CHAPTER 2

Review of Pre-quantum Physics

> I was lucky enough to attend a few lectures of S. S. Chern just before he retired from Berkeley in which he said that the cotangent bundle (differential forms) is the feminine side of analysis on manifolds, and the tangent bundle (vector fields) is the masculine side. From this perspective, Hamiltonian mechanics is the feminine side of classical physics, [and] its masculine side is Lagrangian mechanics.
> —Richard Montgomery ([**83**], p. 352)

This chapter is devoted to a brief review of classical mechanics, special relativity, and electromagnetic theory. Among the many available texts on these subjects, the Feynman Lectures [**41**] are excellent for Newtonian mechanics, relativity, and electromagnetism, and Purcell [**90**] perhaps even better for the latter. The classic physics text on Hamiltonian and Lagrangian mechanics is Goldstein [**55**]; Abraham and Marsden [**1**] and Arnold [**4**] are good treatments of this material by and for mathematicians.

2.1. Mechanics according to Newton and Hamilton

We may as well begin at the beginning with Newton's law:

$$F = ma. \tag{2.1}$$

Familiar as this is, it needs some explanation. We consider a system of k particles with fixed masses $m_1, \dots, m_k$, located at positions $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. The jth particle is acted upon by a force $\mathbf{F}_j$ that depends on the positions $\mathbf{x}_1, \dots, \mathbf{x}_k$ and the time t (and perhaps on other parameters such as the masses m_j). We then concatenate the positions and forces into $3k$-vectors,

$$F = (\mathbf{F}_1, \dots, \mathbf{F}_k) \in \mathbb{R}^{3k}, \qquad x = (\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{R}^{3k}$$

and the masses into a $3k \times 3k$ matrix,

$$m = \begin{pmatrix} m_1 I_3 & 0 & \dots & 0 \\ 0 & m_2 I_3 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & m_k I_3 \end{pmatrix} \qquad (I_3 = 3 \times 3 \text{ identity matrix}),$$

and set

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt}.$$

Then (2.1) is a second-order ordinary differential equation for x when the force F is known, and a solution is determined by initial values $x_0 = x(t_0)$ and $v_0 = v(t_0)$.

Mathematicians look at physical laws such as (2.1) with the expectation that they will contain some universal truth about the world. They do, but not in the

absolute way that the formula $A = \pi r^2$ is a universal truth about circles; they are always burdened with a certain amount of fine print. The law (2.1) is valid in all places at all times, as far as we know, but it has to be modified in relativistic situations where the masses cannot be treated as constants, and it fades into the background in the study of the submicroscopic world where the whole concept of force loses much of its utility. Moreover, (2.1) is valid only in inertial coordinate systems, so it gives the right answers about what goes on in a laboratory on the surface of the earth only to the extent that the rotation of the earth and its motion about the sun can be neglected. It is important to keep in mind that all physical theories have their limitations; the quantum field theories toward which we are heading are no exception.

We shall consider only *autonomous* and *conservative* forces, that is, those F that depend only on the positions x (and not explicitly on t) and for which the line integral $\int_C F \cdot dx$ vanishes for every closed curve C in $\mathbb{R}^{3n}$. In this case, F is the gradient of a function on $\mathbb{R}^{3n}$ denoted by $-V$ and called the *potential energy*. We also have the *kinetic energy* T of the system,

$$T = \tfrac{1}{2} mv \cdot v = \sum \tfrac{1}{2} m_j |\mathbf{v}_j|^2,$$

and the *total energy*,

$$E = T + V.$$

It is to be noted that V, and hence E, is well defined only up to an additive constant. Total energy is conserved:

$$\frac{dE}{dt} = mv \cdot \frac{dv}{dt} + \nabla V \cdot \frac{dx}{dt} = (ma - F) \cdot v = 0.$$

The Hamiltonian reformulation of Newtonian mechanics allows more flexibility and reveals some important mathematical structures. For reasons that will become clearer below, it takes the *momentum*

$$p = mv$$

instead of the velocity v as a primary object. The total energy, considered as a function of position and momentum, is called the *Hamiltonian* and is denoted by H:

$$H(x, p) = \tfrac{1}{2} m^{-1} p \cdot p + V(x).$$

We can rewrite Newton's law $F = ma$ as a first-order system in the variables x and p:

$$\frac{dx}{dt} = v = m^{-1} p, \qquad \frac{dp}{dt} = ma = -\nabla V(x).$$

The key point is that the quantities on the right are derivatives of the Hamiltonian:

$$\frac{dx}{dt} = \nabla_p H, \qquad \frac{dp}{dt} = -\nabla_x H. \tag{2.2}$$

These are *Hamilton's equations*, which tell how position and momentum evolve with time. From them one easily obtains the evolution equations for any function of position and momentum, $f(x, p)$:

$$\frac{df}{dt} = \nabla_x f \cdot \frac{dx}{dt} + \nabla_p f \cdot \frac{dp}{dt} = \{f, H\}, \tag{2.3}$$

where the *Poisson bracket* $\{f, g\}$ of any functions f and g is defined by

$$\{f, g\} = \nabla_x f \cdot \nabla_p g - \nabla_p f \cdot \nabla_x g.$$

Of particular importance are the Poisson brackets of the coordinate functions themselves:

$$\{x_i, p_j\} = -\{p_j, x_i\} = \delta_{ij}, \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0. \tag{2.4}$$

At this point a fundamental mathematical structure comes clearly into view (from the modern perspective): that of a symplectic manifold.

Let us pause for a brief review of this concept. A *symplectic manifold* is a C^∞ manifold M equipped with a differential 2-form Ω that is closed ($d\Omega = 0$) and pointwise nondegenerate, i.e., for each $a \in M$ and $v \in T_aM$, $\Omega_a(v, w) = 0$ for all $w \in T_aM$ only when $w = 0$. Nondegeneracy forces the dimension of M to be even, $\dim M = 2n$, and it is equivalent to the condition that the nth exterior power of Ω is everywhere nonvanishing.

The form Ω gives an identification of 1-forms with vector fields. Namely, if ϕ is a 1-form, the corresponding vector field X_ϕ is defined by

$$\Omega(X_\phi, Y) = \phi(Y)$$

for all vector fields Y. In particular, if f is a smooth function on M, X_{df} is a smooth vector field on M called the *Hamiltonian vector field* of f; by a small abuse of notation, we denote it by X_f rather than X_{df}. Thus, for any vector field Y,

$$\Omega(X_f, Y) = df(Y) = Yf.$$

The *Poisson bracket* of two smooth functions f and g is

$$\{f, g\} = \Omega(X_f, X_g) = X_g f = -X_f g.$$

The Poisson bracket makes $C^\infty(M)$ into a Lie algebra; the Jacobi identity is a consequence of the fact that $d\Omega = 0$ and the formula for the action of the exterior derivative of a form on vector fields. Moreover, the correspondence $f \mapsto X_f$ is a Lie algebra homomorphism:

$$\begin{aligned} X_{\{f,g\}}h = -\{h, \{f, g\}\} &= \{g, \{h, f\}\} + \{f, \{g, h\}\} \\ &= -X_g X_f h + X_f X_g h = [X_f, X_g]h. \end{aligned}$$

A celebrated theorem of Darboux states that for any point a in a symplectic manifold M there is a system of local coordinates $x_1, \ldots, x_n, p_1 \ldots, p_n$ on a neighborhood of a such that $\Omega = \sum dx_j \wedge dp_j$. A coordinate system with this property is called *canonical*, and the coordinates x_j and p_j are said to be *canonically conjugate*. In canonical coordinates, Hamiltonian vector fields and Poisson brackets are given by

$$X_f = \sum \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial p_j} \right), \qquad \{f, g\} = \sum \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right).$$

