CHAPTER II

A Linear Ordering of Braids

In this chapter, we introduce the linear ordering of braids, sometimes called the *Dehornoy ordering*, that is the main subject of this book, and we list its main properties known so far. The construction starts with the notion of a σ-positive braid, and it relies on three basic properties, called **A**, **C**, and **S**, from which the σ-ordering can easily be constructed and investigated. In this chapter, we take Properties **A**, **C**, and **S** for granted, and explore their consequences. The many different proofs of these statements will be found in subsequent chapters.

The chapter is organized as follows. In Section 1, we introduce the σ-ordering and its variant the σ^{Φ}-ordering starting from Properties **A** and **C**. In Section 2, we give many examples of the sometimes surprising behaviour of the σ-ordering, and we introduce Property **S**. In Section 3, we develop global properties of the σ-ordering, involving Archimedian property, discreteness, density, and convex subgroups. Finally, in Section 4, we investigate the restriction of the σ-ordering to the monoid B_n^+ of positive braids, showing that this restriction is a well-ordering, and we give an inductive construction of the σ-ordering of B_n^+ from the σ-ordering of B_{n-1}^+.

Convention. In this chapter and everywhere in this book, when we speak of positive braids, we always mean those braids that lie in the monoid B_∞^+, *i.e.*, those braids that admit at least one expression by a word containing no letter σ_i^{-1}. Such braids are sometimes called Garside positive braids, but we shall not use that name here. So the word "positive" never refers to any of the specific linear orderings we shall investigate hereafter. For the latter case, we shall introduce specific names for the braids that are larger than 1, typically σ-positive and σ^{Φ}-positive in the case of the σ-ordering and of the σ^{Φ}-ordering.

1. The σ-ordering of B_n

In this section we give a first definition of the σ-ordering of braids, based on the notion of a σ-positive braid word—many alternative definitions will be given in subsequent chapters. We explain how to construct the σ-ordering from two specific properties of braids called **A** and **C**. We also introduce a useful variant of the σ-ordering, called the σ^{Φ}-ordering, which is its image under the flip automorphism. Finally, we briefly discuss the algorithmic issues involving the σ–ordering.

1.1. Ordering a group. We start with premilinary remarks about what can be expected here. First, we recall that a *strict ordering* of a set Ω is a binary relation $\prec$ that is antireflexive ($x \prec x$ never holds) and transitive (the conjunction of $x \prec y$ and $y \prec z$ implies $x \prec z$). A strict ordering of Ω is called *linear* (or *total*) if, for all x, x' in Ω, one of $x = x'$, $x \prec x'$, $x' \prec x$ holds. Then, we recall the notion of an orderable group.

Definition 1.1. (i) A *left-invariant ordering*, or *left-ordering*, of a group G is a strict linear ordering $\prec$ of G such that $g \prec h$ implies $fg \prec fh$ for all f, g, h in G. A group G is said to be *left-orderable* if there exists at least one left-invariant ordering of G.

(ii) A *bi-invariant ordering*, or *bi-ordering*, of a group G is a left-ordering of G that is also right-invariant, *i.e.*, $g \prec h$ implies $gf \prec hf$ for all f, g, h in G. A group G is said to be *bi-orderable* if there exists at least one bi-invariant ordering of G.

Proposition 1.2. *For $n \geqslant 3$, the group B_n is not bi-orderable.*

Proof. If $\prec$ is a bi-invariant ordering of a group G, then $g \prec h$ implies $\varphi(g) \prec \varphi(h)$ for each inner automorphism φ of G. Now, in the case of B_n, the inner automorphism Φ_n associated with Garside's fundamental braid Δ_n of (I.4.1) exchanges σ_i and σ_{n-i} for each i. Hence it is impossible to have $\sigma_1 \prec \sigma_{n-1}$ and $\Phi_n(\sigma_1) \prec \Phi_n(\sigma_{n-1})$ simultaneously. □

Therefore, in the best case, we shall be interested in orders that are invariant under multiplication on one side. Then, both sides play symmetric roles, as an immediate verification gives

Lemma 1.3. *Assume that G is a group and $\prec$ is a left-invariant ordering of G. Define $g \mathrel{\tilde{\prec}} h$ to mean $g^{-1} \prec h^{-1}$. Then $\tilde{\prec}$ is a right-invariant ordering of G.*

We shall concentrate hereafter on left-invariant orderings. Specifying such an ordering is actually equivalent to specifying a subsemigroup of a certain type, called a positive cone.

Definition 1.4. A subset P of a group G is called a *positive cone* on G if P is closed under multiplication and $G \setminus \{1\}$ is the disjoint union of P and P^{-1}.

Lemma 1.5. (i) *Assume that $\prec$ is a left-invariant ordering of a group G. Then the set P of all elements in G that are larger than 1 is a positive cone on G, and $g \prec h$ is equivalent to $g^{-1}h \in P$.*

(ii) *Assume that P is a positive cone on a group G. Then the relation $g^{-1}h \in P$ is a left-invariant ordering of G, and P is then the set of all elements of G that are larger than 1.*

The verification is easy. Note that the formula $hg^{-1} \in P$ would define a right-invariant ordering.

1.2. The σ-ordering of braids. We now introduce on B_n a certain binary relation that will turn out to be a left-invariant ordering. The construction involves particular braid words defined in terms of the letters they contain.

Definition 1.6. A braid word w is said to be *σ-positive* (*resp. σ-negative*) if, among the letters $\sigma_i^{\pm 1}$ that occur in w, the one with lowest index occurs positively only, *i.e.*, σ_i occurs but σ_i^{-1} does not (*resp.* negatively only, *i.e.*, σ_i^{-1} occurs but σ_i does not).

For instance, $\sigma_3\sigma_2\sigma_3^{-1}$ is a σ-positive braid word: the letter with lowest index is σ_2 (there is no $\sigma_1^{\pm 1}$), and there is one σ_2 but no σ_2^{-1}. By contrast, the word $\sigma_2^{-1}\sigma_3\sigma_2$, which is equivalent to $\sigma_3\sigma_2\sigma_3^{-1}$, is neither σ-positive nor σ-negative: the letter with lowest index is σ_2 again, but, here, both σ_2 and σ_2^{-1} appear.

DEFINITION 1.7. For β, β' in B_n, we say that $\beta <_n \beta'$ is true if $\beta^{-1}\beta'$ admits an n-strand representative word that is σ-positive.

EXAMPLE 1.8. Let $\beta = \sigma_2$ and $\beta' = \sigma_3\sigma_2$. Among the 4-strand braid words that represent the quotient $(\sigma_2)^{-1}(\sigma_3\sigma_2)$, there is the word $\sigma_2^{-1}\sigma_3\sigma_2$, which is neither σ-positive nor σ-negative, but there is also the word $\sigma_3\sigma_2\sigma_3^{-1}$—and many others. As the latter word is a 4-strand braid word that is σ-positive, $\beta <_4 \beta'$ is true.

Similarly, we have

$$\sigma_1 >_\infty \sigma_2 >_\infty \sigma_3 >_\infty \cdots \tag{1.1}$$

since, for each i, the braid word $\sigma_{i+1}^{-1}\sigma_i$ is σ-positive.

The central property is the following result of [**47**] (see Remark 1.16) which implies the first part of the theorem mentioned in the Introduction:

PROPOSITION 1.9. (i) *For $2 \leqslant n \leqslant \infty$, the relation $<_n$ is a left-invariant ordering of B_n.*

(ii) *For each n, the relation $<_n$ is the restriction of $<_\infty$ to B_n.*

Owing to (ii) above, we shall drop the subscripts and simply write $<$ for $<_n$. The order $<$ will be called the *σ-ordering* of braids, which is coherent with its definition in terms of the generators σ_i.

By definition, the relation $\beta >_n 1$ is true if and only if β admits at least one σ-positive n-strand representative word. According to Lemma 1.5, proving Proposition 1.9(i) amounts to proving that the set of all such braids is a positive cone. The latter result is a consequence of the following two statements:

Property A (Acyclicity). *A σ-positive braid word is not trivial.*

Property C (Comparison). *Every nontrivial braid of B_n admits an n-strand representative word that is σ-positive or σ-negative.*

PROOF OF PROPOSITION 1.9 FROM PROPERTIES **A** AND **C**. (i) Let P_n be the set of all n-strand braids that admit a σ-positive n-strand representative word. We shall prove that P_n is a positive cone in B_n. First, the concatenation of two σ-positive n-strand braid words is a σ-positive n-strand braid word; hence P_n is closed under multiplication.

Then, we claim that $B_n \setminus \{1\}$ is the disjoint union of P_n and P_n^{-1}. Indeed, Property **A** implies $1 \notin P_n$, and therefore $1 \notin P_n^{-1}$ as $1^{-1} = 1$ holds. So $P_n \cup P_n^{-1}$ is included in $B_n \setminus \{1\}$. Now assume $\beta \in P_n \cap P_n^{-1}$. We deduce $\beta^{-1} \in P_n$, whence

$$1 = \beta\beta^{-1} \in P_n \cdot P_n \subseteq P_n,$$

which contradicts $1 \notin P_n$. So P_n and P_n^{-1} must be disjoint. Finally, Property **C** (for B_n) means that $P_n \cup P_n^{-1}$ covers $B_n \setminus \{1\}$.

(ii) Assume $\beta, \beta' \in B_n$. Any σ-positive n-strand braid word representing $\beta^{-1}\beta'$ *a fortiori* witnesses the relation $\beta <_\infty \beta'$, so $\beta <_n \beta'$ implies $\beta <_\infty \beta'$. Conversely, assume $\beta <_\infty \beta'$. As $<_n$ is a linear ordering of B_n, one of $\beta <_n \beta'$ or $\beta \geqslant_n \beta'$ holds. In the latter case, we would deduce $\beta \geqslant_\infty \beta'$, which contradicts the hypothesis $\beta <_\infty \beta'$. So $\beta <_n \beta'$ is the only possibility. □

Property **A** has four different proofs in this text: they can be found on pages 73, 175, 190, and 224. As for Property **C**, no fewer than eight proofs are given, on pages 60, 89, 116, 148, 164, 190, 201, and 205.

In addition to being invariant under left multiplication, the σ-ordering of braids is invariant under the shift endomorphism, defined as follows.

Definition 1.10. For w a braid word, the *shifting* of w is the braid word $\mathrm{sh}(w)$ obtained by replacing each letter σ_i with σ_{i+1}, and each letter σ_i^{-1} with σ_{i+1}^{-1}.

The explicit form of the braid relations implies that the shift mapping induces an endomorphism of B_∞, still denoted sh and called the *shift endomorphism*. The same argument guaranteeing that the canonical morphism of B_{n-1} into B_n is an embedding shows that the shift endomorphism of B_∞ is injective.

Proposition 1.11. *For all braids β, β', the relation $\beta < \beta'$ is equivalent to $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$.*

Proof. The shifting of a σ-positive braid word is a σ-positive braid word, so $\beta < \beta'$ implies $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$. Conversely, as $<$ is a linear ordering, the only possibility when $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$ is true is that $\beta < \beta'$ is true as well, as $\beta \geqslant \beta'$ would imply $\mathrm{sh}(\beta) \geqslant \mathrm{sh}(\beta')$. □

It is straightforward to check that, conversely, the σ-ordering is the only partial ordering on B_∞ that is invariant under multiplication on the left and under the shift endomorphism, and satisfies for all braids β, β' the inequality

$$1 < \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\beta').$$

1.3. Equivalent formulations. Before proceeding, we introduce derived notions in order to restate Properties **A** and **C** in slightly different forms. First, we can refine the notion of a σ-positive braid word by taking into account the specific index i that is involved.

Definition 1.12. A braid word is said to be *σ_i-positive* if it contains at least one letter σ_i, but no σ_i^{-1} and no $\sigma_j^{\pm 1}$ with $j < i$. Similarly, it is said to be *σ_i-negative* if it contains at least one σ_i^{-1}, but no σ_i and no $\sigma_j^{\pm 1}$ with $j < i$. It is said to be *σ_i-free* if it contains no $\sigma_j^{\pm 1}$ with $j \leqslant i$.

So a braid word is σ-positive if and only if it is σ_i-positive for some i. Note that, for $i \geqslant 2$, a word w is σ_i-positive if and only if it is $\mathrm{sh}^{i-1}(w_1)$ for some σ_1-positive word w_1—we recall that sh is the shift mapping of Definition 1.10. Similarly, a braid word w is σ_i-free if and only if it is $\mathrm{sh}^i(w_1)$ for some w_1.

Then Properties **A** and **C** can be expressed in terms of σ_1-positive, σ_1-negative, and σ_1-free words.

Proposition 1.13. *Property* **A** *is equivalent to:*
Property A (second form). *A σ_1-positive braid word is not trivial.*

Proof. Every σ_1-positive braid word is σ-positive, so the first form of Property **A** implies the second form.

Conversely, assume the second form of Property **A**. Let w be a σ-positive word. Then w is σ_i-positive for some i. As observed above, this means that we have $w = \mathrm{sh}^{i-1}(w_1)$ for some σ_1-positive word w_1. By the second form of Property **A**, the word w_1 is not trivial, *i.e.*, it does not represent the unit braid. As the shift endomorphism of B_∞ is injective, this implies that w is not trivial either. So, the first form of Property **A** is satisfied. □

PROPOSITION 1.14. *Property* **C** *is equivalent to:*
Property C (second form). *Every braid of B_n admits an n-strand representative word that is σ_1-positive, σ_1-negative, or σ_1-free.*

PROOF. A σ-positive braid word is either σ_1-positive or σ_1-free, so the first form of Property **C** implies the second form.

Conversely, assume the second form of Property **C**. We prove the first form using induction on $n \geqslant 2$. For $n = 2$, the two forms coincide. Assume $n \geqslant 3$. Let β be a nontrivial n-strand braid. By the second form of Property **C**, we find an n-strand braid word w representing β that is σ_1-positive, σ_1-negative, or σ_1-free. In the first two cases, we are done. Otherwise, let $w_1 = \mathrm{sh}^{-1}(w)$, which makes sense as, by hypothesis, w contains no letter $\sigma_1^{\pm 1}$. As the shift endomorphism of B_∞ is injective, the word w_1 does not represent 1, so the induction hypothesis implies that w_1 is equivalent to some $(n-1)$-strand braid word w_1' that is σ-positive or σ-negative. By construction, the word $\mathrm{sh}(w_1')$ represents β and it is σ-positive or σ-negative. □

On the other hand, it will be often convenient in the sequel to have a name for the braids that admit a σ-positive word representative. So, we introduce the following natural terminology.

DEFINITION 1.15. A braid β is said to be *σ-positive inside B_n* (*resp. σ-negative, σ_i-positive, σ_i-negative, σ_i-free*) if, among all word representatives of β, there is at least one n-strand braid word that is σ-positive (*resp.* σ-negative, σ_i-positive, σ_i-negative, σ_i-free).

We insist that, in Definition 1.15, we only demand that there exists *at least one* word representative with the considered property. So, for instance, the braid $\sigma_2^{-1}\sigma_3\sigma_2$ is σ_2-positive since, among its many word representatives, there is one, namely $\sigma_3\sigma_2\sigma_3^{-1}$, that is σ_2-positive—there are many more: $\sigma_3\sigma_2\sigma_3^{-1}\sigma_3\sigma_3^{-1}$ is another σ_2-positive 4-strand braid word that represents the braid $\sigma_2^{-1}\sigma_3\sigma_2$.

With this terminology, $\beta <_n \beta'$ is equivalent to $\beta^{-1}\beta'$ being σ-positive inside B_n. Similarly, Property **A** means that a σ-positive braid is not trivial, and Property **C** that every nontrivial braid of B_n is σ-positive or σ-negative inside B_n.

REMARK 1.16. By Proposition 1.9(ii), a braid β of B_n satisfies $\beta >_n 1$ if and only if it satisfies $\beta >_\infty 1$, hence β is σ-positive inside B_n if and only if it is σ-positive inside B_∞. In other words, if an n-strand braid admits a word representative that is σ-positive, then it admits a word representative that is σ-positive and is an n-strand braid word, an *a priori* stronger property. Building on this result, we shall often drop the mention "inside B_n", exactly as when we write $<$ for $<_n$. However, a careful distinction has to be made when proving Property **C**. It can be mentioned that the original argument of [**47**] only leads to a proof of Property **C** in B_∞: this is enough to order every braid group B_n, but not to deduce Property **C** in B_n; see Chapter IV.

1.4. The σ^Φ-ordering of braids. If $\prec$ is an ordering of a group G and φ is an automorphism of G, then the relation $\varphi(g) \prec \varphi(h)$ defines a new ordering of G with the same invariance properties as $\prec$. In the case of B_n, the flip automorphism, *i.e.*, the inner automorphism Φ_n associated with the braid Δ_n, plays an important role, and it is natural to introduce the image of the σ-ordering under Φ_n, *i.e.*, the flipped version of the σ-ordering. As will be seen in Section 4, the new ordering so obtained has some nice properties not shared by the original version, particularly in terms of avoiding the infinite descending sequence of (1.1).

We recall from Lemma I.4.4 that Φ_n exchanges σ_i and σ_{n-i} for $1 \leqslant i < n$, thus corresponding to a symmetry in the associated braid diagrams.

Definition 1.17. For $2 \leqslant n < \infty$ and β, β' in B_n, we declare that $\beta <^{\Phi}_n \beta'$ is true if we have $\Phi_n(\beta) < \Phi_n(\beta')$.

Proposition 1.18. *The relation $<^{\Phi}_n$ is a left-invariant ordering of B_n. Moreover, for all β, β' in B_n, the relations $\beta <^{\Phi}_n \beta'$ and $\beta <^{\Phi}_{n+1} \beta'$ are equivalent.*

Proof. The first part is clear as Φ_n is an automorphism of B_n.

