Preface

The quantum groups investigated in this book are quantum enveloping algebras defined by their Drinfeld–Jimbo presentation once a symmetrizable (generalized) Cartan matrix is specified. This presentation is essentially a q-deformation or "quantization" of the familiar presentation (by Chevalley generators and Serre relations) of the universal enveloping algebra of a Kac–Moody Lie algebra associated with a symmetrizable Cartan matrix. Thus, one approach to quantum enveloping algebras closely follows the study of universal enveloping algebras of Lie algebras, the results often amounting to quantizations of their classical counterparts.

There is a well-known procedure for obtaining symmetrizable Cartan matrices from finite (possibly valued) graphs. About two decades before the birth of quantum groups, representations of quivers (i.e., directed graphs) were introduced and developed as part of both a new approach to the representation theory of finite dimensional algebras and a method to deal with problems in linear algebra. P. Gabriel [**118**] showed, for example, that if the underlying graph of a quiver is a (simply laced) Dynkin graph, then the indecomposable representations correspond naturally to the positive roots of the finite dimensional complex semisimple Lie algebra associated with the same Dynkin graph. Over a decade later, V. Kac [**170**] generalized Gabriel's result to an arbitrary quiver, obtaining a one-to-one correspondence between the positive real roots of the associated Lie algebra and certain indecomposable quiver representations, as well as a one-to-many correspondence from the positive imaginary roots to the remaining indecomposable representations. Thus, an essential feature of the structure of a symmetrizable Kac–Moody Lie algebra — namely, its root space decomposition — has an interpretation in terms of representations of finite dimensional algebras.

The birth of quantum groups in the 1980s provided an opportunity for quantizing and deepening the finite dimensional algebra results described above. In 1990, C. M. Ringel [**247**] introduced an algebra, which he called the Hall algebra, but which is now commonly known as the Ringel–Hall algebra, associated with the representation category of a finite dimensional algebra over the finite field $\mathbb{F}_q$. In this work, Ringel established some fundamental relations that turned out to be specializations of the modified quantum Serre relations. Ringel then proved, in the finite type case, that the structure constants of the Ringel–Hall algebra are polynomials in q; the resulting generic Ringel–Hall algebra is isomorphic to the "positive part" of the corresponding quantum enveloping algebra.

With this breakthrough in the realization of quantum enveloping algebras of finite type, the development of the theory reached a new level. First, the geometric approach (via the theory of perverse sheaves) was introduced by G. Lusztig [**206**]. He obtained not only a geometric realization of the $\pm$-parts of quantum enveloping algebras associated with symmetrizable Cartan matrices but also canonical bases for these algebras and their representations as an application. Second, J. A. Green [**136**] established a comultiplication formula for Ringel–Hall algebras of hereditary algebras and extended Ringel's algebraic realization to arbitrary types. Thus, the Gabriel–Kac work at the root system — or skeletal — level can be thought of as having been extended to an actual construction of the full quantum enveloping algebra. Beyond the theory of Ringel–Hall algebras, other developments include Nakajima's quiver varieties [**229**] and the realization of all symmetrizable Kac–Moody Lie algebras by L. Peng and J. Xiao [**238**].

At almost the same time as Ringel's work on Hall algebras, A. Beilinson, G. Lusztig, and R. MacPherson investigated a class of finite dimensional algebras, known as quantum Schur algebras, which they used to give a realization of the *entire* quantum enveloping algebras in the important case of type A, i.e., associated with the general linear Lie algebras $\mathfrak{gl}_n$. This work thus provided another finite dimensional algebra approach to quantum enveloping algebras, completely different from the theory of Ringel–Hall algebras. However, the multiplication formulas that played a key role in this approach result from an analysis of quantum Schur algebras over finite fields, using the geometry of flags on a finite dimensional vector space. A stabilization property derived from the multiplication formula permits the definition of an infinite dimensional algebra as a "limit" of all quantum Schur algebras. In turn, this algebra has a completion that naturally contains the quantum enveloping algebra as a subalgebra. As a bonus, this method leads to an explicit basis, called the BLM basis, for the entire quantum enveloping algebra, and it yields explicit multiplication formulas for any basis element by a generator. It has been proved by J. Du and B. Parshall [**104**] that a triangular part of the BLM basis coincides with the Ringel–Hall algebra basis.