A diffeomorphism $\Phi : M \to M$ that preserves the symplectic structure is called a *canonical transformation* (by physicists) or a *symplectomorphism* (by geometers). Suppose $\{\Phi_t : t \in \mathbb{R}\}$ is a one-parameter group of canonical transformations; then its infinitesimal generator, the vector field X defined by $Xf = (d/dt) f \circ \Phi_t|_{t=0}$, satisfies $L_X\Omega = 0$, where L denotes the Lie derivative. Conversely, if X is a vector field such that $L_X\Omega = 0$, the flow it generates consists of (local) canonical transformations. But $L_X\Omega = i_X(d\Omega) + d(i_X\Omega) = d(i_X\Omega)$, where i_X denotes contraction with X, and $i_X\Omega$ is the 1-form associated to X by Ω, so X is an infinitesimal canonical transformation if and only if its associated 1-form is closed. But this means that $i_X\Omega = df$ for some function f (perhaps well-defined only on a covering

space of M if M is not simply connected), or in other words, that $X = X_f$. Thus, with the understanding that functions and transformations may be only locally well-defined, *Hamiltonian vector fields are precisely the infinitesimal generators of canonical transformations.*

For Hamiltonian mechanics, the symplectic manifolds of primary importance are cotangent bundles. Let N be an n-dimensional manifold, T^*N its cotangent bundle, and $\pi : T^*N \to N$ the natural projection. There is a canonical 1-form ω on T^*N, defined by $\omega(v) = \phi(\pi_* v)$ for $\phi \in T^*N$ and $v \in T_\phi(T^*N)$, and $\Omega = -d\omega$ is a symplectic form on T^*N. (The minus sign is inserted to make this consistent with the preceding discussion and some standard conventions in the literature.) Indeed, any system $\{x_1, \dots, x_n\}$ of local coordinates on an open set $U \subset N$ induces a frame $\{dx_1, \dots, dx_n\}$ on T^*U and hence a system of linear coordinates $\{p_1, \dots, p_n\}$ on each fiber of T^*U (that is, if ϕ is a 1-form on U, $\phi = \sum p_j(\phi) dx_j$), and $\{x_1, \dots, x_n, p_1, \dots, p_n\}$ is a system of local coordinates on T^*U. In these coordinates the canonical 1-form is given by $\omega = \sum p_j \, dx_j$, so $\Omega = \sum dx_j \wedge dp_j$. Thus Ω is indeed a symplectic form, and the coordinates x_j, p_j are canonical.

We return to physics. According to Newton's law, once the forces are given, the motion of the system is completely determined by (i) the position x and (ii) the velocity v or the momentum p at an initial time t_0. We therefore take this data as a complete description of the *state* of the system. To build a general mathematical framework for dealing with these matters, we start with a *configuration space* N, which is taken to be a manifold and is supposed to be a description of the possible "positions" of the system. There is quite a lot of flexibility here. For example, if the system consists of k particles moving in $\mathbb{R}^3$ as discussed previously, N will be $\mathbb{R}^{3k}$, or perhaps $\{(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{R}^{3k} : \mathbf{x}_i \neq \mathbf{x}_j \text{ for } i \neq j\}$. If the motion is subject to some constraints, N could be a submanifold of $\mathbb{R}^{3k}$ instead. For an asymmetric rigid body, however, the appropriate configuration space is $\mathbb{R}^3 \times SO(3)$: three linear coordinates to give the location of the center of mass (or some other convenient point in the body), and three angular coordinates to give the body's orientation in space.

Velocities are taken to be tangent vectors to the configuration space, so the position-velocity state space is TN. On the other hand, the appearance of Poisson brackets in the Hamiltonian formalism leads us to take the position-momentum state space to be the symplectic manifold T^*N. That is, momenta should be considered as cotangent vectors, and in the relation $p = mv$ the mass matrix m should be interpreted as a Riemannian metric on N that mediates between vectors and covectors. Thus in the setting of $N = \mathbb{R}^{3k}$ considered above, the inner product on tangent vectors is given by $\langle v, w \rangle_m = mv \cdot w$; the corresponding inner product on cotangent vectors is given by $\langle p, q \rangle^m = m^{-1} p \cdot q$, and the cotangent vector corresponding to the tangent vector v is $p = mv$. T^*N is traditionally called the *phase space* of the system. (The origin of the name lies in statistical mechanics.)

As a function on T^*N, the Hamiltonian is $H(\phi) = \frac{1}{2}\langle \phi, \phi \rangle^m + V(\pi(\phi))$, where $\langle \cdot, \cdot \rangle^m$ is the Riemannian inner product just described and $\pi : T^*N \to N$ is the natural projection. Hamilton's equations (2.2) say that the Hamiltonian vector field X_H is the infinitesimal generator of the one-parameter group Φ_t of time translations of the system. (If (x, p) is the state at time t_0, $\Phi_t(x, p)$ is the state at time $t + t_0$. Actually Φ_t may be only a flow rather than a full one-parameter group, as particles might collide or escape to infinity in a finite time.) Consequently, the time

translations are canonical transformations: the symplectic structure is an invariant of the dynamics.

In this setting it is easy to derive the connection between symmetries and conserved quantities generally known as *Noether's theorem*. To wit, suppose $\{\Psi_s : s \in \mathbb{R}\}$ is a one-parameter group of canonical transformations of phase space, and suppose the Hamiltonian H is invariant under these transformations. The infinitesimal generator of Ψ_s is a Hamiltonian vector field X_f, and invariance means that $X_f H = 0$. But $X_f H = \{H, f\} = -\{f, H\} = -df/dt$, so f is invariant under time translations, i.e., f is a conserved quantity. For example, if $N = \mathbb{R}^{3k}$ and Ψ_s is spatial translation in the direction of a unit vector $\mathbf{u} \in \mathbb{R}^3$, i.e., $\Psi_s(x,p) = (\mathbf{x}_1 + s\mathbf{u}, \ldots, \mathbf{x}_n + s\mathbf{u}, p)$, then f is the $\mathbf{u}$-component of the total linear momentum, $f(x,p) = (\mathbf{p}_1 + \ldots + \mathbf{p}_n) \cdot \mathbf{u}$. If $N = \mathbb{R}^3$ and Ψ_s is rotation through the angle s about the $\mathbf{u}$ axis, then f is the $\mathbf{u}$-component of the angular momentum, $f(\mathbf{x}, \mathbf{p}) = (\mathbf{x} \times \mathbf{p}) \cdot \mathbf{u}$.

One of the advantages of the Hamiltonian formulation of mechanics is that it allows the possibility of simplifying the problem by performing arbitrary canonical transformations, including ones that mix up the position and momentum variables. As an illustration, we consider a simple but important problem whose quantum analogue will be of fundamental importance later on: the one-dimensional harmonic oscillator. This is a single particle of mass m, moving along the real line subject to a linear restoring force $F(x) = -\kappa x$ ($\kappa > 0$). The associated potential is $V(x) = \frac{1}{2}\kappa x^2$. Of course, Newton's equation $mx'' = -\kappa x$ is easy to solve directly, but the Hamiltonian method allows us to transform this simple differential equation into a completely trivial one.

Let us first observe that canonical transformations on $\mathbb{R}^2$ are precisely those transformations that preserve orientation and area, as the symplectic form $dx \wedge dp$ is just the element of oriented area.

