Chapter 1

Representations of quivers

This chapter opens with the definitions of a quiver $Q = (Q_0, Q_1)$ and its representations over a field $\mathcal{k}$ as well as a demonstration of the fact that representations of Q identify with modules for the path algebra $\mathcal{k}Q$. To each quiver Q with n vertices, we attach an integral bilinear (Euler) form $\langle -, - \rangle$ on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$. The Euler form plays a central role in defining the Weyl group $W(Q)$ and root system $\Phi(Q)$ of the quiver Q. Making use of these important tools, quivers naturally divide into three disjoint classes: Dynkin, tame, and wild. Next, we introduce the reflection functors of Bernstein, Gelfand, and Ponomarev, using them to show that all indecomposable representations of a Dynkin quiver can be produced from simple ones. This approach provides an elegant proof of Gabriel's theorem, which states that Q has finitely many isoclasses (= isomorphism classes) of indecomposable modules if and only if Q has Dynkin type. The proof of this result gives a bijection between the isoclasses of indecomposable representations of a Dynkin quiver and the set of positive roots in its root system.

The last section treats representation varieties of the quiver Q over an algebraically closed field $\mathcal{K}$. (Appendix A provides a sketch of the necessary elementary algebraic geometry.) If a representation V of Q has dimension vector $\mathbf{d}$, V determines a point in the variety $R(\mathbf{d})$ of all representations having dimension vector $\mathbf{d}$. The isoclass of V then defines an orbit $\mathfrak{O}_V$ for a reductive group $\mathrm{GL}_{\mathbf{d}}(\mathcal{k})$ in its natural action on $R(\mathbf{d})$. These ideas are key in a final result on generic extensions of representations which have interesting applications in Chapter 11.

1.1. Quivers and their representations

We begin with the following basic definition.

Definition 1.1. A *quiver* $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$ consists of two sets Q_0, Q_1, and two maps $\mathrm{t}, \mathrm{h}\colon Q_1 \to Q_0$. The elements in Q_0 are called the *vertices* of Q, and those in Q_1 are called the *arrows* of Q. For each ρ in Q_1, $\mathrm{t}\rho$ and $\mathrm{h}\rho$ are called the *tail* and *head* of ρ, respectively. We will indicate an arrow ρ (between $\mathrm{t}\rho$ and $\mathrm{h}\rho$) as

$$\underset{\mathrm{t}\rho}{\bullet} \xrightarrow{\quad\rho\quad} \underset{\mathrm{h}\rho}{\bullet} \qquad \text{or} \qquad \mathrm{t}\rho \xrightarrow{\quad\rho\quad} \mathrm{h}\rho$$

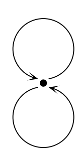
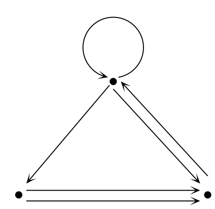
An arrow ρ with $\mathrm{t}\rho = \mathrm{h}\rho$ is called a *loop*. A quiver $Q' = (Q'_0, Q'_1, \mathrm{t}', \mathrm{h}')$ is called a *subquiver* of Q if $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, and t' and h' are the restrictions of t and h, respectively, to Q'_1. Also, Q' is called *proper* if either $Q'_0 \subset Q_0$ or $Q'_1 \subset Q_1$, and it is called *full* if, for any vertices $i, j \in Q'_0$, all arrows in Q between i and j also lie in Q'_1. We often simply write $Q = (Q_0, Q_1)$ instead of $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$.

A *morphism* $\varphi\colon Q \to Q'$ from a quiver $Q = (Q_0, Q_1)$ to a quiver $Q' = (Q'_0, Q'_1)$ consists of a map $\varphi\colon Q_0 \to Q'_0$ and a map $\varphi\colon Q_1 \to Q'_1$, satisfying $\mathrm{t}'\varphi(\rho) = \varphi(\mathrm{t}\rho)$ and $\mathrm{h}'\varphi(\rho) = \varphi(\mathrm{h}\rho)$, for $\rho \in Q_1$. The embedding of a subquiver of Q into Q is an example of a quiver morphism. An *isomorphism* of quivers is a morphism of quivers, bijective on both vertices and arrows. Quivers with *automorphisms* will be considered in Chapter 3.

A quiver Q is called *finite* if both Q_0 and Q_1 are finite sets.

All quivers considered in the sequel are finite, unless otherwise indicated. (For example, the Auslander–Reiten quivers studied in §3.3 may be infinite.)

Example 1.2. The following are examples of quivers.

(a)  (b) 

A *path of length* $d \geqslant 1$ in Q with tail i and head j is a sequence $\rho_d \cdots \rho_2\rho_1$

$$i\ \bullet \xrightarrow{\rho_1} \bullet \xrightarrow{\rho_2} \bullet \longrightarrow \cdots\cdots \longrightarrow \bullet \xrightarrow{\rho_d} \bullet\ j$$

of arrows ρ_s such that $\mathrm{t}\rho_1 = i$, $\mathrm{h}\rho_s = \mathrm{t}\rho_{s+1}$ $(1 \leqslant s \leqslant d-1)$, and $\mathrm{h}\rho_d = j$. In case $i = j$, the path p is said to be an *oriented cycle*. In particular, a loop is an oriented cycle. A quiver Q is called *acyclic* if Q admits no oriented cycles. Besides paths of length $\geqslant 1$, we also consider the "trivial path" e_i, which is the path of length 0 with tail and head $i \in Q_0$.

Definition 1.3. Let $\Bbbk$ be a field. A *representation* $V = (V_i, V_\rho)$ of a quiver Q over $\Bbbk$ consists of a family of $\Bbbk$-vector spaces V_i, for $i \in Q_0$, together with a family of $\Bbbk$-linear maps $V_\rho\colon V_{\mathrm{t}\rho} \to V_{\mathrm{h}\rho}$, for $\rho \in Q_1$. A *morphism* $f\colon V \to W$ between representations V and W is given by $\Bbbk$-linear maps $f_i\colon V_i \to W_i$, for all $i \in Q_0$, satisfying $W_\rho \circ f_{\mathrm{t}\rho} = f_{\mathrm{h}\rho} \circ V_\rho$ (i.e., making the square

$$\begin{CD} V_{\mathrm{t}\rho} @>{V_\rho}>> V_{\mathrm{h}\rho} \\ @V{f_{\mathrm{t}\rho}}VV @VV{f_{\mathrm{h}\rho}}V \\ W_{\mathrm{t}\rho} @>>{W_\rho}> W_{\mathrm{h}\rho} \end{CD}$$

commutative), for each arrow $\rho \in Q_1$.

A representation $V = (V_i, V_\rho)$ is regarded as finite dimensional if all V_i, $i \in Q_0$, are finite dimensional over $\Bbbk$. Let $\mathsf{Rep}_\Bbbk Q$ denote the category of finite dimensional representations of Q over $\Bbbk$. For each representation $V = (V_i, V_\rho)$, the vector $\mathbf{dim}\, V := (\dim V_i)_{i\in Q_0} \in \mathbb{Z}^n$ is called the *dimension vector* of V, and the sum $\sum_{i\in Q_0} \dim V_i$ is called the *dimension* of V. The category $\mathsf{Rep}_\Bbbk Q$ contains a zero object, which is the representation $V = (V_i, V_\rho)$ with all $V_i = 0$ (and thus all $V_\rho = 0$). One may define a subrepresentation $W = (W_i, W_\rho)$ of a representation $V = (V_i, V_\rho)$ of Q. This requires that W_i is a subspace of V_i, for each $i \in Q_0$, and that $W_\rho\colon W_{\mathrm{t}\rho} \to W_{\mathrm{h}\rho}$ is the restriction of V_ρ to $W_{\mathrm{t}\rho}$, for each $\rho \in Q_1$. A representation $V = (V_i, V_\rho)$ of Q is called *nilpotent* if there is a positive integer n (depending on V) such that, for every path $\rho_r \cdots \rho_1$ of length $r \geqslant n$, $V_{\rho_r} \cdots V_{\rho_1}\colon V_{\mathrm{t}\rho_1} \to V_{\mathrm{h}\rho_r}$ is the zero map.

Furthermore, to each $j \in Q_0$ we can attach naturally a one-dimensional representation $S_j = \big((S_j)_i, (S_j)_\rho\big)$ such that $(S_j)_j = \Bbbk$, $(S_j)_i = 0$, for all $j \neq i \in Q_0$, and $(S_j)_\rho = 0$, for all $\rho \in Q_1$. It is obvious that S_j is a simple (or an irreducible) representation in the sense that it has no nontrivial subrepresentation. Also, the S_i are nilpotent.

If Q is acyclic, then $\{S_i \mid i \in Q_0\}$ is a complete set of all simple representations in $\mathsf{Rep}_\Bbbk Q$; see Exercise 1.2(1). However, if Q contains oriented cycles, then there are simple representations of Q other than the S_i. Explicitly, if $\rho_m \cdots \rho_1$ is an oriented cycle, then, for each nonzero $\lambda \in \Bbbk$, the representation $V(\lambda) = (V_i, V_\rho)$ of Q defined by

$$V_i = \begin{cases} \Bbbk, & \text{if } i = \mathrm{t}\rho_s, \text{ for } 1 \leqslant s \leqslant m, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and}$$
$$V_\rho = \begin{cases} \lambda, & \text{if } \rho = \rho_1, \\ 1, & \text{if } \rho = \rho_s, \text{ for } 2 \leqslant s \leqslant m, \\ 0, & \text{otherwise} \end{cases} \tag{1.1.1}$$

is simple; see Exercise 1.2(2). Clearly, $V(\lambda)$ is not nilpotent. Nevertheless, for an arbitrary quiver Q, $\{S_i \mid i \in Q_0\}$ forms a complete set of all simple nilpotent representations in $\mathsf{Rep}_{\Bbbk} Q$; see Exercise 1.2(3).

The direct sum of two representations $V = (V_i, V_\rho)$ and $W = (W_i, W_\rho)$ is, by definition, the representation $U = (U_i, U_\rho)$, where $U_i := V_i \oplus W_i$, for all $i \in Q_0$, and

$$U_\rho = \begin{pmatrix} V_\rho & 0 \\ 0 & W_\rho \end{pmatrix} : U_{\mathrm{t}\rho} = V_{\mathrm{t}\rho} \oplus W_{\mathrm{t}\rho} \longrightarrow V_{\mathrm{h}\rho} \oplus W_{\mathrm{h}\rho} = U_{\mathrm{h}\rho},$$

for all $\rho \in Q_1$. A representation V is said to be *indecomposable* if it is nonzero and not isomorphic to a direct sum of two nonzero representations. For $m \in \mathbb{N}$, we will let $mV = \underbrace{V \oplus \cdots \oplus V}_{m}$ be the direct sum of m copies of V.

Besides having a zero object, subobjects, and direct sums, etc., the category $\mathsf{Rep}_{\Bbbk} Q$ enjoys the other properties of an abelian category.[1] All of these properties can be readily checked, but they also follow from the fact, which we will soon prove that $\mathsf{Rep}_{\Bbbk} Q$ is equivalent to the category of finite dimensional left modules over a $\Bbbk$-algebra; see Proposition 1.7 below.

It turns out that a whole range of problems of linear algebra can be formulated in the context of representations of quivers, as shown in the examples below.

Examples 1.4. (1) Let Q be the quiver consisting of a single vertex 0 and one loop ρ. Each representation V of Q is given by a $\Bbbk$-vector space V_0 together with a $\Bbbk$-linear endomorphism $V_\rho \colon V_0 \to V_0$. If V_0 is finite dimensional, by choosing a basis of V_0, we can view V_ρ as a matrix over $\Bbbk$, and the classification of isoclasses of representations of Q with a fixed dimension d is equivalent to the classification of conjugacy classes of $d \times d$ matrices over $\Bbbk$. In particular, if $\Bbbk$ is algebraically closed, the classification is given by the well-known Jordan–Weierstrass theorem which assigns to each $d \times d$ matrix its Jordan normal form.

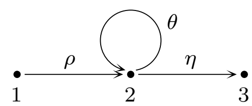
(2) Let Q be the Kronecker quiver $1 \bullet \overset{\rho}{\underset{\eta}{\rightrightarrows}} \bullet 2$. Each representation in $\mathsf{Rep}_{\Bbbk} Q$ is given by two finite dimensional vector spaces V_1, V_2 and two linear maps $V_\rho, V_\eta \colon V_1 \to V_2$. By choosing bases of V_1 and V_2, V_ρ and V_η are given by matrices over $\Bbbk$ of the same size. Thus, the problem of classifying representations of Q with a given dimension vector (m, n) is converted to that of finding a normal form of a pair of matrices $X, Y \in \mathrm{M}_{n\times m}(\Bbbk)$ under simultaneous row and column transformations. Here $\mathrm{M}_{n\times m}(\Bbbk)$ denotes the set of all $n \times m$ matrices with entries in $\Bbbk$.

[1] A detailed discussion of categories, sufficient for this book, is contained in [**BAII**, Ch. 1 and 6] or [**HAII**, Ch. 1]. Many of the categories considered in this book are abelian (e.g., module categories).

(3) A linear time-invariant dynamical system is a system of ordinary differential equations

$$\frac{d(x(t))}{dt} = Yx(t) + Xu(t), \quad y(t) = Zx(t),$$

where $u(t), x(t)$, and $y(t)$ are functions of $t \in \mathbb{R}$ with values in $\mathbb{C}^l$, $\mathbb{C}^m$, and $\mathbb{C}^n$, respectively, and $X \in \mathrm{M}_{m\times l}(\mathbb{C})$, $Y \in \mathrm{M}_m(\mathbb{C})$, $Z \in \mathrm{M}_{n\times m}(\mathbb{C})$. Thus, such a system is associated with a representation V of the quiver Q



with $V_1 = \mathbb{C}^l$, $V_2 = \mathbb{C}^m$, $V_3 = \mathbb{C}^n$, and $V_\rho = X$, $V_\theta = Y$, $V_\eta = Z$.

Given two linear time-invariant dynamical systems, each isomorphism between the associated representations provides a bijection between the spaces of solutions. However, it seems hopeless to give a complete classification of representations of Q.

We can view representations of a quiver as modules over a $\Bbbk$-algebra $\Bbbk Q$, the path algebra of Q over $\Bbbk$, which is defined as follows.

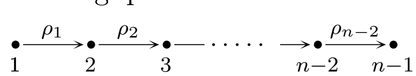
Definition 1.5. The *path algebra* $\Bbbk Q$ is the $\Bbbk$-algebra having as basis the set of all the paths in Q. The product in the algebra is given by linearity and the following product rule for paths $\rho_d \cdots \rho_2\rho_1$ and $\eta_e \cdots \eta_2\eta_1$:

$$(\rho_d \cdots \rho_2\rho_1) \cdot (\eta_e \cdots \eta_2\eta_1) = \begin{cases} \rho_d \cdots \rho_2\rho_1\eta_e \cdots \eta_2\eta_1, & \text{if } \mathrm{h}\eta_e = \mathrm{t}\rho_1; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\Bbbk Q$ is an associative algebra with the identity $1 = \sum_{i\in Q_0} e_i$. The trivial paths e_i, $i \in Q_0$, are pairwise orthogonal idempotents of $\Bbbk Q$. Moreover, the algebra $\Bbbk Q$ is finite dimensional if and only if Q is acyclic.

Examples 1.6. (1) If Q denotes the quiver consisting of a single vertex and n loops, then $\Bbbk Q$ is the free $\Bbbk$-algebra with n generators; see Proposition 0.11(1). In particular, if $n = 1$, $\Bbbk Q$ is the polynomial ring $\Bbbk[T]$ with an indeterminate T; cf. Example 1.4(1).

(2) Let Q be the following quiver:



(1.1.2)

Then the path algebra $\Bbbk Q$ is isomorphic to the lower triangular matrix algebra

$$A := \{(a_{i,j}) \in \mathrm{M}_{n-1}(\Bbbk) \mid a_{i,j} = 0 \text{ if } i < j\}.$$

In fact, under this isomorphism, the standard matrix unit $E_{i,j}$, $i \geqslant j$, corresponds to the path with tail j and head i.

The quiver introduced in this example is usually called a *linear quiver*, and it is often denoted by $\mathscr{L}_{n-1}$. We will discuss this quiver again later in different contexts. (And, to be consistent with later contexts, we consider here $\mathscr{L}_{n-1}$ instead of $\mathscr{L}_n$.)

Finally, we denote by $\mathcal{k}Q$-mod the category of finite dimensional left $\mathcal{k}Q$-modules. We have the following proposition.

Proposition 1.7. *The categories* $\mathsf{Rep}_{\mathcal{k}}Q$ *and* $\mathcal{k}Q$-mod *are equivalent.*

Proof. Let $V = (V_i, V_\rho)$ be a representation of Q. Set $\mathscr{F}(V) = \bigoplus_{i\in Q_0} V_i$. Then $\mathscr{F}(V)$ admits a $\mathcal{k}Q$-module structure defined by

$$p \cdot v = V_{\rho_d} \circ \cdots \circ V_{\rho_1}(v_{\mathrm{t}\rho_1}) \in V_{\mathrm{h}\rho_d} \subseteq \mathscr{F}(V),$$

where $v = \sum_{i\in Q_0} v_i \in \mathscr{F}(V)$, and $p = \rho_d \cdots \rho_1$ is a path in Q. Clearly, each morphism $\varphi\colon V \to W$ between two representations V and W gives rise to a $\mathcal{k}Q$-module homomorphism $\mathscr{F}(\varphi)\colon \mathscr{F}(V) \to \mathscr{F}(W)$. Thus, we obtain a functor $\mathscr{F}\colon \mathsf{Rep}_{\mathcal{k}}Q \to \mathcal{k}Q$-$\mathsf{mod}$.