Assume $\beta, \beta \in B_n$ and $\beta <^{\Phi}_n \beta'$. By definition, we have $\Phi_n(\beta) < \Phi_n(\beta')$, hence $\mathrm{sh}(\Phi_n(\beta)) < \mathrm{sh}(\Phi_n(\beta'))$ by Proposition 1.11. By construction, we have

$$\Phi_{n+1}(\beta) = \mathrm{sh}(\Phi_n(\beta)) \quad \text{and} \quad \Phi_{n+1}(\beta') = \mathrm{sh}(\Phi_n(\beta')),$$

so $\beta <^{\Phi}_{n+1} \beta'$ follows. As $<^{\Phi}_n$ is a linear ordering, this is enough to conclude that $<^{\Phi}_n$ coincides with the restriction of $<^{\Phi}_{n+1}$ to B_n. ☐

Owing to Proposition 1.18, we shall drop the subscripts and simply write $<^{\Phi}$ for the ordering of B_∞ whose restriction to B_n is $<^{\Phi}_n$. For instance, we have

$$1 <^{\Phi} \sigma_1 <^{\Phi} \sigma_2 <^{\Phi} \dots .$$

The flipped order $<^{\Phi}$ is easily described in terms of word representatives.

Definition 1.19. (i) A braid word w is said to be *σ^{Φ}-positive* (*resp.* *σ^{Φ}-negative*) if, among the letters $\sigma_i^{\pm 1}$ that occur in w, the one with *highest* index occurs positively only (*resp.* negatively only).

(ii) A braid β is said to be *σ^{Φ}-positive* (*resp.* *σ^{Φ}-negative*) if it admits at least one braid word representative that is σ^{Φ}-positive (*resp.* σ^{Φ}-negative).

The only difference between a σ-positive and a σ^{Φ}-positive braid word is that, in the former case, we consider the letter σ_i with lowest index, while, in the latter case, we consider the letter σ_i with highest index.

Proposition 1.20. *For all braids β, β', the relation $\beta <^{\Phi} \beta'$ holds if and only if $\beta^{-1}\beta'$ is σ^{Φ}-positive.*

Proof. By construction, an n-strand braid word w is σ^{Φ}-positive if and only if the n-strand braid word $\Phi_n(w)$ is σ-positive. ☐

Thus the flipped order $<^{\Phi}$ is the counterpart of the σ-order $<$ in which the highest index replaces the lowest index, and σ^{Φ}-positive words replace σ-positive words. It is therefore natural to call it the *σ^{Φ}-ordering* of braids.

As the flip Φ_n is an automorphism of the group B_n, the properties of $<$ and $<^{\Phi}$ are similar. However, there are at least two reasons for considering both $<$ and $<^{\Phi}$. First, there is no flip on B_∞, and the two orderings differ radically on B_∞: (1.1) shows that $(B^+_\infty, <)$ has infinite descending sequences, while we shall see in Section 4.1 below that $(B^+_\infty, <^{\Phi})$ is a well-ordering, and, therefore, it has no infinite descending chain. The second reason is that, in subsequent chapters, certain approaches demand that one specific version be used: the original version $<$ in Chapter IV, the flipped version $<^{\Phi}$ in Chapters VII and VIII.

1.5. Algorithmic aspects. The σ-ordering of braids is a complicated object. However, it is completely effective in that there exist efficient comparison algorithms. In this section (and everywhere in the sequel) we denote by $\overline{w}$ the braid represented by a braid word w—but, as usual, we use σ_i both for the letter and for the braid it represents.

PROPOSITION 1.21. *For each n, the σ-ordering of B_n has at most a quadratic complexity: there exists an algorithm that, starting with two n-strand braid words w, w' of length ℓ, runs in time $O(\ell^2)$ and decides whether $\overline{w} < \overline{w'}$ holds.*

At this early stage, we cannot yet describe the algorithms witnessing to the above upper complexity bound. It turns out that most of the proofs of Property **C** alluded to in Section 1.2 provide an effective comparison algorithm. Some of them are quite inefficient—typically the one of Chapter IV—but several lead to a quadratic complexity. This is particularly the case with those based on the Φ-normal form of Chapter VII and on the ϕ-normal form of Chapter VIII: in both cases, the normal form can be computed in quadratic time, and, then, the comparison itself can be made in linear time. This is also the case with the lamination method of Chapter XII: in this case, the coordinates of a braid can be computed in quadratic time, and the comparison (with the unit braid) can then be made in (sub)linear time. Similar results are conjectured in the case of the handle reduction method of Chapter V and the Tetris algorithm of Chapter XI—see Chapter XVI for further discussion.

Let us mention that, for a convenient definition for the RAM complexity of the input braids, the algorithm of Chapter XII even leads to a complexity upper bound which is quadratic independently of the braid index n, *i.e.*, there exists an absolute constant C so that the running time for complexity ℓ input braids in B_∞ is bounded above by $C \cdot \ell^2$.

We also point out that every comparison algorithm for the σ-ordering of braids automatically gives a solution to the braid word problem, *i.e.*, to the braid isotopy problem: indeed, we have $\overline{w} = \overline{w'}$ if and only if we have neither $\overline{w} < \overline{w'}$ nor $\overline{w} > \overline{w'}$. It also leads to a comparison for the flipped version $<^\Phi$ of the σ-ordering, as, if w, w' are n-strand braid words, $\overline{w} <^\Phi \overline{w'}$ is equivalent to $\overline{\Phi_n(w)} < \overline{\Phi_n(w')}$, and the flip automorphism Φ_n can be computed in linear time.

Another related question is that of effectively finding σ-positive representative words, *i.e.*, starting with a braid word w, finding an equivalent braid word w' that is σ-positive, σ-negative, or empty. Property **C** asserts that this is always possible. Every algorithmic solution to that problem gives a comparison algorithm as, by Property **A**, w' being σ-positive implies $\overline{w} = \overline{w'} > 1$, but, conversely, deciding $\overline{w} > 1$ does not require that we exhibit a σ-positive witness.

PROPOSITION 1.22. *The σ-positive representative problem has at most an exponential complexity: there exist a polynomial $P(n, \ell)$ and an algorithm that, starting with an n-strand braid word w of length ℓ, runs in time $2^{P(n,\ell)}$ and returns a braid word of length bounded by $2^{P(n,\ell)}$ that is equivalent to w and is σ-positive, σ-negative, or empty.*

The handle reduction approach of Chapter V gives the precise form of such a polynomial: $P(n, \ell) = n^4\ell$. From the transmission-relaxation approach of Chapter XI, an asymptotically better estimate can be extracted: $P(n, \ell) = \mathrm{const} \cdot n\ell$. However, the algorithm outlined in Chapter XI is just polynomial, but the output of the algorithm is not a braid word in the standard sense but a zipped word, this meaning that, sometimes, instead of writing one and the same subword many times, the algorithm outputs the subword once and specifies the number of repetitions. This allows us to make the size of the output bounded above by a polynomial in n and ℓ though the length of the word after unzipping is not known to be of polynomial size so far.

It is likely that the approach of Chapter VIII leads to much better results: very recently, J. Fromentin announced a new algorithm that solves the σ-positive representative problem with a quadratic time complexity and a linear space complexity, without zipping the output. We refer to Chapter XVI for further discussion.

2. Local properties of the σ-ordering

We shall now list—with or without proof—some properties of the σ-ordering of braids. In this section, we consider properties that can be called local in that they involve finitely many braids at a time.

2.1. Curious examples. We start with a series of examples, including some rather surprising ones, that illustrate the complexity of the σ-ordering. The reader should note that all examples below live in B_3. This shows that, despite its simple definition, even the σ-ordering of 3-strand braids is a quite complicated object.

The first example shows that the σ-ordering is not invariant under multiplication on the right, as was already known from Proposition 1.2.

Example 2.1. Let $\beta = \sigma_1\sigma_2^{-1}$, and $\gamma = \sigma_1\sigma_2\sigma_1$, *i.e.*, $\gamma = \Delta_3$. The word $\sigma_1\sigma_2^{-1}$ contains one occurrence of σ_1 and no occurrence of σ_1^{-1}, so the braid β is σ-positive, and $\beta > 1$ is true. On the other hand, the braid $\gamma^{-1}\beta\gamma$ is represented by the word $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1$, hence also by the equivalent word $\sigma_2\sigma_1^{-1}$, as, by Lemma I.4.4, we have $\Delta_3^{-1}\sigma_i\Delta_3 = \sigma_{3-i}$ for $i = 1, 2$. The word $\sigma_2\sigma_1^{-1}$ contains one letter σ_1^{-1} and no letter σ_1. So, by definition, we have $\gamma^{-1}\beta\gamma < 1$, and, therefore, $\beta\gamma < \gamma$. So $1 < \beta$ does not imply $\gamma < \beta\gamma$.

A phenomenon connected with the noninvariance under right multiplication is that a conjugate of a braid that is larger than 1 may be smaller than 1. Example 2.1 actually gives us an illustration of this situation: in fact, in this case, the conjugate is the inverse.

Example 2.2. Let $\beta = \sigma_1\sigma_2^{-1}$ again. Then β is σ_1-positive, hence larger than 1. By Lemma I.4.4, conjugating by Δ_3 amounts to exchanging σ_1 and σ_2. So we have $\Delta_3\beta\Delta_3^{-1} = \sigma_2\sigma_1^{-1}$, a σ_1-negative braid, hence smaller than 1, *i.e.*, we have $\beta > 1$ and $\Delta_3\beta\Delta_3^{-1} < 1$; however, we shall see in Corollary 3.7 below that the conjugates of a braid β cannot be too far from β.

An easy exercise is that every left-invariant ordering such that $g \prec h$ implies $g^{-1} \succ h^{-1}$ is also right-invariant. As the braid ordering is not right-invariant, there must exist counterexamples, *i.e.*, braids β, γ satisfying $\beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$. Here are examples of this situation.

Example 2.3. Let $\beta = \Delta_3$ and $\gamma = \sigma_2^2\sigma_1$. Then we find $\beta^{-1}\gamma = \sigma_1\sigma_2^{-1}$, a σ_1-positive word, and $\beta\gamma^{-1} = \sigma_1\sigma_2^{-1}$, again a σ_1-positive word, So, in this case, we have $1 < \beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$.

EXAMPLE 2.4. Here is a stronger example. Let $\beta = \sigma_2^{-1}\sigma_1^2\sigma_2$ and $\gamma = \Delta_3$. We now find $\beta^{-1}\gamma = \sigma_1\sigma_2^{-1}\sigma_1$ (see below), a σ_1-positive word, and $\beta\gamma^{-1} = \sigma_2^{-1}\sigma_1\sigma_2^{-1}$, a σ_1-positive word. So we obtain again $1 < \beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$. But there is more. We claim that $\beta^{-p}\gamma = \sigma_1\sigma_2^{-2p+1}\sigma_1$ holds for $p \geqslant 1$. Indeed, for $p = 1$, we have

$$\beta^{-1}\gamma = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_1^{-1}\sigma_2\sigma_1\sigma_2 \cdot \sigma_1 = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_2\sigma_1 \cdot \sigma_1 = \sigma_1\sigma_2^{-1}\sigma_1.$$

For $p \geqslant 2$, applying the induction hypothesis, we find

$$\begin{aligned}\beta^{-p}\gamma &= \sigma_2^{-1}\sigma_1^{-2}\sigma_2 \cdot \beta^{-p+1}\gamma \\ &= \sigma_2^{-1}\sigma_1^{-2}\sigma_2 \cdot \sigma_1\sigma_2^{-2p+3}\sigma_1 = \sigma_1\sigma_2^{-2} \cdot \sigma_2^{-2p+3}\sigma_1 = \sigma_1\sigma_2^{-2p+1}\sigma_1.\end{aligned}$$

As $\sigma_1\sigma_2^{-2p+1}\sigma_1$ is a σ_1-positive word for each p, we have in this case $1 < \beta^p < \gamma$ for each positive p, and $\beta^{-1} < \gamma^{-1}$.

Even more curious situations occur. Assume that β is a σ_1-positive braid. Then the sequence $1, \beta, \beta^2, \ldots$ is strictly increasing, and its entries admit expressions in which more and more letters σ_1 occur. One might therefore expect that, eventually, the braid β^p dominates σ_1, which only contains one letter σ_1. The next example shows this is not the case.

EXAMPLE 2.5. Consider $\beta = \sigma_2^{-1}\sigma_1$. Then $\beta^p < \sigma_1$ holds for each p. The inequality clearly holds for $p \leqslant 0$. For positive p, we will show that $\sigma_1^{-1}\beta^p$ is σ_1-negative. To this end, we prove the equality

$$\sigma_1^{-1}\beta^p = \sigma_2(\sigma_2\sigma_1^{-1})^{p-1}\sigma_1^{-1}\sigma_2^{-1} \tag{2.1}$$

using induction on $p \geqslant 1$. For $p = 1$, (2.1) reduces to $\sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1}$, which directly follows from the braid relation. For $p \geqslant 2$, we find

$$\begin{aligned}\sigma_1^{-1}\beta^p &= (\sigma_1^{-1}\beta^{p-1}) \cdot \sigma_2^{-1}\sigma_1 \\ &= \sigma_2(\sigma_2\sigma_1^{-1})^{p-2}\sigma_1^{-1}\sigma_2^{-1} \cdot \sigma_2^{-1}\sigma_1 \\ &= \sigma_2(\sigma_2\sigma_1^{-1})^{p-2}\sigma_2\sigma_1^{-2}\sigma_2^{-1} = \sigma_2(\sigma_2\sigma_1^{-1})^{p-1}\sigma_1^{-1}\sigma_2^{-1},\end{aligned}$$

using the induction hypothesis and the equality $\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2\sigma_1^{-2}\sigma_2^{-1}$.

It can be observed that, more generally, $\beta^p < \sigma_2^{-q}\sigma_1$ holds for all nonnegative p and q. So the ascending sequence β^p does not even approach σ_1, as it remains below each entry in the descending sequence $\sigma_2^{-q}\sigma_1$.

Our last example will demonstrate that the σ-ordering of B_n is not Conradian.

DEFINITION 2.6. A left-invariant ordering $\prec$ of a group G is *Conradian* if for all g, h in G that are greater than 1, there exists a positive integer p satisfying $h \prec gh^p$.

Conrad used this property in [**38**] to show that such left-ordered groups share many of the properties of bi-orderable groups; see Section XV.5 for more details.

PROPOSITION 2.7. *For $n \geqslant 3$, the σ-ordering of the braid group B_n is not Conradian.*

PROOF. Let $\beta = \sigma_2^{-1}\sigma_1$ and $\gamma = \sigma_2^{-2}\sigma_1$. Clearly, β and γ are σ_1-positive, so $\beta > 1$ and $\gamma > 1$ hold. We claim that $\gamma\beta^p < \beta$ holds for each $p \geqslant 0$. To see that, using induction on $p \geqslant 0$, we prove the equality

$$\beta^{-1}\gamma\beta^p = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-1}\sigma_1^{-2}\sigma_2^{-1}. \tag{2.2}$$

For $p = 0$, using the braid relations, we find

$$\beta^{-1}\gamma = \sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{-1}\sigma_1^{-2}\sigma_2^{-1}.$$

For $p = 1$, we have

$$\beta^{-1}\gamma\beta = \sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2^2\sigma_1^{-2}\sigma_2^{-1}.$$

For $p \geqslant 1$, again using the equality $\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2\sigma_1^{-2}\sigma_2^{-1}$ of Example 2.5, we find

$$\begin{aligned}\beta^{-1}\gamma\beta^p &= (\sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-2}\sigma_1^{-2}\sigma_2^{-1})(\sigma_2^{-1}\sigma_1)\\ &= \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-2}\sigma_1^{-1}\sigma_2\sigma_1^{-2}\sigma_2^{-1} = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-1}\sigma_1^{-2}\sigma_2^{-1}.\end{aligned}$$

For $p \geqslant 1$, the right-hand side of (2.2) is σ_1-negative, and, for $p = 0$, it is equivalent to the σ_1-negative word $\sigma_2\sigma_1^{-1}\sigma_2$, so, in each case, we obtain $\beta < \gamma\beta^p$. □

2.2. Property S. After the many counterexamples of Section 2.1, we turn to positive results.

We have seen in Example 2.1 that the σ-ordering of braids is not invariant under multiplication on the right, and, therefore, that a conjugate of a braid larger than 1 need not be larger than 1. This phenomenon cannot, however, occur with conjugates of positive braids, *i.e.*, of braids that can be expressed using the generators σ_i only, and not their inverses. The core of the question is the last of the three fundamental properties of braids we shall develop here:

Property S (Subword). *Every braid of the form $\beta^{-1}\sigma_i\beta$ is σ-positive.*

Property **S** was first proved by Richard Laver in [**137**]. In this text, proofs of Property **S** appear on pages 83, 152, 193, and 263.

Using the compatibility of $<$ with multiplication on the left and a straighforward induction, we deduce the following result, which explains our terminology:

PROPOSITION 2.8. *Assume that β, β' are braids and some braid word representing β' is obtained by inserting positive letters σ_i in a braid word representing β. Then we have $\beta' > \beta$.*

We recall that B_∞^+ denotes the submonoid of B_∞ generated by the braids σ_i. Another consequence of Property **S** is:

PROPOSITION 2.9. *If β belongs to B_∞^+ and is not 1, then $\beta' > 1$ is true for every conjugate β' of β. More generally, $\beta > 1$ is true for every quasi-positive braid β, the latter being defined as a braid that can be expressed as a product of conjugates of positive braids.*

PROOF. Assume $\beta' = \gamma^{-1}\beta\gamma$ with $\beta \in B_\infty^+$. By definition, β is a product of finitely many braids σ_i, so, in order to prove $\beta' > 1$, it suffices to establish that $\gamma^{-1}\sigma_i\gamma > 1$ holds for each i, and this is Property **S**. □

As was noted by Stepan Orevkov [**166**], the converse implication is not true: the braid $\sigma_2^{-5}\sigma_1\sigma_2^2\sigma_1$ is a non-quasi-positive braid but every conjugate of it is σ-positive.