Let $\omega = \sqrt{\kappa/m}$. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{\kappa}{2}x^2 = \frac{\omega}{2}\left(\frac{1}{\sqrt{\kappa m}}p^2 + \sqrt{\kappa m}\, x^2\right).$$

As a first step, we make the canonical transformation $\widetilde{x} = (\kappa m)^{1/4}x$, $\widetilde{p} = p/(\kappa m)^{1/4}$, which makes $H = (\omega/2)(\widetilde{x}^2 + \widetilde{p}^2)$. Now, the polar-coordinate map $(\widetilde{x}, \widetilde{p}) \mapsto (r, \theta)$ is not area-preserving, but its close relative $(\widetilde{x}, \widetilde{p}) \mapsto (\frac{1}{2}r^2, \theta)$ is, since $r\, dr = d(\frac{1}{2}r^2)$. Thus, the map $(\widetilde{x}, \widetilde{p}) \mapsto (s, \theta)$ is canonical if we take $s = \frac{1}{2}(\widetilde{x}^2 + \widetilde{p}^2)$ and $\theta = \arctan(p/x)$, and in the new coordinates, H is simply ωs. Hamilton's equations therefore boil down to

$$\frac{ds}{dt} = \frac{\partial H}{\partial \theta} = 0, \qquad \frac{d\theta}{dt} = -\frac{\partial H}{\partial s} = -\omega.$$

Thus s is a constant — namely, $s = E/\omega$ where E is the total energy of the system — and $\theta = \theta_0 - \omega t$. Finally, $\widetilde{x} = \sqrt{2s}\cos\theta = \sqrt{2E/\omega}\cos(\omega t - \theta_0)$, so

$$x = \sqrt{\frac{2E}{\kappa}}\cos(\omega t - \theta_0).$$

The constant θ_0 can be chosen to make the initial values $x(t_0)$ and $p(t_0)$ whatever we wish, subject to the condition that $H(x_0, p_0) = E$.

Another canonical transformation that will be useful later is the one that simplifies the two-body problem. Suppose that two particles with masses m_1, m_2, positions $\mathbf{x}_1, \mathbf{x}_2$, and momenta $\mathbf{p}_1, \mathbf{p}_2$ are subject to a conservative force with potential V that depends only on their relative displacement $\mathbf{x}_1 - \mathbf{x}_2$, so the Hamiltonian is

$$H = \frac{|\mathbf{p}_1|^2}{2m_1} + \frac{|\mathbf{p}_2|^2}{2m_2} + V(\mathbf{x}_1 - \mathbf{x}_2).$$

Let

$$M = m_1 + m_2, \qquad m = \frac{m_1 m_2}{m_1 + m_2}, \qquad \mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}, \qquad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2,$$
$$\mathbf{P} = M\frac{d\mathbf{X}}{dt} = \mathbf{p}_1 + \mathbf{p}_2, \qquad \mathbf{p} = m\frac{d\mathbf{x}}{dt} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}.$$

The transformation $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) \mapsto (\mathbf{X}, \mathbf{x}, \mathbf{P}, \mathbf{p})$ is canonical, and in the new coordinates the Hamiltonian becomes

$$H = \frac{|\mathbf{P}|^2}{2M} + \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{x}),$$

in which the coordinates are decoupled. Hamilton's equations then give $d\mathbf{X}/dt = \mathbf{P}/M$, $d\mathbf{P}/dt = 0$, so that $\mathbf{X}$, the center of mass of the two-body system, moves with constant velocity; and $d\mathbf{x}/dt = \mathbf{p}/m$, $d\mathbf{p}/dt = -\nabla V(\mathbf{x})$, so that $m\mathbf{x}''(t) = -\nabla V(\mathbf{x})$. In short, after removing the uniform motion of the center of mass, the problem reduces to that of a single particle of mass m, the *reduced mass* of the system, moving in the potential $V(\mathbf{x})$.

2.2. Mechanics according to Lagrange

We now turn to the other major reworking of Newtonian mechanics, the Lagrangian formulation. On the simplest level of a system of particles moving in a potential V, whose Hamiltonian is $H(x,p) = \frac{1}{2}m^{-1}p \cdot p + V(x)$, the *Lagrangian* is the function of position and velocity given by

$$L(x,v) = mv \cdot v - H(x, mv) = \tfrac{1}{2}mv \cdot v - V(x). \tag{2.5}$$

Stated in a more intrinsic fashion, if we regard H as a function on the cotangent bundle T^*N, the Lagrangian is the function on the tangent bundle TN given by $L(\xi) = \langle j_m\xi, \xi\rangle - H(j_m\xi)$, where $j_m : TN \to T^*N$ is the isomorphism provided by the inner product m, and $\langle\cdot,\cdot\rangle$ is the pairing of cotangent vectors with tangent vectors. Returning to (2.5), we have

$$\nabla_x L = -\nabla_x H = \nabla V, \qquad \nabla_v L = mv,$$

so Newton's law becomes *Lagrange's equation*

$$\frac{d}{dt}\nabla_v L = \nabla_x L.$$

The significance of this is that it is the Euler-Lagrange equation for a certain problem in the calculus of variations. Namely, given a path $t \mapsto x(t)$, $t_0 \le t \le t_1$, in the configuration space N, we consider the *action*

$$S = \int_{t_0}^{t_1} L(x(t), x'(t))\, dt,$$

and the problem is to *minimize the action over all paths $x(t)$ that begin and end at two given points $x_0 = x(t_0)$ and $x_1 = x(t_1)$*. Indeed, suppose we replace $x(t)$ by a

slightly different path $x(t) + \delta x(t)$, where $\delta x(t_0) = \delta x(t_1) = 0$. To first order in δx, the change in the Lagrangian is

$$\delta L = L(x + \delta x,\, x' + \delta x') - L(x, x') = \nabla_x L \cdot \delta x + \nabla_v L \cdot \delta x',$$

so the change in the action is

$$\delta S = \int_{t_0}^{t_1} \delta L \, dt = \int_{t_0}^{t_1} \left[\nabla_x L - \frac{d}{dt} \nabla_v L \right] \cdot \delta x \, dt,$$

where the endpoint terms in the integration by parts vanish since δx vanishes at the endpoints. At a minimum, δS must vanish, and δx is arbitrary, so Lagrange's equation must hold.

This result is commonly known as the *principle of least action*; it will come back to haunt us in Chapter 8. Of course, this name is somewhat inaccurate: a solution of Newton's equations is a path that is a critical point, not necessarily a minimizer, of the action.

The calculation that led us from Hamiltonian mechanics to Lagrangian mechanics is easily reversible. Indeed, suppose we start with the Lagrangian $L(x, v) = \frac{1}{2} mv \cdot v - V(x)$. We define the Hamiltonian H by

$$H(x, p) = m^{-1} p \cdot p - L(x, m^{-1} p) = \tfrac{1}{2} m^{-1} p \cdot p + V(x).$$

Then $\nabla_p H = m^{-1} p = v = dx/dt$, and $\nabla_v L = mv = p$, so by Lagrange's equation, $-\nabla_x H = \nabla_x L = (d/dt)(\nabla_v L) = dp/dt$, and we have Hamilton's equations.

The Lagrangian and Hamiltonian machinery can be used in many situations other than the simple mechanical systems we have mentioned. The most common paradigm is to formulate a physical problem as a variational problem for a Lagrangian L defined on the tangent bundle of some configuration space N. The general procedure for converting such a problem to Hamiltonian form is as follows.

Let $x_1, \ldots, x_n$ be local coordinates on N, and let $v_1, \ldots, v_n$ be the corresponding linear coordinates on the fibers of TN; thus L is a function of x and v. For the moment, we fix a point $a \in N$ and think of L as a function on $T_a N$. Its differential $dL = \sum (\partial L / \partial v_j) dv_j$ is a 1-form on $T_a N$. If we identify the vector space $T_a N$ with its own tangent spaces, we can think of dL as a map that assigns to each $v \in T_a N$ a linear functional on $T_a N$, i.e., a map from $T_a N$ to $T_a^* N$. Now letting a vary too, we obtain a map $W : TN \to T^* N$.

Let us write this out in terms of local coordinates. Given coordinates $x_1, \ldots, x_n$ on $U \subset N$, we identify TU and T^*U both with $U \times \mathbb{R}^n$; then L is a function of x and the fiber coordinates v on TU, and $W(x, v) = (x, \nabla_v L)$. The component $p_j = \partial L / \partial v_j$ of $\nabla_v L$ is called the *conjugate momentum* to the position variable x_j with respect to L.