Conversely, for each module M in $\mathcal{k}Q$-mod, define $M_i = e_iM$, for each $i \in Q_0$, where e_i is the trivial path at i. The $\mathcal{k}Q$-module structure on M induces $\mathcal{k}$-linear maps $M_\rho\colon M_{\mathrm{t}\rho} \to M_{\mathrm{h}\rho}$, $v \mapsto \rho v$, for $\rho \in Q_1$. This gives a representation $\mathscr{G}(M) = (M_i, M_\rho)$ of the quiver Q. Furthermore, a $\mathcal{k}Q$-module homomorphism $\psi\colon M \to N$ induces a morphism $\mathscr{G}(\psi) = (\psi_i)\colon \mathscr{G}(M) \to \mathscr{G}(N)$, where $\psi_i\colon M_i = e_iM \to e_iN = N_i$ is the restriction of ψ, $i \in Q_0$. In this way, we have also defined a functor $\mathscr{G}\colon \mathcal{k}Q$-$\mathsf{mod} \to \mathsf{Rep}_{\mathcal{k}}Q$.

From the construction, it is easily seen that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathsf{Rep}_{\mathcal{k}}Q} \quad\text{and}\quad \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{k}Q\text{-}\mathsf{mod}},$$

where $\mathrm{id}_{\mathsf{Rep}_{\mathcal{k}}Q}$ and $\mathrm{id}_{\mathcal{k}Q\text{-}\mathsf{mod}}$ denote the identity functors of the categories $\mathsf{Rep}_{\mathcal{k}}Q$ and $\mathcal{k}Q$-mod, respectively. □

As a consequence of this proposition (or, more simply, by a direct verification), the category $\mathsf{Rep}_{\mathcal{k}}Q$ is abelian. If $\mathcal{k}Q$ is finite dimensional (i.e., Q is acyclic), then $\mathsf{Rep}_{\mathcal{k}}Q$ has enough projective and injective objects. In addition, the Krull–Schmidt theorem [**BAII**, p. 115] holds for $\mathsf{Rep}_{\mathcal{k}}Q$, i.e., every object in $\mathsf{Rep}_{\mathcal{k}}Q$ has a direct sum decomposition into indecomposable objects, which are unique up to isomorphism. One of the main tasks in the representation theory of quivers is to classify all indecomposable representations, up to isomorphism.

In the sequel, we will always identify $\mathcal{k}Q$-modules with representations of Q over $\mathcal{k}$.

1.2. Euler forms, Cartan matrices, and the classification of quivers

Let $Q = (Q_0, Q_1)$ be a quiver. This section introduces a bilinear form on $\mathbb{Z}Q_0$, the free abelian group with basis Q_0. This form leads to the definition of Cartan matrix of Q (in the case where Q has no loops), and to a classification of quivers into three types: Dynkin, tame, and wild quivers.

The elements in $\mathbb{Z}Q_0$ will be written as either $\mathtt{x} = (x_i)$ or $\mathtt{x} = \sum_{i \in Q_0} x_i i$. Define a bilinear form $\langle -, - \rangle \colon \mathbb{Z}Q_0 \times \mathbb{Z}Q_0 \to \mathbb{Z}$, called the *Euler form* of Q, by

$$\langle \mathtt{x}, \mathtt{y} \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\rho \in Q_1} x_{\mathrm{t}\rho} y_{\mathrm{h}\rho}, \quad \text{for } \mathtt{x} = (x_i),\ \mathtt{y} = (y_i) \in \mathbb{Z}Q_0.$$

Its symmetrization

$$(\mathtt{x}, \mathtt{y}) := \langle \mathtt{x}, \mathtt{y} \rangle + \langle \mathtt{y}, \mathtt{x} \rangle$$

is called the *symmetric* Euler form of Q. It is easily checked that

$$(\mathtt{x}, \mathtt{y}) = \mathtt{x}^{\mathsf{T}} C_Q \mathtt{y},$$

where elements in $\mathbb{Z}Q_0$ are considered as column vectors, $\mathtt{x}^{\mathsf{T}}$ denotes the transpose of $\mathtt{x}$, and $C_Q = (c_{i,j})_{i,j \in Q_0}$ is a symmetric $|Q_0| \times |Q_0|$-matrix with entries in $\mathbb{Z}$ given by

$$c_{i,j} = \begin{cases} 2 - 2|\{\text{loops at } i\}|, & \text{if } i = j; \\ -|\{\text{arrows between } i \text{ and } j\}|, & \text{if } i \neq j. \end{cases}$$

The following easy lemma relates the matrix C_Q with Cartan matrix defined in Definition 0.1.

Lemma 1.8. *Let Q be a quiver.*

(1) *The matrix C_Q of the symmetric Euler form is symmetric and independent of the orientation of Q.*

(2) *If Q contains no loops, then C_Q is a symmetric Cartan matrix. Conversely, any symmetric Cartan matrix can be realized in this way.*

Proof. Statement (1) is clear. If Q contains no loops, then $c_{i,i} = 2$, for all i. This, together with (1) and the fact $c_{i,j} \leqslant 0$, for $i \neq j$, implies that C_Q is a symmetric Cartan matrix. Using the graph realization of Cartan matrices given in §0.1, we also see that every symmetric Cartan matrix can be realized as C_Q, for a quiver Q, proving (2). □

The Euler form admits a homological description given in (2) of the following proposition.[2]

[2] The group $\mathrm{Ext}^1_{\Bbbk Q}(V, W)$ can be defined by means of a projective resolution of V if $\Bbbk Q$ is finite dimensional. In all cases, the theory of Yoneda extensions can be used. See [**HAII**, §3.4] for the relevant material on Ext^1-groups.

Proposition 1.9. *Let $V = (V_i, V_\rho)$ and $W = (W_i, W_\rho)$ be finite dimensional representations of Q.*

(1) *There exists an exact sequence of $\Bbbk$-vector spaces:*

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\Bbbk Q}(V,W) &\xrightarrow{\ \iota\ } \bigoplus_{i\in Q_0} \operatorname{Hom}_{\Bbbk}(V_i,W_i) \\ &\xrightarrow{\ \delta\ } \bigoplus_{\rho\in Q_1} \operatorname{Hom}_{\Bbbk}(V_{\mathrm{t}\rho},W_{\mathrm{h}\rho}) \xrightarrow{\ \varepsilon\ } \operatorname{Ext}^1_{\Bbbk Q}(V,W) \longrightarrow 0. \end{aligned} \tag{1.2.1}$$

(2) *We have*

$$\langle \mathbf{dim}\, V, \mathbf{dim}\, W\rangle = \dim \operatorname{Hom}_{\Bbbk Q}(V,W) - \dim \operatorname{Ext}^1_{\Bbbk Q}(V,W). \tag{1.2.2}$$

Proof. We give the definitions of the involved linear maps in (1): ι is the inclusion map, δ maps $f = (f_i)_{i\in Q_0}$ to $(f_{\mathrm{h}\rho}V_\rho - W_\rho f_{\mathrm{t}\rho})_{\rho\in Q_1}$, and ε sends $(\varphi_\rho)_{\rho\in Q_1}$ to the equivalence class of the exact sequence $0 \to W \to E \to V \to 0$ with $E_i = W_i \oplus V_i$ and $E_\rho = \begin{pmatrix} W_\rho & \varphi_\rho \\ 0 & V_\rho \end{pmatrix}$, for all $i \in Q_0$ and $\rho \in Q_1$. The routine verification of exactness is left to the reader in Exercise 1.4.

For (2), since $\dim V_i \dim W_j = \dim \operatorname{Hom}_{\Bbbk}(V_i, W_j)$,

$$\langle \mathbf{dim}\, V, \mathbf{dim}\, W\rangle = \dim \bigoplus_{i\in Q_0} \operatorname{Hom}_{\Bbbk}(V_i,W_i) - \dim \bigoplus_{\rho\in Q_1} \operatorname{Hom}_{\Bbbk}(V_{\mathrm{t}\rho},W_{\mathrm{h}\rho}),$$

which, in turn, equals $\dim \operatorname{Hom}_{\Bbbk Q}(V,W) - \dim \operatorname{Ext}^1_{\Bbbk Q}(V,W)$, by taking the alternating sum of the dimensions of the vector spaces in (1.2.1). □

We deduce, in particular, from the exactness of (1.2.1) that $\operatorname{Ext}^1_{\Bbbk Q}(V,W)$ is finite dimensional, for finite dimensional representations V and W of a (finite) quiver Q.

Now consider the case $V = S_i$ and $W = S_j$ in (1.2.1), for $i, j \in Q_0$. Obviously, ι is an isomorphism, so

$$\dim \operatorname{Ext}^1_{\Bbbk Q}(S_i, S_j) = |\{\text{arrows from } i \text{ to } j\}|,$$

yielding the following result.

Corollary 1.10. *Let Q be a quiver, and let S_i be the simple $\Bbbk Q$-module corresponding to $i \in Q_0$. Then the entries of the matrix C_Q can be interpreted as*

$$c_{i,j} = \begin{cases} 2 - 2\dim \operatorname{Ext}^1_{\Bbbk Q}(S_i,S_i), & \text{if } i = j; \\ -\dim \operatorname{Ext}^1_{\Bbbk Q}(S_i,S_j) - \dim \operatorname{Ext}^1_{\Bbbk Q}(S_j,S_i), & \text{if } i \neq j. \end{cases} \tag{1.2.3}$$

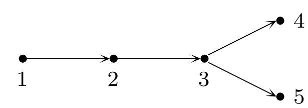
The quadratic form

$$\mathtt{q}_Q(\mathtt{x}) := \langle \mathtt{x},\mathtt{x}\rangle = \sum_{i\in Q_0} x_i^2 - \sum_{\rho\in Q_1} x_{\mathrm{t}\rho}x_{\mathrm{h}\rho} = \frac{1}{2}(\mathtt{x},\mathtt{x}) = \frac{1}{2}\mathtt{x}^{\mathsf{T}} C_Q \mathtt{x}$$

associated with the Euler form $\langle -, - \rangle$ is called the *Tits form* on $\mathbb{Z}Q_0$. It is clear that $\mathtt{q}_Q$ is independent of the orientation of Q.

Recall that a quadratic form $\mathtt{q}$ is called *positive definite* (resp., *positive semidefinite*) if $\mathtt{q}(\mathtt{x}) > 0$ (resp., $\mathtt{q}(\mathtt{x}) \geqslant 0$), for all $\mathtt{x} \neq 0$ (see, for example, [**BAI**, Ch. 6]). We will call a quadratic form *indefinite* if it is neither positive definite nor positive semidefinite.

For example, let Q be the following quiver



Then

$$\mathtt{q}_Q(\mathtt{x}) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_3x_5,$$

which is positive definite since

$$\mathtt{q}_Q(\mathtt{x}) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(x_2 - x_3)^2 + \left(\frac{1}{2}x_3 - x_4\right)^2 + \left(\frac{1}{2}x_3 - x_5\right)^2.$$

As indicated by the next result (and the remark following it), quivers can be conveniently classified according to the nature of $\mathtt{q}_Q$. The two classes of graphs given in the theorem (cf. Figs. II and III) turn out to play a fundamental role in this situation.

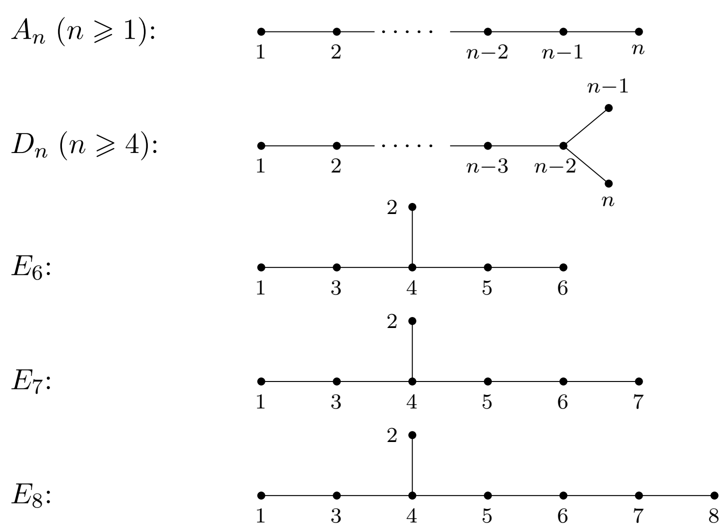


Fig. II. *Dynkin graphs*

Theorem 1.11. *Let Q be a connected quiver without loops.*

(1) *The Tits form q_Q is positive definite if and only if the underlying graph Σ_Q of Q is one of the graphs listed in Fig. II. (These graphs are called* Dynkin graphs.*)*

(2) *The Tits form q_Q is positive semidefinite, but not definite, if and only if the underlying graph Σ_Q of Q is one of the graphs listed in Fig. III. (These graphs are called* extended Dynkin graphs.*)*

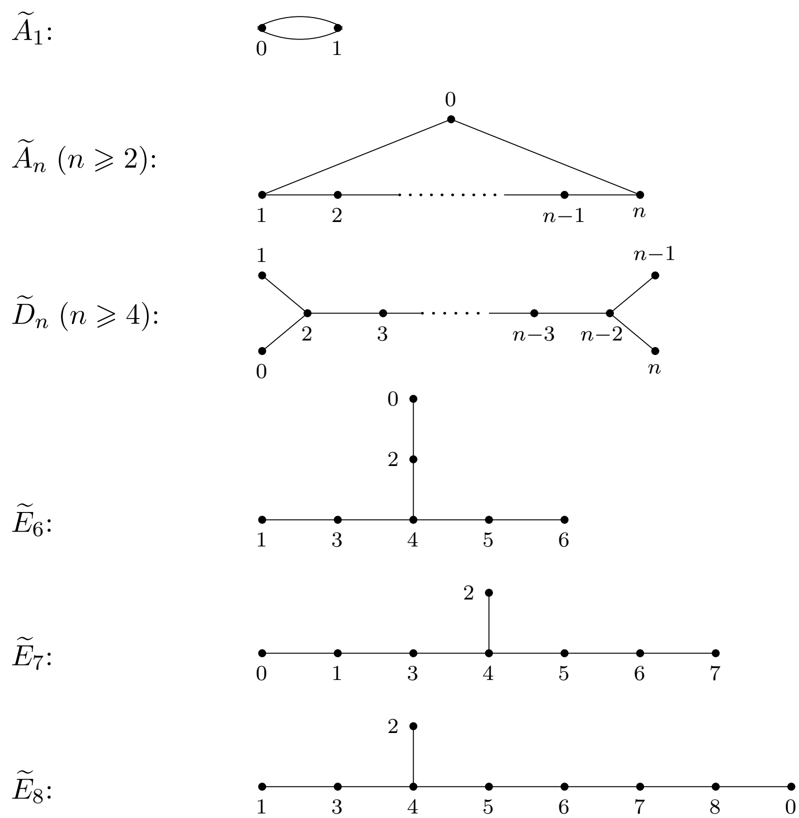


Fig. III. *Extended Dynkin graphs*

Proof. Let Q be a connected quiver without loops, and let Σ_Q denote the underlying graph of Q. We first make an easy observation:

Observation: *If Σ_Q is not a Dynkin graph, it must contain an extended Dynkin graph as a subgraph.*

In fact, if Σ_Q has a cycle, it contains a subgraph of type $\widetilde{A}$. If Σ_Q contains at most 1 branch vertex (= vertex connected to at least 3 different vertices by edges), it must contain a subgraph of type $\widetilde{E}$, since Σ_Q is not a Dynkin graph. And, if Σ_Q has at least 2 branch vertices, then it contains a subgraph of type $\widetilde{D}$.

Next, we describe the radical of a positive semidefinite $\mathtt{q}_Q$. Suppose that Q_0 has m vertices. We number vertices of Q in a some fixed order, and write, for simplicity, $Q_0 = \{1, 2, \dots, m\}$ or $Q_0 = \{0, 1, \dots, m-1\}$ if the latter is more desirable. (See the extended Dynkin graphs listed in Fig. III.) Let $C_Q = (c_{i,j})$ be the Cartan matrix associated with Q. Recall that, by definition, $-c_{i,j}$ is the number of arrows between i and j, for $i \neq j$.

We define the *radical* of the quadratic form $\mathtt{q}_Q$ to be

$$\mathrm{rad}(\mathtt{q}_Q) = \{\mathtt{x} \in \mathbb{Z}Q_0 \mid (\mathtt{x}, \mathtt{y}) = 0, \text{ for all } \mathtt{y} \in \mathbb{Z}Q_0\}.$$

It is easy to see that $\mathtt{x} = (x_i) \in \mathrm{rad}(\mathtt{q}_Q)$ if and only if

$$\sum_j c_{i,j} x_j = 2x_i + \sum_{j \neq i} c_{i,j} x_j = 0, \quad \text{for all } i.$$

Call a vector $\mathtt{x} = (x_i) \in \mathbb{Z}Q_0$ *sincere* if $x_i \neq 0$, for all i.

Claim: *Assume there exists a nonzero vector* $\mathtt{x} = (x_i) \in \mathrm{rad}(\mathtt{q}_Q) \cap \mathbb{N}Q_0$. *Then* $\mathtt{x}$ *is sincere and* $\mathtt{q}_Q$ *is positive semidefinite. Moreover, in this case,*

$$\mathrm{rad}(\mathtt{q}_Q) = \{\mathtt{y} \in \mathbb{Z}Q_0 \mid \mathtt{q}_Q(\mathtt{y}) = 0\} = \mathbb{Z}(\mathtt{x}/d), \tag{1.2.4}$$

where d is the greatest common divisor of the x_i.

