 0$ is preserved under the Ricci flow.)

To prove the claim, consider any $(x_2,t_2) \in \mathcal{M}\times(0,t_0]$ and let ω_1 and ω_2 be unit 2-forms at (x_2,t_2) which are eigenvectors for $\operatorname{Rm}(g(x_2,t_2))$ corresponding to $\mu_1(\operatorname{Rm})$ and $\mu_2(\operatorname{Rm})$, respectively. Parallel translate ω_1 and ω_2 along geodesics emanating from x_2 with respect to $g(t_2)$ to define ω_1 and ω_2 in a space-time neighborhood of (x_2,t_2), where ω_1 and ω_2 are independent of time. We have at (x_2, t_2)

$$\begin{aligned}
0 &\geq \frac{\partial}{\partial t}\left(\operatorname{Rm}(\omega_1,\omega_1)+\operatorname{Rm}(\omega_2,\omega_2)\right) \\
&= \left(\frac{\partial}{\partial t}\operatorname{Rm}\right)(\omega_1,\omega_1)+\left(\frac{\partial}{\partial t}\operatorname{Rm}\right)(\omega_2,\omega_2) \\
&= \left(\Delta\operatorname{Rm}+\operatorname{Rm}^2+\operatorname{Rm}^{\#}\right)(\omega_1,\omega_1) \\
&\quad+\left(\Delta\operatorname{Rm}+\operatorname{Rm}^2+\operatorname{Rm}^{\#}\right)(\omega_2,\omega_2) \\
&= \Delta\left(\operatorname{Rm}(\omega_1,\omega_1)+\operatorname{Rm}(\omega_2,\omega_2)\right) \\
&\quad+\mu_1(\operatorname{Rm})^2+\mu_2(\operatorname{Rm})^2 \\
&\quad+\left(\operatorname{Rm}^{\#}\right)(\omega_1,\omega_1)+\left(\operatorname{Rm}^{\#}\right)(\omega_2,\omega_2) \\
&\geq \mu_1(\operatorname{Rm})^2+\mu_2(\operatorname{Rm})^2,
\end{aligned}$$

where to obtain the last inequality we used (11.96) with $\mathbf{R}$ replaced by Rm and we used the fact that $\operatorname{Rm}(\omega_1,\omega_1)+\operatorname{Rm}(\omega_2,\omega_2)\geq 0$ while at (x_2,t_2) we have $=0$. Hence

$$\mu_1\left(\operatorname{Rm}\left(g\left(x_2,t_2\right)\right)\right)=\mu_2\left(\operatorname{Rm}\left(g\left(x_2,t_2\right)\right)\right)=0.$$

□

5.3. Invariance under the ODE of $\ell_{a,b}(\mathfrak{C})$. The following is the main result of this section. This result establishes a 1-parameter family of $\mathrm{O}(n)$-invariant closed convex cones preserved by the ODE starting with the cone of 2-nonnegative algebraic curvature operators and ending with a cone contained in the cone of positive algebraic curvature operators.

PROPOSITION 11.47 ($\ell_{a,b}$ of the cone of 2-nonnegative operators is preserved by the ODE). *Let $\mathfrak{C}$ denote the cone of 2-nonnegative curvature operators and let $\bar{b} \doteqdot \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}$. Assume $n\geq 4$,*

$$b\in\left[0,\bar{b}\right], \quad \text{and} \quad 2a=2b+(n-2)b^2. \tag{11.98}$$

Then we have the following.

(i) *$\ell_{a,b}(\mathfrak{C})$ is preserved by the ODE $\frac{d}{dt}\mathbf{R}=\mathbf{R}^2+\mathbf{R}^{\#}$.*
(ii) *If in addition $b>0$, then the vector field $\mathbf{R}^2+\mathbf{R}^{\#}$ is transverse to the boundary of $\ell_{a,b}(\mathfrak{C})$ at points where $\mathbf{R}\neq 0$.*
(iii) *The cone $\ell_{a,\bar{b}}(\mathfrak{C}\setminus\{0\})$ lies inside the cone of positive curvature operators. Note that $\ell_{0,0}(\mathfrak{C})=\mathfrak{C}$.*

REMARK 11.48. Regarding the $b=0$ case in relation to part (ii), note that it is easy to see that $Q(\mathbf{R})=\mathbf{R}^2+\mathbf{R}^{\#}$ is not in general transverse to

$\partial\mathfrak{C}$. For example, when $n = 3$, if

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $Q(\mathbf{R}) = \mathbf{R}$, which is not transverse to $\partial\mathfrak{C}$.

PROOF. We again adopt the convention of omitting the subscripts a, b of $\ell_{a,b}$, $D_{a,b}$, and $X_{a,b}$.

(i) By Lemma 11.26, in order to show that $\ell(\mathfrak{C})$ is preserved by the ODE, we need to show that for any $\mathbf{R} \in \partial\mathfrak{C}$ the vector

$$X(\mathbf{R}) = \ell^{-1}(\ell(\mathbf{R})^2 + \ell(\mathbf{R})^{\#}) = \mathbf{R}^2 + \mathbf{R}^{\#} + D(\mathbf{R})$$

is contained in the tangent cone of $\mathfrak{C}$. Since, by Proposition 11.42, $\mathbf{R}^2 + \mathbf{R}^{\#}$ lies inside the tangent cone of $\mathfrak{C}$, it suffices to show that $D(\mathbf{R})$ also lies inside the tangent cone of $\mathfrak{C}$. However this is true for the following reasons. It follows from Lemma 11.40(ii) that $D(\mathbf{R})$ lies in the cone of geometrically nonnegative operators.[23] By Exercise 11.43, $D(\mathbf{R})$ is contained in the tangent cone of $\mathfrak{C}$.

(ii) First observe that $\mathbf{R}^2 + \mathbf{R}^{\#}$ being transverse to the boundary of $\ell(\mathfrak{C})$ is equivalent to

$$X(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^{\#} + D(\mathbf{R})$$

being transverse to $\partial\mathfrak{C}$. We have $\mathbf{R}^2 + \mathbf{R}^{\#}$ is contained in the tangent cone of $\mathfrak{C}$. Thus, to show that $X(\mathbf{R})$ is transverse to $\partial\mathfrak{C}$, it suffices to show that $D(\mathbf{R})$ is a positive curvature operator.

Recall that by (11.66),

$$\begin{aligned} D(\mathbf{R}) &= 2a \operatorname{Rc} \wedge \operatorname{Rc} + 2b^2 \operatorname{Rc}_0^2 \wedge \operatorname{id} \\ &\quad + \frac{\sigma b^2}{1 + 2(n-1)a} \left(-n(n-2)b^2 - 4b + 2 \right) \mathbf{I}, \end{aligned}$$

where $\sigma = \frac{1}{n} \| \operatorname{Rc}_0(\mathbf{R}) \|^2$.[24] We first assume that $0 < b < \bar{b}$ so that

$$-n(n-2)b^2 - 4b + 2 > 0.$$

Case 1. $\sigma > 0$. Then $D(\mathbf{R})$ is a positive curvature operator since $2a \operatorname{Rc} \wedge \operatorname{Rc}$ and $2b^2 \operatorname{Rc}_0^2 \wedge \operatorname{id}$ are nonnegative and the coefficient in front of $\mathbf{I}$ is positive. This implies $X(\mathbf{R})$ is transverse to $\partial\mathfrak{C}$.

Case 2. $\sigma = 0$. Then $\operatorname{Rc}_0 = 0$, so that $\operatorname{Rc} = \frac{1}{n} \operatorname{Scal} \cdot \operatorname{id} > 0$ since $\mathbf{R} \neq 0$ implies $\operatorname{Scal} > 0$. Hence

$$D(\mathbf{R}) = 2a \operatorname{Rc} \wedge \operatorname{Rc} = \frac{2a}{n^2} (\operatorname{Scal})^2 \operatorname{id} \wedge \operatorname{id}$$

is a positive curvature operator.

Hence $X(\mathbf{R})$ is transverse to $\partial\mathfrak{C}$ when $b < \bar{b}$.

[23] Hence $D(\mathbf{R})$ lies in the cone of nonnegative curvature operators.

[24] Note that $D(\mathbf{R})$ is geometrically nonnegative.

Now we consider the case where $b = \bar{b}$. In this case, by part (iii) of the present proposition (whose proof is independent of part (ii)), we have $\ell_{a,\bar{b}}(\mathbf{R})$ is a positive curvature operator since $\mathbf{R} \neq 0$. This implies

$$X_{a,\bar{b}}(\mathbf{R}) = \ell_{a,\bar{b}}^{-1}(\ell_{a,\bar{b}}(\mathbf{R})^2 + \ell_{a,\bar{b}}(\mathbf{R})^{\#})$$

is a positive curvature operator and hence transverse to $\partial\mathfrak{C}$.

(iii) Finally we show that $\ell_{a,b}(\mathbf{R})$ is a positive algebraic curvature operator when $\mathbf{R} \not\equiv 0$ and b takes the end value $\bar{b}$. We divide the proof of this statement into two cases. Let μ_1 be the smallest eigenvalue of $\mathbf{R}$. Recall

$$\ell_{a,\bar{b}}(\mathbf{R}) = \mathbf{R} + 2(a-\bar{b})\bar{\lambda}\mathbf{I} + 2\bar{b}\,\mathrm{Rc}\wedge\mathrm{id}\,.$$

Case (iii-a). Suppose $\mu_1 \geq 0$. Then

$$\ell_{a,\bar{b}}(\mathbf{R}) > 0$$

since $a > \bar{b}$, $\bar{b} > 0$, and $\mathrm{Rc}\wedge\mathrm{id}$ is nonnegative.

Case (iii-b). Suppose $\mu_1 < 0$. Let $\mu_1(\tilde{\mathbf{R}})$ denote the smallest eigenvalue of the algebraic curvature operator $\tilde{\mathbf{R}}$. From formula (11.92) we have $\mathrm{Rc} \geq (n-3)\,|\mu_1|$. Hence from (11.94) we have $\mu_1\,(\mathrm{Rc}\wedge\mathrm{id}) \geq (n-3)\,|\mu_1|$. Now we have

$$\begin{aligned}
\mu_1(\ell_{a,\bar{b}}(\mathbf{R})) &> \mu_1 + 2(a-\bar{b})\bar{\lambda} + 2\bar{b}(n-3)|\mu_1| \\
&\geq \mu_1 + 2\bar{b}|\mu_1| + 2(a-\bar{b})\bar{\lambda} \\
&= (2\bar{b}-1)|\mu_1| + (n-2)\bar{b}^2\frac{2\,\mathrm{trace}(\mathbf{R})}{n},
\end{aligned}$$

where here we have used that $n \geq 4$ and $\bar{\lambda} = \frac{\mathrm{Scal}(\mathbf{R})}{n} = \frac{2\,\mathrm{trace}(\mathbf{R})}{n}$. By elementary computations we have $\bar{b} < \frac{\sqrt{2}}{\sqrt{n(n-2)}} \leq \frac{1}{2}$. We may further estimate using (11.91) that

$$\begin{aligned}
\mu_1(\ell_{a,\bar{b}}(\mathbf{R})) &> -(1-2\bar{b})\frac{2\,\mathrm{trace}(\mathbf{R})}{n(n-1)-4} + (n-2)\bar{b}^2\frac{2\,\mathrm{trace}(\mathbf{R})}{n} \\
&= \left(\frac{2}{n^2} - \frac{1}{n(n-1)-4}\right)(1-2\bar{b})2\,\mathrm{trace}(\mathbf{R})
\end{aligned}$$

since $\left(n(n-2)\bar{b}+2\right)^2 = 2n(n-2)+4$, so that $2(1-2\bar{b}) = n(n-2)\bar{b}^2$. That $\ell_{a,\bar{b}}(\mathbf{R})$ is a positive algebraic curvature operator now follows from the fact that $\frac{2}{n^2} - \frac{1}{n(n-1)-4} \geq 0$ if $n \geq 4$. □

PROBLEM 11.49. Determine a larger set of values of a and b for which the sets $\ell_{a,b}(\mathfrak{C})$, where $\mathfrak{C}$ is the cone of 2-nonnegative curvature operators, are preserved by the ODE.