Now suppose that W is invertible. From the Lagrangian L on TN we construct the Hamiltonian H on $T^* N$ by

$$H(\xi) = \langle \xi, W^{-1} \xi \rangle - L(W^{-1} \xi),$$

where the first term on the right is the pairing of a covector with a vector. A calculation similar to the one we performed above then shows that the Euler-Lagrange equation for L is equivalent to Hamilton's equations for H. Exactly the same recipe can be used to go from H back to L; it is called the *Legendre transformation*.

In the general situation, this procedure for going from H to L and back may be problematic because of the question of the invertibility of W. However, it clearly

coincides with the transformation we considered earlier for the case where $L(x,v) = \frac{1}{2}mv\cdot v - V(x)$. More generally, suppose that for each $a \in N$ the restriction of L to T_aN is a quadratic function whose pure second-order part is positive definite. Then W is indeed invertible: it is an affine-linear isomorphism of T_aN and T_a^*N for each a. This is the most important case in practice, and the only one that will concern us in the sequel.

The Lagrangian formulation of Noether's theorem about symmetries and conserved quantities (which is the original version) is as follows: Suppose $\{\psi_s : s \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms of the configuration space N that preserve the Lagrangian:

$$L(\psi_s(x), (\psi_s)_*(v)) = L(x,v).$$

Then the quantity $I(x,v) = \nabla_v L(x,v)\cdot d\psi_s(x)/ds$ is a constant of the motion. (See Arnold [**4**] or Goldstein [**55**].)

Our real interest is not in mechanical systems with finitely many degrees of freedom but in fields, which can be regarded as continuum mechanical systems with infinitely many degrees of freedom. Let us illustrate the transition to this situation with a simple example.

Consider a long elastic rod that is subject to longitudinal vibrations (or, if you prefer, a long air column such as an organ pipe). As a simplified model, we chop the rod up into small bits of equal size, consider each bit as a point particle, and replace the elastic forces within the rod by ideal springs connecting the particles. More precisely, we take each particle to have mass Δm and the separation between adjacent particles when the system is at rest to be Δx. Let u_j be the displacement of the jth particle from its position at rest. Then Newton's law gives the following system of equations for the u_j's:

$$\Delta m\frac{d^2u_j}{dt^2} = k(u_{j+1}-u_j) - k(u_j - u_{j-1}) = k(u_{j+1} - 2u_j + u_{j-1}), \tag{2.6}$$

where k is the common spring constant, and the Lagrangian is

$$L = T - V = \tfrac{1}{2}\sum \Delta m\left(\frac{du_j}{dt}\right)^2 - \tfrac{1}{2}\sum k(u_{j+1}-u_j)^2. \tag{2.7}$$

Let us divide (2.6) by the equilibrium separation Δx,

$$\frac{\Delta m}{\Delta x}\frac{d^2u_j}{dt^2} = (k\Delta x)\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2},$$

and rewrite (2.7) as

$$L = \tfrac{1}{2}\sum\left[\frac{\Delta m}{\Delta x}\left(\frac{du_j}{dt}\right)^2 - (k\Delta x)\left(\frac{u_{j+1}-u_j}{\Delta x}\right)^2\right]\Delta x.$$

Now pass to the continuum limit: $\Delta m/\Delta x$ becomes the mass density μ, $k\Delta x$ becomes the Young's modulus Y, the displacements u_j become a displacement function $u(x)$, and we obtain

$$\mu\frac{\partial^2 u}{\partial t^2} = Y\frac{\partial^2 u}{\partial x^2} \tag{2.8}$$

from (2.6) and

$$L(u, \partial_t u, \partial_x u) = \tfrac{1}{2} \int_a^b \left[\mu \left(\frac{\partial u}{\partial t} \right)^2 - Y \left(\frac{\partial u}{\partial x} \right)^2 \right] dx \tag{2.9}$$

from (2.7), where $[a, b]$ is the interval occupied by the rod. Equation (2.8) is the familiar *wave equation* which is indeed the standard model for elastic vibrations, and (2.9) expresses the Lagrangian as the integral of the *Lagrangian density*

$$\mathfrak{L}(u, \partial_t u, \partial_x u) = \tfrac{1}{2} \left[\mu \left(\frac{\partial u}{\partial t} \right)^2 - Y \left(\frac{\partial u}{\partial x} \right)^2 \right]. \tag{2.10}$$

The wave equation (2.8) can be derived from the Lagrangian (2.9) by the calculus of variations just as before. That is, one considers the action

$$S(u) = \int_{t_0}^{t_1} L \, dt = \int_{t_0}^{t_1} \int_a^b \mathfrak{L}(u, \partial_x u, \partial_t u) \, dx \, dt,$$

and requires $\delta S = S(u + \delta u) - S(u)$ to vanish to first order in δu for appropriate δu. To make this precise, one must consider what happens at the ends of the rod. One possibility is to consider an infinitely long rod, $[a, b] = \mathbb{R}$. The vibrations $u(x, t)$ are then assumed to have compact support in x or vanish rapidly as $x \to \pm\infty$, so that the Lagrangian makes sense, and one imposes the same requirement on δu, together with $\delta u = 0$ at $t = t_0, t_1$. Likewise, for a rod with fixed ends or an air column with closed ends, one requires u and δu to vanish at $x = a, b$.

A little more interesting is the case of a rod with free ends or an air column with open ends. Here there is no *a priori* restriction on u and δu at the endpoints; let us see what happens. With L given by (2.9), to first order in δu we have

$$\delta S = \int_{t_0}^{t_1} \int_a^b \left[\mu \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} - Y \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} \right] dx \, dt.$$

Integration by parts yields

$$\delta S = \int_{t_0}^{t_1} \int_a^b \left[-m \frac{\partial^2 u}{\partial t^2} + Y \frac{\partial^2 u}{\partial x^2} \right] \delta u \, dx \, dt - Y \int_{t_0}^{t_1} \left[\frac{\partial u}{\partial x} \delta u \right]_{x=a}^{b} dt.$$

(We always assume δu vanishes at $t = t_0, t_1$, so the terms with t-derivatives contribute no boundary terms.) Thus if we want $\delta S = 0$ for arbitrary δu, u must satisfy not only the wave equation but the boundary conditions $\partial u / \partial x = 0$ at $x = a, b$. These are in fact the physically correct boundary conditions for a rod with free ends or an air column with open ends, and they yield a well-posed boundary value problem for the wave equation.

Let us be clear about the subtle shift of notation that has taken place here. For a system with finitely many degrees of freedom, we had position variables x_j and velocity variables v_j. Here the index j has become the continuous variable x, x_j has become $u(x, t)$, and v_j has become $\partial_t u$. It is not worthwhile to try to maintain the geometric language of tangent and cotangent vectors at this point, but we can still speak of the *canonically conjugate momentum density* to the field u,

$$p = \frac{\partial \mathfrak{L}}{\partial (\partial_t u)} = \mu \frac{\partial u}{\partial t},$$

and the *Hamiltonian density*,

$$\mathcal{H} = p\partial_t u - \mathcal{L} = \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + Y\left(\frac{\partial u}{\partial x}\right)^2\right].$$

The *Hamiltonian* $H = \int \mathcal{H}\, dx$ still represents the total energy, and it is a conserved quantity.

Similar ideas pertain to the wave equation in higher dimensions. Beginning with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left[c^{-2}(\partial_t u)^2 - |\nabla_x u|^2\right] \tag{2.11}$$

for a function $u(x,t)$ with $x \in \Omega$, an open subset of $\mathbb{R}^n$, and $t \in \mathbb{R}$, one forms the action integral

$$S = \int_{t_0}^{t_1} \int_\Omega \mathcal{L}\, dx\, dt.$$

Setting the first variation of S equal to zero yields the wave equation

$$c^{-2}\partial_t^2 u - \nabla^2 u = 0.$$

If Ω is a bounded domain and one imposes no restrictions on the perturbation δu on the boundary $\partial\Omega$, one obtains the Neumann boundary conditions $\partial u/\partial n = 0$ for free. Alternatively, one can take $\Omega = \mathbb{R}^n$, $t_0 = -\infty$, and $t_1 = +\infty$, with some implicit or explicit assumptions on the vanishing of the fields at infinity, and obtain the wave equation on $\mathbb{R}^n$.