In fact, let $\mathtt{x} = (x_i) \in \mathrm{rad}(\mathtt{q}_Q) \cap \mathbb{N}Q_0$. Assume $\mathtt{x} \neq 0$. If $x_i = 0$, for some i, then

$$\sum_{j \neq i} c_{i,j} x_j = 0.$$

This implies $x_j = 0$ whenever $j \neq i$ and $c_{i,j} \neq 0$ (i.e., whenever there is an arrow between i and j). Since Q is connected, this would imply $\mathtt{x} = 0$, a contradiction. Thus, $\mathtt{x}$ is sincere. Since C_Q is symmetric, we have, for $\mathtt{y} = (y_i) \in \mathbb{Z}Q_0$, that

$$\begin{aligned}
0 &\leqslant -\sum_{i<j} c_{i,j} \frac{x_i x_j}{2} \Big(\frac{y_i}{x_i} - \frac{y_j}{x_j}\Big)^2 \\
&= -\sum_{i<j} c_{i,j} \frac{x_j}{2x_i} y_i^2 + \sum_{i<j} c_{i,j} y_i y_j - \sum_{i<j} c_{i,j} \frac{x_i}{2x_j} y_j^2 \\
&= -\sum_{i \neq j} c_{i,j} \frac{x_j}{2x_i} y_i^2 + \sum_{i<j} c_{i,j} y_i y_j = \sum_i \Big(-\sum_{j \neq i} c_{i,j} x_j\Big) \frac{y_i^2}{2x_i} + \sum_{i<j} c_{i,j} y_i y_j \\
&= \sum_i 2x_i \frac{y_i^2}{2x_i} + \sum_{i<j} c_{i,j} y_i y_j = \sum_i y_i^2 + \sum_{i<j} c_{i,j} y_i y_j = \mathtt{q}_Q(\mathtt{y}).
\end{aligned}$$

Therefore, $\mathtt{q}_Q$ is positive semidefinite. It is clear that

$$\mathbb{Z}(\mathtt{x}/d) \subseteq \mathrm{rad}(\mathtt{q}_Q) \subseteq \{\mathtt{y} \in \mathbb{Z}Q_0 \mid \mathtt{q}_Q(\mathtt{y}) = 0\}.$$

Conversely, let $\mathtt{y} \in \mathbb{Z}Q_0$ satisfy $\mathtt{q}_Q(\mathtt{y}) = 0$. The above calculation for $\mathtt{q}_Q(\mathtt{y})$ shows that $y_i/x_i = y_j/x_j$ whenever $c_{i,j} \neq 0$. Since Q is connected, $\mathtt{y} = a\mathtt{x}$, for some $a \in \mathbb{Q}$. This implies that $\mathtt{y} \in \mathbb{Z}(\mathtt{x}/d)$, hence (1.2.4) holds as does the claim.

Now suppose that Σ_Q is a Dynkin graph. By embedding Σ_Q into an extended Dynkin graph, the claim implies that $\mathtt{q}_Q(\mathtt{x}) > 0$, for all $0 \neq \mathtt{x} \in \mathbb{Z}Q_0$, that is, $\mathtt{q}_Q$ is positive definite. Conversely, if Σ_Q is not a Dynkin graph, then Q contains a subquiver whose underlying graph is an extended Dynkin graph. Thus, $\mathtt{q}_Q$ cannot be positive definite. This proves (1).

To prove (2), let Σ_Q be an extended Dynkin graph. It is easy to check that

$$\delta_Q = \begin{cases} (1,1,1,\ldots,1,1)^{\mathsf{T}}, & \text{if } \Sigma_Q \text{ is of type } \widetilde{A}_n; \\ (1,1,2,\ldots,2,1,1)^{\mathsf{T}}, & \text{if } \Sigma_Q \text{ is of type } \widetilde{D}_n; \\ (1,1,2,2,3,2,1)^{\mathsf{T}}, & \text{if } \Sigma_Q \text{ is of type } \widetilde{E}_6; \\ (1,2,2,3,4,3,2,1)^{\mathsf{T}}, & \text{if } \Sigma_Q \text{ is of type } \widetilde{E}_7; \\ (1,2,3,4,6,5,4,3,2)^{\mathsf{T}}, & \text{if } \Sigma_Q \text{ is of type } \widetilde{E}_8, \end{cases} \tag{1.2.5}$$

lies in $\mathrm{rad}(\mathtt{q}_Q)$. Hence, by the claim, $\mathtt{q}_Q$ is positive semidefinite, but not definite.

Conversely, it suffices to show that if Σ_Q is neither a Dynkin nor an extended Dynkin graph, then $\mathtt{q}_Q$ is indefinite. In this case, Q contains a proper subquiver $Q' = (Q'_0, Q'_1)$ such that $\Sigma_{Q'}$ is an extended Dynkin graph. If $Q_0 = Q'_0$, then there is an arrow in $Q_1 \backslash Q'_1$. It follows that $\mathtt{q}_Q(\mathtt{x}) < 0$, for $0 \neq \mathtt{x} \in \mathrm{rad}(\mathtt{q}_{Q'})$ since $\mathtt{x}$ is sincere by the claim. Hence, $\mathtt{q}_Q$ is indefinite. If Q'_0 is a proper subset of Q_0, choose j in $Q_0 \backslash Q'_0$ and $i \in Q'_0$ such that there is an arrow between i and j. Thus, $c_{i,j} \neq 0$. Fix a nonzero $\mathtt{x} \in \mathrm{rad}(\mathtt{q}_{Q'}) \cap \mathbb{N}Q_0$, and define $\mathtt{y} \in \mathbb{N}Q_0$ by $y_j = 1$, $y_k = 2x_k$, for $k \in Q'_0$, and $y_k = 0$, for $k \notin Q'_0 \cup \{j\}$. We get

$$\mathtt{q}_Q(\mathtt{y}) = 4\mathtt{q}_{Q'}(\mathtt{x}) + 1 + \sum_{k \in Q'_0} 2c_{k,j}x_k \leqslant 1 + 2c_{i,j} < 0.$$

Thus, $\mathtt{q}_Q$ is indefinite. This completes the proof. □

A connected quiver whose underlying graph is a graph shown in Fig. II (resp., Fig. III) is called a *Dynkin* (resp., *tame* or *affine*) quiver. By convention, tame quivers include the quiver consisting of a single vertex and one loop. In this case, $\mathtt{q}_Q = 0$ identically on $\mathbb{Z}$. A connected quiver which is neither a Dynkin quiver nor a tame quiver is called a *wild quiver*.

1.3. Weyl groups and root systems

In this section, we define the Weyl group $W(Q)$ and the root system $\Phi(Q)$ of a quiver $Q = (Q_0, Q_1)$. Then we give characterizations of Dynkin, tame, and wild quivers in terms of root systems.

For simplicity, we write $Q_0 = \{1, 2, \ldots, n\}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the corresponding basis for $\mathbb{Z}Q_0 (\cong \mathbb{Z}^n)$. For each vertex $i \in Q_0$, we define an element $r_i \in \mathrm{Aut}(\mathbb{Z}Q_0)$ by

$$r_i(\mu) = \mu - (\mu, \alpha_i)\alpha_i, \quad \text{for } \mu \in \mathbb{Z}Q_0, \tag{1.3.1}$$

or, simply by

$$r_i(\alpha_j) = \alpha_j - c_{i,j}\alpha_i, \ \text{ for } j \in Q_0.$$

Notice that r_i fixes the set $\{\mu \in \mathbb{Z}Q_0 \mid (\mu, \alpha_i) = 0\}$. In general, r_i does not have order 2. But, if there are no loops at the vertex i, then $r_i(\alpha_i) = -\alpha_i$, and r_i does have order 2. In this case, we call r_i a *simple reflection*[3] on $\mathbb{Z}Q_0$, and we call α_i a *simple root*, and denote by $\Pi = \Pi_Q$ the set of all simple roots. It is easy to see that, for each simple reflection r_i, we have $(r_i(\mu), r_i(\nu)) = (\mu, \nu)$, for any $\mu, \nu \in \mathbb{Z}Q_0$.

For the remainder of this section, we discuss the Weyl group and root system of a general quiver Q. When Q is a Dynkin quiver, it will be evident that these two concepts agree with those introduced in §0.6 in the review there of finite dimensional semisimple Lie algebras. However, the development below does not use the material of §0.6. See the remark at the end of this section for further discussion.

Define the *Weyl group* $W(Q)$ of the quiver Q to be the subgroup of $\mathrm{Aut}(\mathbb{Z}Q_0)$ generated by the simple reflections r_i. Because the bilinear form $(-,-)$ is independent of the orientation of Q, the group $W(Q)$ is independent of the orientation of Q. The set

$$\Phi_{\mathrm{re}}(Q) = \bigcup_{w \in W(Q)} w(\Pi) \subset \mathbb{Z}Q_0$$

is called the set of *real roots* of Q. Obviously, each $\mu \in \Phi_{\mathrm{re}}(Q)$ satisfies $\mathfrak{q}_Q(\mu) = \langle \mu, \mu \rangle = 1$.

For each $\mu = \sum_{i=1}^{n} \mu_i \alpha_i \in \mathbb{Z}Q_0$, let

$$\operatorname{supp} \mu = \{i \in Q_0 \mid \mu_i \neq 0\} \tag{1.3.2}$$

be the support of μ. We say that $\operatorname{supp} \mu$ is connected if the full subquiver of Q with vertex set $\operatorname{supp} \mu$ is connected.

[3] In fact, r_i extends to a linear transformation of the real vector space $\mathbb{Z}Q_0 \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}Q_0$, which is a reflection in the hyperplane perpendicular to α_i with respect to the Euler form $(-,-)$.

The *fundamental set* $\mathscr{F}_Q$ of Q is defined to be the set

$$\mathscr{F}_Q = \{0 \neq \mu \in \mathbb{N}^n \mid (\mu, \alpha_i) \leqslant 0, \text{ for all } \alpha_i \in \Pi, \text{ and } \operatorname{supp}\mu \text{ is connected}\}.$$

Then the set

$$\Phi_{\mathrm{im}}(Q) = \bigcup_{w \in W(Q)} w(\mathscr{F}_Q) \cup w(-\mathscr{F}_Q)$$

is called the set of *imaginary roots*. Clearly, each $\mu \in \Phi_{\mathrm{im}}(Q)$ satisfies $\mathfrak{q}_Q(\mu) \leqslant 0$. If Q has a loop at $i \in Q_0$, then $\alpha_i \in \mathscr{F}_Q$. Thus, α_i is an imaginary root.

Finally, the *root system* of Q is defined as

$$\Phi(Q) = \Phi_{\mathrm{re}}(Q) \cup \Phi_{\mathrm{im}}(Q).$$

An element $\mu \in \Phi(Q) \cap \mathbb{N}^n$ is called a *positive root*. We denote by $\Phi^+(Q)$ (resp., $\Phi^+_{\mathrm{re}}(Q)$, $\Phi^+_{\mathrm{im}}(Q)$) the set of all positive (resp., positive real, positive imaginary) roots. In fact, we have

$$\Phi_{\mathrm{re}}(Q) \cap \Phi_{\mathrm{im}}(Q) = \emptyset \text{ and } \Phi(Q) = \Phi^+(Q) \cup (-\Phi^+(Q)).$$

(See Exercise 1.9.)

Examples 1.12. (1) Suppose that Q is a quiver consisting of a single loop. Then C_Q is the 1×1 zero matrix, $(\alpha_1, \alpha_1) = 0$ and $\Pi_Q = \emptyset$. Thus, $W(Q) = \{1\}$, the trivial group. In this case, $\Phi_{\mathrm{im}}(Q) = \mathbb{Z}\alpha_1 \backslash \{0\}$.

(2) Let Q denote the linear quiver $\mathscr{L}_{n-1}$ in Example 1.6(2) (see (1.1.2)). Then Q is a Dynkin quiver having underlying Dynkin graph A_{n-1} in Fig. II, and C_Q is the Cartan matrix of type A_{n-1} defined in Example 0.5. The Weyl group $W(Q)$ is the symmetric group on n letters. Moreover, let $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$ for $i \neq j$, in the notation of Example 0.5. Then $\alpha_{j,i} = -\alpha_{i,j}$ and $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, for any $i < j$. As in Example 0.5, the $\alpha_i (= \alpha_{i,i+1})$, for $1 \leqslant i \leqslant n-1$, make up the simple roots. It can be checked that $\Phi_{\mathrm{im}}(Q) = \emptyset$, and that

$$\Phi(Q) = \Phi_{\mathrm{re}}(Q) = \{\alpha_{i,j} \mid i \neq j\}.$$

(3) Suppose that Q has underlying graph of type $\widetilde{A}_1$ shown in Fig. III. Then $W(Q)$ is the isomorphic to the infinite dihedral group D_∞ (i.e., it is the free product of two cyclic groups of order 2). Also, $\Phi_{\mathrm{re}}(Q) = \{m\alpha_1 + n\alpha_2 \mid m \in \mathbb{Z}, n = m \pm 1\}$, while $\Phi_{\mathrm{im}}(Q) = \mathbb{Z}(\alpha_1 + \alpha_2) \backslash \{0\}$.

We leave the detailed verification of these examples to the reader in Exercise 1.10.

In general, the root system of a quiver has the following characterization.

Theorem 1.13. *Let Q be a connected quiver without loops. Then*

(1) *Q is a Dynkin quiver $\Longleftrightarrow$ $\Phi(Q)$ is finite $\Longleftrightarrow$ $\Phi_{\mathrm{im}}(Q) = \emptyset$;*

(2) *Q is a tame quiver* $\Longleftrightarrow$ $\Phi_{\mathrm{im}}(Q) = \mathbb{Z}\delta\backslash\{0\}$, *for some* $\delta \in \mathbb{N}Q_0$; *and*

(3) *Q is a wild quiver* $\Longleftrightarrow$ *there exists* $\mu \in \Phi^+(Q)$ *such that* $\operatorname{supp}\mu = Q$ *and such that* $(\mu, \alpha_i) < 0$, *for all* $i \in Q_0$.

In proving the theorem, we require the following lemma, which is verified using only elementary calculus. For completeness, we include a proof here; however, the reader is welcome simply to assume this result and go directly to the proof of Theorem 1.13.

Lemma 1.14. *Given real numbers* $a_{i,j}$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$, *let* X_i *denote the linear function*

$$X_i \colon \mathbb{R}^n \longrightarrow \mathbb{R},\ \mathtt{x} = (x_1, x_2, \ldots, x_n)^{\mathsf{T}} \longmapsto \sum_{j=1}^{n} a_{i,j} x_j.$$

Then the system of linear inequalities

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n > 0,\ \ 1 \leqslant i \leqslant m, \tag{1.3.3}$$

has a real solution if and only if the only nonnegative real numbers $b_1, b_2, \ldots, b_m$ *satisfying*

$$b_1X_1 + b_2X_2 + \cdots + b_mX_m = 0$$

are $b_1 = \ldots = b_n = 0$.

Proof. The necessity is obvious. For the sufficiency, consider the subset

$$\Delta = \left\{ \sum_{i=1}^{m} b_i \mathtt{a}_i \ \middle|\ b_i \geqslant 0,\ \sum_{i=1}^{m} b_i = 1 \right\}$$

of $\mathbb{R}^n$, where $\mathtt{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})^{\mathsf{T}}$, $1 \leqslant i \leqslant m$. It is clear that Δ is a compact and convex subset of $\mathbb{R}^n$. Now assume that $(0, 0, \ldots, 0)^{\mathsf{T}}$ does not lie in Δ. The proof is completed by showing that the inequalities (1.3.3) have a common solution. The compactness of Δ implies that there is a $\xi \in \Delta$ such that $\|\xi\| = \sqrt{\xi \cdot \xi}$ is minimal and positive, where $\mathtt{x} \cdot \mathtt{y}$ denotes the usual inner product of $\mathtt{x}, \mathtt{y}$ in $\mathbb{R}^n$. Write $\xi = (c_1, c_2, \ldots, c_n)^{\mathsf{T}}$. We will show that $(c_1, c_2, \ldots, c_n)$ is a solution of (1.3.3) or, equivalently, $\mathtt{a}_i \cdot \xi > 0$, for all $1 \leqslant i \leqslant m$. Since all $\mathtt{a}_i$ belong to Δ, it suffices to show that $\mathtt{x} \cdot \xi > 0$, for all $\mathtt{x} \in \Delta$. Choose an arbitrary $\mathtt{x} \in \Delta$. If $\mathtt{x} = \xi$, then $\mathtt{x} \cdot \xi = \|\xi\|^2 > 0$. Suppose now $\mathtt{x} \neq \xi$. Since Δ is convex, $t\mathtt{x} + (1-t)\xi \in \Delta$, for all $0 \leqslant t \leqslant 1$. This implies that

$$\|t\mathtt{x} + (1-t)\xi\| \geqslant \|\xi\|.$$

From equalities

$$\|t\mathtt{x} + (1-t)\xi\|^2 = \|t(\mathtt{x} - \xi) + \xi\|^2 = t^2\|\mathtt{x} - \xi\|^2 + 2t(\mathtt{x} - \xi) \cdot \xi + \|\xi\|^2,$$

it follows that, for $0 \leqslant t \leqslant 1$,

$$t^2\|\mathtt{x} - \xi\|^2 + 2t(\mathtt{x} - \xi) \cdot \xi \geqslant 0.$$

Consider the real function $f(t) := t^2\|\mathtt{x}-\xi\|^2+2t(\mathtt{x}-\xi)\cdot\xi$. Since $\|\mathtt{x}-\xi\|^2 > 0$, $f(t)$ takes the unique minimal value at the zero $t_0 = -(\mathtt{x}-\xi)\cdot\xi/\|\mathtt{x}-\xi\|^2$ of its derivative $f'(t) = 2t\|\mathtt{x}-\xi\|^2 + 2(\mathtt{x}-\xi)\cdot\xi$. The facts that $f(0) = 0$ and $f(t) \geqslant 0$, for $0 \leqslant t \leqslant 1$, force $t_0 \leqslant 0$, since the graph $y = f(t)$ is a parabola. Hence, $(\mathtt{x}-\xi)\cdot\xi \geqslant 0$, that is, $\mathtt{x}\cdot\xi \geqslant \xi\cdot\xi = \|\xi\|^2 > 0$. This completes the proof. □

We make some conventions for use in the proof of Theorem 1.13 as well as later in the sequel. Write $\mathtt{x} \geqslant 0$, for a vector $\mathtt{x} = (x_i) \in \mathbb{R}^n$ if all $x_i \geqslant 0$; write $\mathtt{x} > 0$ if $\mathtt{x} \geqslant 0$ and $\mathtt{x} \neq 0$. In the latter case, $\mathtt{x}$ is called *positive*. We will also use the notation $\mathtt{x} \gg 0$ when all $x_i > 0$. Thus, $\mathtt{x} \ll 0$ means that $-\mathtt{x} \gg 0$. For two vectors $\mathtt{x}, \mathtt{y} \in \mathbb{Z}Q_0$, we write $\mathtt{x} \geqslant \mathtt{y}$ if $\mathtt{x}-\mathtt{y} \geqslant 0$.

