Introduction

The 26 sporadic simple groups have played an exceptional role in the theory and classification[1] of the finite simple groups, ever since the discovery of the first few such groups in the late 1800s. In the last decade or so, various authors have also observed exceptional behavior in the mod p group cohomology of some of the sporadic groups, especially at the prime $p = 2$; see for example Benson [**24, 25, 26**], Benson and Wilkerson [**29**], Milgram [**92, 93, 94**]. Determining (or at least describing) the cohomology of sporadic groups represents an area of considerable current activity; and such study combines methods of algebraic topology with information and techniques from finite group theory.

In this work, we provide for each sporadic group a description of a 2-completed classifying space, in terms of classifying spaces of suitable 2-local subgroups. In particular, this leads to an additive decomposition of the mod 2 group cohomology. Similar decompositions can be obtained for odd primes p; but the indexing set of p-local subgroups is usually smaller and often less interesting.

Although our focus here is on sporadic groups, we emphasize that this work should be also be viewed in relation to work on cohomology of the other simple groups. For example, our decompositions for sporadic groups are largely inspired by a standard decomposition (see Example 5.1.18) of the mod p cohomology for a group of Lie type of characteristic p. However, even in that model case for Lie type groups, it remains a major open problem to give an explicit description of the individual component terms and their cohomology. In contrast, the cohomology and completed classifying spaces of groups of Lie type in coprime characteristic are rather well understood in the light of the extraordinary work of Quillen [**103**] for the general linear groups, and subsequent work along the same lines by Fiedorowicz and Priddy [**60**] for the remaining classical groups and Kleinerman [**81**] for the groups of exceptional Lie type. Cohomology and classifying spaces of symmetric and alternating groups are understood to some extent (Nakaoka [**99**], Mui [**97**], Mann [**89, 90**], Gunawardena, Lannes and Zarati [**70**], Ha and Lesh [**71**], etc).

The context of this work. Naturally the structure of the p-local subgroups[2] in a finite group G should provide an important ingredient in the determination of its mod-p group cohomology. One p-local approach to cohomology, often used as a tool in modern work on individual sporadic groups, is provided by the topological literature on *cohomology decompositions*. In particular, we mention Dwyer's analysis [**51, 53**] of *ample* collections $\mathcal{C}$ of p-subgroups of G, namely collections which

[1] The reader who is uneasy with the classification of the finite simple groups may insert "known" before "sporadic simple groups" here. The classification is not used in our analysis in this work, which studies only the properties of those 26 individual sporadic groups.

[2] A *p-local subgroup* is the normalizer of a non-identity p-subgroup.

afford an additive decomposition of the mod p group cohomology of G, where the components are given by the cohomology of certain p-local subgroups of G (for example, the normalizers of the p-groups in $\mathcal{C}$). The origins of these decompositions go back to work of Brown, Quillen, and Webb in the 1970s and 1980s. More recently, treatments of certain ample collections (notably by Jackowski, McClure, Oliver, Dwyer, and Grodal) consider *homotopy decompositions*, namely corresponding decompositions not at the level of cohomology, but instead at the deeper topological level of the classifying space BG. Here the decomposition is in fact afforded by a homotopy colimit, in which the component terms are classifying spaces of suitable subgroups of G, and the terms are indexed by a category determined by the collection $\mathcal{C}$. For these purposes, there are various standard ample collections in the literature, such as $\mathcal{S}_p(G)$ defined by all nontrivial p-subgroups; but often the standard collections are too large to be useful for practical computation, so that it is also of interest to determine ample subcollections which are as *small* as possible.

Several years ago, the first author initiated the project that became this work: namely the construction, via a homotopy decomposition over subgroups, of a 2-completed classifying space $BG_2^\wedge$ for each sporadic group G. One motivation was the observation that for a number of the sporadic groups, the group theory literature (especially that on *2-local geometries*) already contained information sufficient to determine a small ample collection, and hence a homotopy decomposition—for example, collections corresponding to the 2-local geometries treated in Smith and Yoshiara [**116**], and the geometry used in Benson [**24**]. For the remaining sporadic groups, progress was initially slower, since the larger groups require substantially more complicated analysis. However, recent work of various authors on the 2-local structure of large sporadic groups has led for example to the completion in Yoshiara [**131**] of the determination of the 2-radical subgroups of the remaining sporadic groups. So, combining such results in group theory with the homotopy colimit theorems indicated above, in the present work we exhibit for each of the 26 sporadic groups a small ample collection related to a 2-local geometry, and the corresponding homotopy decomposition. In each case, since the group acts flag-transitively on the geometry, the relevant homotopy colimit is over an indexing category given by a *simplex*; so that the diagram of classifying spaces is that of a pushout n-cube in the relevant dimension n. (Occasionally cancellations simplify the calculation to an even smaller subdiagram.) We show furthermore that each decomposition is *sharp*, in Dwyer's sense of affording an alternating sum decomposition of group cohomology via the formula of Webb—this simplification arises from the collapse of an underlying spectral sequence. In particular, these results confirm a conjecture (see [**116**, Conj. 1, p. 376]) concerning such a connection between group theory and algebraic topology. That conjecture was necessarily somewhat vague, since the 2-local geometries in the literature were defined in a number of different ways; however, in this work we do indicate for each group a 2-local geometry suitable for our purposes.