By applying the flip automorphism Φ_n, we immediately deduce from Property **S** that every braid of the form $\beta^{-1}\sigma_i\beta$ is also σ^Φ-positive, and that the counterpart of Proposition 2.8 involving the ordering $<^\Phi$ is true. A direct application is the following result, which is important for analysing the restriction of $<^\Phi$ to B_∞^+:

PROPOSITION 2.10. *For each n, the set B_n^+ is the initial segment of $(B_\infty^+, <^\Phi)$ determined by σ_n, i.e., we have $B_n^+ = \{\beta \in B_\infty^+ \mid \beta <^\Phi \sigma_n\}$.*

PROOF. By definition, $\beta <^{\Phi} \sigma_n$ holds for every β in B_n^+. Indeed, if w is any n-strand braid word representing β, then $w^{-1}\sigma_n$ is a σ_n^{Φ}-positive word representing $\beta^{-1}\sigma_n$.

Conversely, assume that β is a positive braid satisfying $\beta <^{\Phi} \sigma_n$. Let w be a positive braid word representing β, and let σ_i be the generator with highest index occurring in w. By the counterpart of Proposition 2.8, we have $\beta \geqslant^{\Phi} \sigma_i$, and, therefore, $i \geqslant n$ would contradict the hypothesis $\beta <^{\Phi} \sigma_n$. □

Another application of Property **S** is the following property from [**51**]. We recall that sh denotes the shift endomorphism of B_∞ that maps σ_i to σ_{i+1} for every i.

PROPOSITION 2.11. *For each braid β, we have $\beta < \mathrm{sh}(\beta)\,\sigma_1$.*

PROOF. Let β be an arbitrary braid in B_n. We claim that the braid $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is σ_1-positive. To see that, we write, inside B_{n+1},

$$\beta^{-1}\mathrm{sh}(\beta)\sigma_1 = (\underline{\beta^{-1}\sigma_2 \ldots \sigma_n\beta}) \cdot (\underline{\sigma_n^{-1} \ldots \sigma_2^{-1}}) \cdot (\underline{\sigma_2 \ldots \sigma_n\beta^{-1}\sigma_n^{-1} \ldots \sigma_2^{-1}}) \cdot \mathrm{sh}(\beta)\sigma_1.$$

The first underlined fragment is a conjugate of the positive braid $\sigma_2 \ldots \sigma_n$, so, by Property **S**, it is σ-positive, hence either σ_1-positive or σ_1-free. The second underlined fragment is σ_1-free. Next, it is easy to check with a picture that the third underlined fragment is equal to $\sigma_1^{-1}\mathrm{sh}(\beta^{-1})\sigma_1$. Putting things together, we obtain

$$\beta^{-1}\mathrm{sh}(\beta)\sigma_1 = \beta' \cdot \underline{\sigma_1^{-1}\mathrm{sh}(\beta^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)\sigma_1},$$

where β' is a braid that is either σ_1-positive or σ_1-free. But, now, we see that the underlined expression is a conjugate of σ_1, so, by Property **S**, it is σ-positive, hence σ_1-positive or σ_1-free. We deduce that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ itself is σ_1-positive or σ_1-free.

Finally, it is impossible that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ be σ_1-free. Indeed, let π be the permutation of $\{1, \ldots, n\}$ induced by β. Then the initial position of the strand that finishes at position 1 in any diagram representing $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is $\pi^{-1}(\pi(1) + 1)$, which cannot be 1.

So the only possibility is that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is σ_1-positive, hence σ-positive. □

3. Global properties of the σ-ordering

We turn to more global properties, involving infinitely many braids at a time. Here we successively consider the Archimedean property, the question of density and the associated topology, and convex subgroups.

3.1. The Archimedean property. We shall show that the σ-ordering and, more generally, any left-invariant ordering of B_n fails to be Archimedean for $n \geqslant 3$. However, certain partial Archimedean properties involving the central elements Δ_n^2 are satisfied.

DEFINITION 3.1. A left-ordered group $(G, \prec)$ is said to be *Archimedean* if, for all g, h larger than 1 in G, there exists a positive integer p for which $g \prec h^p$ holds.

In other words, the powers of any nontrivial element are cofinal in the ordering. For example, an infinite cyclic group, with either of the two possible orderings, is Archimedean. On the other hand, $\mathbb{Z} \times \mathbb{Z}$ with lexicographic ordering is not Archimedean, whereas Archimedean orderings for the same group do exist, by embedding $\mathbb{Z} \times \mathbb{Z}$ in the additive real numbers, sending the generators to rationally independent numbers, and taking the induced ordering.

Proposition 3.2. *The σ-ordering of B_n is not Archimedian for $n \geqslant 3$.*

Proof. For every positive integer p, we have $1 < \sigma_2^p < \sigma_1$. □

One can say more.

Proposition 3.3. *For $n \geqslant 3$, every left-invariant ordering of B_n fails to be Archimedian.*

This follows from the the fact that B_n is not Abelian for $n \geqslant 3$ and from a result of P. Conrad [**38**] generalizing the classical theorem of Hölder [**111**]: any left-invariant Archimedian ordering of a group must also be right-invariant, and the group embeds, simultaneously in the algebraic and order senses, in the additive real numbers. In particular, such a group is Abelian.

By contrast to the previous negative result, there is a partial Archimedian property involving the central element Δ_n^2, namely that every braid is dominated by some power of the braid Δ_n^2.

The results we shall establish turn out to be true not only for the σ-ordering, but also for any left-invariant ordering of B_n. So, for the rest of this section, we consider this extended framework. When $\prec$ denotes a strict ordering, $\preccurlyeq$ denotes the corresponding nonstrict ordering, *i.e.*, $x \preccurlyeq y$ stands for "$x \prec y$ or $x = y$".

Lemma 3.4. *Assume that $\prec$ is a left-invariant ordering of B_n. Then $\Delta_n^{2p} \prec \beta$ implies $\beta^{-1} \prec \Delta_n^{-2p}$, and the conjunction of $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$ implies $\Delta_n^{2p+2q} \prec \beta\gamma$. The same implications hold for $\preccurlyeq$.*

Proof. Assume $\Delta_n^{2p} \prec \beta$. Multiplying by β^{-1} on the left, we get $\beta^{-1}\Delta_n^{2p} \prec 1$, which is also $\Delta_n^{2p}\beta^{-1} \prec 1$. Multiplying by Δ_n^{-2p} on the left, we deduce $\beta^{-1} \prec \Delta_n^{-2p}$.

Now assume $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$. By multiplying the first inequality by Δ_n^{2q} on the left, we obtain $\Delta_n^{2p+2q} \prec \Delta_n^{2q}\beta = \beta\Delta_n^{2q}$. By multiplying the second inequality by β on the left, we obtain $\beta\Delta_n^{2q} \prec \beta\gamma$. We deduce $\Delta_n^{2p+2q} \prec \beta\gamma$. □

Lemma 3.5. *Assume that $\prec$ is a left-invariant ordering of B_n satisfying $1 \prec \Delta_n$. Then, for each i in $\{1, ..., n-1\}$, we have $\Delta_n^{-2} \prec \sigma_i \prec \Delta_n^2$.*

Proof. By Lemma I.4.4, we have $\delta_n^n = \Delta_n^2$, so the hypothesis $1 \prec \Delta_n$ implies $1 \prec \Delta_n^2 = \delta_n^n$, hence $1 \prec \delta_n$, and, therefore, $1 \prec \delta_n \prec \delta_n^2 \prec ... \prec \delta_n^n = \Delta_n^2$.

Assume that $\Delta_n^2 \preccurlyeq \sigma_i$ holds for some i. Let j be any element of $\{1, ..., n-1\}$. By formulas (I.4.3) and (I.4.4), we can find p with $0 \leqslant p \leqslant n-1$ satisfying $\sigma_j = \delta_n^{-p}\sigma_i\delta_n^p$. Then we obtain

$$1 \prec \delta_n^{n-p} = \delta_n^{-p}\Delta_n^2 \preccurlyeq \delta_n^{-p}\sigma_i \preccurlyeq \delta_n^{-p}\sigma_i\delta_n^p = \sigma_j.$$

So $1 \prec \sigma_j$ holds for each generator σ_j. Applying Lemma 3.4, we deduce that, if a braid β can be represented by a positive braid word that contains at least one letter σ_i, then $\Delta_n^2 \preccurlyeq \beta$ holds. This applies in particular to Δ_n, and we deduce $\Delta_n^2 \preccurlyeq \Delta_n$, which contradicts the assumption $1 \prec \Delta_n$.

Similarly, assume that $\sigma_i \preccurlyeq \Delta_n^{-2}$ holds. Consider again any σ_j. If p is as above, we also have $\sigma_j = \delta_n^{n-p}\sigma_i\delta_n^{p-n}$, since δ_n^n lies in the center of B_n. Then we find

$$\sigma_j = \delta_n^{n-p}\sigma_i\delta_n^{p-n} \prec \delta_n^{n-p}\sigma_i \preccurlyeq \delta_n^{n-p}\Delta_n^{-2} = \delta_n^{-p} \preccurlyeq 1.$$

This time, $\sigma_j \prec 1$ holds for each j. As Δ_n is a positive braid, this implies $\Delta_n \prec 1$, which contradicts the assumption $1 \prec \Delta_n$. □

Gathering the results, we immediately deduce:

PROPOSITION 3.6. *Assume $\prec$ is a left-invariant ordering of B_n and $1 \prec \Delta_n$ holds. Then, for each braid β in B_n, there exists a unique integer p for which $\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2}$ is true. Moreover, if β can be represented by a braid word of length ℓ, we have $|p| \leqslant \ell$.*

PROOF. Lemma 3.5 implies that each generator σ_i lies in the interval $(\Delta_n^{-2}, \Delta_n^2)$. Then Lemma 3.4 implies that every braid that can be represented by a word of length ℓ lies in the interval $[\Delta_n^{-2\ell}, \Delta_n^{2\ell})$. As this interval is the disjoint union of the intervals $[\Delta_n^{2p}, \Delta_n^{2p+2})$ for $-\ell \leqslant p < \ell$, the result of the proposition follows. □

In this way, we obtain a decomposition of $(B_n, \prec)$ into a sequence of disjoint intervals of size Δ_n^2, as suggested in Figure 1.

As noted by A. Malyutin and N.Yu. Netsvetaev in [**150**], the previous result implies that the action of conjugacy cannot move a braid too far.

COROLLARY 3.7 (Figure 1). *Assume that $\prec$ is a left-invariant ordering of B_n satisfying $1 \prec \Delta_n$. Then, if β and β' are conjugate,*

$$\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2} \quad \textit{implies} \quad \Delta_n^{2p-2} \preccurlyeq \beta' \prec \Delta_n^{2p+4}. \tag{3.1}$$

So, in particular, $\beta\Delta_n^{-4} \prec \beta' \prec \beta\Delta_n^4$ is always true.

PROOF. Assume $\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2}$ and $\beta' = \gamma\beta\gamma^{-1}$. By Proposition 3.6, we have $\Delta_n^{2q} \preccurlyeq \gamma \prec \Delta_n^{2q+2}$ for some q. Lemma 3.4 first implies $\Delta_n^{-2q-2} \prec \gamma^{-1} \preccurlyeq \Delta_n^{-2q}$, and then

$$\Delta_n^{2q+2p-2q-2} \prec \gamma\beta\gamma^{-1} \prec \Delta_n^{2q+2+2p+2-2q},$$

which gives $\Delta_n^{2p-2} \prec \beta' \prec \Delta_n^{2p+4}$. □

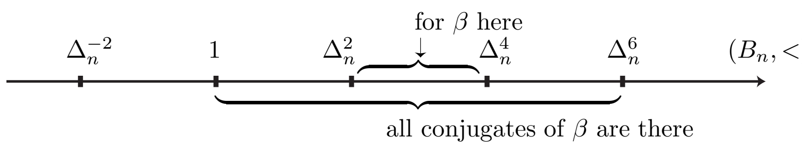

FIGURE 1. Powers of Δ_n^2 and the action of conjugacy on $(B_n, <)$.

All the previous results apply to the σ-ordering, as it is a left-invariant ordering of B_n and $1 < \Delta_n$ is satisfied. Note that, in this case, Corollary 3.7 is optimal in the sense that we cannot replace intervals of length Δ_n^2 with intervals of length Δ_n in Lemma 3.4: for instance, we have $1 < \sigma_1^2\sigma_2 < \Delta_3$ and $\Delta_3^2 < \Delta_3\sigma_1^2\sigma_2 < \Delta_3^3$.

3.2. Discreteness and density. Left-invariant orderings of a group have a sort of homogeneity—the ordering near any two group elements has similar order properties, because of invariance under left translation. In particular, there is a basic dichotomy between discrete and dense orders.

DEFINITION 3.8. A left-invariant ordering of a group is said to be *discrete* if its positive cone has a least element; it is said to be *dense* if the positive cone does not have a least element.

Equivalently, a left-invariant ordering of a group is discrete if every group element has an immediate successor and predecessor, and it is dense if between any two group elements one can find another element of the group. One verifies easily that, in a discretely left-ordered group, with least element ε larger than 1, the immediate successor of a group element g is $g\varepsilon$ and its immediate predecessor is $g\varepsilon^{-1}$.

The braid orderings display both types.

PROPOSITION 3.9. *The σ-ordering of B_n is discrete, with least σ-positive element σ_{n-1}.*

PROOF. Clearly σ_{n-1} is σ-positive. Conversely, assume that β belongs to B_n and is σ-positive. If β is σ_i-positive for some i with $i \leqslant n-2$, then $\sigma_{n-1}^{-1}\beta$ is σ_i-positive as well, so $\sigma_{n-1} < \beta$ holds. On the other hand, if β is σ_{n-1}-positive, it must be σ_{n-1}^p for some $p \geqslant 1$, and we find $\sigma_{n-1}^{-1}\beta = \sigma_{n-1}^{p-1}$, hence $\sigma_{n-1} \leqslant \beta$. □

As the flip automorphism Φ_n is an isomorphism of $(B_n, <)$ to $(B_n, <^\Phi)$, the flipped version $<^\Phi$ of the σ-ordering is also discrete on B_n, and σ_1 is the least σ^Φ-positive element. In the inclusions $B_n \subseteq B_{n+1}$, the σ^Φ-ordering has the pleasant property that the same element σ_1 is least σ-positive in each braid group. For this reason, we see a difference in the two orderings in the limit. The reader may easily verify the following.

PROPOSITION 3.10. *The σ-ordering of B_∞ is dense, whereas the σ^Φ-ordering of B_∞ is discrete, with σ_1 being the least element larger than* 1.

COROLLARY 3.11. *The ordered set $(B_\infty, <)$ is order-isomorphic to $(\mathbb{Q}, <)$.*

PROOF. A well-known result of Cantor says that any two countable linearly ordered sets that are dense—there always exists an element between any two elements—and unbounded—there is no minimal or maximal element—are isomorphic: assuming that the sets are $\{a_n \mid n \in \mathbb{N}\}$ and $\{b_n \mid n \in \mathbb{N}\}$, one alternatively defines $f(a_0)$, $f^{-1}(b_0)$, $f(a_1)$, $f^{-1}(b_1)$, etc. so as to keep f order-preserving.

Here the rationals are eligible, and the set B_∞ is countable. So, in order to apply Cantor's criterion, it suffices to prove that $(B_\infty, <)$ is dense and unbounded. The former result is Proposition 3.10. The latter is clear: for every braid β, we have $\beta\sigma_1^{-1} < \beta < \beta\sigma_1$. □

Of course, the order-isomorphism of Corollary 3.11 could not be an isomorphism in the algebraic sense, as B_∞ is non-Abelian.

Every linearly ordered set has an order topology, with open intervals forming a basis for the topology. If the ordering is discrete, as is the case for the σ-ordering of B_n for $n < \infty$, then the topology is also discrete. Since B_∞, with the σ-ordering, is order isomorphic with the rational numbers, its order topology is metrizable. In fact, it has a natural metric, as follows.

PROPOSITION 3.12. *For $\beta \neq \beta'$ in B_∞, define $d(\beta, \beta')$ to be 2^{-p} where p is the greatest integer satisfying $\beta^{-1}\beta' \in \mathrm{sh}^p(B_\infty)$, completed with $d(\beta, \beta) = 0$. Then d is a distance on B_∞, and the topology of B_∞ associated with the linear order $<$ is the topology associated with d.*

PROOF. It is routine to verify that d is a distance. The open disk of radius 2^{-p} centered at β is the left coset $\beta\,\mathrm{sh}^p(B_\infty)$, *i.e.*, the set of all braids of the form $\beta\,\mathrm{sh}^p(\gamma)$.

Assume now that β_1, β, β_2 lie in B_n and $\beta_1 < \beta < \beta_2$ holds. We will show that the open d-disk around β of radius 2^{-n+1} is included in the interval (β_1, β_2). Indeed, if $d(\beta, \gamma) < 2^{-n+1}$, then $\beta^{-1}\gamma$ belongs to $\mathrm{sh}^n(B_\infty)$. The hypothesis $\beta_1 < \beta$ implies that $\beta_1^{-1}\beta$ is σ_i-positive for some $i \leqslant n-1$. Writing $\beta_1^{-1}\gamma = (\beta_1^{-1}\beta)(\beta^{-1}\gamma)$, we see that $\beta_1^{-1}\gamma$ is also σ_i-positive and, therefore, $\beta_1 < \gamma$ is true. A similar argument gives $\gamma < \beta_2$.