This last point leads to a glimpse of the importance of Lagrangians for quantum field theory. As we shall see, the transition from classical mechanics to quantum mechanics proceeds most easily by adopting the Hamiltonian point of view. However, since this approach singles out the time axis for special attention, it lacks relativistic invariance. A Lagrangian density, on the other hand, or the action which is its integral over space-time, employs the space and time variables on an equal footing, and it may have relativistic invariance built in — as (2.11) does, for example. It therefore serves as a useful starting point for building relativistic field theories.

When we return to these ideas in later chapters, we shall often be cavalier about dropping the word "density"; that is, we shall often refer to $\mathcal{L}$ as the Lagrangian and $\mathcal{H}$ as the Hamiltonian; the meaning will always be clear from the context.

2.3. Special relativity

It is taken as axiomatic that the fundamental laws of physics should be invariant under the Euclidean group of translations and rotations of space, as well as under translations of time. Newtonian physics and its immediate descendants are also invariant under the *Galilean transformations*

$$(t, \mathbf{x}) \mapsto (t,\, \mathbf{x} + t\mathbf{v}),$$

for any $\mathbf{v} \in \mathbb{R}^3$. The fundamental equations governing electromagnetism, however, are not, as we shall see in the next section. Rather, they are invariant under translations and under the Lorentz group $O(1,3)$ (see §1.3). For the time being we adopt notation that displays the speed of light c explicitly, so $O(1,3)$ is the group of linear transformations of $\mathbb{R}^4$ that preserve the Lorentz bilinear form

$$\Lambda((t,\mathbf{x}),\,(t',\mathbf{x}')) = c^2tt' - \mathbf{x}\cdot\mathbf{x}' = c^2tt' - xx' - yy' - zz'.$$

Einstein's great insight to realize that the laws of mechanics should be modified so as to be invariant under this group too.

A typical Lorentz transformation that mixes space and time variables is the boost along the x-axis with velocity v ($|v| < c$):

$$(t, x, y, z) \mapsto \left(\frac{t + (vx/c^2)}{\sqrt{1 - (v/c)^2}},\ \frac{x + vt}{\sqrt{1 - (v/c)^2}},\ y,\ z \right). \tag{2.12}$$

(This description of the boost is related to the one in §1.3 by the substitution $v/c = \tanh s$.) The factors of $\sqrt{1 - (v/c)^2}$ in (2.12) are what account for the "Fitzgerald contraction" and "time dilation" effects in relativistic motion. (The fact that the word "contraction" is used in one case and "dilation" is used in the other is purely a matter of psychology. The factor $\sqrt{1 - (v/c)^2}$ is in the denominator for both x and t.) The feature of these phenomena that is perplexing to the intuition is that the inverse transformation of (2.12) is of exactly the same form, with v replaced by $-v$, so the *same* factors of $\sqrt{1 - (v/c)^2}$ appear in the inverse, not their reciprocals. Nonetheless, everything works out consistently.

For example, suppose an observer A fires a bomb at an object O that is at rest with respect to A, and an observer B rides along with the bomb like Slim Pickens in *Dr. Strangelove*. A and B synchronize their clocks so that the bomb is launched at time 0. Suppose the bomb is set to go off 1 second after launch, the speed of the bomb is $0.6c$, and the distance from A to O is 2.25×10^8 meters (= 0.75 light-second). Result: The bomb goes off when it reaches O. From A's point of view, the bomb takes 1.25 seconds to reach O, and it takes that long to go off because the clock on board runs slow by a factor of $\sqrt{1 - (v/c)^2} = 0.8$. From B's point of view, the bomb takes 1 second to reach O, but the distance from A to O is only 1.8×10^8 meters because it is shortened by a factor of 0.8. (This is why muons created by cosmic rays in the upper atmosphere, around 10^{-4} light-second from earth, manage to reach the earth's surface by traveling at almost the speed of light, even though the half-life of a muon is only about 10^{-6} second. We think the muons' clocks run slow; they think the distance is short.) On the other hand, from B's point of view, A's clock runs slow, so (according to B) A's clock only reads 0.8 second when the bomb goes off. The point here is that the events of the explosion and the reading of 0.8 second on A's clock are simultaneous for B but not for A. This is possible because the two events happen at different places: the transformation (2.12) does not take equal times to equal times when $x \neq 0$. In constrast, A and B agree that B's clock reads 1 second when the bomb goes off; these two events happen at the same place, namely O, so their simultaneity is independent of the reference frame.

An elementary derivation of the relativistic formulas for momentum, energy, etc., can be found in Feynman [**41**]. Here we take a different approach, via Lagrangian mechanics, as a simple exercise in the sort of thought processes that go into finding the laws of quantum field theory.

We consider a free particle and wish to derive its laws of motion from an action functional, $S = \int_{t_0}^{t_1} L(\mathbf{x}, \mathbf{v})\, dt$, where $\mathbf{x} = \mathbf{x}(t)$ is a path with given endpoints $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, and $\mathbf{v}(t) = \mathbf{x}'(t)$. (We implicitly assume that $|\mathbf{v}(t)| < c$ and that $c^2(t_1 - t_0)^2 > |\mathbf{x}_1 - \mathbf{x}_0|^2$.) We wish the action to be Lorentz invariant, i.e., to be unchanged if $(\mathbf{x}_0, t_0)$ and $(\mathbf{x}_1, t_1)$ are subjected to the same Lorentz transformation, and we also wish it to yield Newtonian mechanics for a free particle in the limit of small velocities. The first requirement narrows down the possibilities

enormously; the simplest one is to take the integrand to be a constant multiple of the Lorentz analogue of the differential of arc length,

$$a\,ds = a\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}, \tag{2.13}$$

which in a given reference frame becomes

$$a\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}\,c\,dt.$$

Thus we try the Lagrangian

$$L(\mathbf{x}, \mathbf{v}) = L(\mathbf{v}) = ac\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}.$$

To figure out the right value of a, observe that when $|\mathbf{v}| \ll c$ we have

$$L(\mathbf{v}) \approx ac - \frac{a|\mathbf{v}|^2}{2c}.$$

Adding or subtracting a constant such as ac to the Lagrangian does not affect the equations of motion, and the second term $-a|\mathbf{v}|^2/2c$ becomes the classical (nonrelativistic) Lagrangian $m|\mathbf{v}|^2/2$ for a free particle of mass m if we take $a = -mc$. Thus we are led to the Lagrangian

$$L = -mc^2\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}. \tag{2.14}$$

The corresponding conjugate momentum is

$$\mathbf{p} = \nabla_{\mathbf{v}} L = \frac{m\mathbf{v}}{\sqrt{1 - |\mathbf{v}^2|/c^2}}.$$

Again, this is approximately the classical momentum $m\mathbf{v}$ when $\mathbf{v}$ is small. In general, it is $M\mathbf{v}$ where

$$M = M(\mathbf{v}) = \frac{m}{\sqrt{1 - |\mathbf{v}|^2/c^2}}.$$

We are led to the conclusion that *the effective mass of a particle of (rest) mass m moving with velocity $\mathbf{v}$ is $M(\mathbf{v})$*. Thus mass is subject to the same dilation effect as time and distance.

The Euler-Lagrange equation for the Lagrangian (2.14) is

$$0 = \frac{d}{dt}\nabla_{\mathbf{v}} L - \nabla_{\mathbf{x}} L = \frac{d\mathbf{p}}{dt},$$

which gives motion in a straight line with constant speed as expected. (In general, if forces are present, Newton's law remains valid if "$m\mathbf{a}$" is reinterpreted as $d\mathbf{p}/dt$. But the question of a proper relativistic interpretation of forces is in general problematic.)