This book provides an introduction to the two algebraic approaches briefly described above, with an emphasis on the structure and realization of quantum enveloping algebras. The treatment is largely elementary and combinatorial. In so far as possible, we have written the book to be accessible to graduate students and to mathematicians who are not experts in the field. Apart from some standard material (e.g., [**BAII**], [**LAII**]), our treatment is entirely self-contained with two notable exceptions: a positivity result for Hecke algebras (in Chapter 7), which requires the use of perverse sheaves, and a theorem of Lusztig used in the proof of Green's theorem (in Chapter 12), which requires the representation theory of Kac–Moody Lie algebras. For the more advanced geometric approach using the theory of perverse sheaves, see Lusztig's book [**209**].

Although the present book centers on the finite dimensional algebra approach to quantum groups, it also takes up two other, important, related topics. First, following [**59**], we use Frobenius morphisms on algebras to link representations of a quiver directly to representations of a species (called a modulated quiver in this book) without specifically working with the species. In the language of Lie theory, a quiver determines a *symmetric* generalized Cartan matrix, while a species corresponds to a symmetrizable one. As Cartan matrices, these two cases are linked by a graph automorphism. A quiver automorphism (i.e., a graph automorphism preserving arrows) gives rise naturally to a Frobenius morphism on the path algebra of the quiver whose fixed-point algebra can be interpreted as the tensor algebra of a species. Thus, the Ringel–Hall algebras associated with the representation categories of quivers with automorphisms cover all the quantum enveloping algebras associated with symmetrizable Kac–Moody Lie algebras.

The second related topic is the Kazhdan–Lusztig theory for (Iwahori–) Hecke algebras and cells. Playing an important role in Chevalley group theory [**159**], Hecke algebras are quantum deformations of group algebras of Coxeter groups. In 1979, D. Kazhdan and G. Lusztig [**177**] discovered a remarkable basis for a Hecke algebra, known as the *Kazhdan–Lusztig* or *canonical basis*, which has important applications in the representation theory of Hecke algebras, algebraic groups, finite groups of Lie type, and quantum groups. We use the same idea in the construction of canonical bases for quantum enveloping algebras of finite type in Chapter 11. As a noteworthy crown to the whole theory, we present the modern cell approach to the representations of symmetric groups and the structure of quantum Schur algebras. The latter is fundamental in the BLM approach to the realization of the entire quantum enveloping algebra of $\mathfrak{gl}_n$.

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The book consists of 14 chapters arranged in 5 parts, complemented by a leading Chapter 0 — that outlines the main features of the book — as well as three appendices. Chapter 0 begins with the two realizations of Cartan matrices: the graph realization and the root datum realization, which lead up to the theories of quiver representations and quantum enveloping algebras, respectively. The main objects discussed in the book are certain algebraic structures — Coxeter groups, associative and Lie algebras, etc. — which are often presented with generators and relations. We set down in §0.2 the relevant notations for presentations. When an algebraic structure is presented by generators and relations, the immediate question arises of a description in some concrete way. For example, Coxeter groups are defined by means of a presentation, but, as J. Tits has shown, have an elegant explicit description as "reflection groups." (See §4.1.) In general, this question is the so-called *realization problem.* In this book, our main focus will be the two beautiful realizations of quantum enveloping algebras. However, as a first taste, we discuss the problem through some relatively simple examples in §0.3 and §0.6. In §0.4, the so-called quantumization process is introduced to explain the phenomenon that counting over finite fields often leads to certain generic objects over a polynomial ring. We shall see that Hecke algebras, quantum Schur algebras, and Ringel–Hall algebras of finite type can all be produced through this process. Finally, as one of the main topics in the book, the crude model of the canonical basis theory, i.e., the elementary matrix construction of canonical bases, is discussed in §0.5.