Outline of the work. Our treatment is fairly lengthy, so we indicate some of its main features in overview.

Chapter 1 gives a brief summary of our main results.

Since we wish to make our account accessible both to group theorists and to homotopy theorists, we then present in Part 1 an extensive exposition for non-experts of some of the background material involved in each area.

Most of Part 1 is devoted to a review of selected topological material leading up to, and into, the modern literature[3] on decompositions of group cohomology: Chapter 2 recalls some basics of the group cohomology of a finite group G; including aspects of the topological approach via the classifying space BG, and of "approximating" BG—via the Borel construction on a suitable G-space. In the classical literature, these spaces of interest are viewed as topological spaces; but in Chapter 3, we will indicate the more modern viewpoint on spaces as simplicial sets—emphasizing the standard equivalence (due to Quillen) of those two viewpoints, in terms of the homotopy categories they determine. In Chapter 4, we then adopt the viewpoint of simplicial sets—since that is the appropriate general context for our description there of two important constructions due to Bousfield and Kan, namely completions and homotopy colimits. Chapter 5 then reviews some of the more specific literature on homotopy decompositions of the classifying space, given in Dwyer's viewpoint via a homotopy colimit indexed by a collection of p-subgroups; the chapter then examines some of the standard particular collections which are ample in the sense of affording such a decomposition.

Part 1 concludes with Chapter 6, which reviews various notions from the group theory literature, primarily relevant to 2-local geometries for sporadic groups; especially from the viewpoint of their observed connections with the ample collections of 2-subgroups and decompositions described in Chapter 5.

Part 2 then presents the main results of our work. Chapter 7 contains a section on each of the individual sporadic groups G: We indicate a homotopy decomposition for the 2-completed classifying space $BG^{\wedge}_2$, in terms of a small ample collection determined by a suitable 2-local geometry—with a corresponding additive decomposition of the mod 2 group cohomology of G. We also add, where available, further remarks on the status of knowledge in the literature about the full ring structure of the mod 2 cohomology. In some cases, our results are fairly immediate deductions from results already in the literature; but in the remaining cases where we require a more substantial argument, we sometimes postpone the detailed proofs to sections corresponding to the relevant groups in our final Chapter 8.

We close this Introduction with a brief mention of a topic which we will touch on repeatedly later, but for which we have *not* tried to develop our own exposition in this work:

Foreword on Lie type groups and buildings. Our main results in Chapter 7 are concerned with the 26 sporadic simple groups, viewed via their 2-local geometries; but at various points in our expository Chapters 4–6, it will also be natural to consider examples given instead by the simple groups of Lie type, together with their natural geometries given by Tits buildings. There are a number of reasons for this, among them:

- Buildings are simplicial complexes providing convenient and very natural spaces for a number of the topological constructions we will be examining.

[3] In particular we will frequently quote from a very useful recent exposition on decompositions and their background by Dwyer [**53**]—which we strongly urge readers to consult alongside our exposition.

• Various standard results on groups of Lie type and buildings provide a "model case" for homology decompositions and related research areas—often providing motivation for generalizations to other groups, or even to all finite groups.

• The 2-local geometries for sporadic groups were inspired by certain analogies with buildings.

In particular, our main results for sporadic groups involve a homotopy decomposition indexed by a simplex—and these can be regarded as analogues of the corresponding standard result for Lie type groups, see Example 5.1.18.

When we discuss examples from the theory of Lie type groups, our approach will be to quote the relevant properties from the literature, wherever they arise in our development, for example in Example 4.6.3—rather than attempting to provide a single exposition of that material beforehand; for nowadays a number of fuller introductions to the Lie theory are readily available. (But we do review a fair number of relevant properties at Example 5.1.1.)

In particular, an expository treatment of groups of Lie type, tailored to the general context of our work here, can be found in Section 6.8 of the book [**22**] of the first author. We also mention the expository discussion of buildings given in Chapter 2 of the book [**115**] now in preparation[4] by the second author. These two treatments also list various additional references: on groups of Lie type, we mention especially Carter [**43**] as well as Ch. 2–4 in the third volume [**64**] of the series of Gorenstein, Lyons, and Solomon; and on buildings, Brown [**36**] and Ronan [**105**].

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[4]As of this writing (June 2007), revised Chapters 1–3 are visible on the Web at the URL http://www.math.uic.edu/∼smiths/book/book.ps; further revisions will continue at least through later 2007.