Next, the quantity

$$E = \mathbf{p} \cdot \mathbf{v} - L,$$

considered as a function of $\mathbf{v}$, is a constant of the motion that constitutes the total energy in nonrelativistic mechanics. In our situation,

$$E = \frac{m|\mathbf{v}|^2}{\sqrt{1 - |\mathbf{v}|^2/c^2}} - mc^2\sqrt{1 - |\mathbf{v}|^2/c^2} = \frac{mc^2}{\sqrt{1 - |\mathbf{v}|^2/c^2}} = Mc^2,$$

which is Einstein's formula for the energy of a free particle in relativistic mechanics. When $\mathbf{v}$ is small,

$$E \approx mc^2 + \tfrac{1}{2}m|\mathbf{v}|^2,$$

the sum of the "rest energy" mc^2 and the classical kinetic energy. We observe that

$$\frac{E^2}{c^2} - |\mathbf{p}|^2 = \frac{m^2c^2}{1-|\mathbf{v}|^2/c^2} - \frac{m^2|\mathbf{v}|^2}{1-|\mathbf{v}|^2/c^2} = m^2c^2, \tag{2.15}$$

so that

$$E = c\sqrt{|\mathbf{p}|^2 + m^2c^2}. \tag{2.16}$$

Expressed in this way as a function of $\mathbf{p}$, E is the *Hamiltonian* of the free particle. Again, for small velocities we have $|\mathbf{p}| \ll mc$ and hence

$$E \approx mc^2 + \frac{|\mathbf{p}|^2}{2m},$$

the rest energy plus the classical Hamiltonian.

The momentum $\mathbf{p}$ and the energy E are different in different frames of reference related by Lorentz transformations. However, it is not hard to check that $\mathbf{p}$ and E/c transform as the space and time components of a 4-vector, called the (relativistic) 4-momentum or energy-momentum vector. The relation (2.15) expresses the fact that the Lorentz inner product of this vector with itself is m^2c^2. It is important to note that in the tensor notation explained in §1.1, this vector is p^μ rather than p_μ; otherwise the sign of $\mathbf{p}$ comes out wrong.

2.4. Electromagnetism

Classically, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are vector-valued functions of position $\mathbf{x} \in \mathbb{R}^3$ and time t that may be operationally defined by the *Lorentz force law*: The force on a particle with charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}, \tag{2.17}$$

where c is the speed of light, and $\mathbf{E}$ and $\mathbf{B}$ are evaluated at the location of the particle.[1]

The behavior of $\mathbf{E}$ and $\mathbf{B}$ is governed by *Maxwell's equations*. Using the Heaviside-Lorentz convention on the scale of electric charge (see §1.1), they are[2]

$$\operatorname{div}\mathbf{E} = \rho \tag{2.18}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{2.19}$$

$$\operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.20}$$

$$\operatorname{curl}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\mathbf{j} \tag{2.21}$$

Here ρ is the charge density and $\mathbf{j}$ is the current density, both of which are functions of position and time. Everything here may be interpreted in the sense of

[1] In some texts the factor of c is omitted, which amounts to a redefinition of $\mathbf{B}$. The point of including it is to make $\mathbf{E}$ and $\mathbf{B}$ have the same dimensions, namely, force per unit charge.

[2] Some texts write Maxwell's equations in a way that involves two additional vector fields called $\mathbf{D}$ and $\mathbf{H}$. This is appropriate for the study of electromagnetism in bulk matter, but on the level of fundamental physics, $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$.

distributions, so that one may allow point charges, charge distributions on curves or surfaces, etc.

The quantities ρ and $\mathbf{j}$ are not independent; indeed, by (2.18) and (2.21),

$$\frac{\partial \rho}{\partial t} = \operatorname{div} \frac{\partial \mathbf{E}}{\partial t} = c \operatorname{div}(\operatorname{curl} \mathbf{B}) - \operatorname{div} \mathbf{j} = - \operatorname{div} \mathbf{j},$$

since $\operatorname{div} \operatorname{curl} = 0$. This so-called *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \tag{2.22}$$

is a strong form of the law of conservation of charge. More precisely, the total charge in a region $R \subset \mathbb{R}^3$ is the integral of ρ over R, and by the divergence theorem,

$$\frac{d}{dt} \iiint_R \rho \, dV = - \iiint \operatorname{div} \mathbf{j} \, dV = - \iint_{\partial R} \mathbf{j} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit outward normal to ∂R. Thus the total charge in R can change only to the extent that charge enters and leaves through ∂R.

On the one hand, electromagnetic fields are *produced by* charges and currents, in accordance with Maxwell's equations; on the other, they *act on* charged bodies in accordance with the Lorentz force law. If one combines these things with a specification of the relation between charge and mass distributions, one obtains from (2.17)–(2.21) a system of differential equations that, in principle, completely describes the evolution of the system. It is nonlinear because of the products in (2.17). However, in practice one often considers electromagnetic fields produced by a given system of charges and currents or the motion of some charged particles induced by a given electromagnetic field (produced by some laboratory apparatus, say) without worrying about feedback.

If ρ and $\mathbf{j}$ are taken as given (subject to (2.22)), Maxwell's equations alone provide a *linear* system of differential equations for the fields. The two equations (2.20) and (2.21) that involve time derivatives form a *symmetric hyperbolic system*,

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= c \operatorname{curl} \mathbf{B} - \mathbf{j}, \\ \frac{\partial \mathbf{B}}{\partial t} &= -c \operatorname{curl} \mathbf{E}. \end{aligned}$$

The mathematical theory of such systems is well understood. In particular, the Cauchy problem is well posed: there is a family of theorems that say that if $\mathbf{j}$ is in some nice space $\mathcal{X}$ of functions or distributions on $\mathbb{R}^4$ and the initial data $\mathbf{E}(t_0, \cdot)$ and $\mathbf{B}(t_0, \cdot)$ are in some related space $\mathcal{Y}$ of functions or distributions on $\mathbb{R}^3$, there is a unique solution $(\mathbf{E}, \mathbf{B})$ of this system in another related space $\mathcal{Z}$. Versions of this result can be found, for example, in Taylor [**115**], §6.5, and Treves [**120**], §15. As for the other two Maxwell equations, observe that

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{div} \mathbf{B} &= \operatorname{div} \frac{\partial \mathbf{B}}{\partial t} = -c \operatorname{div} \operatorname{curl} \mathbf{E} = 0, \\ \frac{\partial}{\partial t} \operatorname{div} \mathbf{E} &= \operatorname{div} \frac{\partial \mathbf{E}}{\partial t} = c \operatorname{div} \operatorname{curl} \mathbf{B} - \operatorname{div} \mathbf{j} = \frac{\partial \rho}{\partial t}. \end{aligned}$$

Hence if the equations (2.18) and (2.19) are satisfied at the initial time t_0, they are satisfied at all other times too, so they are merely restrictions on the initial values of $\mathbf{E}$ and $\mathbf{B}$.

In what follows we assume that the quantities in Maxwell's equations are defined on all of space-time $\mathbb{R}^4$, although they may have singularities, in which case the equations are to be interpreted in the sense of distributions. (However, one should *not* think of the spatial component $\mathbb{R}^3$ as representing the entire universe. Rather, one should regard it as a model for some region of which the phenomena in which one is interested are largely concentrated in some compact subregion. Depending on the phenomena one is studying, the actual size of such a region could be anything from a fraction of a meter to many light-years. But one almost always has in mind that the fields and charge distributions vanish at infinity.)

By (2.19), $\mathbf{B}$ is the curl of a vector field $\mathbf{A}$ known as the *vector potential*. The equation (2.20) then can be rewritten as $\operatorname{curl}(\mathbf{E} + c^{-1}\partial_t \mathbf{A}) = 0$, so $\mathbf{E} + c^{-1}\partial_t \mathbf{A}$ is the gradient of a function $-\phi$, the *scalar potential*. Of course $\mathbf{A}$ and ϕ are far from unique: given any (smooth) function χ on $\mathbb{R}^4$, one can replace $\mathbf{A}$ by $\mathbf{A} - \nabla_{\mathbf{x}}\chi$ and ϕ by $\phi + c^{-1}\partial_t \chi$ without changing $\mathbf{E}$ and $\mathbf{B}$. Such adjustments to $\mathbf{A}$ and ϕ are called *gauge transformations* (for reasons that will be explained in §9.1), and the choice of a particular χ to make $\mathbf{A}$ and ϕ satisfy some desired condition is called a *choice of gauge*.

