Introduction

Braid theory is a beautiful subject which combines the visual appeal and insights of topology with the precision and power of algebra. It is relevant not only to algebraists and topologists, but also to scientists working in many disciplines. It even touches upon such diverse fields as polymer chemistry, molecular biology, cryptography and robotics.

The theory of braids has been an exceptionally active mathematical subject in recent decades. The field really caught fire in the mid 1980's with the revolutionary discoveries of Vaughan Jones [**115**], providing strong connections with operator theory, statistical mechanics and utilizing many ideas which originated from mathematical physics.

That braids have a natural ordering, compatible with their algebraic structure, was discovered a decade later by one of the authors (P.D.), and since then it has been intensively studied and generalized by many mathematicians, including the authors. That phenomenon is the subject of this book.

One of the exciting aspects of this work is the rich variety of mathematical techniques that come into play. In these pages, one will find subtle combinatorics, applications of hyperbolic geometry, automata theory, laminations and triangulations, dynamics, even unprovability results, in addition to the more traditional methods of topology and algebra.

A meeting of two classical subjects

It was an idea whose time was overdue—the marriage of braid theory with the theory of orderable groups. The braid groups B_n were introduced by Emil Artin [**4**] in 1925—see also [**5**]. Indeed, many of the ideas date back to the nineteenth century in the works of Hurwicz, Klein, Poincaré, Riemann, and certainly other authors. One can even find a braid sketched in the notebooks of Gauss [**97**]—see [**177**] for a discussion about Gauss and braids, including a reproduction of the picture he drew in his notebook.

The n-strand braid group B_n has a well-known presentation—other definitions will be given later:

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1\rangle.$$

We use B_n^+ for the monoid with the above presentation, which is called the n-strand braid monoid. The monoid B_n^+ is included in a larger submonoid B_n^{+*} of B_n, called the dual braid monoid, which is associated with the presentation of B_n given by Birman, Ko and Lee in [**15**]—details may be found in Chapter VIII.

To each braid, there is an associated permutation of the set $\{1, ..., n\}$, with σ_i sent to $(i, i+1)$, defining a homomorphism of B_n onto the symmetric group $\mathfrak{S}_n$. The kernel of this mapping is the *pure* braid group PB_n.

The theory of ordered groups is also well over a hundred years old. One of the basic theorems of the subject is Hölder's theorem, published in 1902 [**111**], that characterizes the additive reals as the unique maximal Archimedean ordered group. It is remarkable, and somewhat puzzling, that it has taken so long for these two venerable subjects to come together as they now have.

A group or a monoid G is *left-orderable* if there exists a linear, *i.e.*, strict total, ordering $\prec$ of its elements which is left-invariant, *i.e.*, $g \prec g'$ implies $hg \prec hg'$ for all g, g', h in G. A group is right-orderable if and only if it is left-orderable, but the orderings are generally different; both choices appear in the literature with roughly equal frequency. If there is a left ordering of G which also invariant under multiplication on the right, we say that G is *orderable*, or for emphasis, *bi-orderable*.

The main theme of this book is proving and explaining the following.

Theorem. *The Artin braid group B_n is left-orderable, by an ordering which is a well-ordering when restricted to the braid monoid B_n^+—and even to the dual braid monoid B_n^{+*}.*

Despite the high degree of interest in braid theory, the importance of the left-orderability of the braid groups, announced in 1992 [**47**], was not widely recognized at first. A possible explanation for this is that the methods of proof were rather unfamiliar to most topologists, the people most interested in braid theory. As will be seen in Chapter IV, that proof involves rather delicate combinatorial and algebraic constructions, which were partly motivated by (while being logically independent of) questions in set theory—see [**121**] for a good introduction. Subsequent combinatorial work brought new results and proposed new approaches: David Larue established in [**132, 131**] results anticipating those of [**83**], Richard Laver proved in [**137**] that the restriction of the braid ordering to B_n^+ is a well-ordering, Serge Burckel gave an effective version of the latter result in [**26, 27**]. However, these results were also not widely known for several years.