6. A pinching family of convex cones in the space of algebraic curvature operators

In this section we show that there exists a continuously changing 1-parameter family of closed convex cones in the space of algebraic curvature operators which are preserved by the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$ and which join the cone of 2-nonnegative curvature operators to the half-line $\mathbb{R}_+\mathbf{I}$. (See Corollary 11.52 below, which hinges on Theorem 11.51 below.) The geometric significance of $\mathbb{R}_+\mathbf{I}$ is that since it is the set of positive multiples of the identity operator, it corresponds to Riemannian metrics with constant positive sectional curvature.

In this section we again let $\bar{b} \doteqdot \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}$ be the end value and let $\mathfrak{C}$ be the cone of 2-nonnegative curvature operators.

6.1. Definition of two pinching families of convex cones.

The following generalizes a notion of Hamilton used to prove his convergence theorem for closed 3-manifolds with positive Ricci curvature.

Definition 11.50 (Pinching family of convex cones). We call a continuous family $C(s) \subset S_B^2(\mathfrak{so}(n))$, $s \in [0,1)$, of top-dimensional closed convex cones a **pinching family** (with respect to the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$) if

1. $C(s)$ is $\mathrm{O}(n)$-invariant for every $s \in [0,1)$;
2. each $\mathbf{R} \in C(s) \setminus \{0\}$ has positive scalar curvature;
3. $\mathbf{R}^2 + \mathbf{R}^\#$ lies in the *interior* of the tangent cone of $C(s)$ at $\mathbf{R}$ for all $\mathbf{R} \in C(s) \setminus \{0\}$ and $s \in (0,1)$;
4. the cones $C(s)$ converge in the pointed Hausdorff topology to the 1-dimensional cone $\mathbb{R}_+\mathbf{I}$ as $s \to 1$ (where the base points are the origin).[25]

The following is the main result of Böhm and Wilking [**43**].

Theorem 11.51 (Existence of a pinching family of convex cones). *There exists a pinching family $\{C_\spadesuit(s)\}_{s\in[0,1)}$ of closed convex cones so that $C_\spadesuit(0)$ is the cone of nonnegative curvature operators.*

We first show how to obtain, from the theorem, a pinching family starting at the cone of 2-nonnegative curvature operators. For this purpose, in view of (11.98), let

$$a_0(b) \doteqdot b + \frac{n-2}{2}b^2$$

and note that the intersection of two $\mathrm{O}(n)$-invariant closed convex cones is also an $\mathrm{O}(n)$-invariant closed convex cone.[26] By Proposition 11.47, $\ell_{a_0(\bar{b}),\bar{b}}(\mathfrak{C})$ is an $\mathrm{O}(n)$-invariant closed convex cone preserved by the ODE and contained

[25] For the definition of pointed Hausdorff convergence see [**54**] for example.

[26] Note that if $a = a_0(b)$, then a and b are related as in (11.98) of Proposition 11.47.

in the cone of nonnegative curvature operators. We define for $s \in [0,1)$ the family of $\mathrm{O}\,(n)$-invariant closed convex cones

$$C_{\heartsuit}(s) \doteqdot C_{\spadesuit}(s) \cap \ell_{a_0(\bar{b}),\bar{b}}(\mathfrak{C}), \tag{11.99}$$

where the sets $C_{\spadesuit}(s)$ are given by Theorem 11.51. Since $C_{\spadesuit}(0)$ is the cone of nonnegative curvature operators, we have $\ell_{a_0(\bar{b}),\bar{b}}(\mathfrak{C}) \subset C_{\spadesuit}(0)$, and hence

$$C_{\heartsuit}(0) = \ell_{a_0(\bar{b}),\bar{b}}(\mathfrak{C}).$$

Using this identification to concatenate the family $\{C_{\heartsuit}(s)\}_{s\in[0,1)}$ with the family $\left\{\ell_{a_0(b),b}(\mathfrak{C})\right\}_{b\in[0,\bar{b}]}$ given by Proposition 11.47, we obtain a pinching family where $\ell_{a_0(0),0}(\mathfrak{C}) = \ell_{0,0}(\mathfrak{C}) = \mathfrak{C}$ is the cone of 2-nonnegative curvature operators. Hence we obtain the following.

COROLLARY 11.52 (Existence of a pinching family starting at 2-positive). *There exists a pinching family $\{C(s)\}_{s\in[0,1)}$ of $\mathrm{O}\,(n)$-invariant closed convex cones such that $C(0)$ is the cone of 2-nonnegative curvature operators.*

In the case of $n = 3$, both of the above results follow from the previous work of Hamilton. We describe this case first. Let $\mu_1 \leq \mu_2 \leq \mu_3$ denote the ordered eigenvalues of the curvature operator $\mathbf{R}$. Then the system of ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^{\#}$ reduces to the diagonal case:

$$\begin{aligned}\frac{d\,\mu_1}{dt} &= \mu_1^2 + \mu_2\mu_3,\\ \frac{d\,\mu_2}{dt} &= \mu_2^2 + \mu_1\mu_3,\\ \frac{d\,\mu_3}{dt} &= \mu_3^2 + \mu_1\mu_2.\end{aligned}$$

LEMMA 11.53 (Preserved cones in dimension 3). *The following closed convex cones are preserved by the* ODE*:*

(1) $\mu_1 + \mu_2 \geq 0$,

(2) $\mu_2 + \mu_3 \leq C(\mu_1 + \mu_2)$ *for any given* $C > 0$,

(3) $\mu_3 - \mu_1 \leq K_1(\mu_1 + \mu_2 + \mu_3)$ *for any given* $K_1 > 0$.

PROOF. The preservation of the cones follows from showing that the time-derivatives $\frac{d}{dt}$ of $\mu_1 + \mu_2$, $\log\left(\frac{\mu_1+\mu_2}{\mu_2+\mu_3}\right)$, and $\log\left(\frac{\mu_1+\mu_2+\mu_3}{\mu_3-\mu_1}\right)$ are nonnegative. The details can be found in [**245**]. □

The family of closed convex cones

$$C(s) \doteqdot \{\mathbf{R} : \mu_1 + \mu_2 \geq 0,\ \mu_3 - \mu_1 \leq (1-s)\,(\mu_1 + \mu_2 + \mu_3)\},$$

where $s \in [0,1)$, is a pinching family. Note that

$$C(0) \doteqdot \{\mathbf{R} : \mu_1 + \mu_2 \geq 0,\ 2\mu_1 + \mu_2 \geq 0\}$$

contains the cone of nonnegative curvature operators $\{\mathbf{R} : \mu_1 \geq 0\}$.

REMARK 11.54. The construction of $\{C(s)\}_{s\in[0,1)}$ in dimension 4 can be found in [**245**].

6.2. Proof of Theorem 11.27 assuming Propositions 11.55 and 11.56. *We now proceed to give the proof of Theorem 11.51.* Given $b \in [0, \frac{1}{2}]$, let

$$p_1(b) \doteqdot \frac{(n-2)b^2}{1+(n-2)b^2}. \tag{11.100}$$

Note that $p_1(0) = 0$, $p_1\left(\frac{1}{2}\right) = \frac{n-2}{n+2}$, and $p_1(b) < 1$ for all b. Define

$$C_b \doteqdot \left\{ \mathbf{R} \in S_B^2(\mathfrak{so}(n)) : \mathbf{R} \geq 0,\ \mathrm{Rc}(\mathbf{R}) \geq p_1(b) \frac{\mathrm{tr}\,(\mathrm{Rc}\,(\mathbf{R}))}{n} \right\} \tag{11.101}$$

to be the $\mathrm{O}\,(n)$-invariant closed convex cone of *nonnegative algebraic curvature operators with pinched Ricci curvatures.* In particular, $C_0 \doteqdot \{\mathbf{R} : \mathbf{R} \geq 0\}$ is the cone of nonnegative curvature operators.

Next we let[27]

$$a_1(b) \doteqdot \frac{(n-2)b^2 + 2b}{2 + 2(n-2)b^2}. \tag{11.102}$$

Here we note that $a_1(0) = 0$ and $a_1\left(\frac{1}{2}\right) = \frac{1}{2}$. Hence the family of $\mathrm{O}\,(n)$-invariant closed convex cones

$$\left\{ \ell_{a_1(b),b}(C_b) \right\}_{b\in[0,\frac{1}{2}]}$$

joins the cone of nonnegative curvature operators $\ell_{0,0}(C_0) = \{\mathbf{R} : \mathbf{R} \geq 0\}$ with the cone $\ell_{\frac{1}{2},\frac{1}{2}}(C_{\frac{1}{2}})$.

To summarize our discussion above, we have constructed a 1-parameter family of $\mathrm{O}(n)$-invariant closed convex cones in $S_B^2(\mathfrak{so}(n))$ by taking the cone of nonnegative curvature operators with a Ricci pinching condition depending on the parameter and starting from no pinching with pinching increasing in the parameter and then conjugating these cones by linear transformations of $S_B^2(\mathfrak{so}(n))$ which start at the identity and push the cones more toward the ray $\mathbb{R}_+\mathbf{I}$. To obtain the family we desire, we shall actually concatenate two families of cones.

The following two delicate propositions give the construction of the family $C_\spadesuit(s)$ in Theorem 11.51 for general dimensions. These two propositions will be proved in the next two subsections. First we have

PROPOSITION 11.55 (The cone $\ell_{a_1(s),s}(C_s)$ is preserved by the ODE). *Suppose $n \geq 4$. For $s \in [0, \frac{1}{2}]$ the sets*

$$\ell_{a_1(s),s}(C_s) \text{ are preserved by the ODE } \frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#.$$

Moreover, for $s \in (0, \frac{1}{2}]$, the vector field $\mathbf{R}^2 + \mathbf{R}^\#$ is transverse to the boundary of $\ell_{a_1(s),s}(C_s)$, provided $\mathbf{R} \in \partial\ell_{a_1(s),s}(C_s) - \{0\}$.

[27]Note that our previous choice of a, in (11.98), was $a = \frac{(n-2)b^2+2b}{2}$.