One of the commonly imposed gauge conditions is

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0, \tag{2.23}$$

which can be achieved by starting with any $\mathbf{A}$ and ϕ and replacing them with $\mathbf{A} - \nabla\chi$ and $\phi + c^{-1}\partial_t \chi$ where χ satisfies the inhomogeneous wave equation $\nabla^2\chi - c^{-2}\partial_t^2\chi = \operatorname{div} \mathbf{A} + c^{-1}\partial_t \phi$. Potentials $\mathbf{A}$ and ϕ that satisfy (2.23) are said to be in the *Lorentz gauge* or *Landau gauge*. In terms of such potentials, Maxwell's equations take a particularly appealing form. The equations (2.19) and (2.20) are already embodied in the definition of $\mathbf{A}$ and ϕ. For (2.18), we have

$$\rho = \operatorname{div} \mathbf{E} = -\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t} \operatorname{div} \mathbf{A} = -\nabla^2 \phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2},$$

and for (2.21), since $\operatorname{curl}(\operatorname{curl} \mathbf{A}) = \nabla(\operatorname{div} \mathbf{A}) - \nabla^2 \mathbf{A}$ we have

$$\begin{aligned} \frac{1}{c}\mathbf{j} &= \operatorname{curl}(\operatorname{curl} \mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A}. \end{aligned}$$

In short, denoting the wave operator or d'Alembertian by $\Box$,

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

we see that Maxwell's equations boil down to

$$\Box \phi = \rho, \qquad \Box \mathbf{A} = \mathbf{j} \tag{2.24}$$

in the presence of the Lorentz gauge condition (2.23).

The situation here is best understood by thinking relativistically and putting space and time on an equal footing. Henceforth we adopt the space-time coordinates

$$x^0 = ct, \quad (x^1, x^2, x^3) = \mathbf{x}$$

to get rid of the factors of c, and we adopt the tensor notation explained in §1.1. We define the *electromagnetic 4-potential* by

$$A^\mu = (\phi, \mathbf{A}), \quad \text{or} \quad A_\mu = (\phi, -\mathbf{A}). \tag{2.25}$$

Gauge transformations now take the form

$$A^\mu \mapsto A^\mu + \partial^\mu \chi, \quad \text{or} \quad A_\mu \mapsto A_\mu + \partial_\mu \chi.$$

The electric and magnetic fields are combined into the *electromagnetic field tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \text{or} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.26}$$

that is,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

The Maxwell equations (2.19) and (2.20) become

$$\partial_\kappa F_{\mu\nu} + \partial_\mu F_{\nu\kappa} + \partial_\nu F_{\kappa\mu} = 0, \tag{2.27}$$

which is an immediate consequence of the form (2.25) of $F_{\mu\nu}$, and the equations (2.18) and (2.21) become

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{2.28}$$

where the *4-current density* j^μ is defined by

$$j^\mu = (\rho, \mathbf{j}).$$

The Lorentz gauge condition (2.23) becomes

$$\partial_\mu A^\mu = 0,$$

and since

$$\partial_\mu F^{\mu\nu} = (\partial_\mu \partial^\mu) A^\nu - \partial^\nu (\partial_\mu A^\mu),$$

the Lorentz condition together with (2.28) yields the wave equations (2.24):

$$\partial^2 A^\mu = j^\mu.$$

Moreover, the continuity equation (2.22) becomes

$$\partial_\mu j^\mu = 0. \tag{2.29}$$

Fans of the language of differential geometry may prefer to see this restated in terms of differential forms. The potential and current density are expressed as 1-forms:

$$A = A_\mu \, dx^\mu = \phi \, dt - A_x \, dx - A_y \, dy - A_z \, dz, \quad j = j_\mu \, dx^\mu = \rho \, dt - j_x \, dx - j_y \, dy - j_z \, dz$$

(with t measured in units so that $c = 1$). The electromagnetic field tensor F is the negative of the exterior derivative of A:

$$F = -dA = (E_x \, dx + E_y \, dy + E_z \, dz) \wedge dt + B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy.$$

We have

$$dF = -ddA = 0,$$

which is equivalent to (2.27). The other Maxwell equation (2.28) and the continuity equation (2.29) become

$$*d*F = j, \qquad d*j = 0,$$

where $*$ is the Hodge star operator relative to the Lorentz form. (It is defined by

$$\alpha \wedge *\beta = \Lambda(\alpha, \beta) dt \wedge dx \wedge dy \wedge dz,$$

where $\Lambda(\alpha, \beta)$ denotes the bilinear functional on differential forms induced by the Lorentz form.)

In terms of the potential A, we have

$$-*d*dA = j.$$

In the presence of the Lorentz gauge condition

$$d*A = 0$$

this is equivalent to

$$-(*d*dA + d*d*)A = j.$$

But $-(*d*d + d*d*)$ is the "Lorentz Laplacian" — that is, the d'Alembertian ∂^2, acting componentwise; hence we recover (2.24) in the form

$$\partial^2 A = j \qquad (d*A = 0).$$

We now turn to the Lagrangian formulation of the laws of electromagnetism. We first derive the Lagrangian for a charged particle with rest mass m and charge[3] q, moving in a given electromagnetic field with potential $A^\mu = (\phi, \mathbf{A})$. Relativistic invariance severly restricts the possibilities for a Lorentz-invariant action functional, and the simplest one is

$$S = \int_{(t_0, \mathbf{x}_0)}^{(t_1, \mathbf{x}_1)} \left(-m\, ds - q A_\mu\, dx^\mu\right), \tag{2.30}$$

where ds is as in (2.13) and the integral is taken over a path in spacetime with the given endpoints. The first term gives the action for a free particle; the second one is meant to represent the interaction with the electromagnetic field; and as with the coefficient $-m$ for the free particle, the coefficient $-q$ for the interaction is dictated by the need to obtain the correct equations of motion. Notice that the gauge ambiguity in A_μ is irrelevant: replacing A_μ by $A_\mu + \partial_\mu \chi$ subtracts the exact differential $d\chi$ from the integrand and the constant $\chi(t_1, \mathbf{x}_1) - \chi(t_0, \mathbf{x}_0)$ from the action, which does not affect the dynamics.

Choosing a particular reference frame with coordinates $(t, \mathbf{x})$ (and units chosen so that $c = 1$) and rewriting the action as an integral in t with $\mathbf{v} = d\mathbf{x}/dt$ gives

$$S = \int_{t_0}^{t_1} \left(-m\sqrt{1 - |\mathbf{v}|^2} + q\mathbf{A} \cdot \mathbf{v} - q\phi\right) dt,$$

so the Lagrangian is

$$L = -m\sqrt{1 - |\mathbf{v}|^2} + q\mathbf{A} \cdot \mathbf{v} - q\phi.$$

At this point there are two different momenta to be considered. We shall refer to the ordinary mass-times-velocity momentum $m\mathbf{v}/\sqrt{1 - |\mathbf{v}|^2}$ as the *kinematic* or *mechanical* momentum and denote it by $\mathbf{p}_\kappa$. On the other hand, $\mathbf{p}$ will denote the *canonical* momentum associated to the Lagrangian L:

$$\mathbf{p} = \nabla_{\mathbf{v}} L = \frac{m\mathbf{v}}{\sqrt{1 - |\mathbf{v}|^2}} + q\mathbf{A} = \mathbf{p}_\kappa + q\mathbf{A}.$$

[3]It is an experimental fact that charge, unlike mass, does not depend on velocity.