The challenge of finding a topological proof of left-orderability of B_n led to the five-author paper [**83**] giving a completely different construction of an ordering of B_n as a mapping class group. Remarkably, it leads to exactly the same ordering as [**47**]. Soon after, a new technique [**183**] was applied to yield yet another proof of orderability of the braid groups—and many other mapping class groups—using ideas of hyperbolic geometry, and moreover giving rise to many possible orderings of the braid groups. This argument, pointed out by William Thurston, uses ideas of Nielsen [**164**] from the 1920's, and it applies to many other mapping class groups. It is interesting to speculate whether Nielsen himself might have solved the problem, if asked whether braid groups are left-orderable in the following language: Does the mapping class group of an n-punctured disk act effectively on the real line by order-preserving homeomorphisms? Nielsen had laid all the groundwork for an affirmative answer.

More recently, a new topological approach using laminations was proposed in [**73**]. In common with the Mosher normal form of [**158**], it relies on using triangulations as a sort of coordinate system. Also, a combinatorial interpretation of the results of [**183**] was proposed by Jonathon Funk in [**93**], including a connection with the theory of topoi.

Braid groups are known to be automatic [**189**]. Without burdening the reader with technical details, it should be mentioned that the ordering of B_n and certain other surface mapping class groups (nonempty boundary) can be considered automatic as well, meaning roughly that it may be determined by some finite-state automaton [**182**].

Very recently [**62, 90**], the alternating decomposition—described in Chapter VII—has greatly improved our understanding of the well-ordering of the monoid B_n^+ and allowed for its extension to the dual braid monoid B_n^{+*}.

Unlike the full braid groups, the *pure* braid groups, PB_n, can be bi-ordered [**124**], by an ordering which also well-orders pure positive braids—but proves to be much simpler than the above well-ordering of B_n^+. The argument relies on a completely different approach, namely using the Magnus representation of a free group, and the fact that PB_n is a semidirect product of free groups. Subsequent work has shown that the braid groups B_n and the pure braid groups PB_n are very different from the point of view of orderability: in particular, for $n \geqslant 5$, no left-ordering of B_n can bi-order a subgroup of finite index, such as PB_n. This was proved independently in [**180**] and [**71**].

A convergence of approaches

As will be recalled in Chapter III, the orderability of a group implies various structural consequences about that group and derived objects. The fact that B_n is left-orderable implies that it is torsion-free, which had been well known. However, it also implies that the group ring $\mathbb{Z}B_n$ has no zero-divisors, which was a natural open question. Bi-orderability of PB_n shows that $\mathbb{Z}PB_n$ embeds in a skew field. In addition, it easily implies that the group PB_n has unique roots, a result proved in [**8**] by complicated combinatorial arguments, and which is definitely not true for B_n.

One may argue that such general results did not dramatically change our understanding of braid groups. The main point of interest, however, is not—or not only—the mere existence of orderings on braid groups, but the particular nature and variety of the constructions we shall present. Witness the beautiful way the ordering of PB_n is deduced from the Magnus expansion in Chapter XV, the fascinating connection between the uncountable family of orderings on B_n constructed in Chapter XIII and the Nielsen–Thurston theory, and, chiefly, the specific properties of one particular ordering on B_n. Here we refer to the ordering of B_n sometimes called the Dehornoy ordering in literature, which will be called the σ-ordering in this text.

Typically, it is the specific form of the braids greater than 1 in the σ-ordering that led to the new, efficient algorithms for the classical braid isotopy problem described in Chapters V, VII, and VIII, and motivated the further study of the algorithms described in Chapters X and XII. But what appears to be of the greatest interest here is the remarkable convergence of many approaches to one and the same object: many different points of view end up with the σ-ordering of braids, and this, in our opinion, is the main hint that this object has an intrinsic interest. Just to let the reader feel the flavour of some of the results, we state below various characterizations of the σ-ordering—the terms will be defined in the appropriate place. So, the braid β is smaller than the braid β' in the σ-ordering if and only if

• in terms of braid words—the braid $\beta^{-1}\beta'$ has a braid word representative where the generator σ_i with smallest index appears only positively (no σ_i^{-1});

• in terms of an action on a self-distributive system—for every ordered LD-system $(S, *, \prec)$, and for every sequence $\boldsymbol{x}$ in S, we have $\boldsymbol{x} \bullet \beta \prec^{\text{Lex}} \boldsymbol{x} \bullet \beta'$;