Proof of Theorem 1.13. For simplicity, write $Q_0 = \{1, 2, \ldots, n\}$. Let $C = C_Q = (c_{i,j})$ be the $n \times n$ Cartan matrix associated with Q.

By the proof of Theorem 1.11, Q is a Dynkin quiver if and only if $\mathscr{F}(Q) = \emptyset$, i.e., $\Phi_{\mathrm{im}}(Q) = \emptyset$; see Exercise 1.12. Thus, if Q is a Dynkin quiver, $\mathsf{q}_Q(\mathtt{x}) = 1$, for each $\mathtt{x} \in \Phi(Q) = \Phi_{\mathrm{re}}(Q)$. Since q_Q is positive definite, $\Phi(Q)$ is finite; see Exercise 1.11. Further, we see from the definition that $\mathtt{x} \in \mathscr{F}(Q)$ implies $a\mathtt{x} \in \mathscr{F}(Q)$, for all $0 < a \in \mathbb{Z}$. Hence, the fact that $\Phi(Q)$ is finite implies $\Phi_{\mathrm{im}}(Q) = \emptyset$. This proves (1).

To prove (2) and (3), we first make some preliminary observations:

(i) *If* $0 \leqslant \mathtt{x} \in \mathbb{R}^n$ *and* $C\mathtt{x} \geqslant 0$*, then either* $\mathtt{x} = 0$ *or* $\mathtt{x} \gg 0$*.* This statement follows from the definition of C and the fact that Q is connected. (See the proof of the claim in argument for Theorem 1.11.)

(ii) *Suppose that there exists* $\mu \gg 0$ *in* $\mathbb{R}^n$ *such that* $C\mu > 0$*. Then any* $\mathtt{x} \in \mathbb{R}^n$ *such that* $C\mathtt{x} \geqslant 0$ *satisfies* $\mathtt{x} \gg 0$ *or* $\mathtt{x} = 0$*.* This assertion follows from (i) since, if $\mathtt{x} \neq 0$ satisfies $C\mathtt{x} \geqslant 0$ but has a negative component, then, for some positive real number r, $\mathtt{y} = \mathtt{x} + r\mu$ satisfies $\mathtt{y} \geqslant 0$ and $C\mathtt{y} > 0$, but $\mathtt{y}$ has a zero component, which contradicts statement (i).

(iii) *Statements* (i) *and* (ii) *hold if* C *is replaced by* $C + rI_n$*, for any positive real number* r*.* (Here I_n denotes the $n \times n$ identity matrix.)

(iv) *If a vector* $\mu \in \mathbb{R}^n$ *exists satisfying the hypothesis of* (ii)*, then* C *is nonsingular.* If $\mathtt{x} \neq 0$, but $C\mathtt{x} = 0$, then $\mathtt{y} = \mu + \lambda\mathtt{x}$ has a negative component for some $\lambda \in \mathbb{R}$, while $C\mathtt{y} = C\mu > 0$. This contradicts (ii).

(v) *Suppose there is* $0 \neq \mathtt{x} \in \mathbb{R}^n$ *satisfying the hypothesis of (i). Then* C *is positive semidefinite.* In fact, $C + rI_n$ is nonsingular for all $r \in \mathbb{R}_+$ (the positive real numbers). Thus, C has only nonnegative eigenvalues, i.e., C is positive semidefinite.

(vi) *The quiver Q is wild if and only if there exists $\mu \in \mathbb{Z}Q_0$ such that $\mu \gg 0$ and $C\mu \ll 0$.* If such a μ exists, then $\mathtt{q}_Q(\mu) < 0$, so Q is wild by definition. Conversely, assume that Q is wild. Given $\mathtt{x} > 0$, if $C\mathtt{x} \geqslant 0$, then we know by (v) that C is positive semidefinite. So we can assume that $C\mathtt{x} \not\geqslant 0$, for all $\mathtt{x} > 0$. Let $\{\mathtt{e}_1^*, \ldots, \mathtt{e}_n^*\}$ be the dual basis of the standard basis $\{\mathtt{e}_1, \ldots, \mathtt{e}_n\}$ of $\mathbb{R}^n$ and define linear functions $\chi_i \colon \mathbb{R}^n \to \mathbb{R}$ by $\chi_i = -\sum_{j=1}^n c_{i,j}\mathtt{e}_j^*$, for $1 \leqslant i \leqslant n$. Now consider the linear map

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^{2n}, \ \mathtt{x} \longmapsto (\mathtt{e}_1^*(\mathtt{x}), \ldots, \mathtt{e}_n^*(\mathtt{x}), \chi_1(\mathtt{x}), \ldots, \chi_n(\mathtt{x}))^{\mathsf{T}}. \qquad (1.3.4)$$

Then $\operatorname{Im} f \cap \mathbb{R}_+^{2n} \neq \emptyset$, for otherwise, by Lemma 1.14, there exists a vector $\mathtt{x} = (a_1, \ldots, a_n, b_1, \ldots, b_n)^{\mathsf{T}} \in \mathbb{R}^{2n}$ such that $\mathtt{x} > 0$ and

$$a_1\mathtt{e}_1^* + \cdots + a_n\mathtt{e}_n^* + b_1\chi_1 + \cdots + b_n\chi_n = 0.$$

Since $C = (c_{i,j})$ is symmetric, this would imply $C\mathtt{b} = \mathtt{a} \geqslant 0$, where $\mathtt{a} = (a_1, \ldots, a_n)^{\mathsf{T}}$ and $\mathtt{b} = (b_1, \ldots, b_n)^{\mathsf{T}}$. Thus, $a_i = b_i = 0$, for all $1 \leqslant i \leqslant n$, since $C\mathtt{x} \not\geqslant 0$, for all $0 < \mathtt{x} \in \mathbb{R}^n$. This is a contradiction. Hence, $\operatorname{Im} f \cap \mathbb{R}_+^{2n} \neq \emptyset$. Choose an $\mathtt{x} \in \operatorname{Im} f \cap \mathbb{R}_+^{2n}$. Since $f(\mathtt{x}) \in \mathbb{R}_+^{2n}$, $\mathtt{x} \gg 0$ and $C\mathtt{x} \ll 0$. Moreover, since all the entries $c_{i,j}$ of C are integers, the existence of such an $\mathtt{x}$ implies that there is $\mu \in \mathbb{Z}Q_0$ such that $\mu \gg 0$ and $C\mu \ll 0$, as required by (vi).

Now we prove (2). If Q is tame, then the claim in the proof of Theorem 1.11 implies that $\Phi_{\mathrm{im}}(Q) = \mathbb{Z}\delta\backslash\{0\}$. If Q is wild, let μ be as in (vi). For any $\gamma \in \mathbb{Z}Q_0$, $\gamma - m\mu \in \Phi_{\mathrm{im}}^+(Q)$, for some suitably large positive integer m. Thus, $\Phi_{\mathrm{im}}(Q)$ cannot equal $\mathbb{Z}\delta\backslash\{0\}$, for any single δ. Thus, the reverse implication in (2) holds.

Finally, statement (3) is also clear from (vi). □

Corollary 1.15. *If Q is a Dynkin quiver, then*

(1) *the Weyl group $W(Q)$ is finite; and*

(2) $\Phi(Q) = \{\mathtt{x} \in \mathbb{Z}Q_0 \mid \mathtt{q}_Q(\mathtt{x}) = 1\}$.

Proof. (1) By the theorem above, $\Phi(Q) = \Phi_{\mathrm{re}}(Q)$ is finite. Thus, each element in $W(Q)$ induces a permutation on the set $\Phi(Q)$ of roots. The fact that $\Phi(Q)$ contains the basis $\{\alpha_i \mid i \in Q_0\}$ of $\mathbb{Z}Q_0$ implies that $W(Q)$ can be embedded into the permutation group of $\Phi(Q)$. Hence, $W(Q)$ is finite.

(2) Obviously, each $\alpha \in \Phi(Q)$ satisfies $\mathtt{q}_Q(\alpha) = 1$. Now suppose $\mathtt{x} = (x_i) \in \mathbb{Z}Q_0$ satisfies $\mathtt{q}_Q(\mathtt{x}) = 1$. First, we show that either $\mathtt{x}$ or $-\mathtt{x}$ is positive. Otherwise, we define $\mathtt{x}^+ = (x_i^+), \mathtt{x}^- = (x_i^-) \in \mathbb{Z}Q_0$ by setting

$$x_i^+ = \begin{cases} x_i, & \text{if } x_i > 0; \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad x_i^- = \begin{cases} x_i, & \text{if } x_i < 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathtt{x}^+ > 0$, $-\mathtt{x}^- > 0$ and $\mathtt{x} = \mathtt{x}^+ + \mathtt{x}^-$. Since $\langle \alpha_i, \alpha_j \rangle \leqslant 0$, for $i \neq j$,

$$\begin{aligned} \mathfrak{q}_Q(\mathtt{x}) &= \langle \mathtt{x}, \mathtt{x} \rangle = \langle \mathtt{x}^+, \mathtt{x}^+ \rangle + \langle \mathtt{x}^+, \mathtt{x}^- \rangle + \langle \mathtt{x}^-, \mathtt{x}^+ \rangle + \langle \mathtt{x}^-, \mathtt{x}^- \rangle \\ &\geqslant \langle \mathtt{x}^+, \mathtt{x}^+ \rangle + \langle \mathtt{x}^-, \mathtt{x}^- \rangle \geqslant 2, \end{aligned}$$

a contradiction. So, we can suppose $\mathtt{x}$ is positive. We now proceed by induction on $l_\mathtt{x} := \sum_{i \in Q_0} x_i$. If $l_\mathtt{x} = 1$, then $\mathtt{x} = \alpha_i$, for some $i \in Q_0$, as required. Now let $l_\mathtt{x} > 1$. We claim that there is an $i \in Q_0$ such that $0 < r_i\mathtt{x} < \mathtt{x}$. Suppose this is not the case, that is, for each $i \in Q_0$, either $0 \not< r_i\mathtt{x}$ or $r_i\mathtt{x} \not< \mathtt{x}$. The equalities

$$2 = 2\mathfrak{q}_Q(\mathtt{x}) = 2\langle \mathtt{x}, \mathtt{x} \rangle = (\mathtt{x}, \mathtt{x}) = \sum_{i \in Q_0} x_i(\mathtt{x}, \alpha_i)$$

imply that there is an $i_0 \in Q_0$ such that $x_{i_0} > 0$ and $(\mathtt{x}, \alpha_{i_0}) > 0$. Hence, $r_{i_0}\mathtt{x} = \mathtt{x} - (\mathtt{x}, \alpha_{i_0})\alpha_{i_0} < \mathtt{x}$. This implies $0 \not< r_{i_0}\mathtt{x}$, i.e., $x_{i_0} - (\mathtt{x}, \alpha_{i_0}) < 0$. On the other hand,

$$\begin{aligned} 0 &\leqslant \mathfrak{q}_Q(\mathtt{x} - x_{i_0}\alpha_{i_0}) = \mathfrak{q}_Q(\mathtt{x}) + \mathfrak{q}_Q(x_{i_0}\alpha_{i_0}) + (\mathtt{x}, -x_{i_0}\alpha_{i_0}) \\ &= 1 + x_{i_0}^2 - x_{i_0}(\mathtt{x}, \alpha_{i_0}) = 1 + x_{i_0}(x_{i_0} - (\mathtt{x}, \alpha_{i_0})). \end{aligned}$$

This forces $x_{i_0} = 1$ and $x_{i_0} - (\mathtt{x}, \alpha_{i_0}) = -1$. Thus, $\mathfrak{q}_Q(\mathtt{x} - x_{i_0}\alpha_{i_0}) = 0$, i.e., $\mathtt{x} = x_{i_0}\alpha_{i_0} = \alpha_{i_0}$. This contradicts the fact that $l_\mathtt{x} > 1$. Therefore, there is an $i \in Q_0$ such that $0 < r_i\mathtt{x} < \mathtt{x}$. By the induction hypothesis, $r_i\mathtt{x} \in \Phi(Q)$ and, hence, $\mathtt{x} \in \Phi(Q)$. □

Remark 1.16. In case Q contains no loops, the root system $\Phi(Q)$ of Q coincides with the root system $\Phi(C_Q)$ of the Kac–Moody Lie algebra $\mathfrak{g}(C_Q)$ associated with the Cartan matrix C_Q; see §0.6 and the references given in the Notes for §§1.2–1.3. Also, the Weyl group $W(Q)$ coincides with the Weyl group $W(C_Q)$ defined through a root datum realization of the Cartan matrix C_Q in §0.1; cf. Remark 4.4(1) and Exercise 4.7. In particular, if Q is a Dynkin quiver of type A_n ($n \geqslant 1$), D_n ($n \geqslant 4$), E_6, E_7, or E_8, then the number of roots in $\Phi(Q)$ is $n(n+1)$, $2n(n-1)$, 72, 126, or 240, respectively. See [**LAII**, p. 66].

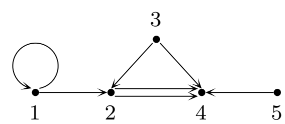
1.4. Bernstein–Gelfand–Ponomarev reflection functors

A basic tool in the study of the representation theory of a quiver Q is the notion of a reflection functor. These functors will be denoted by $\mathscr{R}_k^+$ and $\mathscr{R}_k^-$, for $k \in Q_0$. Because they were first introduced by Bernstein, Gelfand, and Ponomarev, they are commonly called *BGP reflection functors*. The representations of quivers considered in this section are defined over a fixed field $\Bbbk$. For a representation V of Q over $\Bbbk$, we simply write $\mathrm{End}(V)$ for the endomorphism algebra $\mathrm{End}_{\Bbbk Q}(V)$.

Let Q be a quiver. For each vertex $k \in Q_0$, we denote by $\mathfrak{s}_k Q$ the quiver obtained from Q by reversing the direction of all arrows ρ satisfying $\mathrm{t}\rho = k$

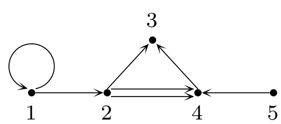
or $\mathrm{h}\rho = k$. If $i\bullet\xrightarrow{\rho}\bullet j$ is an arrow in Q with $i = k$ or $j = k$, we denote by $j\bullet\xrightarrow{\rho^{\mathrm{op}}}\bullet i$ the corresponding reversed arrow in $\mathfrak{s}_k Q$.

Example 1.17. Let Q be the following quiver:

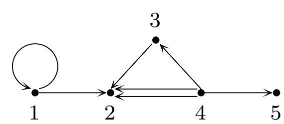


Then $\mathfrak{s}_3 Q$ and $\mathfrak{s}_4 Q$ are given as follows:

$\mathfrak{s}_3 Q$



$\mathfrak{s}_4 Q$



To define BGP reflection functors, we need the notions of a sink and a source in a quiver $Q = (Q_0, Q_1)$. A vertex $k \in Q_0$ is called a *sink* (resp., a *source*) if there is no arrow ρ with $\mathrm{t}\rho = k$ (resp., $\mathrm{h}\rho = k$).

For example, the quiver Q in Example 1.17 has two sources 3 and 5, and one sink 4.

Clearly, if $k \in Q_0$ is a sink (resp., a source), then it is a source (resp., a sink) of the quiver $\mathfrak{s}_k Q$.

Now let k be a sink in Q. For each representation $V = (V_i, V_\rho) \in \mathsf{Rep}_{\Bbbk} Q$, we define a representation $\mathscr{R}_k^+ V = (W_i, W_\eta) \in \mathsf{Rep}_{\Bbbk} \mathfrak{s}_k Q$ as follows. For all $i \neq k$, set $W_i = V_i$, and define W_k to be the kernel of the map

$$\xi_k \colon \bigoplus_{\substack{\rho \in Q_1 \\ \mathrm{h}\rho = k}} V_{\mathrm{t}\rho} \longrightarrow V_k, \quad (x_{\mathrm{t}\rho})_\rho \longmapsto \sum_\rho V_\rho(x_{\mathrm{t}\rho}).$$

For each arrow η in $\mathfrak{s}_k Q$, if $\mathrm{t}\eta \neq k$, set $W_\eta = V_\eta$. If $\mathrm{t}\eta = k$, then η is realized by reversing an arrow $\delta \in Q_1$ with $\mathrm{h}\delta = k$, and we define W_η to be the composition

$$W_k = \operatorname{Ker} \xi_k \xrightarrow{\iota_k} \bigoplus_{\substack{\rho \in Q_1 \\ \mathrm{h}\rho = k}} V_{\mathrm{t}\rho} \xrightarrow{\pi_\delta} V_{\mathrm{t}\delta} = W_{\mathrm{h}\eta},$$

where ι_k denotes the canonical inclusion, and π_δ is the canonical projection.