Conversely, let us start with an arbitrary open d-disk $\beta\,\mathrm{sh}^p(B_\infty)$. Let β' be a braid in this disk; we have to find an open $<$-interval containing β' which lies entirely in the disk. By hypothesis, we have $\beta' = \beta\,\mathrm{sh}^p(\gamma)$ for some γ of B_∞. Let γ_1 and γ_2 be any braids satisfying $\gamma_1 < \gamma < \gamma_2$. Then the interval $(\beta\,\mathrm{sh}^p(\gamma_1), \beta\,\mathrm{sh}^p(\gamma_2))$ contains $\beta\,\mathrm{sh}^p(\gamma)$ and is included in the disk, because $\mathrm{sh}^p(B_\infty)$ is convex—see Proposition 3.17 below. This completes the proof that the topologies associated with $<$ and with d coincide. □

3.3. Dense subgroups. It is clear that densely ordered groups can have subgroups which are discretely ordered (by the same ordering)—witness $\mathbb{Z}$ in $\mathbb{Q}$. But the reverse can happen, too. For example, the lexicographic ordering on $\mathbb{Q} \times \mathbb{Z}$ is discrete—with least positive element $(0, 1)$—whereas the subgroup $\mathbb{Q} \times \{0\}$ is densely ordered. This latter phenomenon happens quite naturally also for the braid groups.

Note that, if one allows the generators σ_i to commute, the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ implies that σ_i and σ_{i+1} become equal. From this one sees that the Abelianization of B_n is infinite cyclic, and the Abelianization map $B_n \to \mathbb{Z}$ can be identified with the sum of the exponents of a word in the σ_i generators. The commutator subgroup $[B_n, B_n]$ consists exactly of braids expressed in the generators σ_i with exponent sum zero.

Proposition 3.13 ([**37**]). *For $n \geqslant 3$, the commutator subgroup $[B_n, B_n]$ is densely ordered under the σ-ordering.*

Proof. For simplicity, we will prove this just for $n = 3$, referring the reader to [**37**] for the general case, whose proof is similar.

For contradiction, suppose $[B_3, B_3]$ has a least σ-positive element β. We consider the braid $\beta\sigma_2\beta^{-1}$. There are three possibilities:

Case 1: $\beta\sigma_2\beta^{-1}$ is σ_1-positive. Then β must be σ_1-positive. So is $\beta\sigma_2\beta^{-1}\sigma_2^{-1}$ and we have $1 < \beta\sigma_2\beta^{-1}\sigma_2^{-1}$. On the other hand, as β is σ_1-positive, $\sigma_2\beta^{-1}\sigma_2^{-1}$ is σ_1-negative, and we have $\sigma_2\beta^{-1}\sigma_2^{-1} < 1$ and $\beta\sigma_2\beta^{-1}\sigma_2^{-1} < \beta$. So the commutator $\beta\sigma_2\beta^{-1}\sigma_2^{-1}$ is a smaller σ-positive element of $[B_3, B_3]$ than β, contradicting the hypothesis on β.

Case 2: $\beta\sigma_2\beta^{-1}$ is σ_1-negative. A similar argument gives $1 < \beta\sigma_2^{-1}\beta^{-1}\sigma_2 < \beta$, again a contradiction.

Case 3: $\beta\sigma_2\beta^{-1}$ is σ_2^p for some p. Counting the exponents, we see that the only possibility is $p = 1$, *i.e.*, β commutes with σ_2. It is shown in [**84**] that the centralizer of the subgroup of B_3 generated by σ_2 is the subgroup (isomorphic to $\mathbb{Z} \times \mathbb{Z}$) generated by σ_2 and Δ_3^2, so we must have $\beta = (\sigma_1\sigma_2\sigma_1)^{2q}\sigma_2^r$ for some integers q, r. But, since β is σ_1-positive and a commutator, we have $q > 0$ and $6q + r = 0$. Now, consider $\beta' = \sigma_1\sigma_2^{-1}$. We have $\beta' > 1$ and $\beta' \in [B_3, B_3]$, and an easy calculation gives $\beta' < \beta$, again contradicting the hypothesis on β. □

Other subgroups of B_n with $n \geqslant 3$ which are shown to be densely ordered by the σ-ordering in [**37**] include the following:

• $[PB_n, PB_n]$, the commutator subgroup of the pure braid group; but PB_n itself is discretely ordered, with least positive element σ_{n-1}^2;

• the subgroup of Brunnian braids—defined as braids such that, for every strand, its removal results in a trivial braid;

• the subgroup of homotopically trivial braids, as considered in [**99**];

• kernels of the Burau representation for those n for which this representation is unfaithful—it is known to be unfaithful for $n \geqslant 5$ and faithful for $n \leqslant 3$.

The method of proof is to identify explicitly which braids can possibly be the least σ-positive elements of a given normal subgroup of B_n.

3.4. Convex subgroups. Convex subgroups play an important role in the theory of orderable groups.

Definition 3.14. If $(G, \prec)$ is a left-ordered group, a subgroup H of G is said to be *convex* if, for all h, h' in H and g in G satisfying $h \prec g \prec h'$, one has $g \in H$.

An equivalent criterion for convexity of H is the conjunction of $1 \prec g \prec h$, $g \in G$, and $h \in H$ implies $g \in H$. It is easy to verify that the collection of convex subgroups of a given group is linearly ordered by inclusion. Moreover, if N is a normal convex subgroup of the left-ordered group G, then the quotient group G/N is left-orderable by ordering cosets according to their representatives.

If the ordering of G is discrete, and H is a convex subgroup distinct from $\{1\}$, then the ordering on H is also discrete, and H contains the minimal positive element of G, which is also minimal positive in H.

We shall see that there are rather few convex subgroups in the braid groups under the σ-ordering.

Proposition 3.15. *The group B_n has no proper normal convex subgroup.*

Proof. Suppose H is a normal and convex subgroup of B_n distinct of $\{1\}$. As remarked above, the minimal positive element σ_{n-1} of B_n belongs to H by convexity. Since H is normal, σ_1 also belongs to H, as the Garside braid Δ_n conjugates it to σ_{n-1}. All the other σ_i generators are positive and less than σ_1, so they must also be in H, and therefore we have $H = B_n$, alternatively, we can observe that all generators σ_i are conjugated to σ_{n-1} in B_n, as seen in Lemma I.4.4. □

Proposition 3.16. *For i in $\{1, \dots, n-1\}$, let H_i be the subgroup of B_n generated by $\sigma_i, \dots, \sigma_{n-1}$. Then each subgroup H_i is convex in B_n and these are the only nontrivial convex subgroups.*

Proof. First, we verify that H_i is convex. Suppose $1 < \gamma < \beta$ with $\beta \in H_i$ and $\gamma \in B_n$. Note that the σ-positive elements of H_i are exactly the σ_j-positive braids in B_n with $j \geqslant i$. So β is σ_j-positive for some $j \geqslant i$. By hypothesis, γ is σ_k-positive for some k in $\{1, \dots, n-1\}$. If we had $k < j$, then $\beta^{-1}\gamma$ would be σ_j-positive, implying $\beta < \gamma$ and contradicting the hypothesis. Therefore, we have $k \geqslant j \geqslant i$ and γ lies in H_i.

It remains to show that there are no other nontrivial convex subgroups. Assume that C is a convex subgroup of B_n distinct of $\{1\}$. Let i be the least positive integer such that C contains a σ_i-positive braid, say β. We will show that $C = H_i$. First note that C contains each σ_j with $j > i$, because $\sigma_j^{-1}\beta$ is σ_i-positive and we have $1 < \sigma_j < \beta \in C$.

Now we may write $\beta = \beta_0\sigma_i\beta_1\sigma_i \dots \sigma_i\beta_m$ for some $m \geqslant 1$ and some β_i belonging to H_{i+1}, hence to C. Since C is a subgroup and β_0 belongs to C, the braid β' defined by $\beta' = \sigma_i\beta_1\sigma_i \dots \sigma_i\beta_m$ also belongs to C. In case $m > 1$, we conclude $\sigma_i^{-1}\beta'$ is also σ_i-positive and therefore we have $1 < \sigma_i < \beta'$. On the other hand, if $m = 1$ holds, we have $\beta' = \sigma_i\beta_1$. In either case, we conclude that σ_i belongs to C. We have shown that C is included in H_i. If the inclusion were proper, then C would contain a braid which is σ_j-positive for some $j < i$, contradicting our choice of i. □

Almost exactly the same argument shows the following.

PROPOSITION 3.17. *The nontrivial convex subgroups of B_∞ are exactly those of the form $\mathrm{sh}^i(B_\infty)$. None of these is normal.*

Finally, using the flip automorphism Φ_n, we see that, when the σ^Φ-ordering $<^\Phi$ replaces the σ-ordering, then the convex subgroups of B_n are the groups B_i with $i \leqslant n$. The same holds for B_∞.

4. The σ-ordering of positive braids

In this section, we review some results about the restriction of the orderings $<$ and $<^\Phi$ to the braid monoids B_n^+, most of which will be further developed in Chapters VII and VIII. As the many examples of Section 2.1 showed, the σ-ordering is a quite complicated ordering. By contrast, its restriction to the monoid B_n^+ is a simple ordering, namely a well-ordering. In particular, every nonempty set of positive braids has a least element, and, if it is bounded, it has a least upper bound.

We give two proofs of the well-order property for the σ-ordering of B_n^+. Due to Laver [**137**] and based on Property **S**, the first one uses Higman's subword lemma, and it is not constructive. Then, we give another argument, which is constructive and much more precise. It is based on Serge Burckel's approach in [**27**]. Here we follow the new description of [**62**], which relies on an operation called the Φ_n-splitting of a braid. It shows that the ordering of B_n^+ is a sort of lexicographical extension of the ordering of B_{n-1}^+.

Most of the properties described in this section for the monoids B_n^+ extend to the case of the so-called dual braid monoids B_n^{+*}. Introduced by Birman, Ko, and Lee in [**15**], the dual monoid B_n^{+*} is a submonoid of B_n that properly includes B_n^+. Interestingly, the proofs turn out to be easier in the case of B_n^{+*} than in the case of B_n^+. We refer to Chapter VIII for details.

4.1. The well-order property. Restricting a linear ordering to a proper subset always gives a linear ordering, but the properties of the initial ordering and of its restriction may be very different—we already saw examples in Section 3.3. This is what happens with the σ-ordering of B_n and its restriction to B_n^+. For instance, we saw in Proposition 3.9 that $(B_n, <)$ is discrete, and that every braid β has an immediate predecessor, namely $\beta\sigma_{n-1}^{-1}$. The situation is radically different with B_n^+. In particular, $(B_n^+, <)$ has limit points: for instance, in $(B_3^+, <)$, the braid σ_1 is the least upper bound of the increasing sequence $(\sigma_2^p)_{p \geqslant 0}$; see Figure 2.

We recall that a linear ordering is called a *well-ordering* if every nonempty subset has a least element, or, equivalently, provided some very weak form of the Axiom of Choice is assumed, if it admits no infinite descending sequence. A direct consequence of Property **S** is the following important result.

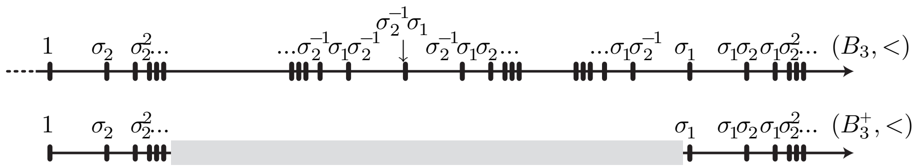

FIGURE 2. Restricting to positive braids completely changes the ordering: for instance, in $(B_3^+, <)$, the braid σ_1 is the limit of σ_2^p, whereas, in $(B_3, <)$, it is an isolated point with immediate predecessor $\sigma_2^{-1}\sigma_1$; the grey part in B_3 includes infinitely many braids, such as $\sigma_2^{-1}\sigma_1$ and its neighbours—and much more—but none of them lies in B_3^+.

PROPOSITION 4.1. *For every n, the restriction of $<$ to B_n^+ is a well-ordering.*

PROOF. A theorem of Higman [**108**], known as Higman's subword lemma, says: An infinite set of words over a finite alphabet necessarily contains two elements w, w' such that w' can be obtained from w by inserting intermediate letters (in not necessarily adjacent positions). Let $\beta_1, \beta_2, ...$ be an infinite sequence of braids in B_n^+. Our aim is to prove that this sequence is not strictly decreasing. For each p, choose a positive braid word w_p representing β_p. There are only finitely many n-strand braid words of a given length, so, for each p, there exists $p' > p$ such that $w_{p'}$ is at least as long as w_p. So, inductively, we can extract a subsequence $w_{p_1}, w_{p_2}, ...$ in which the lengths are nondecreasing. If the set $\{w_{p_1}, w_{p_2}, ...\}$ is finite, there exist k, k' such that w_{p_k} and $w_{p_{k'}}$ are equal, and then we have $\beta_{p_k} = \beta_{p_{k'}}$. Otherwise, by Higman's theorem, there exist k, k' such that w_{p_k} is a subword of $w_{p_{k'}}$, and, by construction, we must have $p_k < p_{k'}$. By Property **S**, this implies $\beta_{p_k} \leqslant \beta_{p_{k'}}$ in B_n^+. So, in any case, the sequence $\beta_1, \beta_2, ...$ is not strictly decreasing. ☐

The previous proof actually shows more.

PROPOSITION 4.2. *Assume that M is a submonoid of B_∞ generated by finitely many braids, each of which is a conjugate of some σ_i—hence of σ_1. Then the restriction of $<$ to M is a well-ordering.*

PROOF. In the proof of Proposition 4.1, Property **S** is used to ensure that, if a word w in the generators σ_i of B_n is a subword of another word w', then we have $\overline{w} \leqslant \overline{w'}$, where $\overline{w}$ denotes the braid represented by w. Now the same property holds for the generators of M, as each of them is a conjugate of some σ_i. Indeed, inserting a pattern of the form $v\sigma_i v^{-1}$ after w_1 in a braid word $w_1 w_2$ amounts to inserting σ_i in the equivalent braid word $w_1 v v^{-1} w_2$, and, therefore, the braid represented by $w_1 \cdot v\sigma_i v^{-1} \cdot w_2$ is larger than the braid represented by $w_1 w_2$. ☐

Typically, the dual braid monoids investigated in Chapter VIII are eligible for Proposition 4.2.

REMARK 4.3. The hypothesis that the monoid M is finitely generated is crucial in Proposition 4.2. For instance, we already observed that the submonoid B_∞^+ of B_∞ is not well-ordered by the σ-ordering, as we have an infinite descending sequence $\sigma_1 > \sigma_2 > ...$. Such phenomena already occur inside B_3: for instance, the submonoid of B_3 generated by all conjugates $\sigma_2^{-p}\sigma_1\sigma_2^p$ of σ_1—and, more generally, the submonoid of all quasi-positive n-strand braids, defined to be the submonoid of B_n generated by all conjugates of $\sigma_1, \dots, \sigma_{n-1}$—contains the infinite descending sequence $\sigma_1 > \sigma_2^{-1}\sigma_1\sigma_2 > \sigma_2^{-2}\sigma_1\sigma_2^2 > \dots$.

Being a well-ordering has strong consequences. In particular, in contrast to what the examples of Section 2.1 showed, the well-order property implies the most general form of the phenomenon observed in Figure 2:

Corollary 4.4. *Every nonempty subset of B_n^+ is either cofinal or it has a least upper bound inside $(B_n^+, <)$.*

Indeed, for X included in B_n^+, unless X is unbounded in B_n^+, the set of all upper bounds of X is nonempty, hence it admits a least element.

4.2. The recursive construction of the ordering on B_n^+. We gave above a quick proof for Proposition 4.1, but the latter is not constructive, and it gives no direct description of the well-ordering $(B_n^+, <)$. We shall now give such a description, based on a recursive construction that connects $(B_{n-1}^+, <)$ and $(B_n^+, <)$. This approach leads in particular to considering the ordering of B_n^+ as an iterated extension of the ordering of B_2^+, *i.e.*, of the standard ordering of natural numbers.

To explain the results, it is crucial to use the flipped version of the σ-ordering, *i.e.*, the ordering $<^\Phi$ defined from σ^Φ-positive braids. The reason is that, although $(B_n^+, <)$ and $(B_n^+, <^\Phi)$ are isomorphic, the pairs $(B_n^+, B_{n-1}^+, <)$ and $(B_n^+, B_{n-1}^+, <^\Phi)$ are not, and the connection between B_n^+ and B_{n-1}^+ is more easily described in the case of $<^\Phi$.

The starting point of the approach is the following result from [**62**]. We recall that Φ_n denotes the flip automorphism (both of B_n and of B_n^+) that exchanges σ_i and σ_{n-i} for $1 \leqslant i \leqslant n-1$.

Proposition 4.5. *Assume $n \geqslant 3$. Then, for each braid β in B_n^+, there exists a unique sequence $(\beta_p, \dots, \beta_1)$ in B_{n-1}^+ such that β admits the decomposition*

$$\beta = \Phi_n^{p-1}(\beta_p) \cdot \dots \cdot \Phi_n(\beta_2) \cdot \beta_1, \tag{4.1}$$

and for each r the only generator σ_i that right divides $\Phi_n^{p-r}(\beta_p) \cdot \dots \cdot \beta_r$ is σ_1. The sequence $(\beta_p, \dots, \beta_1)$ is called the Φ_n-splitting of β.