• in terms of braid word combinatorics—any sequence of handle reductions from any braid word representing $\beta^{-1}\beta'$ ends up with a σ-positive word;

• in terms of Φ-splittings, assuming that β, β' belong to B_n^+—the Φ_n-splitting of β is `ShortLex`-smaller than that of β';

• in terms of ϕ-splittings, assuming that β, β' belong to B_n^{+*}—the ϕ_n-splitting of β is `ShortLex`-smaller than that of β';

• in terms of automorphisms of a free group—for some i, the automorphism associated with $\beta^{-1}\beta'$ maps x_j to x_j for $j < i$, and it maps x_i to a word that ends with x_i^{-1};

• in terms of free group ordering—we have $\beta(z_n) \lhd \beta'(z_n)$ in $F_\infty \setminus \{1\}$;

• in terms of mapping class groups—the standardized curve diagram associated with β' first diverges from the one associated with β towards the left;

• in terms of $\mathbb{Z}^{2n}$-coordinates—the first nonzero coefficient of odd index in the sequence $(0, 1, \ldots, 0, 1) \bullet \beta^{-1}\beta'$ is positive;

• in terms of Mosher's normal form—the last flip of the Mosher normal form of $\beta^{-1}\beta'$ occurs in the upper half-sphere;

• in terms of hyperbolic geometry—the endpoint of the lifting of $\beta(\Gamma_a)$ is smaller (as a real number) than the endpoint of the lifting of $\beta'(\Gamma_a)$.

Even if the various constructions of the σ-ordering depend on choosing a particular family of generators for the braid groups, namely the Artin generators σ_i, this convergence might suggest calling this ordering canonical or, at least, standard. This convergence is the very subject of this text: our aim here is not to give a complete study of any of the different approaches, but to try to let the reader feel the flavour of these different approaches. More precisely—and with the exceptions of Chapters XIII and XIV which deal with more general orderings, and of Chapter XV which deals with ordering pure braids—our aim will be to describe the σ-ordering of braids in the various possible frameworks: algebraic, combinatorial, topological, geometric, and to see which properties can be established by each technique.

As explained in Chapter II, exactly three properties of braids, called **A**, **C**, and **S** here, are crucial to prove that the σ-ordering exists and to establish its main properties. Roughly speaking, each chapter of the subsequent text—except Chapters XIII, XIV, and XV—will describe one possible approach to the question of ordering the braids, and, in each case, explain which of the properties **A**, **C**, and **S** can be proved: some approaches are relevant for establishing all three properties, while others enable us only to prove one or two of them, possibly assuming some other one already proved. We emphasize that, although these properties are established in various contexts and by very different means, there is certainly no circular reasoning involved.

However, the point of this book is not merely to prove and reprove the existence of the braid ordering. Each of the chapters gives a different viewpoint which adds new colours to our description and provides further results.

Organization of the text

Various equivalent definitions of the braid groups are described in Chapter I. The σ-ordering of braids is introduced in Chapter II, where its general properties are discussed. A number of curious examples are presented, showing that the σ-ordering has some quite unexpected properties. The well-ordering of B_n^+ is also introduced in this chapter.

Chapter III presents various applications of the braid ordering. This includes purely algebraic consequences of orderability, such as the zero-divisor conjecture, but also more specific applications following from the specific properties of the σ-ordering, such as a faithfulness criterion of representations and efficient solutions to the word problem. We point out that the braid groups provide interesting examples and counterexamples in the theory of ordered groups. In addition, we outline some applications to knot theory, the theory of pseudo-characters, and certain unprovability results arising in braid theory.

The chapters which follow contain various approaches to the orderability phenomenon. The combinatorial approaches are gathered in Chapters IV through IX, while the topological approaches are presented in Chapters X to XIV.

Chapter IV introduces left self-distributive algebraic systems (LD-systems) and the action of braids upon such systems. This is the technique whereby the orderability of braids was first demonstrated and the σ-ordering introduced. The chapter sketches a self-contained proof of left-orderability of B_n by establishing Properties **A**, **C** and **S** with arguments utilizing LD-systems. Here we consider colourings of the strands of the braids and observe that the braid relations dictate the self-distributive law among the colours. Then we can order braids by choosing orderable LD-systems as colours, a simple idea, although the existence of an orderable LD-system requires a sophisticated argument. The chapter concludes with a short discussion of the historical origins of orderable LD-systems in set theory.