Next, with each morphism $f = (f_i) \colon V \to V'$ in $\mathsf{Rep}_{\Bbbk} Q$, associate a morphism $\mathscr{R}_k^+ f = g = (g_i) \colon \mathscr{R}_k^+ V \to \mathscr{R}_k^+ V'$ in $\mathsf{Rep}_{\Bbbk} \mathfrak{s}_k Q$ defined by setting $g_i = f_i$, for $i \neq k$. The g_k is defined as a restriction map by the following natural commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Ker}\xi_k & \xrightarrow{\iota_k} & \bigoplus\limits_{\substack{\rho\in Q_1\\ \mathrm{h}\rho=k}} V_{\mathrm{t}\rho} & \xrightarrow{\xi_k} & V_k \\
{\scriptstyle g_k}\downarrow & & \downarrow{\scriptstyle \bigoplus f_{\mathrm{t}\rho}} & & \downarrow{\scriptstyle f_k} \\
\operatorname{Ker}\xi'_k & \xrightarrow{\iota'_k} & \bigoplus\limits_{\substack{\rho\in Q_1\\ \mathrm{h}\rho=k}} V'_{\mathrm{t}\rho} & \xrightarrow{\xi'_k} & V'_k
\end{array}$$

In this way, we obtain a functor

$$\mathscr{R}_k^+ \colon \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk}\mathfrak{s}_k Q.$$

Dually, let k be a source in Q. With each representation $V=(V_i,V_\rho)\in \mathsf{Rep}_{\Bbbk}Q$, associate a representation $\mathscr{R}_k^- V=(W_i,W_\eta)\in \mathsf{Rep}_{\Bbbk}\mathfrak{s}_kQ$ as follows. For all $i\neq k$, $W_i=V_i$. Define W_k to be the cokernel of the map

$$\gamma_k \colon V_k \longrightarrow \bigoplus_{\substack{\rho\in Q_1\\ \mathrm{t}\rho=k}} V_{\mathrm{h}\rho}, \quad x\longmapsto (V_\rho(x))_\rho.$$

For each arrow η in $\mathfrak{s}_kQ$, if $\mathrm{h}\eta\neq k$, define $W_\eta=V_\eta$. If $\mathrm{h}\eta=k$, then η is realized as δ^{op}, for some $\delta\in Q_1$ with $\mathrm{t}\delta=k$. Then define W_η to be the composition

$$W_{\mathrm{t}\eta}=V_{\mathrm{h}\delta}\xrightarrow{l_\delta}\bigoplus_{\substack{\rho\in Q_1\\ \mathrm{t}\rho=k}} V_{\mathrm{h}\rho}\xrightarrow{p_k} W_k,$$

where l_δ denotes the canonical inclusion, and p_k is the canonical quotient map. Similarly, for each morphism $f\colon V\to V'$ in $\mathsf{Rep}_{\Bbbk}Q$, we can define a morphism $\mathscr{R}_k^- f\colon \mathscr{R}_k^- V\to \mathscr{R}_k^- V'$ in $\mathsf{Rep}_{\Bbbk}\mathfrak{s}_kQ$. In this way, we obtain a functor

$$\mathscr{R}_k^- \colon \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk}\mathfrak{s}_k Q.$$

It is easy to see that, for a representation V of Q, $\mathscr{R}_k^+V=0$ or $\mathscr{R}_k^-V=0$ if and only if V is isomorphic to a direct sum of copies of the simple representation S_k.

The following nice result relates the reflection functors $\mathscr{R}_k^\pm$ to the reflection r_k defined in (1.3.1).

Theorem 1.18 (Bernstein–Gelfand–Ponomarev)**.** *Let Q be a quiver and V a representation of Q.*

(1) *Let k be a sink of Q. There is a canonical monomorphism*

$$\varphi\colon \mathscr{R}_k^-\mathscr{R}_k^+V\longrightarrow V.$$

Moreover, φ splits, and V is isomorphic to a direct sum of $\mathscr{R}_k^-\mathscr{R}_k^+V$ and various copies of S_k. In particular, if V is indecomposable, then either

$V \cong S_k$ or φ is an isomorphism with

$$\operatorname{End}(\mathscr{R}_k^+ V) \cong \operatorname{End}(V) \ \ \textit{and} \ \ \mathbf{dim}\, \mathscr{R}_k^+ V = r_k(\mathbf{dim}\, V).$$

(2) *Let k be a source of Q. There is a canonical epimorphism*

$$\psi \colon V \longrightarrow \mathscr{R}_k^+ \mathscr{R}_k^- V.$$

Moreover, ψ splits, and V is isomorphic to a direct sum of $\mathscr{R}_k^+ \mathscr{R}_k^- V$ and various copies of S_k. In particular, if V is indecomposable, then either $V \cong S_k$ or ψ is an isomorphism with

$$\operatorname{End}(\mathscr{R}_k^- V) \cong \operatorname{End}(V) \ \ \textit{and} \ \ \mathbf{dim}\, \mathscr{R}_k^- V = r_k(\mathbf{dim}\, V).$$

Proof. (1) The construction gives the following diagram with the top row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathscr{R}_k^+ V)_k & \xrightarrow{\iota_k = \gamma_k} & \bigoplus\limits_{\substack{\rho \in Q_1 \\ \mathrm{h}\rho = k}} V_{\mathrm{t}\rho} & \xrightarrow{p_k} & (\mathscr{R}_k^- \mathscr{R}_k^+ V)_k & \longrightarrow & 0 \\ & & & & \Big\downarrow \xi_k & \swarrow_{\varphi_k} & & & \\ & & & & V_k & & & & \end{array}$$

Then there is a unique $\Bbbk$-linear map $\varphi_k \colon (\mathscr{R}_k^- \mathscr{R}_k^+ V)_k \to V_k$ such that $\xi_k = \varphi_k p_k$. It is obvious that φ_k is injective and that $\varphi = (\varphi_i)_i$ is a monomorphism from $\mathscr{R}_k^- \mathscr{R}_k^+ V$ to V, where $\varphi_i = \mathrm{id}_{V_i}$, for all $i \neq k$. Further, φ splits and induces an isomorphism

$$V \cong \mathscr{R}_k^- \mathscr{R}_k^+ V \oplus m S_k,$$

where $m = \dim V_k - \dim \operatorname{Im} \xi_k$. Thus, if V is indecomposable, then either $m = 1$ and $V \cong S_k$ or ξ_k is surjective. In the latter case, $\varphi \colon \mathscr{R}_k^- \mathscr{R}_k^+ V \to V$ is an isomorphism. This also implies that $\mathscr{R}_k^+ \varphi \colon \mathscr{R}_k^+ \mathscr{R}_k^- \mathscr{R}_k^+ V \to \mathscr{R}_k^+ V$ is an isomorphism. Hence, the composition of any two maps in

$$\operatorname{End}(V) \xrightarrow{\mathscr{R}_k^+} \operatorname{End}(\mathscr{R}_k^+ V) \xrightarrow{\mathscr{R}_k^-} \operatorname{End}(\mathscr{R}_k^- \mathscr{R}_k^+ V) \xrightarrow{\mathscr{R}_k^+} \operatorname{End}(\mathscr{R}_k^+ \mathscr{R}_k^- \mathscr{R}_k^+ V)$$

is a bijection. Consequently, $\operatorname{End}(\mathscr{R}_k^+ V) \cong \operatorname{End}(V)$.

Finally, the surjectivity of ξ_k yields the equality

$$\dim(\mathscr{R}_k^+ V)_k = \sum_{\substack{\rho \in Q_1 \\ \mathrm{h}\rho = k}} \dim V_{\mathrm{t}\rho} - \dim V_k.$$

This implies that $\mathbf{dim}\, \mathscr{R}_k^+ V = r_k(\mathbf{dim}\, V)$ since $\dim(\mathscr{R}_k^+ V)_i = \dim V_i$ for all $i \neq k$.

(2) This can be proved in a way similar to the method used in (1) by considering the diagram

$$\begin{array}{ccccccccc} & & & & V_k & & & & \\ & & & \overset{\psi_k}{\swarrow} & \downarrow{\scriptstyle \gamma_k} & & & & \\ 0 \longrightarrow & (\mathscr{R}_k^+ \mathscr{R}_k^- V)_k & \xrightarrow[\xi_k]{} & & \bigoplus_{\substack{\rho\in Q_1 \\ \mathrm{t}\rho = k}} V_{\mathrm{h}\rho} & \xrightarrow[p_k]{} & (\mathscr{R}_k^- V)_k & \longrightarrow 0 \end{array}$$

□

This theorem gives rise to the following two useful corollaries. For each vertex k of Q, denote by $\mathsf{Rep}_{k} Q\langle k\rangle$ the full subcategory of $\mathsf{Rep}_{k} Q$ consisting of representations which do not have a summand isomorphic to S_k. The category $\mathsf{Rep}_{k} Q\langle k\rangle$ is an additive category, but not usually abelian, since morphisms need not have kernels or cokernels.

Corollary 1.19. *If k is a sink in Q, then $\mathscr{R}_k^+$ and $\mathscr{R}_k^-$ induce mutually inverse equivalences $\mathsf{Rep}_{k} Q\langle k\rangle \to \mathsf{Rep}_{k} \mathfrak{s}_k Q\langle k\rangle$. In particular, there is a one-to-one correspondence between the isoclasses of indecomposable representations of Q and those of $\mathfrak{s}_k Q$.*

Proof. The first assertion follows obviously from Theorem 1.18. The one-to-one correspondence is induced by

$$V \longmapsto \mathscr{R}_k^+ V, \quad S_k \longmapsto S_k,$$

where $V \in \mathsf{Rep}_{k} Q$ is indecomposable and $V \ncong S_k$. □

A sequence $i_1, i_2, \ldots, i_m$ of vertices (possibly with repetitions) in Q is called (+)-*admissible* if i_1 is a sink in Q, and, for each $2 \leqslant s \leqslant m$, i_s is a sink in $\mathfrak{s}_{i_{s-1}} \cdots \mathfrak{s}_{i_1} Q$. A (−)-admissible sequence is defined dually.

Corollary 1.20. *Let $i_1, i_2, \ldots, i_m$ be a (+)-admissible sequence in Q, and let V be an indecomposable representation of Q. For each $0 \leqslant s \leqslant m$, set*

$$V(s) = \mathscr{R}_{i_s}^+ \cdots \mathscr{R}_{i_1}^+ V \quad \textit{and} \quad \mathtt{x}(s) = r_{i_s} \cdots r_{i_1}(\mathbf{dim}\, V).$$

Suppose there is $0 \leqslant t < m$ such that $\mathtt{x}(t) > 0$, but $\mathtt{x}(t+1) \not> 0$. Then, for each $0 \leqslant s \leqslant t$, $V(s)$ is indecomposable, $\mathbf{dim}\, V(s) = \mathtt{x}(s)$, and $V(t+1), \ldots, V(m)$ are the zero representation. Moreover, $V(t)$ is isomorphic to the simple representation $S_{i_{t+1}}$ of $\mathfrak{s}_{i_t} \cdots \mathfrak{s}_{i_1} Q$ corresponding to the vertex i_{t+1}. In particular, $V \cong \mathscr{R}_{i_1}^- \cdots \mathscr{R}_{i_t}^- S_{i_{t+1}}$.

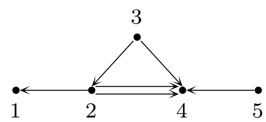
Suppose that Q is an acyclic quiver. Then Q has a sink, say i_1. By deleting i_1 and arrows with head i_1 in Q, we obtain a full subquiver of Q which again admits a sink, say i_2. Repeating this process, we finally obtain a numbering $i_1, i_2, \ldots, i_n$ of all vertices of Q. Clearly, $i_1, i_2, \ldots, i_n$

is (+)-admissible and $i_n, \dots, i_2, i_1$ is (−)-admissible. Moreover, each arrow in $\mathfrak{s}_{i_n} \cdots \mathfrak{s}_{i_2}\mathfrak{s}_{i_1}Q$ is obtained from an arrow in Q by changing its orientation exactly twice, so $\mathfrak{s}_{i_n} \cdots \mathfrak{s}_{i_2}\mathfrak{s}_{i_1}Q = Q$. Thus, we obtain two functors

$$\mathscr{C}^+ := \mathscr{R}^+_{i_n} \cdots \mathscr{R}^+_{i_2}\mathscr{R}^+_{i_1} \text{ and } \mathscr{C}^- := \mathscr{R}^-_{i_1}\mathscr{R}^-_{i_2} \cdots \mathscr{R}^-_{i_n}$$

from $\mathsf{Rep}_{\Bbbk} Q$ to $\mathsf{Rep}_{\Bbbk} Q$, called *Coxeter functors*. The Coxeter functors depend on the orientation of Q, but not on the choice of admissible sequence (see the example below and Exercise 1.17).

Example 1.21. Let Q be the following quiver



It is easy to see that both the sequences $1, 4, 2, 3, 5$ and $4, 5, 1, 2, 3$ are (+)-admissible. They give two functors

$$\mathscr{R}^+_5\mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_4\mathscr{R}^+_1,\ \mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_1\mathscr{R}^+_5\mathscr{R}^+_4 : \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk} Q.$$

Since 1 and 5 are sinks of $\mathfrak{s}_4 Q$ and since there are no arrows connecting them, we have by definition

$$\mathscr{R}^+_1\mathscr{R}^+_5 = \mathscr{R}^+_5\mathscr{R}^+_1 : \mathsf{Rep}_{\Bbbk}\mathfrak{s}_4 Q \longrightarrow \mathsf{Rep}_{\Bbbk}\mathfrak{s}_1\mathfrak{s}_5\mathfrak{s}_4 Q.$$

This implies that

$$\mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_1\mathscr{R}^+_5\mathscr{R}^+_4 = \mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_5\mathscr{R}^+_1\mathscr{R}^+_4 : \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk} Q.$$

Repeating this argument, we finally get

$$\begin{aligned}\mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_1\mathscr{R}^+_5\mathscr{R}^+_4 &= \mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_5\mathscr{R}^+_1\mathscr{R}^+_4 = \mathscr{R}^+_3\mathscr{R}^+_5\mathscr{R}^+_2\mathscr{R}^+_1\mathscr{R}^+_4 \\ &= \mathscr{R}^+_5\mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_1\mathscr{R}^+_4 = \mathscr{R}^+_5\mathscr{R}^+_3\mathscr{R}^+_2\mathscr{R}^+_4\mathscr{R}^+_1.\end{aligned}$$

Finally, we mention that in §11.4 the BGP reflection functors will be extended to include quivers with automorphisms.

1.5. Gabriel's theorem

A quiver Q is said to be of *finite representation type* or *representation-finite* (over a field $\Bbbk$) if, up to isomorphism, there are only finitely many indecomposable representations in $\mathsf{Rep}_{\Bbbk} Q$. Although finiteness is defined with respect to a specified field $\Bbbk$, a famous theorem of Gabriel, Theorem 1.23, shows that finiteness is independent of the choice of field. More precisely, Q is of finite representation type if and only if Q is a Dynkin quiver, and hence if and only if its Tits form $\mathfrak{q}_Q$ is positive definite.

The proof of this theorem will use the BGP reflection functors introduced in the previous section. Throughout this section, Q is assumed to be a connected quiver with n vertices containing no loops. Therefore, for each vertex i, the simple reflection $r_i \in W(Q)$ is defined. To begin with, at least, all representations of Q are defined over a fixed field $\Bbbk$.

For any ordered listing $i_1, i_2, \ldots, i_n$ of the vertices in Q, the element $c := r_{i_n} \cdots r_{i_2} r_{i_1} \in W(Q) \subseteq \mathrm{Aut}(\mathbb{Z}Q_0)$ is called a *Coxeter transformation.* Of course, c depends on how the elements of Q_0 are listed.

Lemma 1.22. *Let Q be a Dynkin quiver.*

(1) *The Coxeter transformation c has no nonzero fixed vectors; that is, $c\mathtt{x} \neq \mathtt{x}$, for every $0 \neq \mathtt{x} \in \mathbb{Z}Q_0$.*

(2) *For each $0 < \mathtt{x} \in \mathbb{Z}Q_0$, there is an integer $m \geqslant 1$ such that $c^m\mathtt{x}$ is not positive.*

Proof. (1) Suppose $\mathtt{x} = (x_i) \in \mathbb{Z}Q_0$ and $c\mathtt{x} = \mathtt{x}$. Since $r_{i_2}, \ldots, r_{i_n}$ do not change the i_1th coordinate of $r_{i_1}\mathtt{x}$, we have $(r_{i_1}\mathtt{x})_{i_1} = (c\mathtt{x})_{i_1} = x_{i_1}$. Hence, $r_{i_1}\mathtt{x} = \mathtt{x}$. Inductively, $r_{i_j}\mathtt{x} = \mathtt{x}$, for all $1 \leqslant j \leqslant n$. From the definition of r_{i_j} it follows that $(\mathtt{x}, \alpha_i) = 0$, for all $i \in Q_0$. This implies that $\mathtt{x} = 0$.

(2) By Corollary 1.15(1), $W(Q)$ is finite. Thus, there is some $h \geqslant 1$ such that $c^h = 1$. If $\mathtt{x}, c\mathtt{x}, \ldots, c^{h-1}\mathtt{x}$ are all positive, then $\mathtt{y} := \mathtt{x} + c\mathtt{x} + \cdots + c^{h-1}\mathtt{x}$ is positive, but $c\mathtt{y} = \mathtt{y}$. This contradicts (1). Hence, there is some $m \geqslant 1$ such that $c^m\mathtt{x}$ is not positive. □

Theorem 1.23 (Gabriel)**.** *Let Q be a connected quiver (and let $\Bbbk$ be an arbitrary field). Then:*

(1) *Q is of finite representation type if and only if Q is a Dynkin quiver.*

(2) *When the equivalent conditions of* (1) *are satisfied, the correspondence $V \mapsto \mathbf{dim}\,V$ induces a bijection between the set of isoclasses of indecomposable representations of Q and the set $\Phi^+(Q)$ of positive roots in the root system of Q.*

Proof. First, we assume that Q is a Dynkin quiver. We will show that Q is of finite representation type and that the conclusion of (2) holds.

For simplicity, set $Q_0 = \{1, 2, \ldots, n\}$ and assume that the sequence $1, 2, \ldots, n$ is (+)-admissible.