The result easily follows from the fact that every positive braid β of B_n^+ admits a unique maximal right divisor that lies in B_{n-1}^+. The unusual enumeration of the sequence from the right emphasizes that the construction starts from the right and involves right divisors.

Now, the main result says that, through the Φ_n-splitting, the ordering of B_n^+ is just a lexicographical extension of the ordering of B_{n-1}^+, more exactly a `ShortLex`-extension in the sense of [**77**], *i.e.*, the variant of the lexicographical extension in which the length is first taken into account.

Proposition 4.6. *Assume $n \geqslant 3$. Let β, β' belong to B_n^+, and let $(\beta_p, \dots, \beta_1)$ and $(\beta'_{p'}, \dots, \beta'_1)$ be their Φ_n-splittings. Then $\beta <^\Phi \beta'$ holds if and only if $(\beta_p, \dots, \beta_1)$ is smaller than $(\beta'_{p'}, \dots, \beta'_1)$ for the `ShortLex`-extension of $(B_{n-1}^+, <^\Phi)$, i.e., we have either $p < p'$, or $p = p'$ and there exists $q \leqslant p$ satisfying $\beta_r = \beta'_r$ for $r > q$ and $\beta_q <^\Phi \beta'_q$.*

The result appears as Corollary VII.4.6, and it is also a consequence of Corollary VIII.3.3, with a disjoint argument.

The Φ_n-splitting of a positive braid can be computed easily, and a direct outcome of Proposition 4.6 is the existence, already mentioned in Section 1.5, of a quadratic upper bound for the complexity of the σ- and σ^Φ-orderings.

Corollary 4.7. *For each n, the orderings $<^\Phi$ and $<$ of B_n can be recognized in quadratic time.*

Proof. We use induction on $n \geqslant 2$. Let w be an n-strand braid word of length ℓ. By Proposition I.4.6, we can obtain in time $O(\ell)$ two positive n-strand braid words w_1, w_2 such that w is equivalent to $w_1^{-1}w_2$. Then $\overline{w} >^\Phi 1$ is equivalent to $\overline{w_2} >^\Phi \overline{w_1}$. The Φ_n-splittings of the braids $\overline{w_1}$ and $\overline{w_2}$ can be computed in time $O(\ell^2)$; see Chapter VII. The induction hypothesis implies that the comparison of the sequences so obtained can be done in time $O(\ell^2)$ as well. The argument is similar for the σ-ordering as the shift automorphism Φ_n is computable in linear time. □

4.3. The length of $(B_n^+, <^\Phi)$. Contrary to an arbitrary linear ordering, a well-ordering is completely determined up to isomorphism by a unique parameter, namely its length, usually specified by an ordinal number. In the case of the braid ordering on B_n^+, the length easily follows from the recursive characterization of Proposition 4.6.

We recall that ordinals are a transfinite continuation of the sequence of natural numbers: after the natural numbers comes ω, the first infinite ordinal, then $\omega + 1$, $\omega + 2$, etc. For our purposes, it is enough to know that ordinals come equipped with a well-ordering and with arithmetic operations (addition, multiplication, exponentiation) that extend those of $\mathbb{N}$. For more background information about ordinals, we refer to any textbook in set theory, for instance [**138**].

Proposition 4.8. *For each n, the ordered set $(B_n^+, <^\Phi)$ has ordinal type $\omega^{\omega^{n-2}}$.*

In other words, the length of $(B_n^+, <^\Phi)$ is the ordinal $\omega^{\omega^{n-2}}$. The proof is an easy induction on n.

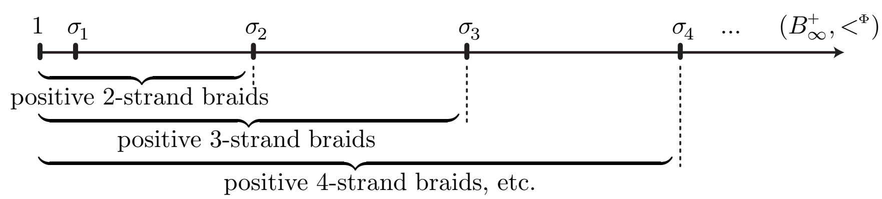

Figure 3. The well-ordered set $(B_\infty^+, <^\Phi)$: an increasing union of end-extensions; for each n, the subset B_n^+ is the initial interval determined by σ_n.

By Proposition 2.10, the ordered set $(B_\infty^+, <^\Phi)$ is the increasing union of the sets $(B_n^+, <^\Phi)$, each set B_n^+ being an initial segment of the next one; see Figure 3. It is easy to deduce

Proposition 4.9. *The ordered set $(B_\infty^+, <^\Phi)$ is a well-ordering with ordinal type ω^{ω^ω}.*

As the flip automorphism Φ_n preserves B_n^+ globally, the results about $(B_n^+, <^\Phi)$ translate into similar results about $(B_n^+, <)$. In particular, Proposition 4.8 implies

COROLLARY 4.10. *For each* n, *the well-ordering* $(B_n^+, <)$ *has ordinal type* $\omega^{\omega^{n-2}}$.

However, we have no counterpart of Proposition 4.9 for $<$: the set B_n^+ is not an initial segment of $(B_\infty^+, <)$, and the latter is not a well-ordered set since it contains the infinite descending sequence of (1.1).

4.4. The rank of a positive braid. One of the nice features when an ordering $\prec$ of a set Ω is a well-ordering is that, for $x \in \Omega$, the position of x in $(\Omega, \prec)$ is unambiguously specified by an ordinal number, called the *rank* of x, namely the order type of the initial segment $\{y \in \Omega \mid y \prec x\}$. The rank function establishes an isomorphism between $(\Omega, \prec)$ and an initial segment of the sequence of ordinals: by construction, $x \prec x'$ is true if and only if the rank of x is smaller than the rank of x'.

So, in our current case, every positive braid β in B_n^+ is associated with a well-defined ordinal number, the rank of β, that specifies its position in $(B_n^+, <^\Phi)$. Moreover, Proposition 2.10 (or simply Figure 3) shows that the rank of β in $(B_n^+, <^\Phi)$ coincides with its rank in $(B_\infty^+, <^\Phi)$, and we can forget about the braid index.

Some values of the rank function are easily computed. For instance, the rank of the braid σ_i is the ordinal $\omega^{\omega^{i-2}}$ for $i \geqslant 2$: indeed, it is the ordinal type of the initial interval determined by σ_i. By Proposition 2.10, the latter is B_i, which, by Proposition 4.8, has ordinal type $\omega^{\omega^{i-2}}$. More values can be read in Figure 4.

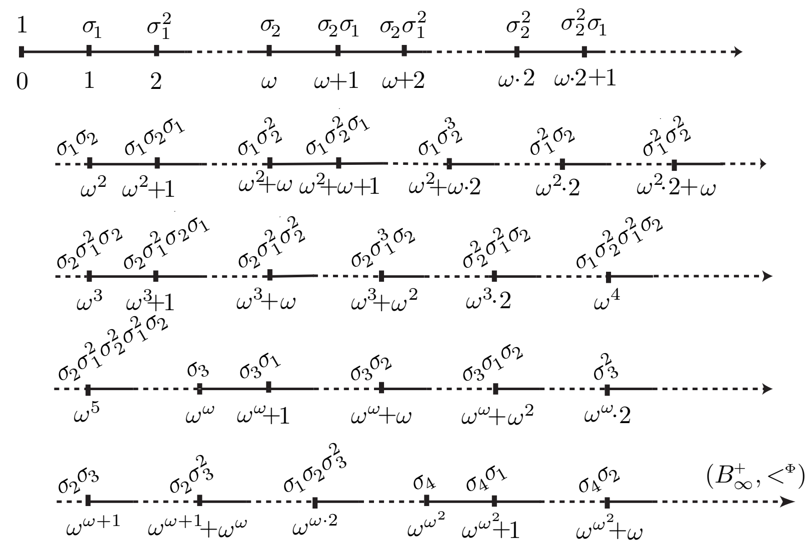

FIGURE 4. Ranks in the well-ordering $(B_\infty^+, <^\Phi)$: the position of each braid is unambiguously specified by an ordinal number that measures the length of the initial interval it determines.

REMARK 4.11. By construction, the rank mapping provides an order-isomorphism between positive braids and ordinals. Except for 2-strand braids, this mapping is *not* an algebraic homomorphism with respect to the ordinal sum: in general, the rank of $\beta_1\beta_2$ is not the sum of the ranks of β_1 and β_2. This happens to be true for $\beta_2 = \sigma_1$, which has rank 1, but, for instance, we can read in Figure 4 that the rank of σ_2 is ω, while that of $\sigma_1\sigma_2$ is ω^2, which is not $1+\omega$.

Arguably, an optimal description of $(B_\infty^+, <^\Phi)$ would consist of a closed formula explicitly computing, for each positive braid β, the rank of β, *i.e.*, determining the absolute position of β in $(B_\infty^+, <^\Phi)$. An algorithmic method has been described in [**28**], but, so far, it leads to no closed formula in the general case. However, in the case of 3-strand braids, such a formula exists. It relies on identifying distinguished word representatives called Φ *-normal*, from which the rank can be directly read.

Definition 4.12. A nonempty positive 3-strand braid word $\sigma_{[p]}^{e_p} \dots \sigma_2^{e_2}\sigma_1^{e_1}$ is said to be Φ *-normal* if the inequalities $e_p \geqslant 1$ and $e_r \geqslant e_r^{\min}$ for $r < p$ are satisfied, where we set $e_1^{\min} = 0$, $e_2^{\min} = 1$, and $e_r^{\min} = 2$ for $r \geqslant 3$, and use $[p]$ to denote 1 for odd p, and 2 for even p.

So the criterion is that a positive 3-strand braid word w is Φ-normal if the successive blocks of letters σ_1 and σ_2 in w, enumerated from the right, and insisting that the rightmost block is a (possibly empty) block of σ_1, have a minimal legal size prescribed by the absolute numbers $e_r^{\min}$. It is easy to check that every nontrivial braid β of B_3^+ is represented by a unique Φ-normal word, naturally called its Φ-*normal form*. Then we have the following explicit formula for the rank.

Proposition 4.13. *For each braid β in B_3^+, the rank of β in $(B_\infty^+, <^\Phi)$ is*

$$\omega^{p-1} \cdot e_p + \sum_{p>r\geqslant 1} \omega^{r-1} \cdot (e_r - e_r^{\min}), \tag{4.2}$$

where $\sigma_{[p]}^{e_p} \dots \sigma_2^{e_2}\sigma_1^{e_1}$ is the Φ -normal form of β.

This makes the description of the ordered set $(B_3^+, <^\Phi)$ complete.

Example 4.14. The Φ-normal form of Δ_3 is $\sigma_1\sigma_2\sigma_1$, as the latter word satisfies the defining inequalities, contrary to $\sigma_2\sigma_1\sigma_2$, *i.e.*, $\sigma_2^1\sigma_1^1\sigma_2^1\sigma_1^0$, in which the third exponent from the right, namely 1, is smaller than the minimal legal value $e_3^{\min} = 2$. So, in this case, the sequence $(e_p, \dots, e_1)$ is $(1,1,1)$, and, applying (4.2), we deduce that the rank of Δ_3 in $(B_3^+, <^\Phi)$ is $\omega^2 \cdot 1 + \omega \cdot (1-1) + 1 \cdot (1-0)$, *i.e.*, $\omega^2 + 1$. The reader can check that, more generally, the flip normal form of Δ_3^d corresponds to the length $d+2$ exponent sequence $(1, 2, \dots, 2, 1, d)$, implying that the rank of Δ_3^d is the ordinal $\omega^{d+1} + d$. More values can be read in Figure 4.

4.5. Connection between positive and arbitrary braids. By Proposition I.4.6, every braid is a quotient of two positive braids. It follows that, in theory, the ordering of arbitrary braids is determined by its restriction to positive braids.

Proposition 4.15. *Let $\beta_1, \dots, \beta_p$ be a finite family of braids in B_n. Then, for d large enough, $\Delta_n^d\beta_1, \dots, \Delta_n^d\beta_p$ lie in B_n^+, and the mutual positions of $\beta_1, \dots, \beta_p$ in $(B_n, <)$ are the same as the mutual positions of the positive braids $\Delta_n^d\beta_1, \dots, \Delta_n^d\beta_p$ in $(B_n^+, <)$.*

The result is clear, as the braid ordering $<$ is left-invariant. A similar result holds for $<^\Phi$.

However, it turns out that this result is of little help in establishing global properties of the braid ordering, and so far there is not much to say about the connection. We just mention two easy remarks involving the left numerators and denominators introduced in Proposition I.4.9 and their right counterpart.

PROPOSITION 4.16. *For each braid β, the right denominator $D_R(\beta)$ (resp. the left denominator $D_L(\beta)$) is the $<$-minimal positive braid β_1 such that $\beta\beta_1$ (resp. $\beta_1\beta$) is positive.*

PROOF. By construction, we have $\beta \cdot D_R(\beta) = N_R(\beta)$ and $D_L(\beta) \cdot \beta = N_L(\beta)$, and both $N_R(\beta)$ and $N_L(\beta)$ are positive braids.

Conversely, assume that β_1 and $\beta\beta_1$ lie in B_∞^+. Then we have $\beta = (\beta\beta_1)\beta_1^{-1}$. By the right counterpart of Proposition I.4.9, we have $\beta_1 = D_R(\beta)\gamma$ for some γ in B_∞^+. Necessarily γ is trivial or σ-positive, and, therefore, we have both $\beta_1 \geqslant D_R(\beta)$ and $\beta_1 \geqslant^\Phi D_R(\beta)$.

Symmetrically, assume that β_1 and $\beta_1\beta$ lie in B_∞^+. Then we have $\beta = \beta_1^{-1}(\beta_1\beta)$. By Proposition I.4.9, there exists γ in B_∞^+ satisfying $\beta_1 = \gamma D_L(\beta)$. As γ belongs to B_∞^+, Property **S** implies both $\beta_1 \geqslant D_L(\beta)$ and $\beta_1 \geqslant^\Phi D_L(\beta)$. ☐

PROPOSITION 4.17. *For each braid β, the relations $\beta > 1$ and $N_L(\beta) > D_L(\beta)$ are equivalent. Similarly, $\beta >^\Phi 1$ and $N_L(\beta) >^\Phi D_L(\beta)$ are equivalent.*

The verification is straightforward as $<$ and $<^\Phi$ are left-invariant. Note that no such relation exists with the right numerators and denominators: for instance, for $\beta = \sigma_2^{-1}\sigma_1$, we have $\beta > 1$, but $N_R(\beta) = \sigma_1\sigma_2 < D_R(\beta) = \sigma_2\sigma_1$.

The previous observations are rather trivial and do not shed much light on the structure of $(B_n, <)$. The point is that the fractionary decompositions defines two injections ι_L and ι_R of B_n into a subset of $B_n^+ \times B_n^+$, but neither of them preserves the ordered structure. On the other hand, we can easily define a well-ordering on $B_n^+ \times B_n^+$ by using a lexicographical extension of the ordering of B_n^+, and, appealing to ι_L or ι_R, deduce a well-ordering of B_n, but the latter will not be invariant under left (or right) multiplication.

CHAPTER XVI

Open Questions and Extensions

In this chapter we mention further results and discuss open questions connected with the various aspects of braid orderings considered in this book.

We should start, however, with a very general remark. There are many approaches to braid groups that have not been considered in this book. In fact, braid groups play a role in many areas of mathematics that have not even been mentioned here—*e.g.*, algebraic geometry or mathematical physics. We can therefore still hope that new, illuminating perspectives on braid orderings will emerge in the future.

The chapter is organized as follows. In Section 1, we list some general questions about the σ-ordering and related topics. Then in Section 2, we discuss more specific questions that arise in the context of the successive chapters of this book, taken in the order in which they appear. Finally, we address in Section 3 some of the many extensions of braid groups from the point of view of order properties.

1. General questions

We begin with three types of questions involving the σ-ordering in general, namely its uses, its structure, and the problem of finding σ-positive representatives.

1.1. Uses of the braid ordering. In Chapter III we listed several applications of the orderability of braid groups and of the more specific properties of the σ-ordering of braids. However, up to now, the applications are not so plentiful and not so strong. This situation contrasts with the seemingly deep and, at the least, sophisticated properties of the σ-ordering explained in this text, which may appear as a promising sign for potentially powerful applications. So, although vague, the first open question is the following.

QUESTION 1.1. *How to use the braid ordering?*

In particular, one of the deepest properties of the σ-ordering of B_n known so far is the fact that its restriction to the braid monoid B_n^+, and even to the dual braid monoid B_n^{+*}, is a well-ordering. As emphasized in Section III.3.1, the well-order property is a very strong condition which enables one to distinguish one element in each nonempty subset—so, typically, in each conjugacy class or each Markov class. But, so far, this observation was of no use because we had no effective way to identify such minimal elements in practice, for instance in the case of the conjugacy problem. Thus, a special case of Question 1.1 is

QUESTION 1.2. *How to take advantage of the fact that the σ-ordering restricted to B_n^+ and B_n^{+*} is a well-ordering?*

To raise less fuzzy questions, we may think more specifically of the conjugacy problem. Let us say that two positive braids β, β' are positively conjugate if there exists a positive braid γ satisfying $\beta\gamma = \gamma\beta'$. As Δ_n^2 is central and multiplying any n-strand braid with a sufficient power of Δ_n^2 yields a positive braid, solving the conjugacy problem of the group B_n is algorithmically equivalent to solving the positive conjugacy problem of the monoid B_n^+. Now, for each positive braid β, the positive conjugacy class of β is a nonempty subset of B_n^+, hence, by the well-order property, it admits a $<$-least element.