Second, we have the following. Let

$$a_2(s) \doteqdot \frac{1+s}{2} \quad \text{and} \quad p_2(s) \doteqdot 1 - \frac{4}{n+2+4s} \tag{11.103}$$

for $s \geq 0$ and let $b = \frac{1}{2}$. Clearly $p_2(s) \in [\frac{n-2}{n+2}, 1)$. Define

$$C'_s \doteqdot \left\{ \mathbf{R} \in S^2_B(\mathfrak{so}(n)) : \mathbf{R} \geq 0,\ \mathrm{Rc}(\mathbf{R}) \geq p_2(s) \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n} \right\}. \tag{11.104}$$

Notice that

$$a_2(0) = \frac{1}{2} = a_1\left(\frac{1}{2}\right), \qquad p_2(0) = \frac{n-2}{n+2} = p_1\left(\frac{1}{2}\right),$$

$\lim_{s\to\infty} a_2(s) = \infty$, and $\lim_{s\to\infty} p_2(s) = 1$. Let $\mathrm{Rc} \doteqdot \mathrm{Rc}(\mathbf{R})$. Hence

$$C'_0 = \left\{ \mathbf{R} \in S^2_B(\mathfrak{so}(n)) : \mathbf{R} \geq 0,\ \mathrm{Rc} \geq \frac{n-2}{n+2} \frac{\mathrm{tr}(\mathrm{Rc})}{n} \right\} = C_{1/2}$$

and

$$\lim_{s\to\infty} C'_s = \left\{ \mathbf{R} \in S^2_B(\mathfrak{so}(n)) : \mathbf{R} \geq 0,\ \mathrm{Rc} = \frac{\mathrm{tr}(\mathrm{Rc})}{n} \right\}.$$

PROPOSITION 11.56. *For each $\bar{s} \in [0,\infty)$, the set*

$$\ell_{a_2(\bar{s}),\frac{1}{2}}\left(C'_{\bar{s}}\right) \text{ is preserved by the ODE } \frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^{\#}.$$

Moreover, $\mathbf{R}^2 + \mathbf{R}^{\#}$ is transverse to the boundary of $\ell_{a_2(\bar{s}),\frac{1}{2}}(C_{\bar{s}})$ at all points on the boundary with $\mathbf{R} \neq 0$.

REMARK 11.57. Apparently it is unknown whether any cones of the form $\left\{\mathbf{R} : \mathbf{R} \geq 0,\ \mathrm{Rc} \geq p\frac{\mathrm{tr}(\mathrm{Rc})}{n}\right\}$, where $p > 0$, are preserved by the ODE. Applying an appropriate linear transformation to this set solves this problem.

PROBLEM 11.58. Determine sufficient conditions on a, b, p so that the set $\ell_{a,b}\left(\left\{\mathbf{R} : \mathbf{R} \geq 0,\ \mathrm{Rc} \geq p\frac{\mathrm{tr}(\mathrm{Rc})}{n}\right\}\right)$ is preserved by the ODE.

HINT: First read the proofs of the two propositions (see subsections 6.3 and 6.4).

Assuming the above two propositions, we give the

PROOF OF THEOREM 11.51. We consider the concatenated family

$$\left\{\ell_{a_1(s),s}(C_s)\right\}_{s\in[0,\frac{1}{2}]} \cup \left\{\ell_{a_2\left(\frac{2s-1}{1-s}\right),\frac{1}{2}}\left(C'_{\frac{2s-1}{1-s}}\right)\right\}_{s\in[\frac{1}{2},1)}, \tag{11.105}$$

where a_1 and a_2 are defined by (11.102) and (11.103). Since

$$\ell_{a_1(\frac{1}{2}),\frac{1}{2}}(C_{\frac{1}{2}}) = \ell_{\frac{1}{2},\frac{1}{2}}(C_{\frac{1}{2}}) = \ell_{a_2(0),\frac{1}{2}}\left(C'_0\right),$$

this family is continuous. Moreover the family joins the cone of nonnegative curvature operators $\{\mathbf{R} : \mathbf{R} \geq 0\} = \ell_{0,0}(C_0)$ to the limit of $\ell_{a_2(\bar{s}),\frac{1}{2}}\left(C'_{\bar{s}}\right)$ as $\bar{s} \to \infty$ (which corresponds to $s \to 1$ provided $\bar{s} = \frac{2s-1}{1-s}$).

Observing that for any $\mathbf{R} \geq 0$,

$$\begin{aligned}
&\frac{1}{a_2(\bar{s})}\ell_{a_2(\bar{s}),\frac{1}{2}}(\mathbf{R}) \\
&= \frac{1}{a_2(\bar{s})}\mathbf{R_W} + \left(\frac{1}{a_2(\bar{s})} + 2(n-1)\right)\mathbf{R_I} + \frac{1+\frac{1}{2}(n-2)}{a_2(\bar{s})}\mathbf{R}_{\mathrm{Rc}_0} \\
&\to 2(n-1)\mathbf{R_I}
\end{aligned}$$

as $\bar{s} \to \infty$, we see that $\ell_{a_2(\bar{s}),\frac{1}{2}}(C'_{\bar{s}})$ converges to $\mathbb{R}_+\mathbf{I}$ as $\bar{s} \to \infty$. Therefore the combination of the two propositions implies the theorem. □

6.3. Proof of Proposition 11.55. We now prove Proposition 11.55. For simplicity, let $a_1 = a_1(b)$ and $p_1 = p_1(b)$. Clearly the sets C_b defined in (11.101) are $\mathrm{O}(n)$-invariant closed convex cones. For each b, C_b is the intersection of two $\mathrm{O}(n)$-invariant closed convex cones: $\{\mathbf{R} : \mathbf{R} \geq 0\}$ and $\left\{\mathbf{R} : \mathrm{Rc}(\mathbf{R}) \geq p_1 \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n}\right\}$. By Lemma 11.26, it suffices to show that for any $\mathbf{R} \in \partial C_b$, the vector

$$X_{a_1,b}(\mathbf{R}) = \ell_{a_1,b}^{-1}(\ell_{a_1,b}(\mathbf{R})^2 + \ell_{a_1,b}(\mathbf{R})^{\#})$$

defined by (11.59) lies in the tangent cone of C_b at $\mathbf{R}$. The boundary ∂C_b consists of those $\mathbf{R} \in C_b$ such that

(1) there exists $\phi \in \mathfrak{so}(n)$ such that $\mathbf{R}(\phi) = 0$ or

(2) there exists $V \in \mathbb{R}^n$ with $|V| = 1$ such that

$$\mathrm{Rc}(\mathbf{R})(V) = p_1 \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n}.$$

For $\mathbf{R} \in \partial C_b$, the tangent cone of C_b at $\mathbf{R}$ is the set of $\mathbf{S}$ such that

$$\mathbf{S} \geq \mathbf{0} \quad \text{and} \quad \mathrm{Rc}(\mathbf{S})(V) \geq p_1 \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{S}))}{n}$$

for all $V \in \mathbb{R}^n$ with $|V| = 1$ and $\mathrm{Rc}(\mathbf{R})(V) = p_1 \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n}$.

We shall show, using Corollary 11.29, that for $\mathbf{R} \in \partial C_b - \{0\}$ we have

(11.106)
$$\text{(I) } X_{a_1,b}(\mathbf{R}) > 0 \quad \text{and} \quad \text{(II) } \mathrm{Rc}(X_{a_1,b}(\mathbf{R}))(V) > p_1 \frac{\mathrm{tr}(\mathrm{Rc}(X_{a_1,b}(\mathbf{R})))}{n}$$

for all $V \in \mathbb{R}^n$ with $|V| = 1$ and $\mathrm{Rc}(\mathbf{R})(V) = p_1 \frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n}$. Hence $X_{a_1,b}(\mathbf{R})$ lies in the interior of the tangent cone of C_b at $\mathbf{R}$.

Proof of (I). $X_{a_1,b}(\mathbf{R}) > 0$. Noticing that $\mathbf{R} \geq 0$ implies $\mathbf{R}^2 + \mathbf{R}^{\#} \geq 0$, it suffices to show that $D_{a_1,b} > 0$. By (11.67), the eigenvalues of $D_{a_1,b}$ are given by

$$d_{ij} = \mathrm{I}_{ij} + \mathrm{II}_{ij},$$

where

(11.107)
$$\begin{aligned}
\mathrm{I}_{ij} &\doteqdot \left((n-2)b^2 - 2(a_1 - b)\right)\lambda_i\lambda_j + 2a_1(\lambda_i + \bar{\lambda})(\lambda_j + \bar{\lambda}) + b^2(\lambda_i^2 + \lambda_j^2) \\
&= \left((n-2)b^2 + 2b\right)\lambda_i\lambda_j + 2a_1\bar{\lambda}^2 + 2a_1(\lambda_i + \lambda_j)\bar{\lambda} + b^2(\lambda_i^2 + \lambda_j^2)
\end{aligned}$$

and

$$\mathrm{II}_{ij} \doteqdot \frac{\sigma}{1+2(n-1)a_1}\left(nb^2(1-2b)-2(a_1-b)(1-2b+nb^2)\right). \tag{11.108}$$

We shall show that $\mathrm{I}_{ij}>0$ and $\mathrm{II}_{ij}\geq 0$ for all $i\neq j$. By (11.102),

$$2(a_1-b)=\frac{1-2b}{1+(n-2)b^2}(n-2)b^2,$$

so that (11.108) implies

$$\begin{aligned}\mathrm{II}_{ij} &= \frac{(1-2b)\sigma}{1+2(n-1)a_1}\left(nb^2-\frac{1-2b+nb^2}{1+(n-2)b^2}(n-2)b^2\right)\\ &= \frac{(1-2b)b^2\sigma}{1+2(n-1)a_1}\frac{2+2(n-2)b}{1+(n-2)b^2}\geq 0.\end{aligned}$$

Now we show the positivity of I_{ij}. Note that from the definition (11.100) of p_1,

$$1-p_1=\frac{1}{1+(n-2)b^2}=\frac{2a_1}{(n-2)b^2+2b}$$

(or, equivalently, $(n-2)b^2+2b=\frac{2a_1}{1-p_1}$). Note also that $b>0$ implies $a_1>0$ and $p_1>0$. Hence if $\mathbf{R}\neq 0$ (which implies $\bar{\lambda}>0$), then (11.107) implies

$$\begin{aligned}\mathrm{I}_{ij} &= \frac{2a_1}{1-p_1}\lambda_i\lambda_j+2a_1\bar{\lambda}^2+2a_1(\lambda_i+\lambda_j)\bar{\lambda}+b^2(\lambda_i^2+\lambda_j^2)\\ &= 2a_1\left(\frac{1}{1-p_1}\lambda_i\lambda_j+(1-p_1)\bar{\lambda}^2+(\lambda_i+\lambda_j)\bar{\lambda}\right)+2a_1p_1\bar{\lambda}^2+b^2(\lambda_i^2+\lambda_j^2)\\ &> \frac{2a_1}{1-p_1}\left(\lambda_i+(1-p_1)\bar{\lambda}\right)\left(\lambda_j+(1-p_1)\bar{\lambda}\right)\geq 0\end{aligned}$$

since $\mathrm{Rc}\geq p_1\frac{\mathrm{tr}(\mathrm{Rc})}{n}$ implies $\lambda_i+\bar{\lambda}\geq p_1\bar{\lambda}$ for all i.