Part 1 (Chapters 1–3) presents the theory of finite dimensional algebras, with an emphasis on representations of quivers with automorphisms. Chapter 1 begins with the basics of quiver representations and proves the theorem of Gabriel mentioned earlier using Bernstein–Gelfand–Ponomarev (BGP) reflection functors. It also lays out the relations between quivers, Euler forms, root systems, Weyl groups, and representation varieties.

Chapter 2 treats the general theory of representations of algebras with Frobenius morphisms. A Frobenius morphism F on a finite dimensional algebra A defined over the algebraic closure $\mathcal{K}$ of the finite field $\mathbb{F}_q$ is a ring automorphism satisfying $F(\lambda a) = \lambda^q a$, for all $\lambda \in \mathcal{K}$ and $a \in A$. It induces a functor on the category of finite dimensional A-modules, called the Frobenius twist functor. If the Frobenius twist of a module is isomorphic to itself, then the module is called an F-stable module. We show that the subcategory of F-stable modules with morphisms compatible with F-stability is equivalent to the module category of the F-fixed point algebra A^F. Thus, the determination of indecomposable A^F-modules is equivalent to that of indecomposable F-stable modules. Additionally, this method provides a relation between almost split sequences for A^F-modules and A-modules in the Auslander–Reiten theory. In preparation for those results, Chapter 2 contains a brief and self-contained introduction to almost split sequences and irreducible morphisms.

In Chapter 3, we apply the general theory to the path algebra A of a quiver Q with automorphism σ. If F is the Frobenius morphism on A induced from σ, the F-fixed point algebra A^F is a hereditary algebra over the finite field $\mathbb{F}_q$ and is the tensor algebra of the species associated with (Q, σ). Up to Morita equivalence, every finite hereditary algebra arises in this way. We further extend the folding relation associated with the quiver automorphism to a folding relation between the Auslander–Reiten quivers of A and A^F. Finally, we study representations of affine quivers with automorphisms and describe their Frobenius twists explicitly as an example of the applications of the theory. The formulas for the number of indecomposable representations of the associated F-fixed point algebra are also presented.

Part 2 (Chapters 4–6) constructs, via generators and relations, the algebras that play an important role throughout the book. It opens in Chapter 4 with the basic theory of Coxeter groups. Symmetric groups and affine Weyl groups provide important examples, which we look at in some detail. A modification of the defining relations for a Coxeter group leads naturally to the construction of the associated Hecke algebra, the properties of which are also rather fully explored. Chapter 4 concludes with a further example showing that Hecke algebras for the symmetric groups arise in a quantumization process that starts with the endomorphism algebra of the complete flag variety of a finite general linear group.

Chapter 5 begins with a brief tour of the basics of Hopf algebras. It continues with the fundamental example of universal enveloping algebras, emphasizing Kac–Moody Lie algebras and their symmetry structure. These results serve as a template for quantum enveloping algebras. The chapter ends with a discussion of the simplest quantum enveloping algebra, quantum $\mathfrak{sl}_2$.

Chapter 6 is devoted to quantum enveloping algebras — defined by means of the Drinfeld–Jimbo presentation — associated with symmetrizable Cartan matrices. There, we first show that these algebras are infinite dimensional and carry Hopf algebra structures. Actions of suitable braid groups on these algebras lead to the definition of root vectors for arbitrary roots as well as to the construction of PBW-type bases in the finite type case.

Part 3 (Chapters 7–9) presents a modern approach to the ordinary representation theory of symmetric groups and the associated Hecke algebras. Chapter 7 is concerned with the combinatorial part of Kazhdan–Lusztig theory — the calculus of Hecke algebras and cells. After introducing the canonical bases for Hecke algebras, we develop Kazhdan–Lusztig polynomials, dual bases, inverse Kazhdan–Lusztig polynomials, and Knuth, cell, and Vogan equivalence relations. We prove that the Knuth equivalence is finer than the left cell equivalence which is, in turn, finer than Vogan equivalence. We conclude with a brief explanation of the geometric meaning of the Kazhdan–Lusztig polynomials, including the positivity property and its applications.