Let V be an indecomposable representation of Q. Then $\mathbf{dim}\,V > 0$. By Lemma 1.22(2), there is an $m \geqslant 1$ such that $c^m(\mathbf{dim}\,V) \not> 0$, where $c = r_n \cdots r_2 r_1$ is a Coxeter transformation. Consider the (+)-admissible sequence

$$(i_1, i_2, \ldots, i_n, i_{n+1}, \ldots, i_{mn}) = (1, 2, \ldots, n, 1, 2, \ldots, \ldots, n, 1, 2, \ldots, n),$$

where the sequence $1, 2, \dots, n$ is repeated m times. Let $0 \leqslant t < mn$ be such that $r_{i_s} \cdots r_{i_1}(\mathbf{dim}\, V) > 0$, for all $s \leqslant t$, but

$$r_{i_{t+1}} r_{i_t} \cdots r_{i_1}(\mathbf{dim}\, V) \ngtr 0.$$

From Corollary 1.20 it follows that

$$V \cong \mathscr{R}^-_{i_1}\mathscr{R}^-_{i_2} \cdots \mathscr{R}^-_{i_t} S_{i_{t+1}} \quad \text{and} \quad \mathbf{dim}\, V = r_{i_1} r_{i_2} \cdots r_{i_t}(\alpha_{i_{t+1}}) \in \Phi^+(Q),$$

where $S_{i_{t+1}}$ is the simple representation of $\mathfrak{s}_{i_t} \cdots \mathfrak{s}_{i_2}\mathfrak{s}_{i_1} Q$ corresponding to the vertex i_{t+1}. This gives a map

$$\varphi \colon \mathcal{I} \longrightarrow \Phi^+(Q),\ [V] \longmapsto \mathbf{dim}\, V,$$

where $\mathcal{I}$ denotes the set of isoclasses of indecomposable representations V of Q, and $[V]$ denotes the isoclass of V.

Now let α be a positive root. Again by Lemma 1.22(2), there exists an $m \geqslant 1$ such that $c^m\alpha \ngtr 0$. As above, there is a $0 \leqslant t \leqslant mn$ such that $r_{i_s} \cdots r_{i_1}\alpha > 0$, for all $s \leqslant t$, but $r_{i_{t+1}} r_{i_t} \cdots r_{i_1}\alpha \ngtr 0$. This implies $r_{i_t} \cdots r_{i_1}\alpha = \alpha_{i_{t+1}}$, that is, $\alpha = r_{i_1} \cdots r_{i_t}(\alpha_{i_{t+1}})$. Set $V(\alpha) = \mathscr{R}^-_{i_1}\mathscr{R}^-_{i_2} \cdots \mathscr{R}^-_{i_t} S_{i_{t+1}} \in \mathsf{Rep}_{\Bbbk} Q$. Then, by Theorem 1.18, $V(\alpha)$ is indecomposable, and $\mathbf{dim}\, V(\alpha) = \alpha$. Since t only depends on α, we get a map

$$\psi \colon \Phi^+(Q) \longrightarrow \mathcal{I},\ \alpha \longmapsto [V(\alpha)].$$

It is easy to see that $\psi \circ \varphi = \mathrm{id}_{\mathcal{I}}$ and $\varphi \circ \psi = \mathrm{id}_{\Phi^+(Q)}$. Thus, φ is a bijection. This implies that Q is of finite representation type since $\Phi^+(Q)$ is finite.

Next, we assume that Q is of finite representation type (over $\Bbbk$). We will show that Q is a Dynkin quiver.

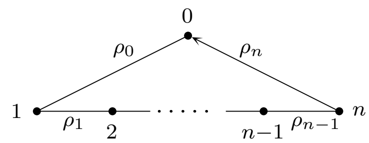
Observe that, for any subquiver Q' (not necessarily full) of a quiver Q, each representation of Q' can be naturally extended to a representation of Q by adding zero spaces and zero maps. In this way we can view $\mathsf{Rep}_{\Bbbk} Q'$ as a full subcategory of $\mathsf{Rep}_{\Bbbk} Q$; see the details in Exercise 1.1.

Let $\Bbbk[x]$ be the polynomial ring over $\Bbbk$ with one indeterminate x. Then each $\Bbbk[x]$-module is given by a pair (U, f), where U is a $\Bbbk$-vector space and f is an endomorphism of U. Denote by $\Bbbk[x]$-mod the category of finite dimensional $\Bbbk[x]$-modules. Given a $\Bbbk[x]$-module U (with x acting via an endomorphism f), the indecomposable summands of U correspond to the Jordan blocks of f; therefore, $\Bbbk[x]$-mod admits infinitely many nonisomorphic indecomposable modules, up to isomorphism.

Suppose now that Q is of finite representation type. Then, obviously, each subquiver of Q is again of finite representation type.

First, the underlying graph of Q contains no cycles, for if it did, Q would contain a subquiver Q' of the following form ($n \geqslant 0$, the arrows ρ_i,

$0 \leqslant i \leqslant n-1$, can have any orientation):

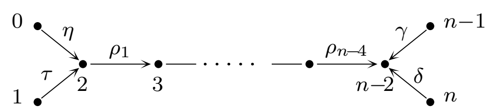


We thus obtain an additive functor

$$\mathscr{V} : \Bbbk[x]\text{-mod} \longrightarrow \mathsf{Rep}_{\Bbbk} Q', \ (U, f) \longmapsto V = (V_i, V_\rho) \tag{1.5.1}$$

by defining $V_i = U$, for $0 \leqslant i \leqslant n$, $V_{\rho_n} = f$, and $V_{\rho_i} = 1_U$, for $0 \leqslant i \leqslant n-1$, where 1_U denotes the identity map on U. Clearly, this functor is fully faithful; see Exercise 1.21. Consequently, there are infinitely many indecomposable representations of Q', a contradiction. This implies particularly that Q contains no multiple arrows.

Second, Q contains no subquiver of type $\widetilde{D}_n$ ($n \geqslant 4$). To see this, let Q' be a subquiver of Q of type $\widetilde{D}_n$. We will show that Q' has infinitely many pairwise nonisomorphic indecomposable representations. Using Corollary 1.19 repeatedly, we may suppose that Q' has the following orientation:



Then there is an additive fully faithful functor

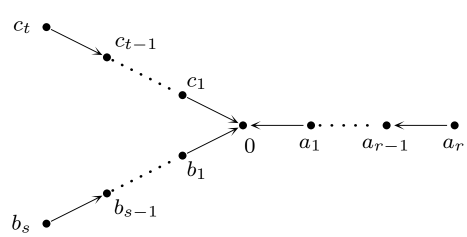
$$\mathscr{V} : \Bbbk[x]\text{-mod} \longrightarrow \mathsf{Rep}_{\Bbbk} Q', \quad (U, f) \longmapsto V, \tag{1.5.2}$$

where $V = (V_i, V_\rho)$ is given by

$$\begin{aligned}
&V_0 = V_1 = V_{n-1} = V_n = U,\\
&V_i = U \oplus U, \quad \text{for } 2 \leqslant i \leqslant n-2;\\
&V_\eta = \begin{pmatrix} 1_U \\ 0 \end{pmatrix}, \ V_\tau = \begin{pmatrix} 0 \\ 1_U \end{pmatrix}, \ V_\gamma = \begin{pmatrix} 1_U \\ 1_U \end{pmatrix}, \ V_\delta = \begin{pmatrix} 1_U \\ f \end{pmatrix},\\
&V_{\rho_i} = \begin{pmatrix} 1_U & 0 \\ 0 & 1_U \end{pmatrix}, \quad \text{for } 1 \leqslant i \leqslant n-4.
\end{aligned}$$

See Exercise 1.21. This is again a contradiction.

Third, Q contains no subquiver of type $\widetilde{E}_n$ ($n = 6, 7, 8$). Otherwise, let Q' be such a subquiver of Q. Again, using Corollary 1.19, we may suppose that Q' has the following orientation:



where $(r, s, t) = (2, 2, 2), (3, 3, 1)$, or $(5, 2, 1)$, that is, Q' is of type $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, respectively. In these three cases, set $\omega = c_1, b_2, a_4$, respectively, and denote the unique arrow directed towards ω by $\omega' \bullet \xrightarrow{\rho} \bullet\, \omega$. By deleting the vertex ω' (hence the arrow ρ) in Q', we obtain a subquiver Q'' of Q' which is of type E_n. It is known [**LAI**, pp. 275–285] that the root system of Q'' contains a unique positive root $\mu = (\mu_i)_i \in \mathbb{Z}Q''_0$ such that $\mu_\omega = 2$. (Indeed, μ is obtained from $\delta_{Q'}$ listed in (1.2.5) by deleting the component corresponding to ω'.) By (1), Q'' has a unique indecomposable representation $Z = (Z_i, Z_\rho)$ having dimension vector μ. Thus, $\dim Z_\omega = 2$. This gives rise to an additive fully faithful functor (see Exercise 1.21)

$$\mathscr{V}\colon \Bbbk[x]\text{-mod} \longrightarrow \mathsf{Rep}_{\Bbbk} Q',\ (U, f) \longmapsto V \tag{1.5.3}$$

such that $V_{\omega'} = U$, $V_i = Z_i \otimes U$, $i \in Q''_0$, $V_\rho = Z_\rho \otimes 1_U$, $\rho \in Q''_0$, and

$$V_\rho = \begin{pmatrix} 1_U \\ f \end{pmatrix}\colon V_{\omega'} = U \longrightarrow U \oplus U = Z_\omega \otimes U = V_\omega,$$

where V_ω is identified with $U \oplus U$ by fixing a basis of Z_ω. This implies that there are infinitely many nonisomorphic indecomposable representations of Q', hence of Q, a contradiction.

In conclusion, we obtain that Q is a Dynkin quiver. □

Remark 1.24. By the theorem, the set of dimension vectors of indecomposable representations of a Dynkin quiver Q coincides with the set of positive roots of Q and, hence, is independent of the ground field $\Bbbk$. For each $\alpha \in \Phi^+(Q)$, let $M(\alpha) = M_{\Bbbk}(\alpha)$ be the indecomposable representation of Q over $\Bbbk$ corresponding to α. Hence, each representation of Q over $\Bbbk$ is isomorphic to

$$M_{\Bbbk}(\lambda) := \bigoplus_{\alpha \in \Phi^+(Q)} \lambda(\alpha) M_{\Bbbk}(\alpha),$$

where λ is a function $\Phi^+(Q) \to \mathbb{N}$.

Example 1.25. Consider again the linear quiver $Q = \mathscr{L}_{n-1}$ (see (1.1.2)). For each pair (i, j) with $1 \leqslant i < j \leqslant n$, define a representation $M_{i,j} = (V_l^{i,j}, V_{\rho_s}^{i,j})$ of Q by

$$V_l^{i,j} = \begin{cases} \Bbbk, & \text{if } i \leqslant l < j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad V_{\rho_s}^{i,j} = \begin{cases} \mathrm{id}, & \text{if } i \leqslant s < j-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{dim}\, M_{i,j} = \alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$; see Example 1.12(2). Moreover, for $1 \leqslant i \leqslant j < l \leqslant n$, there is a canonical inclusion $\iota = (\iota_r)\colon M_{j,l} \to M_{i,l}$ defined by

$$\iota_r = \begin{cases} \mathrm{id}, & \text{if } j \leqslant r \leqslant l-1; \\ 0, & \text{otherwise.} \end{cases}$$

If $i = j$, then ι is the identity map on $M_{i,l}$. Dually, for $1 \leqslant i < j \leqslant l \leqslant n$, there is a canonical projection $\pi = (\pi_r)\colon M_{i,l} \to M_{i,j}$ defined by

$$\pi_r = \begin{cases} \mathrm{id}, & \text{if } i \leqslant r \leqslant j-1; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that

$$\mathrm{Hom}_A(M_{i,j}, M_{s,t}) = \begin{cases} \mathcal{k}(\iota \circ \pi), & \text{if } s \leqslant i < t \leqslant j; \\ 0, & \text{otherwise,} \end{cases} \tag{1.5.4}$$

where $\pi\colon M_{i,j} \to M_{i,t}$ and $\iota\colon M_{i,t} \to M_{s,t}$. In particular, each $M_{i,j}$ is indecomposable since $\mathrm{End}_{\mathcal{k}Q}(M_{i,j}) \cong \mathcal{k}$. In other words, $M_{i,j}$ is the indecomposable representation of Q corresponding to the root $\alpha_{i,j}$. Thus, we also write $M_{i,j}$ as $M(\alpha_{i,j}) = M_{\mathcal{k}}(\alpha_{i,j})$. By Theorem 1.23, the $M_{i,j}$ form a complete set of indecomposable representations of Q.

Remark 1.26. Gabriel's theorem was generalized to arbitrary quivers by Kac in 1980. In the case where Q is not a Dynkin quiver and $\mathcal{k}$ is an algebraically closed field, there is a unique (up to isomorphism) indecomposable representation in $\mathsf{Rep}_{\mathcal{k}}Q$ corresponding to each positive real root. But, there are infinitely many nonisomorphic indecomposable representations corresponding to each positive imaginary root. (See the Notes at the end of the chapter for references.)

1.6. Representation varieties and generic extensions

Let $\mathcal{K}$ be an *algebraically closed* field. Given a quiver Q, we construct a family of algebraic varieties over $\mathcal{K}$, called the *representation varieties* of Q. These varieties are indexed by vectors in $\mathbb{N}Q_0$ which serve as dimension vectors of representations of Q. Each representation V of Q determines a point in the appropriate variety associated with its dimension vector. This setting suggests the important notion of the degeneration of a representation of Q, and, when Q is a Dynkin quiver, the notion of the generic extension of two given representations of Q.

We will make use of elementary algebraic geometry and algebraic group theory. The necessary material is reviewed in §§A.1–A.3 in Appendix A.

Let $Q = (Q_0, Q_1)$ be a quiver. For a vector $\mathbf{d} = (d_i)_i \in \mathbb{N}Q_0$, define the *representation variety* associated with $\mathbf{d}$ to be the affine space

$$R(\mathbf{d}) = R(Q, \mathbf{d}) := \prod_{\rho \in Q_1} \mathrm{Hom}_{\mathcal{K}}(\mathcal{K}^{d_{\mathrm{t}\rho}}, \mathcal{K}^{d_{\mathrm{h}\rho}}) \cong \prod_{\rho \in Q_1} \mathrm{M}_{d_{\mathrm{h}\rho} \times d_{\mathrm{t}\rho}}(\mathcal{K}).$$

Thus, a point $x = (x_\rho)_\rho$ of $R(\mathbf{d})$ determines a representation $V(x)$ of Q in which $V(x)_i = \mathcal{K}^{d_i}$, for $i \in Q_0$, and $V(x)_\rho = x_\rho$, for $\rho \in Q_1$. The algebraic group $\mathrm{GL}_{\mathbf{d}}(\mathcal{K}) := \prod_{i\in Q_0}\mathrm{GL}_{d_i}(\mathcal{K})$ acts regularly on $R(\mathbf{d})$, in the sense of Definition A.27, by conjugation

$$(g_i)_i \cdot (x_\rho)_\rho = (g_{\mathrm{h}\rho}x_\rho g_{\mathrm{t}\rho}^{-1})_\rho,$$

and the $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})$-orbits $\mathfrak{O}_x$ in $R(\mathbf{d})$ correspond bijectively to the isoclasses $[V(x)]$ of representations of Q with dimension vector $\mathbf{d}$. Since $\dim \mathrm{GL}_d(\mathcal{K}) = d^2$ and $\dim \mathrm{M}_{d\times d'}(\mathcal{K}) = dd'$, the Tits form can be rewritten as

$$\mathsf{q}_Q(\mathbf{d}) = \dim \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) - \dim R(\mathbf{d}). \tag{1.6.1}$$

The stabilizer $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})_x = \{g \in \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) \mid g\cdot x = x\}$ of any $x \in R(\mathbf{d})$ is naturally identified with the group $\mathrm{Aut}_{\mathcal{K}Q}(V)$ of automorphisms of $V := V(x)$. By Exercise A.10, $\mathrm{Aut}_{\mathcal{K}Q}(V)$ is a Zariski open subset of $\mathrm{End}_{\mathcal{K}Q}(V)$, and hence $\dim \mathrm{End}_{\mathcal{K}Q}(V) = \dim \mathrm{Aut}_{\mathcal{K}Q}(V)$. It follows from Theorem A.29 that the orbit $\mathfrak{O}_V := \mathfrak{O}_x$ of V has dimension

$$\dim \mathfrak{O}_V = \dim \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) - \dim \mathrm{End}_{\mathcal{K}Q}(V). \tag{1.6.2}$$

We now use these ideas to provide an elegant proof (due to Tits) of the necessity of Theorem 1.23(1). Let Q be a connected quiver which admits only finitely many pairwise nonisomorphic indecomposable representations. Then, for each $0 \neq \mathbf{d} \in \mathbb{N}Q_0$, there are finitely many isoclasses of representations having dimension vector $\mathbf{d}$, i.e., $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})$ has finitely many orbits on $R(\mathbf{d})$. Hence, for any fixed $\mathbf{d}$, there exists a dense orbit $\mathfrak{O}_V$ in $R(\mathbf{d})$, so that $\dim \mathfrak{O}_V = \dim R(\mathbf{d})$. This implies that

$$\begin{aligned}\mathsf{q}_Q(\mathbf{d}) &= \dim \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) - \dim R(\mathbf{d}) = \dim \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) - \dim \mathfrak{O}_V \\ &= \dim \mathrm{End}_{\mathcal{K}Q}(V) > 0.\end{aligned}$$

Since, for each $\mathtt{x} = (x_i) \in \mathbb{Z}Q_0$, $\mathsf{q}_Q(\mathtt{x}) \geqslant \mathsf{q}_Q(\mathtt{x}')$, where $\mathtt{x}' = (|x_i|)$, we infer that q_Q is positive definite. By Theorem 1.11, Q is a Dynkin quiver since it is trivially seen that Q has no loops.

Definition 1.27. Given two representations V, W having the same dimension vector $\mathbf{d} \in \mathbb{N}Q_0$, we say that V degenerates to W, or that W is a *degeneration* of V, and write $[W] \leqslant_{\mathrm{dg}} [V]$ (or simply $W \leqslant_{\mathrm{dg}} V$), if $\mathfrak{O}_W \subseteq \overline{\mathfrak{O}_V}$ in $R(\mathbf{d})$, where $\overline{\mathfrak{O}_V}$ denotes the closure of $\mathfrak{O}_V$ in $R(\mathbf{d})$ with respect to the Zariski topology.