Proof of (II). Now we prove that the Rc pinching condition in (11.106) holds. Suppose that

$$\mathrm{Rc}\left(\mathbf{R}\right)_{ii}=p_1\frac{\mathrm{tr}\left(\mathrm{Rc}\left(\mathbf{R}\right)\right)}{n},\quad \text{i.e., } \lambda_i=-(1-p_1)\bar{\lambda} \tag{11.109}$$

for some i. (Otherwise there is nothing to prove. For convenience of notation we have arbitrarily chosen i such that the i-th basis vector is a vector V as in (11.106).) We need to show that

$$\mathrm{Rc}(X_{a_1,b})_{ii}-p_1\frac{\mathrm{Scal}(X_{a_1,b})}{n}>0. \tag{11.110}$$

By (11.108) and

$$\mathrm{I}_{ij}\geq 2a_1p_1\bar{\lambda}^2+b^2(\lambda_i^2+\lambda_j^2),$$

we have

$$\begin{aligned}\operatorname{Rc}(D_{a_1,b})_{ii} = r_i = \sum_{j\neq i} d_{ij} &= \sum_{j\neq i}(\mathrm{I}_{ij} + \mathrm{II}_{ij}) \\ &\geq 2a_1(n-1)p_1\bar{\lambda}^2 + (n-2)b^2\lambda_i^2 + nb^2\sigma \\ &\quad + \frac{(n-1)\sigma}{1+2(n-1)a_1}\left(nb^2(1-2b) - 2(a_1-b)(1-2b+nb^2)\right).\end{aligned}$$

Also, since $\mathbf{R} \geq 0$ (actually we only need $\mathbf{R}$ to have nonnegative sectional curvature) and $\operatorname{Rc}_{kk} \geq p_1\bar{\lambda}$,

$$\operatorname{Rc}\left(\mathbf{R}^2 + \mathbf{R}^\#\right)_{ii} = \sum_k \operatorname{Rc}_{kk}\mathbf{R}_{ikik} \geq p_1\bar{\lambda}\sum_k \mathbf{R}_{ikik} \geq p_1^2\bar{\lambda}^2. \tag{11.111}$$

Since $X_{a_1,b} = D_{a_1,b}(\mathbf{R}) + \mathbf{R}^2 + \mathbf{R}^\#$, by combining the above formulas, we have

(11.112)

$$\begin{aligned}\operatorname{Rc}(X_{a_1,b})_{ii} &\geq p_1^2\bar{\lambda}^2 + 2a_1(n-1)p_1\bar{\lambda}^2 + (n-2)b^2\lambda_i^2 + nb^2\sigma \\ &\quad + \frac{(n-1)\sigma}{1+2(n-1)a_1}\left(nb^2(1-2b) - 2(a_1-b)(1-2b+nb^2)\right) \\ &= p_1^2\bar{\lambda}^2 + 2a_1(n-1)p_1\bar{\lambda}^2 + (n-2)b^2\lambda_i^2 \\ &\quad + \frac{(n-2)^2b^2 + 2(n-1)\left(2a_1b - a_1 + b\right)}{1+2(n-1)a_1}\sigma \qquad (11.113) \\ &= p_1^2\bar{\lambda}^2 + 2a_1(n-1)p_1\bar{\lambda}^2 + (n-2)b^2\lambda_i^2 \\ &\quad + \left(\frac{(1+(n-2)b)^2}{1+2(n-1)a_1} + 2b - 1\right)\sigma. \qquad (11.114)\end{aligned}$$

We next compute the scalar curvature of the algebraic curvature operator $X_{a_1,b}$. By (11.55), i.e., $\operatorname{Rc}\left(\mathbf{R}^2 + \mathbf{R}^\#\right)_{ii} = \sum_k \operatorname{Rc}_{kk}\mathbf{R}_{ikik}$, we have that

$$\begin{aligned}\operatorname{Scal}(\mathbf{R}^2 + \mathbf{R}^\#) &= \sum_k |\operatorname{Rc}_{kk}|^2 = \sum_k (\lambda_k + \bar{\lambda})^2 \\ &= \sum_k \lambda_k^2 + n\bar{\lambda}^2 = n\sigma + n\bar{\lambda}^2. \qquad (11.115)\end{aligned}$$

Using (11.68), we compute

$$\begin{aligned}
\operatorname{Scal}(D_{a_1,b}) &= \sum_i r_i \\
&= \sum_i \left(-2b\lambda_i^2 + 2a_1(n-2)\bar\lambda\lambda_i + 2a_1(n-1)\bar\lambda^2\right) \\
&\quad + \frac{n\sigma}{1+2(n-1)a_1}\left(n^2b^2 - 2(n-1)(a_1-b)(1-2b)\right) \\
&= -2bn\sigma + 2a_1n(n-1)\bar\lambda^2 \qquad (11.116)\\
&\quad + \frac{n\sigma}{1+2(n-1)a_1}\left(n^2b^2 - 2(n-1)(a_1-b)(1-2b)\right) \\
&= 2a_1n(n-1)\bar\lambda^2 - n\sigma + \frac{n(1+(n-2)b)^2}{1+2(n-1)a_1}\sigma. \qquad (11.117)
\end{aligned}$$

Combining this with (11.115), we have

$$\frac{\operatorname{Scal}(X_{a_1,b})}{n} = (1+2(n-1)a_1)\bar\lambda^2 + \frac{(1+(n-2)b)^2}{1+2(n-1)a_1}\sigma. \qquad (11.118)$$

Note that both formulas (11.116) and (11.118) hold for any $a_1 \neq -\frac{1}{2(n-1)}$ and do not use the relation (11.102).

Combining (11.114) and (11.118), we have

$$\begin{aligned}
&\operatorname{Rc}(X_{a_1,b})_{ii} - p_1\frac{\operatorname{Scal}(X_{a_1,b})}{n} \\
&\geq p_1^2\bar\lambda^2 + 2a_1(n-1)p_1\bar\lambda^2 + (n-2)b^2\lambda_i^2 + \left(\frac{(1+(n-2)b)^2}{1+2(n-1)a_1} + 2b - 1\right)\sigma \\
&\quad - p_1(1+2(n-1)a_1)\bar\lambda^2 - p_1\frac{(1+(n-2)b)^2}{1+2(n-1)a_1}\sigma \\
&= p_1\left(p_1-1\right)\bar\lambda^2 + (n-2)b^2\lambda_i^2 + \left(\left(1-p_1\right)\frac{(1+(n-2)b)^2}{1+2(n-1)a_1} + 2b - 1\right)\sigma.
\end{aligned}$$

By the definition (11.100) of p_1,

$$p_1\left(p_1-1\right)\bar\lambda^2 + (n-2)b^2\lambda_i^2 = (n-2)b^2\left(-\left(1-p_1\right)^2\bar\lambda^2 + \lambda_i^2\right).$$

On the other hand, by (11.102) and (11.100), we have

$$\begin{aligned}
&\left(1-p_1\right)\frac{(1+(n-2)b)^2}{1+2(n-1)a_1} + 2b - 1 \\
&= \frac{(1+(n-2)b)^2}{(nb+1)\left((n-2)\,b+1\right)} + 2b - 1 \\
&= \frac{2nb^2}{nb+1}.
\end{aligned}$$

Hence

$$\begin{aligned}&\mathrm{Rc}(X_{a_1,b})_{ii} - p_1 \frac{\mathrm{Scal}(X_{a_1,b})}{n} \\ &\geq (n-2)b^2 \left(-(1-p_1)^2 \bar{\lambda}^2 + \lambda_i^2 \right) + \frac{2nb^2}{nb+1}\sigma.\end{aligned}$$

By (11.109) we have $\lambda_i = -(1-p_1)\bar{\lambda}$ and hence

$$\mathrm{Rc}(X_{a_1,b})_{ii} - p_1 \frac{\mathrm{Scal}(X_{a_1,b})}{n} \geq \frac{2nb^2}{nb+1}\sigma > 0.$$

This proves (11.110) and therefore completes the proof of Proposition 11.55.

6.4. Proof of Proposition 11.56. We now prove Proposition 11.56. That is, we prove $\ell_{a_2(\bar{s}),\frac{1}{2}}(C'_{\bar{s}})$ is preserved by the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$ for $\bar{s} \in [0,\infty)$. By the same reasoning as in Proposition 11.55 it suffices to show that for $s \geq 0$ and $\mathbf{R} \in \partial C'_s - \{0\}$, we have

(I) $D_{a_2(s),\frac{1}{2}}(\mathbf{R}) > 0$ and

(II) if i is such that $\mathrm{Rc}(\mathbf{R})_{ii} = p_2(s)\frac{\mathrm{tr}(\mathrm{Rc}(\mathbf{R}))}{n}$, we have

$$\mathrm{Rc}\left(X_{a_2(s),\frac{1}{2}}(\mathbf{R})\right)_{ii} > p_2(s) \frac{\mathrm{tr}\left(\mathrm{Rc}\left(X_{a_2(s),\frac{1}{2}}(\mathbf{R})\right)\right)}{n}.$$

Proof of (I). $D_{a_2(s),\frac{1}{2}} > 0$. Using Corollary 11.29 with $a = \frac{1+s}{2}$ and $b = \frac{1}{2}$ (so that $2(a-b) = s$ and $1 - 2b = 0$), we have that the eigenvalues of $D_{a_2(s),\frac{1}{2}}$ are given by

$$\begin{aligned}d_{ij} &= \left(\frac{n-2}{4} - s\right)\lambda_i\lambda_j + (1+s)(\lambda_i + \bar{\lambda})(\lambda_j + \bar{\lambda}) + \frac{1}{4}(\lambda_i^2 + \lambda_j^2) \\ &\quad - \frac{ns}{4n + 4(n-1)s}\sigma \\ &= \frac{n+2}{4}\lambda_i\lambda_j + (1+s)\bar{\lambda}^2 + (1+s)(\lambda_i + \lambda_j)\bar{\lambda} + \frac{1}{4}(\lambda_i^2 + \lambda_j^2) \\ &\quad - \frac{ns}{4n + 4(n-1)s}\sigma.\end{aligned} \tag{11.119}$$

Since $\lambda_j + \bar{\lambda} \geq p_2\bar{\lambda}$ for all j, we can estimate $\sigma = \frac{1}{n}\sum_j \lambda_j^2$ from above in terms of $\bar{\lambda}^2$. In particular, consider the optimization problem for $\sigma = \sigma(\lambda_1, \ldots, \lambda_n)$ under the constraint $\sum_j \lambda_j = 0$. Since the functional σ has strictly positive Hessian, it achieves its maximum on the boundary of the set

$$\left\{ (\lambda_1, \ldots, \lambda_n) : \lambda_j + \bar{\lambda} \geq p_2\bar{\lambda} \text{ for all } j \text{ and } \sum_j \lambda_j = 0 \right\}$$

and furthermore at the vertices. Therefore the extremal points of σ are $(-(1-p_2)\bar{\lambda}, \ldots, -(1-p_2)\bar{\lambda}, (n-1)(1-p_2)\bar{\lambda})$ and its permutations. Hence

$$\sigma \leq (n-1)(1-p_2(s))^2\bar{\lambda}^2 = \frac{16(n-1)}{(n+2+4s)^2}\bar{\lambda}^2. \tag{11.120}$$