A combinatorial algorithm called handle reduction is the subject of Chapter V. This procedure, which extends the idea of word reduction in a free group, is a very efficient procedure in practice for determining whether a braid word represents a braid larger than 1, and incidentally gives a rapid solution to the word problem in the braid groups. Handle reduction gives an alternative proof of Property **C**, under the assumption that Property **A** holds.

The deep structure of the braid groups discovered by Garside [**95**], and its connection with the σ-ordering, are discussed in Chapter VI. The relationship is not a simple one, but investigating the ordering of the divisors of Δ_n^d in B_n^+ leads to new insights into Solomon's descent algebras. A complete description is obtained in the case of 3-strand braids, leading to a new proof of Property **C** in that case.

Some quite recent developments in braid orderings are contained in Chapters VII and VIII, which describe normal forms for braids which are more compatible with the braid ordering than the greedy form associated with Garside theory. Chapter VII begins with an inductive scheme called the Φ_n-splitting, which yields a decomposition of every positive n-strand braid into a finite sequence of positive $(n-1)$-strand braids. The main result is that, under this decomposition, the ordering of B_n^+ is a simple lexicographic extension of the ordering of B_{n-1}^+. Most results of the chapter rely on techniques due to S. Burckel in [**27**] for encoding positive braid words by finite trees, but they are used here as a sort of black box, and their proofs are omitted.

The dual braid monoids introduced by Birman, Ko and Lee are at the heart of Chapter VIII. The monoid B_n^{+*}, which properly includes the monoid B_n^+ for $n \geqslant 3$, also has a good divisibility structure. As in Chapter VII, one introduces the notion of ϕ_n-splitting, which yields a decomposition of every braid of B_n^{+*} into a finite sequence of braids of B_{n-1}^{+*}. The main result is again that, under this decomposition, the ordering of B_n^{+*} is a lexicographic extension of the ordering of B_{n-1}^{+*}, a recent result of J. Fromentin. The consequences are similar to those of Chapter VII, including a new proof of Property **C**, and a new quadratic algorithm for comparing braids. The main interest of the approach may be that it allows for direct, elementary proofs, by contrast to Burckel's techniques, which use tricky transfinite induction arguments.

Chapter IX contains an approach to the σ-ordering using a very classical fact, that the braid groups can be realized as a certain group of automorphisms of a free group. As observed by David Larue, this method yields a quick proof of Property **A**, a partial (and not so quick) proof of Property **C**, as well as a simple criterion for recognizing whether a braid is σ-positive in terms of its action on the free group. We also outline in this chapter the interpretation developed by Jonathon Funk, in which a certain linear ordering of words in the free group is preserved under the braid automorphisms.

We begin the topological description of the σ-ordering in Chapter X. Here we realize B_n as the mapping class group of a disk with n punctures. The braid action can be visualized by use of curve diagrams which provide a canonical form for the image of the real line, if the disk is regarded as the unit complex disk. This was the first geometric argument for the left-orderability of the braid groups, and it is remarkable that the ordering described in this way is identical with the original, *i.e.*, with the σ-ordering. An advantage of this approach is that it also applies to more general mapping class groups. We emphasize that Chapters IX and X are based on very similar ideas, except that in the first one these ideas are expressed in a more algebraic language, and in the second in a more geometric one.

In Chapter XI we study in detail a topological technique which we already encountered in Chapter X for explicitly constructing σ-consistent representative braid words of any given element of the braid group, namely the method of untangling, or relaxing, curve diagrams. We discuss two examples of such algorithms: one due to Bressaud in [**19**], which has a fascinating alternative description in terms of finite state automata and word rewriting systems, and another one from [**74**] which leads to a deeper understanding of the connection between the length of a braid word and the complexity of the curve diagram of a braid. This approach can also be interpreted in terms of Teichmüller geometry [**178**].