Question 1.3. *Can one effectively compute the $<$-least element in a positive conjugacy class?*

A similar question can be raised with "Markov equivalence class" replacing "conjugacy class"; a solution would typically associate a computable, well-defined ordinal number with each knot.

The recent developments described in Chapters VII and VIII around the alternating and cycling normal forms of braids have not been exploited thus far, and they might be useful here.

1.2. Structure of the braid ordering. To a large extent, the structure of the σ-ordering of braids remains mysterious. Even in the case of B_3, the examples of Section II.2.1 show that the order $<$ is a complicated object. By contrast, the results of Chapters VII and VIII give a much simpler description for the restriction of $<$ to the submonoids B_n^+ and B_n^{+*} of B_n. The reason why the description is more satisfactory for B_n^+ than for B_n is that we have a simple recursive definition describing how the ordering of B_n^+ can be obtained from that of B_{n-1}^+. It is natural to raise the question of finding similar constructions for B_n, *i.e.*, more precisely, to raise

Question 1.4. *Does there exist a simple recursive definition of the σ-ordering on B_n from the σ-ordering on B_{n-1}?*

Question 1.5. *Does there exists a (computable) unique normal form on B_n so that, for any two braids β, β', whether $\beta < \beta'$ holds can be read directly from the normal forms of β and β'?*

The handle reduction algorithm of Chapter V does not answer Question 1.5, because it does not lead to a unique normal form, and because the result can be used to compare a braid with 1, but not directly to compare two braids. It is natural to wonder whether Bressaud's normal form of Section XI.1 might be useful here. Note that the algorithm based on the Mosher normal form presented in Section XII.3 yields a positive answer to the above question, except that the normal forms are not braid words but sequences of edge flips of singular triangulations.

1.3. Sigma-positive representatives. We mentioned in Proposition II.1.21 that the algorithmic complexity of the σ-ordering is at most quadratic. Several proofs have been given—in particular the stronger version of Chapter XII involving random access machine (RAM) complexity. It seems unlikely that there exist subquadratic algorithms, and the current result might be close to be optimal.

The situation is different with the stronger question of finding σ-positive representatives. Property **C** says that every nontrivial braid admits at least one representative braid word that is σ-positive or σ-negative. The exponential upper bound of Proposition II.1.22 is certainly far from optimal. Actually, all proofs of Property **C** sketched in this text lead to algorithmic methods for finding σ-positive representatives. Some solutions are inefficient: the only upper bound proved for the method of Chapter IV is a tower of exponentials of exponential height. Similarly, the proof of Chapter VII relies on a transfinite induction, and deriving a complexity statement is not easy. On the other hand, some methods, like handle reduction of Chapter V or transmission-relaxation of Chapter XII, seem quite efficient for producing short σ-positive expressions, but no result has been proved so far.

CONJECTURE 1.6. *For every $n > 3$, there exist numbers C_n, C'_n such that every nontrivial n-strand braid represented by a word of length ℓ has a σ-positive or σ-negative representative of length at most $C_n \cdot \ell$. Moreover, such a representative word can be found by an algorithm whose running time is bounded by $C'_n \cdot \ell^2$.*

Very recently, J. Fromentin announced a proof of Conjecture 1.6, with the explicit bounds $C_n = C'_n \leqslant 12n^2$. His method consists in turning the main proof of Section VIII.3 into an algorithm that, running on an arbitrary n-strand braid word w, translates it into a fractionary expression in the $a_{i,j}^{\pm 1}$ letters, then puts the numerator and the denominator in cycling normal form, and returns an explicit σ^{Φ}-positive or σ^{Φ}-negative expression for the braid represented by w.

REMARK 1.7. The braid $\sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}$ has no σ-positive representative of length 6 or less [**83**, Theorem 5.1], so $C_4 \geqslant 7/5$ must hold. Moreover, the word

$$\sigma_1\sigma_2^{-2}\sigma_3^2\sigma_4^{-2} \dots \sigma_{n-1}^{2e}\sigma_{n-2}^{2e}\sigma_{n-3}^{-2e} \dots \sigma_2^2\sigma_1^{-1}, \tag{1.1}$$

with $e = \pm 1$ according to the parity of n, has no σ_1-positive or σ_1-negative representative with fewer than $(n-2)(n+1)$ crossings. As the above n-strand braid word has length $4(n-2)$, it seems that C_n needs to satisfy $C_n \geqslant (n+1)/4$.

Finally, let us mention possible connections with the problem of finding geodesics in braid groups. For β a braid, let us denote by $\ell_\sigma(\beta)$ the minimal length of a braid word representing β. The *geodesic problem* is the question of effectively finding, for each braid word w, an equivalent braid word w' satisfying $\ell(w') = \ell_\sigma(\beta)$, *i.e.*, finding a shortest representative of β.

It is shown in [**169**] that the B_∞-version of the geodesic problem is co-*NP*-complete. However, this result says nothing about the problem in a fixed group B_n, nor about the problem of finding *quasi-geodesics*, *i.e.*, about algorithms that, starting with a braid word w, would produce an equivalent word of length $O(\ell_\sigma(\overline{w}))$. As for the latter problem, the symmetric version of the greedy normal form provides, for each n, a quadratic algorithm returning for each n-strand braid word w an equivalent braid word of length at most $n^2\ell_\sigma(\overline{w})$. *A priori*, the problem of finding short representatives is unconnected with the problem of finding σ-positive representatives. In particular, the examples of (1.1) show that, when n is unbounded, the ratio between the length of the shorterst σ-positive representative and the length of the shortest representative may be at least $n/4$. However, the algorithms solving the latter problem often turn out to also solve the former, at least in part.

2. More specific questions

We turn to more specific questions involving the σ-ordering of braids and the various approaches that have been developed in the text. For simplicity, we organize the questions according to the chapters they refer to.

2.1. Self-distributivity. Many questions about self-distributivity in general, and about the self-distributive structure of braids in particular, remain open. We shall mention one such question here, and refer to [**52**] and [**51**] for many more.

We saw in Section IV.2 that, under the hypothesis that $(S, *)$ is a left-cancellative LD-system, there exists a partial action of B_n on S^n. The action is partial in that $\boldsymbol{x} \bullet \beta$ need not exist for each $\boldsymbol{x}$ in S^n and each braid β. In Proposition IV.2.5, we proved that, for every braid β in B_n, there exist $\boldsymbol{x}$ in S^n such that $\boldsymbol{x} \bullet \beta$ is defined. Reversing the point of view, let us introduce, for $\boldsymbol{x}$ in S^n,

$$D_S(\boldsymbol{x}) = \{\beta \in B_n \mid \boldsymbol{x} \bullet \beta \text{ is defined}\}.$$

As the action of positive braids is always defined, we have $B_n^+ \subseteq D_S(\boldsymbol{x}) \subseteq B_n$. If $(S, *)$ is a rack, the braid action is defined everywhere, and so, for each $\boldsymbol{x}$, we have $D_S(\boldsymbol{x}) = B_n$. On the other hand, if S is the LD-system $(B_\infty, *)$ of Definition IV.1.7, it is easy to see that $D_{B_\infty}(1, \dots, 1)$ never contains σ_i^{-1}, and, therefore, it is a proper subset of B_n. In some cases studied in [**132**], $D_S(\beta_1, \dots, \beta_n)$ coincides with B_n^+, and, then, the restriction of the braid order $<$ to $D_S(\beta_1, \dots, \beta_n)$ is a well-ordering.

Conjecture 2.1 (Laver). *For all braids $\beta_1, \dots, \beta_n$, the subset $D_{B_\infty}(\beta_1, \dots, \beta_n)$ of B_n is well ordered by the σ-ordering.*

Note that the question is a pure problem of braids, in that it involves no objects other than braids.

2.2. Handle reduction. We have seen that handle reduction, as described in Chapter V, is a very efficient solution to the braid word problem in practice—actually, the most efficient known so far; see, for instance, [**35**] for a comparison with the Tetris algorithm of Chapter XI. However, there remains a large gap between the complexity bound established in Proposition V.1.5 and the experimental values of Tables V.1 and V.2. This suggests that the argument of Section V.2 is far from optimal. One may hope that this is the manifestation of some deep and as yet unknown aspect of the geometry of braids.

Conjecture 2.2. *For each n, the handle reduction algorithm for B_n has a quadratic time complexity, and a linear space complexity: starting from a braid word of length ℓ, the running time lies in $O(\ell^2)$ and all words produced during the algorithm have length in $O(\ell)$—so does, in particular, the final reduced word.*

Clearly, Conjecture 2.2 implies Conjecture 1.6. The second statement in Conjecture 2.2 would be a consequence of a positive solution to the following more general conjecture about the subword reversing method—which extends without change to many group presentations; see [**53**] and [**64**].

Conjecture 2.3. *If w is an n-strand braid word length ℓ, and if w' is a freely reduced braid word obtained from w by a sequence of special transformations in the sense of Definition* V.2.7*, each immediately followed by a free reduction, then the length of w' is at most $C_n \cdot \ell$, where C_n is some constant that depends on n.*

It has been demonstrated experimentally in [**161**] that, by combining two handle reductions, namely, starting from a braid word w, first reducing w to w', and then reducing $\Phi_n(w')$ to $\Phi_n(w'')$, one obtains a final word w'' that is a short representative of $\overline{w}$. A. Myasnikov conjectured a positive answer to

Question 2.4. *Does the above double handle reduction yield quasi-geodesics in B_n? In particular, does there exist a constant C_n such that, for w, w'' as above, one obtains $\ell(w'') \leqslant C_n \cdot \ell_\sigma(\overline{w})$?*

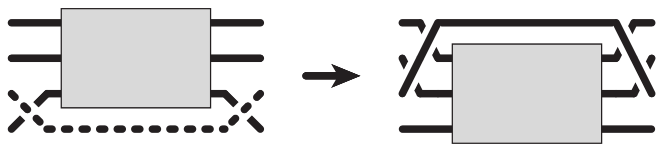

FIGURE 1. Coarse reduction of a σ_1-handle: instead of skirting around the next crossings, we push the strand responsible for the handle over the whole intermediate part.

To conclude with a perhaps easier question, let us come back to the *coarse handle reduction* briefly alluded to at the end of Chapter V. This variant of handle reduction consists in replacing a handle of the form $\sigma_i^e \cdot \mathrm{sh}^i(v) \cdot \sigma_i^{-e}$ with $\sigma_{i+1}^{-e} ... \sigma_{n-1}^{-e} \cdot \mathrm{sh}^{i-1}(v) \cdot \sigma_{n-1}^{e} ... \sigma_{i+1}^{e}$, as illustrated in Figure 1.

QUESTION 2.5. *Does coarse handle reduction converge?*

The arguments of Sections V.2.4 and V.2.5 are still valid, but those of Section V.2.2 are not, as the words obtained using coarse reduction from a word that is drawn in some set $\mathrm{Div}(\beta)$ may escape from $\mathrm{Div}(\beta)$. Experiments suggest that coarse reduction always converges, but the proof is still to be found.

2.3. Connection with the Garside structure. The results of Section VI.3 remain partial, and it is an obvious question to ask for a complete description of the $<$-increasing enumeration $S_{n,d}$ of the divisors of Δ_n^d similar to the one given in Section VI.2 for the case $n = 3$. The general case is probably difficult, but the case of 4-strand braids should be doable. Owing to the recursive rule of Definition VI.1.6, one can expect the generic entry of $S_{4,d-1}$ to have six copies in $S_{4,d}$, but some entries from $S_{4,d-2}$ have three copies in $S_{4,d}$ only.

More promising might be the questions of braid combinatorics to which the approach of Chapter VI leads. Counting problems involving braids have been little investigated, and a number of questions remain open. We saw in Section VI.1 that a crucial role in counting problems connected with the greedy normal form is played by a certain $n! \times n!$ matrix M_n, whose rows and columns are indexed by permutations of $\{1, ..., n\}$, and the (π, π')-entry of M_n is 1 if and only if all descents of π'^{-1} are descents of π, and is 0 otherwise. In particular, the number of positive n-strand braids that divide Δ_n^d is directly connected with the eigenvalues of M_n—and of an equivalent smaller matrix $\widehat{M_n}$ whose size is the number of partitions of n. Table 1 below shows the associated characteristic polynomials for small values of n, immediately leading to

CONJECTURE 2.6. *For each n, the characteristic polynomial of M_{n-1} divides that of M_n. More precisely, the spectrum of M_n is the spectrum of M_{n-1}, plus $p(n) - p(n-1)$ nonzero eigenvalues.*

Very recently, a proof of the first part of the conjecture has been announced by F. Hivert, J.C. Novelli, and J.Y. Thibon in [**109**]. They use the framework of noncommutative symmetric and quasi-symmetric functions connected with combinatorial Hopf algebras, and they construct an explicit derivation that connects M_{n-1} and M_n.

TABLE 1. Characteristic polynomial of M_n up to a power of x—together with the corresponding spectral radius ρ_n and its relative growth.

$$P_{M_1}(x) = x - 1$$
$$P_{M_2}(x) = P_{M_1}(x) \cdot (x - 1)$$
$$P_{M_3}(x) = P_{M_2}(x) \cdot (x - 2)$$
$$P_{M_4}(x) = P_{M_3}(x) \cdot (x^2 - 6x + 3)$$
$$P_{M_5}(x) = P_{M_4}(x) \cdot (x^2 - 20x + 24)$$
$$P_{M_6}(x) = P_{M_5}(x) \cdot (x^4 - 82x^3 + 359x^2 - 260x + 60)$$
$$P_{M_7}(x) = P_{M_6}(x) \cdot (x^4 - 390x^3 + 6,024x^2 - 13,680x + 8,640)$$

n	1	2	3	4	5	6	7	8
ρ_n	1	1	2	5.449	18.717	77.405	373.990	2,066.575
$\rho_n/(n\rho_{n-1})$	-	0.5	0.667	0.681	0.687	0.689	0.690	0.691

Furthermore, for small values of n, all numbers $b_{n,d}(\beta)$—except $b_{n,d}(\Delta_n)$, which is 1—grow like ρ_n^d, where ρ_n is the spectral radius of M_n. Whether this is always true is unknown, but it makes it natural to investigate ρ_n. The trivial upper bound $\#\mathrm{Div}(\Delta_n^d) \leqslant (n!)^d$ suggests that we compare ρ_n with $n\rho_{n-1}$. The values listed in Table 1 may suggest that this ratio tends to $\log 2$.

Finally, it should be clear that all the above questions involving the symmetric groups can be extended to other finite Coxeter groups and to the corresponding braid groups, *i.e.*, the spherical Artin–Tits groups of Section 3.1.

2.4. Alternating decompositions. The recursive characterization of the σ-ordering of B_n^+ by means of the Φ_n-splitting provides a very simple description of this ordering. However, in the current exposition, this description, as well as all results of Section VII.4, is deduced from Burckel's delicate combinatorial methods, which involve in particular transfinite inductions.

QUESTION 2.7. *Does there exist for the recursive characterization of the σ-ordering of B_n^+, and for the other results of Section* VII.4*, a direct proof in the vein of the one described in Chapter* VIII*?*

The two approaches developed in Chapters VII and VIII are quite similar, and answering Question 2.7 in the positive should not be impossible. However, the Artin relations differ from the Birman–Ko–Lee relations in that some of them involve words of length 3, and this small technical difference might make the solution more difficult in the case of B_n^+.

Another natural question—that is problably connected with the previous one—involves the computation of the ordinal rank. With Corollary VII.2.22, we have a simple closed formula that expresses the rank of any positive 3-strand braid in the well-ordering of B_3^+ in terms of its Φ-normal form, which itself is very easily computed. In this way, we arguably obtain an optimal description of the ordering, as we identify the position of any element in an absolute way.

QUESTION 2.8. *Does there exist a similar method for determining the ordinal rank of an arbitrary braid in $(B_n^+, <^\Phi)$?*

A general solution is proposed in [**28**]. It relies on Burckel's notion of reducible words, and consists in counting how many irreducible braid words precede a given one in the tree ordering. The method is algorithmically efficient only in the case of 3 strands, and further investigation is certainly needed in the general case.

2.5. Dual braid monoids. The results of Chapter VIII are quite recent and many open questions remain, in particular the counterpart of Question 2.8. Another natural question would be to determine the ranks of the elements of B_n^+ inside $(B_n^{+*}, <^\Phi)$, hence to compare the ranks of a positive braid in $(B_n^+, <^\Phi)$ and in $(B_n^{+*}, <^\Phi)$.

Another problem is to study the action of conjugacy on B_n^{+*}, in particular in view of Question 1.3. The definition of the cycling normal form suggests the introduction of a cycling operation similar to that used in the Garside-based solution to the conjugacy problem, and one may hope for progress in this direction. A similar approach is of course possible with the alternating decompositions of Chapter VII, but the fact that the family of generators $a_{i,j}$ is closed under conjugacy might make the context of Chapter VIII more suitable.

Other types of questions connected with the Φ- and the ϕ-normal forms involve random walks on the monoids B_n^+ or B_n^{+*} and possible stabilization phenomena, as studied for instance in [**149**]. To state a simple question, we may ask

QUESTION 2.9. *Assume that X is a random walk on B_n^+ (resp. B_n^{+*}). What is the expectation for the Φ_n-breadth (resp. the ϕ_n-breadth) of X?*

In other words, what is the average Φ_n-breadth of a random positive n-strand braid of length ℓ? Experiments suggest a connection with $\sqrt{\ell}$ that is not explained so far.

Another natural question is whether the Φ- and ϕ-normal forms might be connected with an automatic structure on B_n. It is known that the languages of normal words are regular languages, but it is unclear whether any form of the fellow traveler property might be satisfied.