Now also recall that $1-p_2(s) = \frac{4}{n+2+4s}$, so that

$$\lambda_i + \frac{4\bar{\lambda}}{n+2} \geq \lambda_i + \frac{4\bar{\lambda}}{n+2+4s} \geq 0, \tag{11.121}$$

where the second inequality follows from $D_{a_2(s),\frac{1}{2}}(\mathbf{R}) > 0$. Motivated by this, we have by (11.119) and (11.120),

$$\begin{aligned} d_{ij} \geq &\frac{n+2}{4}\left(\lambda_i + \frac{4\bar{\lambda}}{n+2}\right)\left(\lambda_j + \frac{4\bar{\lambda}}{n+2}\right) + \frac{1}{4}(\lambda_i^2 + \lambda_j^2) \\ &+ \left(\frac{n-2}{n+2} + s\right)\bar{\lambda}^2 + s(\lambda_i + \lambda_j)\bar{\lambda} \\ &- \frac{4(n-1)ns}{(n+(n-1)s)(n+2+4s)^2}\bar{\lambda}^2. \end{aligned}$$

Therefore, throwing away the first two terms on the RHS above and using (11.121), we have

(11.122)
$$\begin{aligned} d_{ij} &\geq \left(\frac{n-2}{n+2} + s - \frac{8s}{n+2+4s} - \frac{4(n-1)ns}{(n+(n-1)s)(n+2+4s)^2}\right)\bar{\lambda}^2 \\ &\doteqdot C(n,s)\,\bar{\lambda}^2. \end{aligned}$$

Since $n \geq 3 \geq 1$ and $s \geq 0$,

$$\begin{aligned} C(n,s) &\geq \frac{n-2}{n+2} + s - \frac{8s}{n+2+4s} - \frac{4(n-1)s}{(n+2)(n+2+4s)} \\ &= \frac{4s^2 + \left(n-6-\frac{4}{n+2}\right)s + n - 2}{n+2+4s}. \end{aligned}$$

In particular, if $n=3$, then $4s^2 + \left(n-6-\frac{4}{n+2}\right)s + n - 2 = 4s^2 - \frac{19}{5}s + 1 > 0$ so that $C(3,s) > 0$. Now, given $s \geq 0$, the function $4s^2 + \left(n-6-\frac{4}{n+2}\right)s + n - 2$ is monotone increasing in n. Hence $C(n,s) > 0$ for all $n \geq 3$ and $s \geq 0$. Therefore $d_{ij} > 0$; this completes the proof of (I).

Proof of (II). By formula (11.118) in the proof of Proposition 11.55 with a_1 replaced by $a_2(s)$ and $b = \frac{1}{2}$ (see the comment after formula

(11.118)), we have

$$\frac{\operatorname{Scal}\left(X_{a_2(s),\frac{1}{2}}\right)}{n} = (1+2(n-1)a_2(s))\bar{\lambda}^2 + \frac{\left(1+(n-2)\frac{1}{2}\right)^2}{1+2(n-1)a_2(s)}\sigma$$
$$= (n+(n-1)s)\bar{\lambda}^2 + \frac{n^2\sigma}{4n+4(n-1)s}.$$

Using Corollary 11.29 with $a = a_2(s) = 1 + \frac{s}{2}$ and $b = \frac{1}{2}$, we have

$$r_i = -\lambda_i^2 + (1+s)(n-2)\bar{\lambda}\lambda_i + (1+s)(n-1)\bar{\lambda}^2 + \frac{n^2\sigma}{4n+4(n-1)s}.$$

Using the estimate (note that $\mathbf{R}$ has nonnegative sectional curvature and $\operatorname{Rc}_{kk} \geq p_2(s)\,\bar{\lambda}$)

$$\operatorname{Rc}(\mathbf{R}^2 + \mathbf{R}^{\#})_{ii} = \sum_k \operatorname{Rc}_{kk}\mathbf{R}_{ikik} \geq p_2(s)\,\bar{\lambda}\operatorname{Rc}_{ii} \geq p_2^2(s)\,\bar{\lambda}^2, \tag{11.123}$$

to prove the Rc pinching condition

$$\operatorname{Rc}\left(D_{a_2(s),\frac{1}{2}}\right)_{ii} + \operatorname{Rc}(\mathbf{R}^2 + \mathbf{R}^{\#})_{ii}$$
$$= \operatorname{Rc}\left(X_{a_2(s),\frac{1}{2}}\right)_{ii} > p_2 \frac{\operatorname{Scal}\left(X_{a_2(s),\frac{1}{2}}\right)}{n},$$

where i is such that $\lambda_i = -(1-p_2)\bar{\lambda}$, it is suffices to show that

$$-\lambda_i^2 + (1+s)(n-2)\bar{\lambda}\lambda_i + (1+s)(n-1)\bar{\lambda}^2 + \frac{n^2\sigma}{4n+4(n-1)s} + p_2^2\bar{\lambda}^2$$
$$> p_2\left((n+(n-1)s)\bar{\lambda}^2 + \frac{n^2\sigma}{4n+4(n-1)s}\right).$$

We leave it to the reader to carry out this easy verification. This completes the proof of Proposition 11.56.

7. Obtaining a generalized pinching set from a pinching family and the proof of Theorem 11.2

In this section we use the pinching families of convex cones constructed in the previous section to obtain a generalized pinching set. From the existence of this generalized pinching set we provide the proof of Böhm and Wilking of the convergence of the Ricci flow for solutions with 2-positive curvature operator.

7.1. Generalized pinching sets and its existence. The following concept was introduced by Hamilton in §5 of [**245**].

Definition 11.59 (Pinching set). We say that a subset $Z \subset S_B^2(\mathfrak{so}(n))$ is a **pinching set** if

(1) Z is closed and convex,
(2) Z is $\mathrm{O}(n)$-invariant,

(3) Z is preserved by the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^{\#}$,
(4) there exist $\delta > 0$ and $K < \infty$ such that

$$\left|\widetilde{\mathbf{R}}\right| \leq K \left|\mathbf{R}\right|^{1-\delta}$$

for all $\mathbf{R} \in Z$, where $\widetilde{\mathbf{R}} \doteqdot \mathbf{R} - \frac{1}{N} \operatorname{trace} \mathbf{R} \cdot \mathbf{I}$ is the trace-free part of $\mathbf{R}$.

As we discussed in the paragraph before Proposition 11.44, if $Z \subset S_B^2(\mathfrak{so}(n))$ is a pinching set, then for any Riemannian manifold $(\mathcal{M}^n, g_0)$ we may define a corresponding subset $\bar{Z}$ of $S_B^2 \Lambda^2 T^* \mathcal{M}$ via the isometries $\iota_x : \mathbb{E}^n \to (T_x\mathcal{M}, g_0(x))$ for $x \in \mathcal{M}$. Since Z is $\mathrm{O}(n)$-invariant, the subset $\bar{Z}$ is independent of the choice of isometries ι_x. Furthermore, $\bar{Z}$ is $\mathrm{O}(n)$-invariant, closed, invariant under parallel translation, and fiberwise convex.

REMARK 11.60. In this regard we note that given a solution $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, to the Ricci flow, one way to consider Uhlenbeck's trick (see Section 2 of Chapter 6 in Volume One) is to define time-dependent bundle isometries

$$\iota(t) : (T\mathcal{M}, g(0)) \to (T\mathcal{M}, g(t))$$

by

$$\frac{d}{dt}\iota(t) = \mathrm{Rc}_{g(t)} \circ \iota(t) \quad \text{and} \quad \iota(0) = \mathrm{id}_{T\mathcal{M}}, \tag{11.124}$$

where we raise an index on the Ricci tensor so that $\mathrm{Rc}_{g(t)} : T\mathcal{M} \to T\mathcal{M}$.

By the Weinberger–Hamilton maximum principle for systems, if we have $\mathrm{Rm}(g_0) \subset \bar{Z}$, then the solution of the Ricci flow $g(t)$, with $g(0) = g_0$, satisfies

$$\iota(t)^* \mathrm{Rm}(g(t)) \subset \bar{Z}$$

for $t \geq 0$, where the $\iota(t)$ are defined by (11.124). In particular, we then have the pinching estimate

$$\left|\widetilde{\mathrm{Rm}}\right| \leq K \left|\mathrm{Rm}\right|^{1-\delta}. \tag{11.125}$$

This estimate is sufficient (but, as we shall see below, not necessary) to prove convergence of the normalized Ricci flow to a constant positive sectional curvature metric.

In dimension 3 Hamilton proved that for any compact subset Ω inside the cone of curvature operators with positive Ricci curvature (i.e., inside the cone of 2-positive curvature operators), there exists a pinching set containing Ω. In fact, a simpler proof of the following result in §§9–10 of [**244**], using the Weinberger–Hamilton maximum principle for systems, was given as Theorem 5.3 of [**245**].

THEOREM 11.61 (Existence of a pinching set in dimension 3). *Let*

$$\mu_1 \leq \mu_2 \leq \mu_3$$

denote the ordered eigenvalues of $\mathbf{R}$*. For any constant* $C < \infty$ *there exists* $\delta > 0$ *such that for any* $K < \infty$*, the closed, convex, and* $\mathrm{O}(n)$*-invariant set* Z *defined by:*

(a) $\mu_1 + \mu_2 \geq 0$,
(b) $\mu_2 + \mu_3 \leq C(\mu_1 + \mu_2)$,
(c) $\mu_3 - \mu_1 \leq K(\mu_1 + \mu_2 + \mu_3)^{1-\delta}$

is preserved by the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$.

REMARK 11.62. Note that $\lim_{C,K\to\infty} Z$ is the cone of curvature operators with nonnegative Ricci curvature. (What do we mean by $\lim_{C,K\to\infty} Z$ given that δ depends on C?)

However, Hamilton's notion of *pinching set* does not seem to be sufficiently general in higher dimensions to prove the convergence of the Ricci flow under the assumption of 2-positivity.

REMARK 11.63. On the other hand, we learned from Burkhard Wilking that in principle one can always construct from a pinching family a pinching set in the sense of Hamilton. The difference of the latter from a generalized pinching set is just related to the speed of convergence. However this in turn is just related to the behavior of the ODE in a neighborhood of the identity.

The following general result (see Theorem 4.1 of [**43**]) enables one to obtain a *generalized pinching set* from a *pinching family* of closed convex cones, which suffices to prove Theorem 11.2.[28]

THEOREM 11.64 (Existence of a generalized pinching set). *Let*

$$C_\clubsuit(s)_{s\in[0,1)} \subset S^2_B(\mathfrak{so}(n))$$

be a continuous family of closed convex $\mathrm{O}(n)$*-invariant cones of maximum dimension, such that* $C_\clubsuit(s) \setminus \{0\}$ *is contained in the half-space of curvature operators with positive scalar curvature for all* $s \in [0,1)$*. Suppose that for* $\mathbf{R} \in \partial C_\clubsuit(s) \setminus \{0\}$ *the vector field* $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#$ *lies in the interior of the tangent cone of* $C_\clubsuit(s)$ *at* $\mathbf{R}$ *for all* $s \in (0,1)$*. Then for any* $\varepsilon \in (0,1)$ *and* $h_0 \in (0,\infty)$ *there exists a closed convex* $\mathrm{O}(n)$*-invariant subset* $F \subset S^2_B(\mathfrak{so}(n))$ *with the following properties:*

(1) F *is preserved by the* ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$*;*
(2) $C_\clubsuit(\varepsilon) \cap \{\mathbf{R} : \mathrm{trace}(\mathbf{R}) \leq h_0\} \subset F \subset C_\clubsuit(\varepsilon)$*;*
(3) $\overline{F \setminus C_\clubsuit(s)}$ *is compact (i.e., bounded) for all* $s \in [\varepsilon, 1)$*.*

REMARK 11.65. We call the set F in Theorem 11.64 a **generalized pinching set**. The motivation for property (2) is given by (11.126) below. When $\lim_{s\to 1} C_\clubsuit(s) = \mathbb{R}_+\mathbf{I}$, property (3) implies that the asymptotic cone of F is $\mathbb{R}_+\mathbf{I}$; it is in this sense that F is a generalized pinching set.