Chapter 8 explicitly determines the cells for the symmetric groups and constructs the simple representations of symmetric groups and their associated Hecke algebras. A main tool is the Robinson–Schensted algorithm. For later application to quantum Schur algebras, we adopt a generalized version, known as the Robinson–Schensted–Knuth (RSK) correspondence which associates with each square matrix over $\mathbb{N}$ a pair of semistandard tableaux — the insertion tableau and the recording tableau. Given two elements in a symmetric group, if they are Vogan equivalent, then they have the same recording tableau; hence, they are Knuth equivalent. This completes the decomposition of a symmetric group into left (or right) cells. As a further application of the positivity property, we introduce the asymptotic Hecke algebras and an Artin–Wedderburn decomposition for the type A Hecke algebras.

Chapter 9 takes up the Kazhdan–Lusztig calculus for quantum Schur algebras, or q-Schur algebras, as a natural extension of the theory of Hecke algebras. Beginning with the Dipper–James definition of a quantum Schur algebra as the endomorphism algebra of tensor space, we immediately establish its integral quasi-heredity by showing the existence of a Specht datum in the sense of [**106**]. We then construct canonical bases for these algebras as a natural extension of the counterpart for Hecke algebras. These bases are, in fact, cellular bases in the sense of Graham–Lehrer [**134**] and can be used to establish the integral quasi-hereditary property for quantum Schur algebras. In addition, the duality between Specht and Δ-filtrations is discussed, and tilting module theory is developed. As an application, we establish the integral double centralizer property which will be further extended in Chapter 14 to the integral quantum Schur–Weyl reciprocity.

Part 4 (Chapters 10–12) presents Ringel's Hall algebra approach to quantum enveloping algebras. The story begins in Chapter 10 with the basic definition of the (integral) Hall algebra of a finitely generated algebra over a finite field. We establish that Hall algebras satisfy certain fundamental relations. These become the quantum Serre relations in a Ringel–Hall algebra which is defined in this book as the twisted Hall algebra associated with a quiver with automorphism (and a finite field). It turns out that there is a surjective algebra homomorphism from a triangular part of a quantum enveloping algebra to the generic composition algebra associated with a quiver with automorphism. In the Dynkin quiver case, the existence of Hall polynomials provides a direct definition of the generic Ringel–Hall algebra. In this case, a dimension comparison shows that the algebra homomorphism above is an isomorphism.

Chapter 11 focuses on Ringel–Hall algebras of Dynkin quivers with automorphisms and the construction of bases for the corresponding quantum enveloping algebras of finite type. Starting from a monoid structure and a poset structure on the set of isomorphism classes of representations, we first obtain a systematic construction of monomial bases for quantum enveloping algebras. We then show that BGP reflection functors induce certain isomorphisms of the subalgebras of Ringel–Hall algebras, which are the restrictions of the Lusztig symmetries defined in Chapter 6. This gives a construction of PBW-type bases, which was mentioned in Chapter 6 without proof. Finally, by relating monomial and PBW-type bases, we present an elementary algebraic construction of Lusztig canonical bases for quantum enveloping algebras of finite type.

Chapter 12 deals with a comultiplication defined by Green [**136**] on the Ringel–Hall algebras. The compatibility of multiplication and comultiplication is based on what is called Green's formula. This result, together with a theorem of Lusztig, shows that the surjective algebra homomorphism defined in Chapter 10 is actually an isomorphism. Hence, the Ringel–Hall algebras provide a realization of the triangular parts of all quantum enveloping algebras.