Finally, we saw in Proposition II.4.2 that the restriction of the σ-ordering to every submonoid of B_n generated by finitely many conjugates of the generators σ_i is a well-ordering. So, the results about B_n^+ and B_n^{+*} might extend to more general monoids.

QUESTION 2.10. *Let B_n^{++} be the submonoid of B_n generated by all braids of the form $\beta\sigma_i\beta^{-1}$ with β a simple n-strand braid. What is the order type of the restriction of the σ-ordering to B_n^{++}?*

More generally, the algebraic study of the monoid B_n^{++} is a natural question that has not yet been addressed. It is known that this monoid is not a Garside monoid in the usual sense, but it seems nevertheless to satisfy much of the interesting properties of a Garside monoid, and, in particular, it might be associated with a new automatic structure on B_n.

2.6. Automorphisms of a free group. The study of automorphism groups and outer automorphism groups of free groups is currently an area of intense activity; see for instance [**191**] for an excellent survey. The analogy between $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ on the one hand and mapping class groups on the other is one of the driving forces behind this research. Now, it is very well known, and explained in Chapter IX, that the braid group B_n is a subgroup of $\mathrm{Aut}(F_n)$, so it is natural to ask the following questions.

Question 2.11. *Which subgroups of* $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$ *are left-orderable? Which ones are bi-orderable?*

There is certainly no shortage of torsion-free subgroups, *i.e.*, of candidates for being (left-)orderable. Indeed, let us consider the natural homomorphisms of $\mathrm{Aut}(F_n)$ to $\mathrm{GL}(n, \mathbb{Z})$ and of $\mathrm{Out}(F_n)$ to $\mathrm{GL}(n, \mathbb{Z})$. Using a result of Baumslag and Taylor, one can show that the preimage of any torsion-free finite-index subgroup of $\mathrm{GL}(n, \mathbb{Z})$ under either of these homomorphisms is a torsion-free finite-index subgroup of $\mathrm{Aut}(F_n)$ and of $\mathrm{Out}(F_n)$.

2.7. Curve diagrams. In Chapter X we gave a proof of Property **C** by using a relaxation algorithm for curve diagrams, in the sense explained in Chapter XI. More precisely, the algorithm works by repeatedly sliding a puncture along a so-called useful arc, and relaxing the diagram after each slide. We saw that the length of the σ-consistent output braid could grow exponentially with the length of the input braid word. The reason for this is that the length of each of the useful arcs can grow exponentially with the length of the input, whereas the length of the relaxing braid (the puncture slide) is proportional to the length of the useful arc. So the algorithm in question is inefficient, but it is so for an obvious reason, and it is easy to invent improvements of the algorithm.

In fact, it seems that the idea of relaxing curve diagrams, as explained in Chapter XI, rather tends to lead to algorithms which are very efficient, but whose efficiency is difficult to prove. Hence we have the following very vague problem.

Question 2.12. *Is there a precise, provable statement which expresses the idea that any relaxation type algorithm which does not have an obvious obstruction to being of polynomial complexity has a quadratic time complexity and a linear space complexity?*

2.8. Relaxation algorithms. Question 2.12 applies in particular to the two types of relaxation algorithms discussed in detail in Chapter XI, namely Bressaud's relaxation algorithm and the transmission-relaxation schemes from [**74**].

A more concrete conjecture specifically involves the Tetris algorithm of Section XI.1. Note that the truth of the following conjecture would imply the truth of Conjecture 1.6:

Conjecture 2.13. *For each* n*, the Tetris algorithm for* B_n *has a quadratic time complexity, and a linear space complexity. Starting from a braid word of length* ℓ *in the generators* $\sigma_{i,j,p}$*, the running time lies in* $O(\ell^2)$*, and all words produced during the algorithm have length in* $O(\ell)$*. Moreover, the linear constants in these bounds depend linearly on the braid index* n*.*

As mentioned in Section XI.1, the language of braid words in normal form is recognized by a finite-state automaton, but it fails to be (synchronously) automatic.

Question 2.14. *Is there an automatic structure on the braid group that is conceptually close to the approach of Section XI.1?*

Bressaud's original motivation was related to the study of random walks on the braid group B_n and of its Poisson boundary. This boundary has been identified by Kaimanovich and Masur [**117**] as the space of uniquely ergodic measured foliations on the disk D_n. Bressaud's relaxation procedure from Section XI.1 may be applied to such a foliation just as well as to a curve diagram, yielding an infinite braid word—this is like a continued fraction expansion of the measured foliation [**157**]. Thus, in the context of trying to find a more combinatorial description of the Poisson boundary, one can ask

QUESTION 2.15. *Is Bressaud's normal form stable for random walks on B_n?*

In the context of Section XI.2, the braid group B_n is equipped with the metric which gives to the braid $\Delta_{i,j}^d$ the length $\log_2(|d|+1)$. This approach provides a combinatorial model of the thick part of the Teichmüller space $\mathcal{T}$, equipped with the Teichmüller metric d_{Teich}. Further squashing this metric by giving length one to any nonzero power of a Garside-like braid $\Delta_{i,j}$ yields a combinatorial model of the Teichmüller space, equipped with the Weil–Peterson metric d_{WP}—for both of the above statements; see Rafi [**178**]. Now any word whose letters are of the form $\Delta_{i,j}^d$ represents a path in all three spaces: the Cayley graph of the braid group, the combinatorial model of $(\mathcal{T}, d_{\mathrm{Teich}})$, and the combinatorial model of $(\mathcal{T}, d_{\mathrm{WP}})$.

CONJECTURE 2.16. *The set of braid words produced by the transmission-relaxation algorithm forms a family of parametrized uniform quasi-geodesics in all three spaces.*

If this were true, then this normal form could serve as a very concrete and algorithmically efficient tool in the exploration of these spaces.

2.9. Triangulations. In Chapter XII we studied the Mosher normal form of a braid, which is a sequence of combinatorial types of triangulations on the surface D_n, where each element of the sequence is obtained from the preceding one by an edge flip. The connection between braid groups and triangulation sequences comes from the fact that the complex of triangulations of D_n, where triangulations are adjacent in the complex if they differ by an edge flip, is a refinement of the Cayley graph of the braid group, and, more precisely, is quasi-isometric to it. We saw in Chapter XII that the Mosher normal form is a useful tool for understanding the σ-ordering.

Since Mosher's discovery of the (automatic) normal form, other complexes which are quasi-isometric to the Cayley graph of B_n have greatly contributed to our understanding of braid groups, for instance the train track complex [**104**] and the marking complex [**151**]. Other complexes, like the pants complex [**24**] and the curve complex [**151**], have also been studied in great depth.

QUESTION 2.17. *Can any of the above-mentioned complexes contribute to our understanding of braid orderings?*

2.10. Hyperbolic geometry. We have seen in Chapter XIII how to define an infinite family of distinct left-invariant orderings on B_n whose restriction to B_n^+ is a well-ordering, but we did not address the determination of the length of that well-ordering. The following question may well be quite easy to answer.

QUESTION 2.18. *What are the possible ordinal types for the restriction of $\prec$ to B_n^+ when $\prec$ is an ordering of Nielsen–Thurston type?*

2.11. The space of all orderings of B_n. We have seen in Chapter XIV that there exist many orders on B_n. In connection with Questions 2.18 above and 2.24 below, it is natural to raise:

QUESTION 2.19. *Assume that $\prec$ is a left-invariant ordering of B_n that satisfies the subword property. What are the possible ordinal types for the restriction of $\prec$ to B_n^+?*

As B_n is not bi-orderable, one could imagine that only long orders may exist on it. This is *not* the case, as is shown in the following ordering, which was already considered in Remark XIII.1.9. We recall that ϵ denotes the exponent sum, *i.e.*, the homomorphism of B_∞ to $\mathbb{Z}$ that maps every σ_i to 1.

PROPOSITION 2.20. *For β, β' in B_n, declare that $\beta <_\epsilon \beta'$ is true if we have either $\epsilon(\beta) < \epsilon(\beta')$, or $\epsilon(\beta) = \epsilon(\beta')$ and $\beta < \beta'$. Then $<_\epsilon$ is a left-invariant ordering of B_n whose restriction to B_n^+ is a well-ordering of ordinal type ω.*

PROOF. For each (positive) braid β, each braid γ satisfying $\gamma <_\epsilon \beta$ must satisfy $\epsilon(\gamma) \leqslant \epsilon(\beta)$. For fixed n, there exist only finitely many positive braids γ satisfying this condition. ☐

Note that the ordering $<_\epsilon$ is the lexicographical ordering deduced from the exact sequence

$$1 \to [B_n, B_n] \to B_n \to \mathbb{Z} \to 1$$

using the σ-ordering of the commutator subgroup and the usual ordering of $\mathbb{Z}$.

Other natural questions involve convex subgroups. As already mentioned, the family of convex subgroups of a left-ordered group is linearly ordered under inclusion and closed under unions and intersections. According to Section II.3.4, the σ-ordering of B_n has exactly n convex subgroups, including $\{1\}$ and B_n itself, and only the latter two subgroups are normal.

On the other hand, with respect to the ordering of Proposition 2.20, the commutator subgroup $[B_n, B_n]$ is both convex and normal. Other convex subgroups are $H_k \cap [B_n, B_n]$, where H_k denotes the subgroup of B_n generated by $\sigma_k, \sigma_{k+1}, \dots, \sigma_{n-1}$, but they are not normal.

For a third example, consider the special case $n = 3$. The commutator subgroup $[B_3, B_3]$ is free on two generators, hence it is bi-orderable. In fact, using the Magnus ordering of this free group, one obtains infinitely many convex subgroups, namely the inverse images of the ideals $1 + O(X^k)$. Using this ordering of $[B_3, B_3]$ and the lexicographic ordering as described in the previous paragraph, one can construct a left-ordering of B_3 which has infinitely many distinct convex subgroups.

QUESTION 2.21. *What convex subgroups must a left-invariant ordering of B_n admit? Is there a left-invariant ordering of B_n which has no convex subgroups at all, other than $\{1\}$ and B_n? What about bi-invariant orderings of PB_n?*

2.12. Pure braid groups. The Magnus ordering of the pure braid group PB_n shares with the σ-ordering of B_n the property that its restriction to the monoid B_n^+ of positive braids—and, similarly, to the dual braid monoid B_n^{+*}—is well-ordered. This leads to several problems.

First, every positive pure braid receives a unique ordinal rank that describes its position in the well-ordered set $(PB_n^+, <_{\mathrm{M}})$. As in the case of B_n^+ and the σ-ordering, we can raise

QUESTION 2.22. *Does there exist a practical method for determining the rank of a pure braid in $(PB_n^+, <_{\mathrm{M}})$?*

We observed that the Magnus ordering extends to the pure braid group PB_∞, and we can consider its restriction to the positive monoid PB_∞^+. It is easy to see that $(PB_\infty^+, <_{\mathrm{M}})$ is not a well-ordering as it admits the infinite descending sequence $\sigma_1^2 >_{\mathrm{M}} \sigma_2^2 >_{\mathrm{M}} \dots$. The situation resembles that of the σ-ordering. In the latter case, we obtained a well-ordering of B_∞^+ by considering a flipped version so as to reverse the problematic inequalities. The point is that $\mathrm{sh}(B_{n-1}^+)$ is the initial segment of $(B_n^+, <)$ determined by σ_1, implying that, after the flip, B_{n-1}^+ is the initial segment of $(B_n^+, <^\Phi)$ determined by σ_{n-1}. The counterpart of that property fails for the Magnus ordering of PB_n: every pure braid in $\mathrm{sh}(PB_\infty^+)$ is smaller than σ_1^2, but the converse is false, as we have for instance $1 <_{\mathrm{M}} \sigma_2\sigma_1^2\sigma_2 <_{\mathrm{M}} \sigma_1^2$. The example of $1 <_{\mathrm{M}} \sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3 <_{\mathrm{M}} \sigma_3\sigma_2^2\sigma_3$ shows that $\mathrm{sh}(PB_{n-1}^+)$ is not even convex in PB_n^+. This does not discard the possibility of defining a flipped version of the Magnus ordering on PB_n^+, and then on PB_∞^+, but the structure of the latter is unclear.

Question 2.23. *Let $<_{\mathrm{M},n}^\Phi$ denote the image of the Magnus ordering of PB_n under the flip automorphism Φ_n. Is the induced ordering of PB_∞^+ a well-ordering? If it is, what is its order type? Would some variant of the Magnus ordering be more suitable for such constructions?*

There seems to be a large difference of complexity between the σ-ordering of B_n and the Magnus ordering of PB_n. In particular, this difference is visible in the gap between the order types of the restrictions to B_n^+, namely the relatively large ordinal $\omega^{\omega^{n-2}}$ for the former, to be compared with the modest ordinal ω^{n-1} for the latter. It is natural to wonder whether this difference is essential.

Question 2.24. *Can there exist a bi-invariant ordering of PB_n whose restriction to PB_n^+ is a well-ordering of order type larger than ω^{n-1}?*

The point here is that we consider *bi*-invariant orderings of PB_n: the restriction of the σ-ordering to PB_n is a left-invariant ordering of PB_n, whose restriction to PB_n^+ is a well-ordering whose order type is easily checked to be $\omega^{\omega^{n-2}}$.

More generally, one could wonder whether a bi-invariant ordering on a group G can be as complicated—in a sense to be made precise—as a left-invariant ordering of G.

3. Generalizations and extensions

The braid groups can be generalized in many respects, so extending the results mentioned in this text to other groups is an obvious task. Of course, several types of extensions may be considered: extending orderability, extending the specific σ-ordering of braids, extending the various approaches that lead to that ordering, extending the associated algorithms, etc. Here, we shall briefly review a few results and conjectures involving such extensions, but we shall not try to be exhaustive.

3.1. Artin–Tits groups. Starting from the presentation of B_n, rather than from any geometric description, we can situate braid groups in a larger framework of a completely different nature: they are special cases of Artin–Tits groups, and more specifically spherical Artin–Tits groups, as introduced in [**65, 21**].

An *Artin–Tits* group is, by definition, a group admitting a presentation with finitely many generators $s_1, \dots, s_n$ and relations of the form $s_is_js_is_j\dots = s_js_is_js_i\dots$, where the words on both sides of the equality sign have the same length (finite and at least 2) depending on i and j, and there is at most one relation for each pair $\{i, j\}$.

For instance, finitely generated free groups (no relations) and free Abelian groups (commutation relations between all pairs of generators) are Artin–Tits groups. An Artin–Tits group is said to be *spherical* if the associated Coxeter group, namely the group obtained by adding the relations $s_i^2 = 1$ for $i = 1, \dots, n$, is finite [**112**]. The braid group B_n is then the spherical Artin–Tits group associated with the symmetric group $\mathfrak{S}_n$, thus corresponding with the so-called Coxeter type A_{n-1}.

QUESTION 3.1. *Which Artin–Tits groups are left-orderable or bi-orderable?*

Currently, the only Artin–Tits groups known to be left-orderable are those that embed in mapping class groups. Among the spherical ones, these are all but those of type $\mathrm{E}_6, \mathrm{E}_7$, and E_8 [**193, 173**]. Let us mention that, if the Artin–Tits group of type E_8 is left-orderable, then, due to embedding properties, all spherical Artin–Tits groups are; see [**159**]. Among the nonspherical ones, there is one well-known family of groups that are bi-orderable [**72**], namely the right-angled Artin–Tits groups (also called partially commutative groups), which have only commutation relations. Indeed, these groups embed in pure surface braid groups; see Section 3.2.

The more specific question of extending the σ-ordering of braid groups to other Artin–Tits groups seems rather artificial and is not very promising in general. It is well known that sending s_1 to σ_1^2 and s_i to σ_i for $i \geqslant 2$ defines an embedding of the type B_n Artin–Tits group into the corresponding type A_n group, *i.e.*, into the braid group B_{n+1}. In this way, one obtains an exact counterpart of the σ-ordering for each type B_n Artin–Tits group and, more generally, for every Artin–Tits group that is a product of type A and type B Artin–Tits groups.

In [**186**], Hervé Sibert proves

PROPOSITION 3.2. *The counterpart of Property* **A** *is true in every Artin–Tits group. The counterpart of Property* **C** *is true only for those groups that are products of type* A *and type* B *groups.*

Thus, except in the special cases of types A and B, extending the definition of the σ-ordering leads to a partial ordering only.

Many algebraic properties of spherical Artin–Tits groups extend to a larger class of groups called Garside groups [**53, 55, 63, 175, 174, 185**]. In particular, the latter are known to be torsion-free.

QUESTION 3.3. *Is every Garside group left-orderable?*

3.2. Mapping class groups and surface braid groups. We defined in Section I.3 the mapping class group $\mathcal{MCG}(\mathcal{S}, \mathcal{P})$ of any compact surface $\mathcal{S}$ relative to a finite set of punctures $\mathcal{P}$. Closely related is the *n-strand braid group $B_n(\mathcal{S})$* of a surface $\mathcal{S}$. It can be defined as the fundamental group of the configuration space of n unlabelled points in $\mathcal{S}$. More geometrically, we can fix arbitrarily n distinguished points $P_1, \dots, P_n$ in the interior of $\mathcal{S}$. Then $B_n(\mathcal{S})$ is the group of isotopy classes of braids in $[0,1] \times \mathcal{S}$, where each strand starts at one of the points $\{0\} \times P_i$ and ends at one of the points $\{1\} \times P_j$. For instance, we have $B_1(\mathcal{S}) = \pi_1(\mathcal{S})$ for every surface $\mathcal{S}$. It is a simple fact [**13**] that for all compact surfaces $\mathcal{S}$, the braid group $B_n(\mathcal{S})$ is in a natural way a subgroup of $\mathcal{MCG}(\mathcal{S}, \{P_1, \dots, P_n\})$, except if $\mathcal{S}$ is one of the following: the sphere S^2, the sphere with one or with two points removed, the torus, or the Klein bottle.