[28]One should distinguish between the notions of pinching set (see Definition 11.59), generalized pinching set (see Remark 11.65), and pinching family (see Definition 11.50).

For example, when $n = 3$, Theorem 11.61(a), (b) can be applied to obtain the pinching family

$$C(s) \doteqdot \{\mathbf{R} : \mu_1 + \mu_2 \geq 0;\ \mu_3 - \mu_1 \leq (1-s)(\mu_1 + \mu_2 + \mu_3)\},$$

which by Theorem 11.64 yields a generalized pinching set F. In other words, to prove Theorem 11.2 in dimension 3, one does not need the 'Ricci pinching is *improved*' estimate, one only needs the 'Ricci pinching is *preserved*' estimate and Theorem 11.64. In higher dimensions, we shall apply Corollary 11.52.

In any dimension, we have obtained a pinching family $\{C(s)\}$, where $C(0)$ is the cone of 2-nonnegative curvature operators and $\lim_{s\to 1} C(s) = \mathbb{R}_+\mathbf{I}$. From this we obtain a generalized pinching set which serves the purpose of (11.125). In particular, assuming the theorem above, i.e., the existence of a generalized pinching set, which we prove below, we may now give a proof of Theorem 11.2.

7.2. Proof of the convergence of the Ricci flow under 2-PCO assuming Theorem 11.64.

Proof of Theorem 11.2. By Corollary 11.52, for the ODE $\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#$ on $S_B^2(\mathfrak{so}(n))$, there exists a pinching family $\{C(s)\}_{s\in[0,1)}$ of $\mathrm{O}(n)$-invariant closed convex cones (in the sense of Definition 11.50) such that $C(s) \setminus \{0\} \subset \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) > 0\}$ and $C(0)$ is the cone of 2-nonnegative curvature operators. In particular, $C(s)$ converges in the pointed Hausdorff topology to the 1-dimensional cone $\mathbb{R}_+\mathbf{I}$ as $s \to 1$.

Now let $(\mathcal{M}^n, g_0)$ be a closed Riemannian manifold with 2-positive curvature operator and let $g(t)$, $t \in [0,T)$, be the solution to the Ricci flow on $\mathcal{M}$ with $g(0) = g_0$ and defined on a maximal time interval. Since $\mathcal{M}$ is compact and $\mathrm{Rm}(g_0)$ is strictly 2-positive, there exists $\varepsilon \in (0,1)$ and $h_0 \in (0,\infty)$ such that

$$\mathrm{Rm}(g_0(x)) \in C(\varepsilon) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \leq h_0\} \tag{11.126}$$

for all $x \in \mathcal{M}$ (here we have used an isometry between $(T_x\mathcal{M}, g_0(x))$ and $\mathbb{E}^n$ to consider $C(\varepsilon) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \leq h_0\}$ as contained in $S_B^2\Lambda^2 T_x^*\mathcal{M}$, which is well-defined and independent of the choice of isometry since it is $\mathrm{O}(n)$-invariant). By Theorem 11.64, there exists a closed convex $\mathrm{O}(n)$-invariant subset $F \subset S_B^2(\mathfrak{so}(n))$ preserved by the ODE such that

$$C(\varepsilon) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \leq h_0\} \subset F$$

and $F \setminus C(s)$ is compact for all $s \in [\varepsilon, 1)$. Let $\mathcal{F} \subset S_B^2\Lambda^2 T^*\mathcal{M}$ denote the closed fiberwise convex set which is invariant under parallel translation and such that $\mathcal{F}_x$ is equal to F for all $x \in \mathcal{M}$ (again, since F is $\mathrm{O}(n)$-invariant, $\mathcal{F}$ is well-defined, i.e., independent of the isometries used to identify $(T_x\mathcal{M}, g_0(x))$ with $\mathbb{E}^n$). By the Weinberger–Hamilton maximum principle for systems (Theorem 10.13), we have

$$\iota(t)^* \,\mathrm{Rm}(g(x,t)) \in \mathcal{F} \quad \text{for all } x \in \mathcal{M} \text{ and } t \geq 0,$$

where $\iota(t)$ is defined by (11.124).

Since the scalar curvature of g_0 is positive, a singularity to the Ricci flow forms in finite time, i.e., the singularity time T is finite. Choose a sequence of points and times $\{(x_i, t_i)\}_{i\in\mathbb{N}}$ with $t_i \to T$ such that

$$K_i \doteqdot |\mathrm{Rm}(x_i, t_i)| = \max_{\mathcal{M}\times[0,t_i]} |\mathrm{Rm}|$$

(note $K_i \to \infty$) and consider the sequence of dilated and time-translated solutions $(\mathcal{M}^n, g_i(t), x_i)$, where

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right).$$

By Perelman's no local collapsing theorem and Hamilton's Cheeger–Gromov-type compactness theorem (Theorem 6.58 on p. 256 and Theorem 3.10 on p. 131 in Part I of this volume), there exists a subsequence such that $\{(\mathcal{M}^n, g_i(t), x_i)\}$ converges to a complete ancient solution $(\mathcal{M}^n_\infty, g_\infty(t), x_\infty)$ to the Ricci flow with 2-nonnegative curvature operator and $|\mathrm{Rm}_\infty(x_\infty, 0)| = 1$. This implies that $|\mathrm{Rm}_\infty(x, 0)| \geq c$, where $c > 0$, for all x contained in an open neighborhood $\mathcal{U}$ of x_∞ in $\mathcal{M}_\infty$. Since $F \setminus C(s)$ is compact for all $s \in [\varepsilon, 1)$ and $\lim_{s\to 1} C(s) = \mathbb{R}_+\mathbf{I}$, we conclude that

$$\mathrm{Rm}(g_\infty(x, 0)) \in \mathbb{R}_+ \,\mathrm{id}_{\Lambda^2 T^*\mathcal{M}} \quad \text{for all } x \in \mathcal{U}. \tag{11.127}$$

Indeed, sweeping under the rug the need to use diffeomorphisms in the compactness theorem (we leave it to the reader to make the argument, in particular, the next displayed equation, rigorous), since $x \in \mathcal{U}$, we have

$$K_i^{-1} |\mathrm{Rm}(x, t_i)| \to |\mathrm{Rm}(g_\infty(x, 0))| \geq c > 0,$$

so that $|\mathrm{Rm}(x, t_i)| \to \infty$ as $i \to \infty$. Hence for every $s \in [\varepsilon, 1)$, $\mathrm{Rm}(x, t_i) \in C(s)$ for i sufficiently large. This implies $K_i^{-1}\,\mathrm{Rm}(x, t_i) \in C(s)$, which in turn implies $\mathrm{Rm}(g_\infty(x, 0)) \in C(s)$ for all $s \in [\varepsilon, 1)$. We conclude (11.127).

Now by (11.127) and Schur's lemma $g_\infty(0)$ has constant positive sectional curvature in $\mathcal{U}$. From this we can deduce that $g_\infty(0)$ has constant positive sectional curvature in all of $\mathcal{M}$ (since, wherever $|\mathrm{Rm}(g_\infty(0))| > 0$, we have that the sectional curvature of $g_\infty(0)$ is locally constant). We have established sequential convergence of the Ricci flow. Finally, by the Huisken–Margerin–Nishikawa pinching theorem, we obtain exponential convergence of the Ricci flow in each C^k-norm. □

Finally we give the proof of the existence of a generalized pinching set assuming the existence of a pinching family. This proof is nonconstructive.

7.3. Proof of the existence of a generalized pinching set.

Proof of Theorem 11.64. Fix any $\varepsilon \in (0,1)$ and $h_0 \in (0,\infty)$. Define the set F to be the intersection of all closed convex $\mathrm{O}(n)$-invariant subsets of $S^2_B(\mathfrak{so}(n))$ satisfying properties (1) and (2) in the statement of Theorem 11.64. We want to show that F is the desired generalized pinching set. Since $C_\clubsuit(\varepsilon)$ is such a set, F is a well-defined closed convex $\mathrm{O}(n)$-invariant subset contained in $C_\clubsuit(\varepsilon)$.

(I) Suppose $\mathbf{R}_0 \in F$ and $\mathbf{R}(t)$ is a solution of the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ with $\mathbf{R}(0) = \mathbf{R}_0$. Then for any closed convex $\mathrm{O}(n)$-invariant subset G of $S_B^2(\mathfrak{so}(n))$ satisfying properties (1) and (2),[29] we have $\mathbf{R}(t) \in G$. Hence, by definition, $\mathbf{R}(t) \in F$. Thus property (1) holds for F.

(II) It is also clear that property (2) holds for F, i.e.,

$$C_{\clubsuit}(\varepsilon) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \leq h_0\} \subset F.$$

(III) It remains to prove property (3). Suppose, with the goal of obtaining a contradiction, that property (3) is false for F for some $s \in [\varepsilon, 1)$. Let $s_0 \in [\varepsilon, 1)$ be the infimum of all $s \in [\varepsilon, 1)$ for which property (3) is false for F, i.e.,

$$s_0 \doteqdot \inf \{s \in [\varepsilon, 1) : F \setminus C_{\clubsuit}(s) \text{ is unbounded}\}.$$

Then there exists a sequence $\{s_i\}_{i \in \mathbb{N}}$ with $s_i \to s_0$ such that $F \setminus C_{\clubsuit}(s_i)$ is unbounded for all i. We shall obtain the contradiction from showing that $F \setminus C_{\clubsuit}(s)$ is bounded for any s sufficiently close to s_0.

For $s \in [0, 1)$ and $\delta > 0$ define

$$\begin{aligned} C_\delta(s) \doteqdot \ & \text{the cone over the set} \\ & \{\mathbf{R} \in C_{\clubsuit}(s) : \operatorname{trace}(\mathbf{R}) = 1,\ \operatorname{dist}(\mathbf{R}, \partial C_{\clubsuit}(s)) \geq \delta\}, \end{aligned}$$

which is a slightly smaller cone contained in $C_{\clubsuit}(s)$ (with $\lim_{\delta \to 0} C_\delta(s) = C_{\clubsuit}(s)$). For $\delta, h > 0$ define

$$T_h C_\delta(s) \doteqdot \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in C_\delta(s),\ \operatorname{trace}(\mathbf{R}) \geq h\},$$

which is, as noted in [**43**], a '**truncated shifted cone**'. We leave it to the reader to show that both $C_\delta(s)$ and $T_h C_\delta(s)$ are closed, convex, and $\mathrm{O}(n)$-invariant (convexity is the main issue).[30]

(1) *When δ is sufficiently small, the cones $C_\delta(s)$ are preserved by the* ODE. For any $s \in (0, 1)$ there exists $\delta_0 = \delta_0(s) > 0$ such that for all $\delta \in [0, \delta_0]$ and $\mathbf{R} \in \partial C_\delta(s) - \{0\}$ the vector $Q(\mathbf{R})$ lies in the *interior* of the tangent cone of $C_\delta(s)$.