(This is the momentum that needs to be considered when we pass to quantum mechanics.) The Euler-Lagrange equation is

$$\frac{d\mathbf{p}}{dt} = \nabla_{\mathbf{x}} L = q\nabla_{\mathbf{x}}(\mathbf{A}\cdot\mathbf{v}) - q\nabla\phi.$$

Here L is considered as a function of the *independent* variables $\mathbf{x}$ and $\mathbf{v}$, so $\mathbf{v}$ is treated as a constant in the expression $\nabla_{\mathbf{x}}(\mathbf{A}\cdot\mathbf{v})$; hence,

$$\nabla_{\mathbf{x}}(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla_{\mathbf{x}})\mathbf{A} + \mathbf{v}\times\operatorname{curl}\mathbf{A}.$$

Moreover,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla_{\mathbf{x}})\mathbf{A}.$$

Hence the Euler-Lagrange equation is equivalent to

$$\begin{aligned}\frac{d\mathbf{p}_\kappa}{dt} = \frac{d}{dt}(\mathbf{p} - q\mathbf{A}) &= q\left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right) + q\mathbf{v}\times\operatorname{curl}\mathbf{A}\\ &= q\mathbf{E} + q\mathbf{v}\times\mathbf{B}.\end{aligned}$$

This is the Lorentz force law (2.17) (with $c = 1$), so our choice (2.30) for the action is indeed correct.

The corresponding Hamiltonian $H = \mathbf{p}\cdot\mathbf{v} - L$ is, in terms of the velocity $\mathbf{v}$,

$$H = \frac{m|\mathbf{v}|^2}{\sqrt{1-|\mathbf{v}|^2}}\cdot\mathbf{v} + m\sqrt{1-|\mathbf{v}|^2} + q\phi = \frac{m}{\sqrt{1-|\mathbf{v}|^2}} + q\phi,$$

the sum of the free energy and the Coulomb potential energy. By the same calculation as in the case of a free particle, we find that $(H-q\phi)^2 - |\mathbf{p}_\kappa|^2 = m^2$, so in terms of the canonical momentum $\mathbf{p}$,

$$H = \sqrt{m^2 + |\mathbf{p} - q\mathbf{A}|^2} + q\phi. \tag{2.31}$$

When $\mathbf{v}$ is small we have

$$L \approx -m + \tfrac{1}{2}m|\mathbf{v}|^2 + q\mathbf{A}\cdot\mathbf{v} - q\phi, \qquad H \approx m + \frac{1}{2m}|\mathbf{p} - q\mathbf{A}|^2 + q\phi,$$

which yield the Lagrangian and Hamiltonian for nonrelativistic motion of a particle in an electromagnetic field after subtraction of the rest mass m.

We observed earlier that the energy and momentum $(E, \mathbf{p})$ of a free particle make up a 4-vector p^μ. In the presence of an electromagnetic potential A^μ, so do the total energy given by (2.31) and the *canonical* momentum $\mathbf{p}$, and we still denote this 4-vector by p^μ. Equation (2.31) implies that

$$(E - q\phi)^2 = m^2 + |\mathbf{p} - q\mathbf{A}|^2,$$

which says that the Lorentz inner product of $p^\mu - qA^\mu$ with itself is m^2: $(p-qA)^2 = m^2$. Comparing this with the formula (2.16) for the free energy, we obtain a rule for incorporating the electromagnetic field that will be of basic importance in the quantum theory: *To obtain the energy-momentum vector for a particle with charge q in an electromagnetic field A^μ from that for a free particle, simply replace p^μ by $p^\mu - qA^\mu$.*

Next we consider the Lagrangian form of the field equations for electromagnetism. Here, since we are dealing with fields instead of particles, the action will be a 4-dimensional integral of a Lagrangian density that is a function of the fields and their derivatives. Moreover, the basic field is taken to be the *potential* A^μ rather than the electromagnetic field $F^{\mu\nu}$; the justification for this is that it is the only

thing that works at the quantum level, in spite of the ambiguity in the definition of A^μ. Moreover, there is an implicit assumption that all fields and charges vanish at spatial infinity in such a way that these 4-dimensional integrals converge and all boundary terms in spatial integrations by parts vanish. In the preceding paragraphs we developed the Lagrangian for a particle moving in a given electromagnetic field; it was the sum of the Lagrangian for a free particle and a Lagrangian for the interaction. Similarly, we now develop the Lagrangian for a field in the presence of a given system of charges and currents; it will be the sum of a Lagrangian involving only the field and a Lagrangian for the interaction.

The form of the interaction term is suggested by the $-qA_\mu dx^\mu/dt = -q\phi + q\mathbf{A}\cdot\mathbf{v}$ that we found before. That was for a single charge q; if instead we consider a continuous distribution of charge with charge density ρ and current density $\mathbf{j}$, the obvious analogue is $-A_\mu j^\mu = -\rho\phi + \mathbf{A}\cdot\mathbf{j}$ where $j^\mu = (\rho, \mathbf{j})$ is the 4-current density. Thus, the interaction part of the action functional will be

$$S_{\text{int}} = -\int_{\mathbb{R}^4} A_\mu j^\mu \, d^4x. \tag{2.32}$$

As before, the ambiguity in A^μ is irrelevant: If we change A_μ by adding $\partial_\mu\chi$, where χ vanishes at infinity, the change in the action is

$$-\int (\partial_\mu \chi) j^\mu \, d^4x = \int \chi \partial_\mu j^\mu \, d^4x = 0$$

because of (2.29).

Now, what about the free-field Lagrangian? It should be Lorentz invariant; it should be unaffected by gauge transformations; it should be quadratic so that the field equations will turn out to be linear (Maxwell's equations). The only possibility is a constant multiple of $F_{\mu\nu}F^{\mu\nu}$ (recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$), and the correct constant turns out to be $-\frac{1}{4}$. Thus the full action functional is

$$S = S_{\text{field}} + S_{\text{int}} = \int_{\mathbb{R}^4} \left(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right) \, d^4x. \tag{2.33}$$

(In the classical notation, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{B}|^2)$.)

To confirm the validity of (2.33), let us derive the field equations. If we add a perturbation δA^μ that vanishes at (spatial and temporal) infinity to A^μ, the change in S to first order is

$$\begin{aligned}
\delta S &= \int \left(-\tfrac{1}{2} F^{\mu\nu}\delta F_{\mu\nu} - j^\nu \delta A_\nu\right) \, d^4x \\
&= \int \left(-\tfrac{1}{2} F^{\mu\nu}[\partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)] - j^\nu \delta A_\nu\right) \, d^4x \\
&= \int \left(-F^{\mu\nu}\partial_\mu(\delta A_\nu) - j^\nu \delta A_\nu\right) \, d^4x \\
&= \int \left(\partial_\mu F^{\mu\nu} - j^\nu\right) \delta A_\nu \, d^4x.
\end{aligned}$$

In the first line used the fact that $F_{\mu\nu}\delta F^{\mu\nu} = F^{\mu\nu}\delta F_{\mu\nu}$, and in passing from the second to the third we used the fact that $-F^{\mu\nu}\partial_\nu \delta A_\mu = -F^{\nu\mu}\partial_\mu \delta A_\nu = F^{\mu\nu}\partial_\mu \delta A_\nu$ (by relabeling indices and using the skew symmetry of $F^{\mu\nu}$). Thus the condition $\delta S = 0$ yields the Maxwell equation (2.28). (Recall that the other equation (2.27) is a consequence of the formula $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.)

We have developed a Lagrangian formulation of the evolution equations for a charged particle moving in a given electromagnetic field and for an electromagnetic field in the presence of a given charge-current distribution. An attempt to combine these into a relativistically correct Lagrangian that describes a system of charges and fields interacting with each other is problematic, largely because of the question of finding a suitable form for the pure-matter term (the analogue of $-\int m\,ds$ for a single particle); see Goldstein [**55**]. When we come to quantum electrodynamics, however, this problem will evaporate. The charge-current distribution will be derived from an electron *field*, and the pure-matter term will be its free-field Lagrangian.