Chapter XII continues the discussion of the σ-ordering in terms of mapping classes. However, here the geometric approach is rephrased in combinatorial terms by use of two somewhat different devices involving triangulations. The first approach, developed in [**73**], uses integral laminations. One encodes the action of a braid on the disk by counting intersections of the image of a certain triangulation with a lamination. This leads to the shortest proof of Property **A** known so far, and yet another characterization of braids larger than 1 in the σ-ordering. The second was inspired by the technique employed by Lee Mosher to establish that mapping class groups are automatic. It develops a new canonical form for braids and a method for determining σ-ordering by means of a finite state automaton.

The discussion in Chapter XIII interprets braid orderings in terms of Nielsen–Thurston theory. The key observation is that the universal cover of the punctured disk has a natural embedding in the hyperbolic plane. Thereby, braids act on a family of hyperbolic geodesics, which have a natural ordering. This point of view provides an infinitude of inequivalent orderings of braid groups and many other mapping class groups. The σ-ordering on B_n corresponds to choosing a particular geodesic in $\mathbb{H}^2$.

A recent, and quite different, topological approach to orderability is taken in Chapter XIV. Here, one considers the set of *all* left orderings of a group G, and of B_n in particular. This set is given a natural topology, forming the space $LO(G)$, which in general is compact and totally disconnected. We study the structure of $LO(B_n)$ and use this global topological approach to show that there are uncountably many essentially different left-orderings of B_n for $n \geqslant 3$, all of which provide well-orderings of B_n^+, a phenomenon noted in Chapter XIII by completely different methods.

Chapter XV is an account of an ordering of the *pure* braid groups. Unlike the full braid groups, the groups PB_n of pure braids can be given an ordering which is invariant under multiplication on both sides. The one we investigate here is defined algebraically, using the Artin combing technique, together with a specific ordering of free groups using the Magnus expansion. This ordering—which is *not* the restriction of the σ-ordering to pure braids—has the nice property that non-trivial braids in $PB_n \cap B_n^+$ are larger than 1 and well-ordered.

The final chapter contains a number of open questions related to braid orderings. Various extensions of the ideas presented in the other chapters are also discussed there.

Guidelines to the reader

An attempt has been made to keep the chapters relatively self-contained. So, apart from Chapters I, II, and XVI, all chapters are parallel one to the other rather than logically interdependent; therefore, after the first chapters, the reader can take the chapters essentially in whatever order he or she likes.

We mentioned that three properties of braids play a crucial role, namely those called **A**, **C**, and **S**, whose statements, as well as other basic definitions, are recalled at the end of the book, page 309. One of our main tasks in this text will be to prove these properties using various possible approaches. In spite of the above general remarks, it might be useful that we propose answers to the question, Which of these approaches offers the quickest, or the most elementary, proof of Properties **A**, **C** and **S**? The answer depends of course on the mathematical preferences of the reader. As for Property **A**, the shortest proofs are the one using the automorphisms of a free group in Chapter IX, and—even shorter once the curious formulas (XII.1.1) have been guessed—the one using laminations in Chapter XII.

As for Property **C**, the shortest argument is probably the one involving self-distributivity as outlined in Chapter IV, but one may prefer the approach through the handle reduction method of Chapter V, which uses nothing exotic and gives an efficient algorithm in addition, or the curve diagram approach of Chapter X, which gives a less efficient method and requires considerable effort to be made rigorous, but appeals to a natural geometric intuition.

Finally, for Property **S**, the hyperbolic geometry argument of Chapter XIII is probably the most general one, as it gives the result not only for the σ-ordering, but also for a whole family of different orderings. On the other hand, even if they may appear intricate, the combinatorial approaches of Chapters VII and VIII give the most precise and effective versions.

Although they are conceptually simple, the braid groups are very subtle non-Abelian groups which have given up their secrets only reluctantly over the years. They will undoubtedly continue to supply us with surprises and fascination, and so will in particular their orderings: despite the many approaches and results mentioned in this book, a lot of questions about braid orderings remain open today, and further developments can be expected. For the moment, we hope that this text, which involves techniques of algebra, combinatorics, hyperbolic geometry, topology, and has even a loose connection with set theory, can illuminate some facets of the ordering of braids.

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