This fact follows from

(i) $\lim_{\delta \to 0} \partial C_\delta(s) = \partial C_{\clubsuit}(s)$,

(ii) the continuity of Q,

(iii) our assumption that for $\mathbf{R} \in \partial C_{\clubsuit}(s) \setminus \{0\}$, $Q(\mathbf{R})$ lies in the interior of tangent cone of $C_{\clubsuit}(s)$ at $\mathbf{R}$, and

(iv) the fact that $Q(\mathbf{R})$ is homogeneous of degree 2, i.e., $Q(c\mathbf{R}) = c^2 Q(\mathbf{R})$ for all $c \geq 0$ (which addresses the issue of the noncompactness of the cones).

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[29] Note that we do not use property (2) here.

[30] Note that the set $\Sigma \doteqdot \{\mathbf{R} \in C_{\clubsuit}(s) : \operatorname{trace}(\mathbf{R}) = 1\}$ is convex. To prove the convexity of $C_\delta(s)$, one just needs to show that

$$\{\mathbf{R} \in \Sigma : \operatorname{dist}(\mathbf{R}, \partial \Sigma) \geq \delta\}$$

is convex. The convexity of $T_h C_\delta(s)$ also follows easily.

(2) *Truncated shifted cones are preserved by the* ODE. There exists $\bar{h} \in (0, \infty)$ (in particular, for $\bar{h}$ sufficiently large) such that the set

$$T_{\bar{h}} C_\delta(s_0) = \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in C_\delta(s_0)\} \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \geq \bar{h}\}$$

is preserved by the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ for all $\delta \in [0, \delta_0(s_0)]$.

First note that

$$\partial \left(T_{\bar{h}} C_\delta(s_0)\right) \subset \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in \partial C_\delta(s_0)\} \cup \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}.$$

We have

$$\operatorname{trace} Q(\mathbf{R}) = |\mathrm{Rc}(\mathbf{R})|^2 \geq \frac{4}{n} (\operatorname{trace}(\mathbf{R}))^2 \geq \frac{4}{n} \bar{h}^2 > 0$$

for $\mathbf{R} \in T_{\bar{h}} C_\delta(s_0)$ (note that $\operatorname{trace}(\mathbf{R}) = \frac{1}{2} \operatorname{tr}(\mathrm{Rc}(\mathbf{R}))$).

Case (a). Hence for $\mathbf{R}$ with $\operatorname{trace}(\mathbf{R}) = \bar{h}$ the vector $Q(\mathbf{R})$ lies in the interior of the tangent cone of $\left\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \geq \bar{h}\right\}$. This implies that for

$$\mathbf{R} \in \partial \left(T_{\bar{h}} C_\delta(s_0)\right) \setminus \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in \partial C_\delta(s_0)\} \subset \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\},$$

the vector $Q(\mathbf{R})$ lies in the interior of the tangent cone of $T_{\bar{h}} C_\delta(s_0)$ at $\mathbf{R}$, which is the half-space $\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \geq 0\}$.

Case (b). Now assume

$$\begin{aligned} \mathbf{R} &\in \partial \left(T_{\bar{h}} C_\delta(s_0)\right) \setminus \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\} \\ &\subset \partial \left(T_{\bar{h}} C_\delta(s_0)\right) \cap \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in \partial C_\delta(s_0)\}, \end{aligned}$$

so that in particular $\operatorname{trace}(\mathbf{R}) > \bar{h}$. In this case the tangent cone of $T_{\bar{h}} C_\delta(s_0)$ at $\mathbf{R}$ is the same as the tangent cone of $C_\delta(s_0)$ at $\mathbf{R} + \mathbf{I}$ (after shifting the vertex by $\mathbf{I}$).

Let η_0 be the infimum of the distance from $Q(\mathbf{R}')$ to the boundary of the tangent cone of $C_\delta(s_0)$ among $\mathbf{R}' \in \partial C_\delta(s_0)$ with $\operatorname{trace}(\mathbf{R}') = 1$. Since by part **(1)**, $\eta_0 > 0$, we have that the distance from $Q(\mathbf{R}')$ to the boundary of the tangent cone of $C_\delta(s_0)$ grows quadratically; in particular, it is at least

$$\eta_0 \left(\operatorname{trace}(\mathbf{R}')\right)^2 \geq c\eta_0 |\mathbf{R}'|^2 \tag{11.128}$$

for all $\mathbf{R}' \in \partial C_\delta(s_0)$, where $c > 0$. For $\mathbf{R} \in \partial \left(T_{\bar{h}} C_\delta(s_0)\right) \cap \{\mathbf{R} : \mathbf{R} + \mathbf{I} \in \partial C_\delta(s_0)\}$,

(i) let $d_1(\mathbf{R})$ be the distance of $Q(\mathbf{R})$ to the boundary of the tangent cone of $T_{\bar{h}} C_\delta(s_0)$ at $\mathbf{R}$ and

(ii) let $d_2(\mathbf{R})$ be the distance of $Q(\mathbf{R}+\mathbf{I})$ to the boundary of the tangent cone of $C_\delta(s_0)$ at $\mathbf{R} + \mathbf{I}$, so that by (11.128)

$$d_2(\mathbf{R}) \geq c\eta_0 \, |\mathbf{R} + \mathbf{I}|^2 \geq \frac{1}{2} c\eta_0 \, |\mathbf{R}|^2$$

(with the last inequality holding for $\bar{h}$ large enough). Next observe that

$$\begin{aligned} |Q(\mathbf{R} + \mathbf{I}) - Q(\mathbf{R})| &= |2\mathbf{R} + \mathbf{I} + 2\mathbf{R}\#\mathbf{I} + \mathbf{I}\#\mathbf{I}| \\ &\leq A_1 \, |\mathbf{R}| \end{aligned}$$

for some constant $A_1 < \infty$. Hence, by the triangle inequality and the above estimates,

$$\begin{aligned} d_1(\mathbf{R}) &\geq d_2(\mathbf{R}) - |Q(\mathbf{R}) - Q(\mathbf{R}+\mathbf{I})| \\ &\geq \frac{1}{2}c\eta_0 |\mathbf{R}|^2 - A_1 |\mathbf{R}|, \end{aligned}$$

so that $d_1(\mathbf{R}) > 0$ provided that $|\mathbf{R}|$ is sufficiently large. Thus $Q(\mathbf{R})$ lies in the interior of the tangent cone of $T_{\bar{h}}C_\delta(s_0)$ at $\mathbf{R}$.

Case (c). Finally, if

$$\mathbf{R} \in \{\mathbf{R} : \mathbf{R}+\mathbf{I} \in \partial C_\delta(s_0)\} \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\} \subset \partial\left(T_{\bar{h}}C_\delta(s_0)\right),$$

then by the arguments in proving Cases (a) and (b), it follows that $Q(\mathbf{R})$ lies in the interior of the tangent cone of $T_{\bar{h}}C_\delta(s_0)$ at $\mathbf{R}$.

By Cases (a), (b), and (c), we have shown that for all $\mathbf{R} \in \partial\left(T_{\bar{h}}C_\delta(s_0)\right)$, the vector $Q(\mathbf{R})$ lies in the interior of the tangent cone of $T_{\bar{h}}C_\delta(s_0)$.

(3) Given any $\bar{h} \in (0, \infty)$, there exists $\delta > 0$ (in particular, for δ sufficiently small) such that the set $C_\clubsuit(s_0) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}$ is contained in the *interior* of $T_{\bar{h}}C_\delta(s_0) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}$ as subsets of the hyperplane $\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}$. This is geometrically clear from considering the limit as $\delta \to 0$.

(4) By part **(3)**, for $\delta > 0$ sufficiently small and for s close enough to s_0, $C_\clubsuit(s) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}$ is contained in the interior of $T_{\bar{h}}C_\delta(s_0) \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\}$. We **claim** that by the definition of s_0, for $\delta > 0$ sufficiently small there exists $\bar{s} \in [\varepsilon, s_0)$ (except when $s_0 = \varepsilon$, where we let $\bar{s} = s_0$) and $k \in (0, \infty)$ large enough such that

(11.129)
$$F \cap \left\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = k\bar{h}\right\} \subset C_\clubsuit(\bar{s}) \cap \left\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = k\bar{h}\right\} \subset k \cdot T_{\bar{h}}C_\delta(s_0),$$

that is,

$$\frac{1}{k}F \cap \left\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\right\} \subset C_\clubsuit(\bar{s}) \cap \left\{\mathbf{R} : \operatorname{trace}(\mathbf{R}) = \bar{h}\right\} \subset T_{\bar{h}}C_\delta(s_0).$$

(a) If $s_0 = \varepsilon = \bar{s}$, then the claim is clear since $F \subset C_\clubsuit(\varepsilon)$.

(b) If $s_0 > \varepsilon$, then $F \setminus C_\clubsuit(\bar{s})$ is compact for all $\bar{s} \in [\varepsilon, s_0)$. Therefore, for each $\bar{s} \in [\varepsilon, s_0)$, there exists $k_0 \in (0, \infty)$ such that the first inclusion in (11.129) holds for all $k \geq k_0$.

(5) By part **(4)**, for $\delta > 0$ sufficiently small and $k \in (0, \infty)$ large enough, the set

$$F' \doteqdot \left(F \cap \{\mathbf{R} : \operatorname{trace}(\mathbf{R}) \leq k\bar{h}\}\right) \cup \left(F \cap k \cdot T_{\bar{h}}C_\delta(s_0)\right)$$

is closed, convex, $\mathrm{O}(n)$-invariant, and preserved by the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$. In particular, the convexity of F' follows from the inclusion (11.129). Now clearly $F' \subset F$. On the other hand, by the definition of F (in particular, its minimality), we have $F \subset F'$. Hence for $\delta > 0$ sufficiently small and $k \in (0, \infty)$ large enough,

$$F = F'.$$

(6) Since the set $k \cdot T_{\bar{h}} C_\delta(s_0) \setminus C_{\delta/2}(s_0)$ is bounded, we have that the set $F' \setminus C_{\delta/2}(s_0)$ is bounded. On the other hand, clearly

$$F' \setminus C_{\clubsuit}(s) \subset F' \setminus C_{\delta/2}(s_0)$$

for any s close enough to s_0. Hence the set $F \setminus C_{\clubsuit}(s) = F' \setminus C_{\clubsuit}(s)$ *is bounded* for any s close enough to s_0. This contradicts the definition of s_0 since there exists a sequence $\{s_i\}_{i \in \mathbb{N}}$ with $s_i \to s_0$ such that $F \setminus C_{\clubsuit}(s_i)$ is not bounded for all i. □

8. Summary of the proof of the convergence of Ricci flow

In this section we summarize the proof of Theorem 11.2.

The goal is to prove sequential convergence to a metric of constant positive sectional curvature (exponential convergence then follows from previous work). Since the condition of 2-nonnegative curvature operator is preserved under the Ricci flow, by Hamilton's compactness theorem and Perelman's no local collapsing theorem, we have sequential convergence to a complete ancient solution with bounded 2-nonnegative curvature operator. By Schur's lemma, it suffices to show that this limit solution has the property that the curvature operator is a multiple of the identity wherever the scalar curvature is positive.