Note that $W <_{\mathrm{dg}} V \Longleftrightarrow \mathfrak{O}_W \subseteq \overline{\mathfrak{O}_V}\backslash\mathfrak{O}_V$. In fact, the relation $\leqslant_{\mathrm{dg}}$ defines a partial ordering on the set of isoclasses of representations of Q, called the *degeneration ordering*.

Remark 1.28. In case Q is a Dynkin quiver, the degeneration ordering $\leqslant_{\mathrm{dg}}$ on the isoclasses of representations is independent of the field $\mathcal{K}$. More precisely, let $\mathcal{K}'$ be another algebraically closed field and $\lambda, \mu \colon \Phi^+(Q) \to \mathbb{N}$ be two functions. Then, in the notation of Remark 1.24,

$$M_{\mathcal{K}}(\lambda) \leqslant_{\mathrm{dg}} M_{\mathcal{K}}(\mu) \Longleftrightarrow M_{\mathcal{K}'}(\lambda) \leqslant_{\mathrm{dg}} M_{\mathcal{K}'}(\mu).$$

This fact can be seen as follows. If M, N are $\mathcal{K}Q$-modules satisfying $\mathbf{dim}\, M = \mathbf{dim}\, N$, then

$$\begin{aligned} &N \leqslant_{\mathrm{dg}} M \\ &\quad \Longleftrightarrow \dim \mathrm{Hom}_{\mathcal{K}Q}(X, N) \geqslant \dim \mathrm{Hom}_{\mathcal{K}Q}(X, M), \text{ for all } X \in \mathsf{Rep}_{\mathcal{K}} Q. \end{aligned}$$

See Exercises 1.24 and 11.2 for sketch of a proof. On the other hand, the dimension of the $\mathcal{K}$-space $\mathrm{Hom}_{\mathcal{K}Q}(M_{\mathcal{K}}(\lambda), M_{\mathcal{K}}(\mu))$ depends only on λ and μ, not on the field $\mathcal{K}$; see Remark 3.35. Thus, the set $\mathfrak{P}(Q) := \{\lambda \mid \lambda \colon \Phi^+(Q) \to \mathbb{N}\}$ admits a partial ordering defined by

$$\lambda \leqslant \mu \Longleftrightarrow M_{\mathcal{K}}(\lambda) \leqslant_{\mathrm{dg}} M_{\mathcal{K}}(\mu).$$

We will revisit this ordering in §11.1. This notion turns out to be very useful in describing monomial bases of quantum enveloping algebras of finite type in §11.3.

Let $\mathbf{d}, \mathbf{d}', \mathbf{d}'' \in \mathbb{N}Q_0$ with $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. For subsets $\mathcal{X} \subset R(\mathbf{d}')$ and $\mathcal{Y} \subset R(\mathbf{d}'')$ stable under $\mathrm{GL}_{\mathbf{d}'}(\mathcal{K})$ and $\mathrm{GL}_{\mathbf{d}''}(\mathcal{K})$, respectively, let $E(\mathcal{X}, \mathcal{Y})$ denote the set of all $z \in R(\mathbf{d})$ such that $V(z)$ is isomorphic to an extension of some $M \in \mathcal{X}$ by some $N \in \mathcal{Y}$ (i.e., there is an exact sequence $0 \to N \to V(z) \to M \to 0$ in $\mathsf{Rep}_{\mathcal{K}} Q$).

Proposition 1.29. *If both $\mathcal{X}$ and $\mathcal{Y}$ are irreducible (resp., closed) varieties, then $E(\mathcal{X}, \mathcal{Y})$ is an irreducible (resp., a closed) variety.*

Proof. Consider the subset of $R(\mathbf{d})$

$$\begin{aligned} Z &= Z(\mathcal{X}, \mathcal{Y}) \\ &= \left\{ \left(\begin{pmatrix} y_\rho & \xi_\rho \\ 0 & x_\rho \end{pmatrix} \right)_\rho \,\middle|\, (x_\rho)_\rho \in \mathcal{X},\, (y_\rho)_\rho \in \mathcal{Y},\, \xi_\rho \in \mathrm{M}_{d''_{\mathrm{h}\rho} \times d'_{\mathrm{t}\rho}}(\mathcal{K}) \right\}. \end{aligned}$$

Then Z is irreducible (resp., closed) if $\mathcal{X}$ and $\mathcal{Y}$ are irreducible (resp., closed); see Exercise A.3(4).

Now consider the map

$$\varphi \colon \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) \times Z \longrightarrow R(\mathbf{d}), \quad (g, z) \longmapsto g \cdot z,$$

whose image is the subset $E(\mathcal{X}, \mathcal{Y})$. Since $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})$ is irreducible, $E(\mathcal{X}, \mathcal{Y})$ is irreducible if $\mathcal{X}$ and $\mathcal{Y}$ are irreducible.

Now let $\mathcal{X}$ and $\mathcal{Y}$ be closed subvarieties. By definition, $Z = Z(\mathcal{X}, \mathcal{Y})$ is stable under the action of the parabolic subgroup

$$P := \prod_{i \in Q_0} \begin{pmatrix} \mathrm{GL}_{d_i''}(\mathcal{K}) & \mathrm{M}_{d_i'' \times d_i'}(\mathcal{K}) \\ 0 & \mathrm{GL}_{d_i'}(\mathcal{K}) \end{pmatrix} \subseteq \mathrm{GL}_{\mathbf{d}}(\mathcal{K}).$$

Thus, by Theorem A.38(3), $E(\mathcal{X}, \mathcal{Y}) = \mathrm{GL}_{\mathbf{d}}(\mathcal{K}) \cdot Z$ is a closed subvariety since Z is. □

Now assume Q is a Dynkin quiver. For representations M, N of Q over $\mathcal{K}$, both orbits $\mathfrak{O}_M$ and $\mathfrak{O}_N$ are irreducible. By Proposition 1.29, $E(\mathfrak{O}_M, \mathfrak{O}_N)$ is irreducible, so it contains a unique dense orbit $\mathfrak{O}_G$. In other words, among all the extensions of M by N, G is the unique (up to isomorphism) one with maximal $\dim \mathfrak{O}_G$ or, equivalently, by (1.6.2), with minimal dimension $\mathrm{End}_{\mathcal{K}Q}(G)$. We call G the *generic extension* of M by N, denoted by $M * N$. If $\mathrm{Ext}^1_{\mathcal{K}Q}(M, N) = 0$, then $M * N \cong M \oplus N$.

Proposition 1.30. *Let Q be a Dynkin quiver.*

(1) *Let M, N, X be representations of Q over $\mathcal{K}$. Then $X \leqslant_{\mathrm{dg}} M * N$ if and only if there exist $M' \leqslant_{\mathrm{dg}} M, N' \leqslant_{\mathrm{dg}} N$ such that X is an extension of M' by N'. In particular, the statements $M' \leqslant_{\mathrm{dg}} M$ and $N' \leqslant_{\mathrm{dg}} N$ imply that $M' * N' \leqslant_{\mathrm{dg}} M * N$.*

(2) *Let $\mathcal{M} = \mathcal{M}_Q$ be the set of isoclasses of representations of Q over $\mathcal{K}$ and define multiplication $*$ on $\mathcal{M}$ by $[M] * [N] = [M * N]$, for any $[M], [N] \in \mathcal{M}$. Then $\mathcal{M}$ is a monoid with identity $1 = [0]$.*

Proof. (1) Consider the closures $\overline{\mathfrak{O}_M}$ and $\overline{\mathfrak{O}_N}$; they are irreducible. Thus, by Proposition 1.29, $E(\overline{\mathfrak{O}_M}, \overline{\mathfrak{O}_N})$ is closed and irreducible. Since $\mathfrak{O}_M \times \mathfrak{O}_N$ is open in $\overline{\mathfrak{O}_M} \times \overline{\mathfrak{O}_N}$, we have that $Z(\mathfrak{O}_M, \mathfrak{O}_N)$ is open, thus dense, in $Z(\overline{\mathfrak{O}_M}, \overline{\mathfrak{O}_N})$. This implies that $E(\mathfrak{O}_M, \mathfrak{O}_N)$ is dense in $E(\overline{\mathfrak{O}_M}, \overline{\mathfrak{O}_N})$. But $\mathfrak{O}_{M*N}$ is dense in $E(\mathfrak{O}_M, \mathfrak{O}_N)$, so we conclude that $E(\overline{\mathfrak{O}_M}, \overline{\mathfrak{O}_N}) = \overline{\mathfrak{O}_{M*N}}$.

(2) It suffices to show that the multiplication $*$ is associative, i.e., $(L * M) * N \cong L * (M * N)$ for representations L, M, N of Q. The generic extension $(L * M) * N$ arises from two exact sequences

$$0 \longrightarrow M \longrightarrow L * M \longrightarrow L \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow (L * M) * N \longrightarrow L * M \longrightarrow 0.$$

Form the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & (L*M)*N & \longrightarrow & L*M & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

in which E is obtained as the pull-back of $M \to L * M$ and $(L * M) * N \to L * M$. (See [**HAII**, §3.4].) By definition, $E \leqslant_{\mathrm{dg}} M * N$ and $(L * M) * N \leqslant_{\mathrm{dg}} L * E$. By (1), we conclude that

$$(L * M) * N \leqslant_{\mathrm{dg}} L * E \leqslant_{\mathrm{dg}} L * (M * N).$$

Dually, $L * (M * N) \leqslant_{\mathrm{dg}} (L * M) * N$, and, consequently,

$$(L * M) * N \cong L * (M * N). \qquad \square$$

Exercises and notes

Exercises

§1.1

1.1. Let Q be a quiver and let Q' be a subquiver of Q. For each representation $V = (V_j, V_\tau)$ of Q', define a representation $\widetilde{V} = (\widetilde{V}_i, \widetilde{V}_\rho)$ of Q by

$$\widetilde{V}_i = \begin{cases} V_i, & \text{if } i \in Q'_0; \\ 0, & \text{if } i \notin Q'_0, \end{cases} \qquad \text{and} \qquad \widetilde{V}_\rho = \begin{cases} V_\rho, & \text{if } i \in Q'_1; \\ 0, & \text{if } i \notin Q'_1. \end{cases}$$

(1) Show that, for a morphism $f = (f_j)\colon V \to W$ of representations of Q', $\tilde{f} = (\tilde{f}_i)$ defined by $\tilde{f}_i = f_i$ for $i \in Q'_0$, and $\tilde{f}_i = 0$ otherwise, is a morphism from $\widetilde{V}$ to $\widetilde{W}$.

(2) Show that the functor

$$\mathscr{E}\colon \mathsf{Rep}_{\Bbbk} Q' \longrightarrow \mathsf{Rep}_{\Bbbk} Q; \quad V \longmapsto \widetilde{V},\ f \longmapsto \tilde{f}$$

is fully faithful (i.e., $\mathsf{Rep}_{\Bbbk} Q'$ can be viewed as a full subcategory of $\mathsf{Rep}_{\Bbbk} Q$).

1.2. Let Q be a quiver and let $\Bbbk$ be a field. Prove:

(1) If Q is acyclic, then $\{S_i \mid i \in Q_0\}$ forms a complete set of simple representations in $\mathsf{Rep}_{\Bbbk}Q$.

Hint: Apply the following fact: Let $V = (V_i, V_\rho)$ be a representation of Q. If there is $j \in Q_0$ such that $V_j \neq 0$ and $V_\rho = 0$ whenever $\mathrm{t}\rho = j$, then any nonzero vector $x \in V$ defines a subrepresentation $W = (W_i, W_\rho)$ by setting $W_j = \Bbbk x$, $W_i = 0$, for $i \neq j$, and $W_\rho = 0$, for all $\rho \in Q_1$, and moreover, W is isomorphic to S_j.

(2) If Q contains an oriented cycle $\rho_m \cdots \rho_1$, then the representation $V(\lambda)$, $\lambda \in \Bbbk\backslash\{0\}$, defined in (1.1.1) is simple.

(3) A representation $V \in \mathsf{Rep}_{\Bbbk}Q$ is nilpotent if and only if all the composition factors of V have the form S_i, $i \in Q_0$. Thus, $\{S_i \mid i \in Q_0\}$ forms a complete set of simple nilpotent representations in $\mathsf{Rep}_{\Bbbk}Q$.

1.3. Let Q be an acyclic quiver and let $A = \Bbbk Q$ be the path algebra of Q over a field $\Bbbk$.

(1) For each $i \in Q_0$, let e_i be the trivial path at i and put $P_i = Ae_i$. Prove that $\{P_i \mid i \in Q_0\}$ is a complete set of indecomposable projective modules in A-mod. Moreover, for $i \in Q_0$,

$$\operatorname{rad} P_i \cong \bigoplus_{\substack{\text{arrows } \rho \text{ with} \\ \mathrm{t}\rho = i}} P_{\mathrm{h}\rho},$$

where $\operatorname{rad} P_i$ denotes the Jacobson radical of P_i.

(2) For each $i \in Q_0$, put $I_i = \mathrm{Hom}_{\Bbbk}(e_iA, \Bbbk)$, the dual space of e_iA, and define a left A-module structure on I_i by setting $a \cdot f(x) = f(xa)$, for $a \in A$, $f \in I_i$, and $x \in e_iA$. Prove that $\{I_i \mid i \in Q_0\}$ is a complete set of indecomposable injective modules in A-mod. Moreover, for $i \in Q_0$,

$$I_i/\operatorname{soc} I_i \cong \bigoplus_{\substack{\text{arrows } \rho \text{ with} \\ \mathrm{h}\rho = i}} I_{\mathrm{t}\rho},$$

where $\operatorname{soc} I_i$ denotes the socle of I_i.

§1.2

1.4. Check the exactness of the sequence (1.2.1).

1.5. Let Q be an acyclic quiver and let $A = \Bbbk Q$ be the path algebra of Q over a field $\Bbbk$. For each $i \in Q_0$, let S_i be the corresponding simple A-module and let P_i be the projective cover of S_i. If $C_A = (c'_{i,j})$ denotes the Cartan matrix of A defined as in the Notes for §0.1 (i.e., $c'_{i,j}$ equals to the multiplicity of S_i in P_j as a composition factor), then

$$c'_{i,j} = \dim_{\Bbbk} \mathrm{Hom}_A(P_i, P_j) = |\{\text{paths from } j \text{ to } i\}|.$$

Moreover, the Euler from $\langle -, - \rangle$ of Q can be characterized by

$$\langle \mathtt{x}, \mathtt{y} \rangle = \mathtt{x}^{\mathsf{T}} (C_A^{-1})^{\mathsf{T}}\, \mathtt{y}, \quad \text{for } \mathtt{x},\ \mathtt{y} \in \mathbb{Z}Q_0.$$

In particular, if C_Q denotes the Cartan matrix of Q, then

$$C_Q = C_A^{-1} + (C_A^{-1})^{\mathsf{T}}.$$

(See [**246**, p. 70] and [**8**, p. 90].)

1.6. Let Σ be a finite connected graph with vertex set $\Sigma_0 = \{1, 2, \ldots, n\}$. For $i, j \in \Sigma_0$, let $d_{i,j}$ be the number of edges connecting i and j. Assume that Σ contains no edge loops, that is, $d_{i,i} = 0$, for all $1 \leqslant i \leqslant n$. A function $f\colon \Sigma_0 \to \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is called *subadditive* if, for each $1 \leqslant i \leqslant n$,

$$2f(i) \geqslant \sum_{j=1}^{n} d_{i,j} f(j).$$

Such a function is called *additive* if $2f(i) = \sum_{j=1}^{n} d_{i,j} f(j)$, for all i. Prove:

(1) Σ is a Dynkin graph if and only if there is a subadditive, but not additive, function $f\colon \Sigma_0 \to \mathbb{Z}^+$.

(2) Σ is an extended Dynkin graph if and only if there is an additive function $f\colon \Sigma_0 \to \mathbb{Z}^+$.

(See [**145**].)

§1.3

1.7. Let Q be a quiver without loops and let $C_Q = (c_{i,j})$ be the associated Cartan matrix. Take $i \neq j$ in Q_0. Prove that the order of $r_i r_j$ is 2, 3, or ∞ if $c_{i,j} = 0$, -1, or $\leqslant -2$.

1.8. Let Q be a connected quiver without loops. If the Weyl group $W(Q)$ is finite, then Q is a Dynkin quiver.

Hint: Suppose Q is not a Dynkin quiver. If Q has multiple arrows, it is easy to see that $W(Q)$ has an element of infinite order. If Q contain no multiple arrows, then it contains a full subquiver Q' of tame type $\widetilde{A}$, $\widetilde{D}$, or $\widetilde{E}$. Check in each of these cases that the Weyl group of Q' contains an element of infinite order.

1.9. Let Q be a connected quiver without loops. Prove that

$$\Phi_{\mathrm{re}}(Q) \cap \Phi_{\mathrm{im}}(Q) = \emptyset \quad\text{and}\quad \Phi(Q) = \Phi^+(Q) \cup (-\Phi^+(Q)).$$

Moreover, for each $\alpha \in \Phi(Q)$, $\operatorname{supp}\alpha$ is connected. (See [**172**, Ch. 5].)

1.10. Fill in all the details to Examples 1.12

1.11. If the Tits form $\mathtt{q}_Q$ of a quiver Q is positive definite, then $\Phi(Q)$ is finite.

Hint: Extend $\mathtt{q}_Q\colon \mathbb{Z}Q_0 \to \mathbb{Z}$ to a quadratic form $\tilde{\mathtt{q}}_Q\colon \mathbb{R}Q_0 \to \mathbb{R}$. Then $\tilde{\mathtt{q}}_Q$ is positive definite. Consider the subset $X := \{\mathtt{x} = (x_i) \in \mathbb{R}Q_0 \mid \|\mathtt{x}\| = 1\}$ of $\mathbb{R}Q_0$, where $\|\mathtt{x}\| = \sqrt{\sum_{i \in Q_0} x_i^2}$. The compactness of X implies that there exists $c > 0$ such that $\tilde{\mathtt{q}}_Q(\mathtt{x}) \geqslant c$, for all $\mathtt{x} \in X$. Thus, for each $0 \neq \mathtt{x} \in \mathbb{R}Q_0$,

$$\tilde{\mathtt{q}}_Q(\mathtt{x}) = \|\mathtt{x}\|^2\, \tilde{\mathtt{q}}_Q\Big(\frac{\mathtt{x}}{\|\mathtt{x}\|}\Big) \geqslant \|\mathtt{x}\|^2 c.$$

Since $\Phi(Q) \subset \mathbb{Z}Q_0$ and $\tilde{\mathtt{q}}_Q(\alpha) = \mathtt{q}_Q(\alpha) = 1$, for all $\alpha \in \Phi(Q)$, $\Phi(Q)$ is finite.