PROPOSITION 3.4. *Let $\mathcal{S}$ be any compact surface with nonempty boundary. Then $\mathcal{MCG}(\mathcal{S})$ is left-orderable.*

In this statement the surface may or may not have punctures, and may or may not be orientable. A proof of this fact appeared in [**182**]. It uses a simple generalisation of the curve diagram construction from Section X.1.2. Since subgroups of left-orderable groups are also left-orderable, we deduce the following.

Corollary 3.5. *Let $\mathcal{S}$ be any compact surface with or without punctures, orientable or nonorientable, but necessarily with $\partial\mathcal{S} \neq \emptyset$. Then $B_n(\mathcal{S})$ is left-orderable.*

However, no interesting analogue of the notion of σ-positivity is known in this case. Nevertheless, it would be interesting to generalize the classification of Nielsen–Thurston type orderings encountered in Chapter XIII to this setting.

The situation is much more subtle if $\mathcal{S}$ is a compact surface without boundary. The mapping class groups of such surfaces have torsion, and consequently are not left-orderable.

Question 3.6. *If $\mathcal{S}$ is a compact orientable surface without boundary, is the surface braid group $B_n(\mathcal{S})$ left-orderable?*

Other interesting questions occur when we consider the pure braid groups of a surface. By definition, the pure n-strand braid group in a surface $\mathcal{S}$, denoted $PB_n(\mathcal{S})$, is the fundamental group of the configuration space of n labelled points in the surface $\mathcal{S}$—or, equivalently the group of pure braids in $\mathcal{S} \times [0, 1]$ where each strand has one endpoint in $\mathcal{S} \times \{0\}$ and the other in $\mathcal{S} \times \{1\}$.

Some of these braid groups are quite obviously not bi-orderable. For instance, for $n \geqslant 3$, the pure n-strand braid group of the sphere $PB_n(S^2)$ has torsion. Indeed, if Δ^2 denotes the usual full-twist braid inside an embedded disk in S^2 containing all the punctures, then Δ^2 is nontrivial whereas its square Δ^4 is trivial—this is the famous belt trick.

Similarly, if the surface $\mathcal{S}$ is nonorientable, then, for $n \geqslant 2$, a generator σ_i^2 of $PB_n(\mathcal{S})$ is conjugate to its own inverse—the conjugating element being a pure braid which pushes the two strands involved in σ_i once around an embedded Möbius band. So $PB_n(\mathcal{S})$ has generalized torsion.

J. González-Meneses proved [**100**] that these obvious obstructions to bi-orderability are the only ones:

Proposition 3.7. *If $\mathcal{S}$ is an orientable closed surface of genus $g \geqslant 1$, then, for $n \geqslant 1$, the pure braid group $PB_n(\mathcal{S})$ is bi-orderable.*

The proof works by developing the ideas of Section XV.3, and combining them with some delicate combinatorics in surface braid groups [**101**].

As an immediate consequence of the theorem, we have that all right-angled Artin groups are bi-orderable, because, according to [**39**], they embed in pure surface braid groups. As a further corollary we have that all subgroups of right-angled Artin groups are bi-orderable, and this class of groups is surprisingly rich: it contains, for instance, all graph braid groups, all surface groups except the three simplest nonorientable ones, and certain 3-manifold groups.

Beyond the question of orderability, one may also wish to extend other techniques developed in this book to the more general context of mapping class groups. In this respect, it is especially tempting to try and generalize the ideas of Section XI.2 to other situations. This leads to the following ambitious claim.

CONJECTURE 3.8. *All results and techniques mentioned in Section* XI.2 *can be generalized to mapping class groups of higher genus surfaces.*

One could even speculate whether similar techniques might be applied to the outer automorphism group of free groups, possibly with applications to geometry of the outer space.

3.3. Torelli groups. The *Torelli group* of a surface $\mathcal{S}$ is defined to be the subgroup of $\mathcal{MCG}(\mathcal{S})$ consisting of those elements which act trivially on the homology $H_1(\mathcal{S}, \mathbb{Z})$, *i.e.*, on the Abelianization of $\pi_1(\mathcal{S})$. For a good recent survey on what is known and not known about Torelli groups, see [**81**].

PROPOSITION 3.9. *For each compact surface $\mathcal{S}$, the Torelli group of $\mathcal{S}$ is residually nilpotent, and hence bi-orderable.*

The proof follows from the deep structure theory of Torelli groups, whose fundamental results are due to Dennis Johnson [**114**]. A crucial role in this theory is played by the so-called Johnson filtration, a certain infinite sequence of subgroups different from the lower central series of the Torelli group, such that the quotient of two successive terms is always torsion-free Abelian. The structure of the Johnson filtration is in fact a manifestation of a more general phenomenon; see [**9**].

QUESTION 3.10. *Is the mapping class group $\mathcal{MCG}(\mathcal{S})$ virtually orderable or even virtually bi-orderable? In particular, what are the orderability properties of the kernel of the action on $H_1(\mathcal{S}, \mathbb{Z}/p\mathbb{Z})$, where p is a prime?*

This subgroup of elements acting trivially on homology with $\mathbb{Z}/p\mathbb{Z}$-coefficients is torsion-free [**113**, Chapter 1], but by a result of Hain [**103, 153**] its Abelianization is finite, at least when the genus of $\mathcal{S}$ is 3 or more. These results are related to the well-known question whether the mapping class group of a closed surface virtually surjects to $\mathbb{Z}$, *i.e.*, whether it has a finite index subgroup which has an infinite Abelian quotient.

3.4. Surface groups and 3-manifold groups. It is shown in [**181**] that the fundamental group—or, equivalently, the one-string braid group—of every compact surface, except for the projective plane $\mathbb{R}P^2$, is left-orderable. Moreover, with the further exception of the Klein bottle, all surface fundamental groups are actually bi-orderable.

The situation is more subtle when considering the case of fundamental groups of compact 3-manifolds, which we will refer to simply as 3-manifold groups. A study of these groups is initiated in the paper [**18**], where necessary and sufficient conditions are derived for the left-orderability and bi-orderability of fundamental groups of the important class of Seifert-fibred 3-manifolds (manifolds which are foliated by topological circles). It is also shown there that for each of the eight 3-dimensional geometries, there exist manifolds modelled on that geometry which have left-orderable groups and also there exist examples whose groups are not left-orderable.

Recall that a 3-manifold is called irreducible if every smooth 2-sphere bounds a 3-ball in the manifold. An important general result of [**18**] is that all compact irreducible orientable 3-manifolds with a positive first Betti number have left-orderable groups. In particular, all knot and link groups are left-orderable.

QUESTION 3.11. *Which knot groups are bi-orderable?*

The group of the figure eight knot 4_1 is bi-orderable. The first unknown case is the knot 5_2 in knot tables.

QUESTION 3.12. *Given an automorphism φ of a surface group G (or more generally of any bi-orderable group), under what conditions does there exist a bi-invariant ordering of G which is φ-invariant, meaning $x \prec y$ implies $\varphi(x) \prec \varphi(y)$?*

This is relevant to the study of 3-manifolds which are bundles over S^1, with surface fibres. If φ is the monodromy associated with such a fibration, then a φ-invariant bi-invariant ordering of the fibre's group naturally leads to a bi-invariant ordering of the fundamental group of the total space, and vice versa. In [**172**] this observation, as well as the techniques described in Chapter XV, are used to prove that certain fibred knots with pseudo-Anosov monodromy have bi-orderable groups. By contrast, the group of any torus knot cannot be bi-ordered, because it contains elements which do not commute, while a power of one of those elements commutes with the other, which cannot occur in a bi-orderable group.

CONJECTURE 3.13. *If G is the fundamental group of a closed orientable (irreducible) 3-manifold, then G is virtually bi-orderable, i.e., there exists a subgroup of finite index which is bi-orderable.*

It is shown in [**18**] that Conjecture 3.13 holds for Seifert-fibred 3-manifolds, and more generally for all manifolds with a geometric structure, except possibly hyperbolic manifolds. We do not even know if hyperbolic manifold groups are virtually left-orderable.

To put the difficulty of these questions into perspective, we point out that from general properties of orderable groups and covering space theory one can show that any 3-manifold satisfying Conjecture 3.13 also satisfies a certain well-known conjecture in 3-manifold theory; this conjecture states that any closed, orientable, irreducible 3-manifold $\mathcal{M}$ with an infinite fundamental group has a finite-sheeted cover $\widetilde{\mathcal{M}}$ with positive first Betti number. This conjecture remains open despite Perelman's recent proof of the geometrization conjecture.

In another direction, the pure braid group can be regarded as the fundamental group of the complement of the family of hyperplanes $z_i = z_j$ in the space $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. The analysis of orderability for PB_n applies to many other (but not all) complex hyperplane arrangements.

PROPOSITION 3.14. *The fundamental group of the complement of every hyperplane arrangement of fibre type is bi-orderable.*

For further details and a recent discussion of the fundamental groups of hyperplane arrangements, see [**167**].

3.5. A topological completion. Let us now come back to the specific case of braids and their σ-ordering. Another line of research consists in looking for extensions of that particular ordering to larger spaces. Here we start with a topological completion.

As mentioned in Section II.3.2, the topology on B_∞ associated with the σ-ordering is metrizable, the radius 2^{-n} ball centered at 1 being the shifted subgroup $\mathrm{sh}^n(B_\infty)$. With respect to that topology, a sequence $\beta_1, \beta_2, \dots$ converges to the trivial braid if for each integer n there exists an integer p such that, for $q > p$, all braids β_q belong to $\mathrm{sh}^n(B_\infty)$. The order topology renders B_∞ homeomorphic to $\mathbb{Q}$—in particular, not completely metrizable.

Very recently, P. Fabel announced in [**79**] the following construction of a completion of B_∞. Let D_∞ be the closed unit disk in $\mathbb{C}$ centered at 0 with punctures on the real line at $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$, and let H_∞ be the group of homeomorphisms of D_∞ fixing the boundary pointwise. Let $M(H_\infty)$ be the group of isotopy classes of H_∞. Viewing it as the mapping class group of a disk containing the punctures at $0, \frac{1}{2}, \ldots, 1 - \frac{1}{n}$, one embeds B_n in $M(H_\infty)$. One can equip $M(H_\infty)$ with a metric d by declaring

$$d(\beta_1, \beta_2) = \inf_{h_1, h_2} \sup_{z \in D_n} d_{\mathbb{C}}(h_1(z), h_2(z)) + \inf_{h_1, h_2} \sup_{z \in D_n} d_{\mathbb{C}}(h_1^{-1}(z), h_2^{-1}(z)),$$

where h_i is a homeomorphism of D_∞ representing the isotopy class β_i. The result announced by Fabel is

Proposition 3.15. *The group $(M(H_\infty), d)$ is complete as a metric space, it contains B_∞ as a dense subgroup, and it is left-ordered by an ordering that extends the σ-ordering of B_∞.*

Question 3.16. *Are all completions extending the σ-ordering of B_∞ essentially equivalent?*

3.6. Parenthesized braids. We shall conclude with another seemingly promising extension of the braids and their σ-ordering—one that will at least enable us to end with a nice figure.

Thompson's group F is a finitely presented group which, in many respects, is a cousin of the braid groups. Like the latter, it can be introduced in many different ways, and it has very rich properties involving geometric group theory and dynamical systems; see [**32**] for an introduction. The open question of its possible amenability has provided a strong motivation for studying the group F in recent years. Let us mention that F is nothing but the counterpart of the group G_{LD} of Section IV.3 when the associativity law replaces the self-distributivity law.

For our current purposes, it is enough to know that F admits the presentation

$$\langle a_1, a_2, ... \mid a_i a_{j-1} = a_j a_i \text{ for } j \geqslant i + 2\rangle. \tag{3.1}$$

It has recently been observed that the groups B_∞ and F can be married in a natural way. This was done independently by M. Brin in [**22, 23**] by constructing what was seen as a braided version of the Thompson group, and in [**58, 59**] by constructing what was seen as a Thompson version of B_∞. The groups so obtained are essentially similar; variants also appeared in [**120**] and [**92**].

From our current point of view, the most natural description is probably the one involving *parenthesized braids*.

Definition 3.17. The *group of parenthesized braids $B_\bullet$* is defined by two infinite series of generators $\sigma_1, \sigma_2, ..., a_1, a_2, ...$, subject to the following relations for $i \geqslant 1$ and $j \geqslant i + 2$:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, \qquad \sigma_i a_j = a_j\sigma_i, \qquad a_i a_{j-1} = a_j a_i, \qquad a_i\sigma_{j-1} = \sigma_j a_i, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \sigma_{i+1}\sigma_i a_{i+1} = a_i\sigma_i, \qquad \sigma_i\sigma_{i+1} a_i = a_{i+1}\sigma_i. \end{cases} \tag{3.2}$$

The elements of $B_\bullet$ can be visualized using braid diagrams in which the distances between strands are not uniform. An ordinary braid diagram connects an initial sequence of equidistant positions to a similar final sequence. A parenthesized

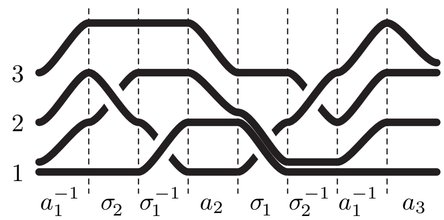

FIGURE 2. The parenthesized braid diagram encoded by the word $a_1^{-1}\sigma_2\sigma_1^{-1}a_2\sigma_1\sigma_2^{-1}a_1^{-1}a_3$. As two strands start from close to 1, the initial positions of the strands can be represented by (••)••, similarly, the final positions correspond to ••(••). The generator σ_i corresponds to the usual crossing between positions i and $i+1$ (but there may be several strands close to these positions), whereas a_i corresponds to shrinking positions from $i+1$ to i.

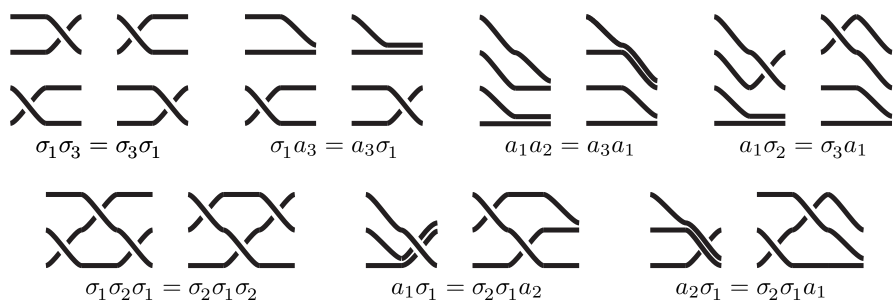

FIGURE 3. Relations of $B_\bullet$ and the corresponding diagrams isotopies: those of the top line are commutations and quasi-commutations; those of the bottom line are braid relations.

braid diagram connects a parenthesized sequence of positions to another possibly different parenthesized sequence of positions, the intuition being that grouped positions are (infinitely) closer than ungrouped ones. A typical example is shown in Figure 2. The generator σ_i corresponds to the usual crossing operator, with the difference that it involves all strands that start in the vicinity of i and $i+1$. The generator a_i corresponds to shrinking all strands that start in the vicinity of i and translating the next ones so as to avoid a gap.

The relations of (3.2) correspond to the isotopies displayed in Figure 3. As can be expected, the elements σ_i generate a copy of B_∞, while the elements a_i generate a copy of Thompson's group F. More precisely, $B_\bullet$ is a group of fractions for a monoid that is a bi-crossed product of B_∞^+ and of the monoid F^+ defined by the presentation of (3.1).

PROPOSITION 3.18. *The group $B_\bullet$ is left-orderable, by an ordering that extends the σ-ordering of B_∞.*

An application of that result—more exactly, of the specific form of the elements larger than 1 in terms of σ-positive expressions—is that the Artin representation of braids extends to $B_\bullet$, and that the latter embeds in the mapping class group of a sphere with a Cantor set of punctures (Figure 4).

Let us also mention that the natural subgroup of $B_\bullet$ corresponding to pure braids was recently shown to be bi-orderable, by an ordering that extends the ordering of PB_∞ constructed in Chapter XV [**30**].

These results do not prove that the parenthesized braid group is an extraordinary object. After all, that two groups can be glued together in a somewhat tricky way is nothing exceptional, so $B_\bullet$ might very well be just an amusing example. However, we think that $B_\bullet$ is really an important object. Once again, the variety of approaches that lead to $B_\bullet$ or to close variants, and, mainly, the unexpected way in which the technical properties fit together, suggest that something interesting is hidden there. In particular, the self-distributive structure of B_∞ described in Chapter IV extends to $B_\bullet$, a surprising result that certainly reflects deep properties. We hope for—and even predict—future applications.

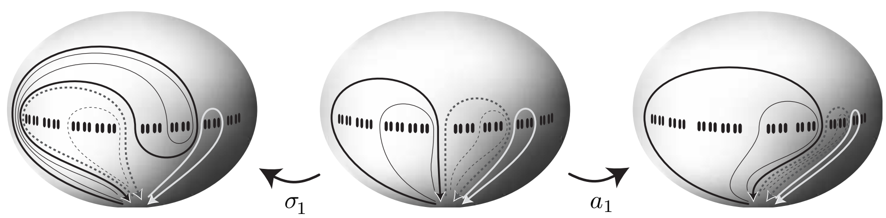

FIGURE 4. Embedding parenthesized braids in the mapping class group of a sphere with a Cantor set of punctures: σ_i acts by the usual half-twist, while a_i acts as a dilatation-translation along the equator.