Hamilton proved that on closed 3-manifolds the condition $\mathrm{Rc} \geq \frac{s}{3} Rg$, where $R \geq 0$, is preserved under the Ricci flow for any $s \in [0,1]$. By the Weinberger–Hamilton maximum principle for systems, this follows from showing that the ODE

$$\frac{d}{dt}\mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\# \doteqdot Q(\mathbf{R})$$

preserves the $\mathrm{O}(3)$-invariant closed convex cone

$$C(s) \doteqdot \left\{ \mathbf{R} \in S^2_B(\mathfrak{so}(3)) : \mu_1(\mathbf{R}) + \mu_2(\mathbf{R}) \geq \frac{s}{3} \operatorname{Scal}(\mathbf{R}) \right\}$$

for any $s \in [0,1]$. Note that $C(0)$ is the cone of algebraic curvature operators with nonnegative Ricci curvature and $C(1)$ is the cone (in particular, half-line) of algebraic curvature operators which are nonnegative multiples of the identity.

Abstracting this construction in higher dimensions, Böhm and Wilking prove the existence of a continuous 1-parameter family of $\mathrm{O}(n)$-invariant closed convex cones $\{C(s)\}_{s \in [0,1)}$, called a 'pinching family', with the following properties:

(1) except for 0, their elements have positive scalar curvature;
(2) the ODE $\frac{d}{dt}\mathbf{R} = Q(\mathbf{R})$ preserves the cones;
(3) the cones limit to $\mathbb{R}_+ \mathbf{I}$ as $s \to 1$;
(4) $C(0)$ is the cone of 2-nonnegative algebraic curvature operators.

We now outline how the existence of this family is proved and why it is sufficient to establish the main theorem.

One proves the existence of

(i) a continuous family of convex cones $\{C_{\spadesuit}(s)\}_{s\in[0,1)}$ joining the cone of nonnegative algebraic curvature operators $\{\mathbf{R} : \mathbf{R} \geq 0\}$ to the half-line $\mathbb{R}_+\mathbf{I}$ and

(ii) a continuous family of convex cones $\{C_{\diamondsuit}(s)\}_{s\in[-1,0]}$ joining the cone of 2-nonnegative algebraic curvature operators to a convex cone inside $\{\mathbf{R} : \mathbf{R} \geq 0\}$.[31]

By concatenating the two families, more specifically, by considering

$$\{C_{\diamondsuit}(s)\}_{s\in[-1,0]} \cup \{C_{\heartsuit}(s)\}_{s\in[0,1)},$$

where $C_{\heartsuit}(s) \doteqdot C_{\spadesuit}(s) \cap C_{\diamondsuit}(0)$ (note that since $C_{\diamondsuit}(0) \subset C_{\spadesuit}(0)$, we have $C_{\diamondsuit}(0) = C_{\heartsuit}(0)$, so that the concatenated family is continuous), we obtain a pinching family.

The idea in proving both (i) and (ii) is to consider $\mathrm{O}(n)$-invariant linear transformations of $S_B^2(\mathfrak{so}(n))$ which preserve the Weyl parts of algebraic curvature operators. Since $\mathrm{O}(n)$-invariant linear transformations are multiples of the identity on each irreducible component of $S_B^2(\mathfrak{so}(3))$, they are parameterized by two real numbers and we denote these transformations by $\ell_{a,b}$ (where $\ell_{0,0} = \mathrm{id}$).

Let $\mathfrak{C}$ denote the cone of 2-nonnegative algebraic curvature operators. For certain values of a and b, the $\mathrm{O}(n)$-invariant closed convex cone $\ell_{a,b}(\mathfrak{C})$ is preserved by the ODE (Proposition 11.47).[32] The proof of this hinges on computing

$$D(\mathbf{R}) \doteqdot \ell_{a,b}^{-1}(Q(\ell_{a,b}(\mathbf{R}))) - Q(\mathbf{R}).$$

The surprising facts are that (see Theorem 11.27, Corollary 11.29, and part (i) of the proof of Proposition 11.47)

(A) the eigenvalues of $D(\mathbf{R})$ are computable in terms of n, a, b, and the eigenvalues of $\mathrm{Rc}(\mathbf{R})$ and

(B) $D(\mathbf{R})$ is nonnegative for certain values of a and b.

Using $\ell_{a,b}(\mathfrak{C})$ for certain values of a and b, one proves (ii) above.

REMARK 11.66. It is rather clear that one can stretch the cone $\mathfrak{C}$ a little, i.e., for values of a and b close to zero we have $\ell_{a,b}(\mathfrak{C})$ is preserved by the ODE. However, without a computable method, it is not clear how to determine for which values of a and b we have that $\ell_{a,b}(\mathfrak{C})$ is preserved by the ODE (see also Problem 11.49). Fortunately, Theorem 11.27 and its consequences provide a computable method.

To prove (i), we first consider a 1-parameter family of cones $\{C_b\}_{b\in[0,1/2]}$ obtained by intersecting the cone of nonnegative curvature operators with those satisfying a Ricci pinching condition, where the coefficient in the pinching condition depends on the parameter. Here C_0 is the cone of nonnegative

[31] Here $\{C_{\diamondsuit}(s)\}_{s\in[-1,0]} \doteqdot \{\ell_{a_0(b),b}(\mathfrak{C})\}_{b\in[0,\bar{b}]}$, where s is linear in b say.

[32] These values include $(a,b) = (0,0)$; note that $\ell_{0,0}(\mathfrak{C}) = \mathfrak{C}$.

curvature operators, e.g., for $b = 0$ the Ricci pinching condition just says $\mathrm{Rc} \geq 0$. For $b \in [0, 1/2]$ and a certain function $a = a_1(b)$ we have that $\ell_{a,b}(C_b)$ is preserved by the ODE (see Remark 11.57). We concatenate this family with a similar family $\ell_{a_2(s),\frac{1}{2}}(C'_s)$, where C'_s is defined analogously to C_b except that the coefficient in the pinching condition depends differently on the parameter. It is this concatenated family which provides a pinching family joining the cone of nonnegative algebraic curvature operators to the half-line $\mathbb{R}_+\mathbf{I}$.

Finally, the reason for why the existence of a pinching family is sufficient to prove the theorem is as follows. By a contradiction argument, given a pinching family $C(s)$ such that $C(0)$ is the cone of 2-nonnegative curvature operators and $\lim_{s\to 1} C(s) = \mathbb{R}_+\mathbf{I}$ and given any compact subset Ω of the set of 2-positive curvature operators, we can construct a generalized pinching set F which contains Ω. The set F is closed, convex, $\mathrm{O}(n)$-invariant, and its asymptotic cone is $\mathbb{R}_+\mathbf{I}$. (See the proof of Theorem 11.64.) By the Weinberger–Hamilton maximum principle for systems, $\mathrm{Rm}(g(t))$ is contained in the corresponding subset of $S^2_B\Lambda^2T^*\mathcal{M}$. Then by applying the compactness theorem and the no local collapsing theorem, we can deduced that a limit of rescalings of the solution $g(t)$ produces a constant positive sectional curvature metric on $\mathcal{M}$.

9. Notes and commentary

9.1. Sphere theorems and spherical space form theorems. The seminal work of Hamilton [**245**] on closed 4-manifolds with positive curvature operator led him to conjecture that the normalized Ricci flow on any closed n-manifold with initial metric g_0 having positive curvature operator converges exponentially in every C^k-norm to a metric of constant sectional curvature. Rauch first conjectured that any closed n-manifold admitting a metric with positive curvature operator is diffeomorphic to a spherical space form. This conjecture is sometimes known as the Rauch–Hamilton spherical space form conjecture.

We should remark that in terms of geometrically characterizing the sphere, there is the well-known Rauch–Berger–Klingenberg 1/4-pinching topological sphere theorem [**417**], [**35**], [**304**], [**305**] which says that a simply-connected complete Riemannian n-manifold $\mathcal{M}$ with sectional curvatures satisfying $1/4 < \mathrm{sect} \leq 1$ is *homeomorphic* to the n-sphere.

Earlier, assuming that the sectional curvatures are sufficiently pinched, i.e., $0.76 < \mathrm{sect} \leq 1$, it was previously proved that $\mathcal{M}$ is *diffeomorphic* to the n-sphere (see Grove, Karcher, and Ruh [**231**]; see pp. 239–240 of Klingenberg's book [**306**] for a discussion of pinching theorems).

Let $K_{\max}(p)$ and $K_{\min}(p)$ denote the maximum and minimum sectional curvatures at a point $p \in \mathcal{M}$. Using the Ricci flow, Huisken [**280**], Margerin [**343**], [**344**], and Nishikawa [**392**], [**393**] (independently) proved that under the pointwise condition that $K_{\max}(p) \leq (1 + \delta_n) K_{\min}(p)$ for all $p \in \mathcal{M}$, where $\delta_n > 0$ depends only on n, and without assuming $\mathcal{M}$ is simply connected, we have that $\mathcal{M}$ is diffeomorphic to a spherical space form.

Recently, using the Ricci flow, Brendle and Schoen [**48**] proved that **positive isotropic curvature** (PIC) is preserved and used this to prove the long-standing pointwise 1/4-pinching spherical space form theorem. We remark that the curvature condition of PIC originally appeared in Micallef and Moore [**347**]. By relating this condition to the stability of minimal 2-spheres, they proved that a simply-connected closed Riemannian n-manifold with PIC is homeomorphic to the n-sphere.

Without assuming $\mathcal{M}$ is simply connected but assuming $\mathcal{M}$ has no essential incompressible 3-dimensional spherical space forms, Hamilton [**255**] originally used the Ricci flow with surgery to approach the classification of closed Riemannian 4-manifolds with positive isotropic curvature.[33] As pointed out by Perelman,[34] there was a gap in Hamilton's proof. The work of Chen and Zhu [**110**] uses Perelman's techniques [**402**], [**403**] to address the issue of this gap.

See also Micallef and Wang [**348**] and Fraser and Wolfson [**193**] for further work on (weakly) positive isotropic curvature.

9.2. Generalized pinching sets. It is unclear what the precise geometries of the sets F defined in the proof of Theorem 11.64 are. However, for heuristic purposes (i.e., to develop some intuition) it is perhaps useful to revisit the example of the paraboloid P in $\mathbb{R}^{n+1}$ defined to be the set of points $(x_0, \vec{x}) = (x_0, x_1, \ldots, x_n)$ such that $x_0 = |\vec{x}|^2 = x_1^2 + \cdots + x_n^2$, which dimension reduces to (and has asymptotic cone) the half-line $\mathbb{R}_+ \times \vec{0}$, where $\vec{0}$ is the origin in $\mathbb{R}^n$. Note that if $(x_0, \vec{x}) \in P$, then

$$\frac{\vec{x}}{|\vec{x}|^2} = \left(1, \frac{x_1}{|\vec{x}|^2}, \ldots, \frac{x_n}{|\vec{x}|^2}\right).$$

In particular, $\lim_{x_0 \to \infty} \frac{\vec{x}}{|\vec{x}|^2} = (1, 0, \ldots, 0) \in \mathbb{R}_+ \times \vec{0}$. An example of a closed convex set with asymptotic cone equal to the half-line but which is closer to being like a cone, i.e., where x_0 grows more slowly and more linearly as a function of $|\vec{x}|$, is given by the set of points $(x_0, \vec{x}) \in \mathbb{R}^{n+1}$ such that $x_0 = \left(|\vec{x}| + \frac{1}{e}\right) \ln\left(|\vec{x}| + \frac{1}{e}\right) + \frac{1}{e}$, defined so that the origin is its vertex.

[33] See the first section of [**255**] for definitions and statements of the main theorem and corollary.

[34] See the introduction to [**403**].