1.12. Let Q be a quiver without loops.

(1) Suppose that $\mu \in \mathbb{N}Q_0$. Let α_i be a simple root which does not belong to $\operatorname{supp}\mu$. Use the definition of the (symmetric) Euler form to conclude that $(\mu, \alpha_i) \leqslant 0$.

(2) Using the elements $\delta_{Q'}$, for Q' a tame quiver, as defined in the proof of Theorem 1.11, verify the claim made in the proof of statement (1) in Theorem 1.13 that Q is Dynkin $\iff \mathscr{F}(Q) = \emptyset$. (The implication $\Longrightarrow$ is obvious.)

§1.4

1.13. Let k be a source of a quiver Q. Prove that $\mathscr{R}_k^- \colon \mathsf{Rep}_{\Bbbk} Q \to \mathsf{Rep}_{\Bbbk} \mathfrak{s}_k Q$ is a functor.

1.14. Let k be a sink of a quiver Q. Show that $\mathscr{R}_k^- \colon \mathsf{Rep}_{\Bbbk} \mathfrak{s}_k Q \to \mathsf{Rep}_{\Bbbk} Q$ is the left adjoint of $\mathscr{R}_k^+ \colon \mathsf{Rep}_{\Bbbk} Q \to \mathsf{Rep}_{\Bbbk} \mathfrak{s}_k Q$.

1.15. Let Q and Q' be two quivers with the same underlying graph $\Sigma_Q = \Sigma_{Q'}$. If $\Sigma_Q = \Sigma_{Q'}$ does not contain cycles, then there is a (+)-admissible sequence $i_1, i_2, \ldots, i_m$ in Q such that

$$Q' = \mathfrak{s}_{i_m} \cdots \mathfrak{s}_{i_2} \mathfrak{s}_{i_1} Q.$$

1.16. (1) Complete the proof of statement (2) in Theorem 1.18.

(2) Fill in the details for the proof of Corollary 1.20.

1.17. (1) Let Q be a quiver. If i and j are sinks of Q, show that

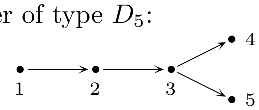
$$\mathscr{R}_j^+ \mathscr{R}_i^+ = \mathscr{R}_i^+ \mathscr{R}_j^+ \colon \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk} \mathfrak{s}_j \mathfrak{s}_i Q.$$

(2) Let Q be an acyclic quiver. If $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ are two ordered listings of all vertices of Q both of which are (+)-admissible, then

$$\mathscr{R}_{i_n}^+ \cdots \mathscr{R}_{i_2}^+ \mathscr{R}_{i_1}^+ = \mathscr{R}_{j_n}^+ \cdots \mathscr{R}_{j_2}^+ \mathscr{R}_{j_1}^+ \colon \mathsf{Rep}_{\Bbbk} Q \longrightarrow \mathsf{Rep}_{\Bbbk} Q.$$

§1.5

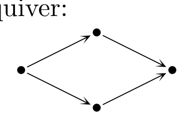
1.18. Let Q be the Dynkin quiver of type D_5:



(1) Describe the Weyl group $W(Q)$ and the root system $\Phi(Q)$ of Q.

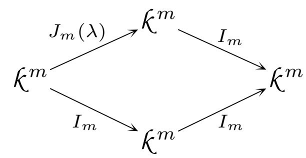
(2) Give a complete classification of indecomposable representations of Q.

1.19. Let Q be the following tame quiver:



(1) Describe the root system $\Phi(Q)$ of Q.

(2) For each $\lambda \in \Bbbk$ and each $m \geqslant 1$, define $V(\lambda, m)$ to be the following representation of Q:



where I_m is the $m \times m$ identity matrix and $J_m(\lambda) = \lambda I_m + J_m$ with J_m the $m \times m$ Jordan block

$$J_m = \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}.$$

Prove that $V(\lambda, m)$, $\lambda \in \Bbbk$, $m \geqslant 1$, are indecomposable and pairwise nonisomorphic. (Comparing with the proof of Theorem 1.23(2).)

1.20. Let Q be a Dynkin quiver and let V be an indecomposable representation of Q over $\Bbbk$. Prove that $\mathrm{End}_{\Bbbk Q}(V) \cong \Bbbk$.

Hint: By the proof of Theorem 1.23 (1), there is a (+)-admissible sequence $i_1, i_2, \ldots, i_t, i_{t+1}$ in Q such that

$$V \cong \mathscr{R}_{i_1}^- \mathscr{R}_{i_2}^- \cdots \mathscr{R}_{i_t}^- S_{i_{t+1}}.$$

Then, by Theorem 1.18(2), $\mathrm{End}_{\Bbbk Q}(V) \cong \mathrm{End}_{\Bbbk Q'}(S_{i_{t+1}}) \cong \Bbbk$, where $Q' = \mathfrak{s}_{i_t} \cdots \mathfrak{s}_{i_1} Q$.

1.21. Prove that the functors $\mathscr{V}$ defined in (1.5.1), (1.5.2) and (1.5.3) are fully faithful.

Hint: To prove that the functor $\mathscr{V}$ in (1.5.3) is full, make use of the fact that $\mathrm{End}_{\Bbbk Q''}(Z) \cong \Bbbk$ by Exercise 1.20, where $Z = (Z_i, Z_\rho)$ is the indecomposable representation of Q'' involved in the definition of $\mathscr{V}$.

§1.6

1.22. Let Q be a quiver, let $\mathcal{K}$ be an algebraically closed field, and let $\mathbf{d} \in \mathbb{N}Q_0$. Show that, for $x \in R(\mathbf{d})$, the orbit $\mathfrak{O}_x$ is open $\Longleftrightarrow$ the corresponding representation $V(x)$ has no self-extension, i.e., $\mathrm{Ext}^1_{\mathcal{K}Q}(V(x), V(x)) = 0$.

Hint: By (1.2.2), (1.6.1), and (1.6.2),

$$\begin{aligned} &\dim \mathrm{End}_{\mathcal{K}Q}(V(x)) - \dim \mathrm{Ext}^1_{\mathcal{K}Q}(V(x), V(x)) \\ &\qquad = \dim \mathrm{End}_{\mathcal{K}Q}(V(x)) + \dim \mathfrak{O}_x - \dim R(\mathbf{d}). \end{aligned}$$

This gives the equality

$$\dim R(\mathbf{d}) - \dim \mathfrak{O}_x = \dim \mathrm{Ext}^1_{\mathcal{K}Q}(V(x), V(x)).$$

Thus,

$$\mathrm{Ext}^1_{\mathcal{K}Q}(V(x), V(x)) = 0 \Longleftrightarrow \dim \mathfrak{O}_x = \dim R(\mathbf{d}).$$

The assertion then follows from the fact that $\mathfrak{O}_x$ is open in its closure $\overline{\mathfrak{O}}_x$. (See Theorem A.29.)

1.23. Let Q be an arbitrary quiver and let $\mathcal{K}$ be an algebraically closed field.

(1) If there is a nonsplit exact sequence $0 \to N \to E \to M \to 0$ in $\mathsf{Rep}_{\mathcal{K}} Q$, then

$$M \oplus N <_{\text{dg}} E.$$

Hint: Let $\mathbf{d}' = \mathbf{dim}\, N$ and $\mathbf{d}'' = \mathbf{dim}\, M$ and write $N \cong V(x)$ and $M \cong V(y)$, for some $x \in R(\mathbf{d}')$ and $y \in R(\mathbf{d}'')$. Then $\mathbf{dim}\, E = \mathbf{d}' + \mathbf{d}''$. Write $E \cong V(z)$ with $z = (z_\rho) \in R(\mathbf{d}' + \mathbf{d}'')$ given by

$$z_\rho = \begin{pmatrix} x_\rho & c_\rho \\ 0 & y_\rho \end{pmatrix}, \quad \text{for each } \rho \in Q_1.$$

For each $t \in \mathcal{K}$, define $z(t) \in R(\mathbf{d}' + \mathbf{d}'')$ by setting

$$z(t)_\rho = \begin{pmatrix} x_\rho & tc_\rho \\ 0 & y_\rho \end{pmatrix}.$$

Then $M \oplus N \cong V(z(0))$ and $V(z(t)) \cong E$, for all $t \neq 0$. Hence, $\mathfrak{O}_{z(0)} \subset \overline{\mathfrak{O}}_z$, i.e., $M \oplus N <_{\text{dg}} E$.

(2) Let $M, N \in \mathsf{Rep}_{\mathcal{K}} Q$ satisfy $M \leqslant_{\text{dg}} N$. Then, for each $X \in \mathsf{Rep}_{\mathcal{K}} Q$,

$$M \oplus X \leqslant_{\text{dg}} N \oplus X.$$

1.24. Let M and N be two representations of a quiver Q over an algebraically closed field $\mathcal{K}$. If $N \leqslant_{\text{dg}} M$, then

$$\dim \operatorname{Hom}_{\mathcal{K}Q}(X, N) \geqslant \dim \operatorname{Hom}_{\mathcal{K}Q}(X, M), \quad \text{for all } X \in \mathsf{Rep}_{\mathcal{K}} Q. \tag{1.E.1}$$

Hint (see [**242**, Prop. 2.1])*:* Let $\mathcal{T}$ be the set of all the triples

$$T = ((i_1, \dots, i_s), (j_1, \dots, j_t), \Phi),$$

where $(i_1, \dots, i_s), (j_1, \dots, j_t)$ are sequences of vertices in Q and $\Phi = (\varphi_{j_l, i_k})$ is a $t \times s$ matrix with $\varphi_{j_l, i_k} \in e_{j_l} \mathcal{K}Q e_{i_k}$, i.e., φ_{j_l, i_k} is a linear combination of paths from i_k to j_l in Q. For a representation $V = (V_i, V_\alpha)$, set

$$V(T) = (V(\varphi_{j_l, i_k}))\colon \bigoplus_{k=1}^{s} V_{i_k} \longrightarrow \bigoplus_{l=1}^{t} V_{j_l},$$

where $V(\varphi_{j_l, i_k})\colon V_{i_k} \to V_{j_l}$ are the $\mathcal{K}$-linear maps induced by the V_α, $\alpha \in Q_1$. Denote the rank of $V(T)$ by $r(V(T))$. The assertion is a direct consequence of the following two claims.

Claim 1: The equalities (1.E.1) hold $\iff r(N(T)) \leqslant r(M(T))$, for all $T \in \mathcal{T}$.

For each $i \in Q_0$, let $P_i = \mathcal{K}Q e_i$. Then each triple $T \in \mathcal{T}$ yields the morphism

$$\mu(T) = (\mu_{i_k, j_l})\colon \bigoplus_{l=1}^{t} P_{j_l} \longrightarrow \bigoplus_{k=1}^{s} P_{i_k},$$

where μ_{i_k,j_l} takes $x \in P_{j_l}$ to $x\varphi_{j_l,i_k} \in P_{i_k}$. Setting $X = X(T) = \operatorname{Coker} \mu(T)$, we have, for each $V \in \mathsf{Rep}_{\mathcal{K}} Q$, the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \to \operatorname{Hom}_{\mathcal{K}Q}(X,V) & \longrightarrow & \bigoplus_{k=1}^{s} \operatorname{Hom}_{\mathcal{K}Q}(P_{i_k},V) & \xrightarrow{\mu(T)^*} & \bigoplus_{l=1}^{t} \operatorname{Hom}_{\mathcal{K}Q}(P_{j_l},V) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow \operatorname{Ker} V(T) & \longrightarrow & \bigoplus_{k=1}^{s} V_{i_k} & \xrightarrow[V(T)]{} & \bigoplus_{l=1}^{t} V_{j_l} \end{array}$$

Consequently,

$$r(V(T)) + \dim \operatorname{Hom}_{\mathcal{K}Q}(X,V) = \sum_{k=1}^{s} \dim V_{i_k}.$$

The claim now follows from the fact $\mathbf{dim}\, M = \mathbf{dim}\, N$ (since $N \leqslant_{\mathrm{dg}} M$) and the fact that each representation of Q is isomorphic to some $X(T)$.

Claim 2: If $N \leqslant_{\mathrm{dg}} M$, then $r(N(T)) \leqslant r(M(T))$, for all $T \in \mathcal{T}$.

Let $\mathbf{dim}\, M = \mathbf{dim}\, N = \mathbf{d}$. For each triple $T \in \mathcal{T}$, consider the map

$$f \colon R(\mathbf{d}) \longrightarrow \operatorname{Hom}_{\mathcal{K}}\left(\bigoplus_{k=1}^{s} \mathcal{K}^{d_{i_k}}, \bigoplus_{l=1}^{t} \mathcal{K}^{d_{j_l}}\right) =: \mathcal{H}_T, \quad V \longmapsto V(T).$$

This is a $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})$-equivariant morphism of varieties, where $\mathrm{GL}_{\mathbf{d}}(\mathcal{K})$ also acts on $\mathcal{H}_T$ by conjugation. For $r \in \mathbb{N}$, the set $\{H \in \mathcal{H}_T \mid r(H) \leqslant r\}$ is a closed subset of $\mathcal{H}_T$, where $r(H)$ is the rank of the matrix H. Hence, the set $\{V \in R(\mathbf{d}) \mid r(V(T)) \leqslant r\}$ is also closed. In particular, if $N \leqslant_{\mathrm{dg}} M$, i.e., $\mathfrak{O}_N \subseteq \overline{\mathfrak{O}_M}$, then $r(N(T)) \leqslant r(M(T))$.

Notes

This chapter is based mainly on [**118**] and [**19**]. The material on Weyl groups and root systems is standard.

§1.1: The *graphic method* — quivers and their representations — was first introduced by Gabriel [**118**]. His work was motivated by the problem of classifying a certain class of algebras of finite representation type (see [**121**, §7.8]). The problem of finding a normal form of a pair of matrices under simultaneous row and column transformations (Example 1.4(2)) was first considered by Kronecker [**185**] in the case where k is algebraically closed; see [**121**, Ch. 1] for a modern treatment. For Example 1.4(3), see [**121**, §2.2].

§§1.2–1.3: The homological interpretation (1.2.2) of the Euler form of Q was first given by Ringel [**244**]. Hence, the Euler form is often also called the Ringel form in the literature. Theorem 1.11 is standard (see [**LAI**, Ch. VI, §4]). Its proof as presented here follows [**50**, §4]. Theorem 1.13 is essentially due to Vinberg [**296**];

the proof given here follows [**172**, Th. 3.4]. The proof of Lemma 1.14 follows [**298**, Prop. 2.1].

A further study of Euler forms for general finite dimensional algebras may be found in [**246**, 2.4] and [**8**, III.3].

The fact that the root system $\Phi(Q)$ of a quiver Q without loops coincides with the root system $\Phi(C_Q)$ of the corresponding Kac–Moody Lie algebra $\mathfrak{g}(C_Q)$ was shown in [**170**].

§1.4: BGP reflection functors were first introduced in [**19**]. The idea of using them to study quadruples of subspaces in a finite dimensional vector space appeared in [**128**]. Theorem 1.18 was formulated and proved in [**19**].

The Coxeter functors are closely related to Auslander–Reiten translation functors (see [**28, 11, 120**]). A module-theoretic interpretation of BGP reflection functors was given in [**11**]. A further remarkable development in this direction was the introduction of tilting theory in [**29, 146**].

§1.5: The proof of the first part of Theorem 1.23 given here follows [**19**], while the proof of the second part was given in the original article of Gabriel [**118**]. There is a short proof of this theorem over an algebraically closed field, which stresses the role of the Tits form and uses elementary facts of algebraic geometry and homological algebra; see, e.g., [**121**, §7].

Soon after the work of Gabriel [**118**], Nazarova [**230**] and Donovan–Freislich [**85**] independently classified indecomposable representations of tame quivers over an algebraically closed field. By dealing with species and their representations, Dlab–Ringel [**74, 75**] generalized the results in [**118, 230, 85**] to all Dynkin and extended Dynkin diagrams corresponding to symmetrizable Cartan matrices of finite and affine type. The general results for representations of wild quivers were achieved by Kac [**170, 171**].

§1.6: The geometric proof of the necessity of Theorem 1.23(1) over an algebraically closed field is due to Tits; see [**118**]. The ordering $\leqslant_{\mathrm{dg}}$ defined here is the opposite of the degeneration ordering used in some of the literature, see, e.g., [**22**]. We follow a traditional notation for the "Chevalley–Bruhat ordering" of a Coxeter group. The description of the degeneration ordering in the Dynkin quiver case in terms of dimensions of homomorphism spaces in Remark 1.28 was given in [**242, 22**]. In general, for an arbitrary finite dimensional algebra A over an algebraically closed field, Riedtmann [**242**] and Zwara [**309**] established the following nice algebraic characterization of degenerations of finite dimensional A-modules: for $M, N \in A\text{-}\mathsf{mod}$, $N \leqslant_{\mathrm{dg}} M$ if and only if there exists an exact sequence

$$0 \longrightarrow N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow 0 \quad \text{or} \quad 0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0,$$

for some $Z \in A\text{-}\mathsf{mod}$.

Propositions 1.29 and 1.30 and their proofs follow Reineke [**239**]. An earlier paper [**22**] by Bongartz defined generic extensions in a more general setting. Generic extensions of nilpotent representations of a cyclic quiver were studied in [**57**]. However, if a connected quiver Q is neither Dynkin nor cyclic, generic extension of two representations of Q may not exist.

For more on the geometry of representations of algebras, see [**51**] and the references therein.