Part 5 (Chapters 13–14) gives a full account of the Beilinson–Lusztig–MacPherson (BLM) construction for the quantum enveloping algebra associated with $\mathfrak{gl}_n$. Chapter 13 derives in an elementary geometric setting some fundamental multiplication formulas for the natural basis elements in quantum Schur algebras. This leads to a new basis for a quantum Schur algebra — the BLM basis — and to the derivation of some multiplication formulas among the new basis elements. The quantum Serre relations in a quantum Schur algebra result from these multiplication formulas. As a byproduct, a certain monomial basis, which is triangularly related to the natural basis, is constructed in order to give a presentation of a quantum Schur algebra.

Finally, in Chapter 14, a further analysis of the fundamental multiplication formulas gives a stabilization property. This prompts the definition of the BLM algebra $\mathbf{K}$ — an infinite dimensional algebra without identity — and some modified fundamental multiplication formulas. By taking a completion of $\mathbf{K}$, we obtain an algebra $\widehat{\mathbf{K}}$ with identity and derive some multiplication formulas from the modified ones. With these formulas, we prove that a certain subspace $\mathbf{V}$ of $\widehat{\mathbf{K}}$ is a subalgebra with quantum Serre relations. We then prove the isomorphism between $\mathbf{V}$ and the entire quantum $\mathfrak{gl}_n$ before closing with the establishment of the integral Schur–Weyl reciprocity. This basic result is obtained by combining the double centralizer property with the surjection from a type of integral Lusztig form to the integral quantum Schur algebras.

In addition to the chapters described above, this book contains three chapter-long appendices. Appendix A outlines basic ideas from algebraic geometry and algebraic group theory that are required in the book and concludes with a brief discussion of some more advanced topics in the representation theory of semisimple groups. Appendix B gives a largely self-contained discussion of quantum matrix spaces and quantum general linear groups — both including standard and multiparametered — and ties them with the theory of quantum Schur algebras given in Chapter 9. Finally, Appendix C provides a short and self-contained account of the theories of quasi-hereditary algebras and cellular algebras which are needed in Part 3. Making use of the results in Appendices B and C, we discuss several of the standard examples of quasi-hereditary algebras and highest weight categories that arise in representation theory.

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As evidenced by the bibliography, this book clearly could not have been written without the work of the many mathematicians who have contributed over the years to this evolving theory. It also draws, at a number of critical points, from previous book-length treatments. For example, Chapters 4 and 6 reflect the influence of Humphreys [**157**], Carter [**35, 36**], Jantzen [**165**], and Lusztig [**209, 213**], while Chapter 8 incorporates and builds on Stanley's development of the Robinson–Schensted–Knuth correspondence in [**281**]. The Notes at the end of each chapter record our indebtedness to this and other work and form a critical part of our exposition.

Each chapter also closes with a series of exercises, some of which are routine, some of which serve to fill in steps in the various arguments, and some of which call attention to the literature by sketching proofs of results to be found there.

Despite its length, there are many important topics that have not been included in this book. The chosen material reflects our own interests and forms what we hope is a coherent whole. Other topics are sometimes briefly mentioned in the Notes (and references).

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Historical notes and acknowledgments: Although the theoretical interrelations between the representation theory of finite dimensional algebras and Lie theory date at least to the 1970s, the interrelations between

finite dimensional algebras and the representation theory of Lie algebras and algebraic groups became truly apparent at the Ottawa–Moosonee Algebra Workshop in 1987. First, Claus Ringel presented his ideas, which led to the development of Ringel–Hall algebras as realizations of the $\pm$-parts of quantum enveloping algebras. Second, the third author of the book and Leonard Scott presented their discovery (with Edward Cline) of quasi-hereditary algebras and highest weight categories. Since then a number of conferences touching on the same theme have been held in Ottawa (1992), Shanghai (1998), Kunming (2001), Toronto (2002), Chengdu and Banff (2004), and Lhasa (2007). The authors would like to thank the organizers of these conferences for the opportunity to observe and participate in the exciting developments in this area. The idea to write a book originated with the second and third authors 15 years ago at the Ottawa meeting. Then the last three authors made an early effort in this direction, but as the project has evolved and the subject matter developed, the first author became a member of the team.

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