CHAPTER 1

Generalized Argument Principle and Rouché's Theorem

In this chapter we review the results of Gohberg and Sigal in [**114**] concerning the generalization to operator-valued functions of two classical results in complex analysis, the *argument principle* and *Rouché's theorem*.

To state the argument principle, we first observe that if f is holomorphic and has a zero of order n at z_0, we can write $f(z) = (z-z_0)^n g(z)$, where g is holomorphic and nowhere vanishing in a neighborhood of z_0, and therefore

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Then the function f'/f has a simple pole with residue n at z_0. A similar fact also holds if f has a pole of order n at z_0, that is, if $f(z) = (z-z_0)^{-n} h(z)$, where h is holomorphic and nowhere vanishing in a neighborhood of z_0. Then

$$\frac{f'(z)}{f(z)} = -\frac{n}{z-z_0} + \frac{h'(z)}{h(z)}.$$

Therefore, if f is holomorphic, the function f'/f will have simple poles at the zeros and poles of f, and the residue is simply the order of the zero of f or the negative of the order of the pole of f.

The argument principle results from an application of the residue formula. It asserts the following.

THEOREM 1.1 (Argument principle). *Let $V \subset \mathbb{C}$ be a bounded domain with smooth boundary ∂V positively oriented and let $f(z)$ be a meromorphic function in a neighborhood of $\overline{V}$. Let P and N be the number of poles and zeros of f in V, counted with their multiplicities. If f has no poles and never vanishes on ∂V, then*

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial V} \frac{f'(z)}{f(z)}\, dz = N - P. \tag{1.1}$$

Rouché's theorem is a consequence of the argument principle [**237**]. It is in some sense a continuity statement. It says that a holomorphic function can be perturbed slightly without changing the number of its zeros. It reads as follows.

THEOREM 1.2 (Rouché's theorem). *With V as above, suppose that $f(z)$ and $g(z)$ are holomorphic in a neighborhood of $\overline{V}$. If $|f(z)| > |g(z)|$ for all $z \in \partial V$, then f and $f+g$ have the same number of zeros in V.*

In order to explain the main results of Gohberg and Sigal in [**114**], we begin with the finite-dimensional case which was first considered by Keldyš in [**152**]; see also [**183**]. We proceed to generalize formula (1.1) in this case as follows. If a matrix-valued function $A(z)$ is holomorphic in a neighborhood of $\overline{V}$ and is invertible in $\overline{V}$ except possibly at a point $z_0 \in V$, then by Gaussian eliminations we can write

$$A(z) = E(z)D(z)F(z) \quad \text{in } V, \tag{1.2}$$

where $E(z), F(z)$ are holomorphic and invertible in V and $D(z)$ is given by

$$D(z) = \begin{pmatrix} (z-z_0)^{k_1} & & 0 \\ & \ddots & \\ 0 & & (z-z_0)^{k_n} \end{pmatrix}.$$

Moreover, the powers $k_1, k_2, \ldots, k_n$ are uniquely determined up to a permutation.

Let tr denote the trace. By virtue of the factorization (1.2), it is easy to produce the following identity:

$$\begin{aligned} &\frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \int_{\partial V} A(z)^{-1} \frac{d}{dz} A(z)\, dz \\ &\quad = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \int_{\partial V} \left(E(z)^{-1} \frac{d}{dz} E(z) + D(z)^{-1} \frac{d}{dz} D(z) + F(z)^{-1} \frac{d}{dz} F(z) \right) dz \\ &\quad = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \int_{\partial V} D(z)^{-1} \frac{d}{dz} D(z)\, dz \\ &\quad = \sum_{j=1}^{n} k_j, \end{aligned}$$

which generalizes (1.1).

In the next sections, we will extend the above identity as well as the factorization (1.2) to infinite-dimensional spaces under some natural conditions.

1.1. Definitions and Preliminaries

In this section we introduce the notation we will use in the text, gather a few definitions, and present some basic results, which are useful for the statement of the generalized Rouché theorem.

1.1.1. Compact Operators. If $\mathcal{B}$ and $\mathcal{B}'$ are two Banach spaces, we denote by $\mathcal{L}(\mathcal{B}, \mathcal{B}')$ the space of bounded linear operators from $\mathcal{B}$ into $\mathcal{B}'$. An operator $K \in \mathcal{L}(\mathcal{B}, \mathcal{B}')$ is said to be compact provided K takes any bounded subset of $\mathcal{B}$ to a relatively compact subset of $\mathcal{B}'$, that is, a set with compact closure.

The operator K is said to be of finite rank if $\operatorname{Im}(K)$, the range of K, is finite-dimensional. Clearly every operator of finite rank is compact.

The next result is called the Fredholm alternative. See, for example, [**164**].

PROPOSITION 1.3 (Fredholm alternative). *Let K be a compact operator on the Banach space $\mathcal{B}$. For $\lambda \in \mathbb{C}, \lambda \neq 0$, $(\lambda I - K)$ is surjective if and only if it is injective.*

1.1.2. Fredholm Operators. An operator $A \in \mathcal{L}(\mathcal{B}, \mathcal{B}')$ is said to be *Fredholm* provided the subspace $\operatorname{Ker} A$ is finite-dimensional and the subspace $\operatorname{Im} A$ is closed in $\mathcal{B}'$ and of finite codimension. Let $\operatorname{Fred}(\mathcal{B}, \mathcal{B}')$ denote the collection of all Fredholm operators from $\mathcal{B}$ into $\mathcal{B}'$. We can show that $\operatorname{Fred}(\mathcal{B}, \mathcal{B}')$ is open in $\mathcal{L}(\mathcal{B}, \mathcal{B}')$.

Next, we define the index of $A \in \mathrm{Fred}(\mathcal{B}, \mathcal{B}')$ to be

$$\mathrm{ind}\, A = \dim \mathrm{Ker}\, A - \mathrm{codim}\, \mathrm{Im}\, A.$$

In finite dimensions, the index depends only on the spaces and not on the operator.

The following proposition shows that the index is stable under compact perturbations [**164**].

PROPOSITION 1.4. *If $A : \mathcal{B} \to \mathcal{B}'$ is Fredholm and $K : \mathcal{B} \to \mathcal{B}'$ is compact, then their sum $A + K$ is Fredholm, and*

$$\mathrm{ind}\,(A + K) = \mathrm{ind}\, A.$$

Proposition 1.4 is a consequence of the following fundamental result about the index of Fredholm operators.

PROPOSITION 1.5. *The mapping $A \mapsto \mathrm{ind}\, A$ is continuous in* $\mathrm{Fred}(\mathcal{B}, \mathcal{B}')$*; i.e.,* ind *is constant on each connected component of* $\mathrm{Fred}(\mathcal{B}, \mathcal{B}')$.

1.1.3. Characteristic Value and Multiplicity. We now introduce the notions of characteristic values and root functions of analytic operator-valued functions, with which the readers might not be familiar. We refer, for instance, to the book by Markus [**175**] for the details.

Let $\mathfrak{U}(z_0)$ be the set of all operator-valued functions with values in $\mathcal{L}(\mathcal{B}, \mathcal{B}')$ which are holomorphic in some neighborhood of z_0, except possibly at z_0.

The point z_0 is called a *characteristic value* of $A(z) \in \mathfrak{U}(z_0)$ if there exists a vector-valued function $\phi(z)$ with values in $\mathcal{B}$ such that

(i) $\phi(z)$ is holomorphic at z_0 and $\phi(z_0) \neq 0$,
(ii) $A(z)\phi(z)$ is holomorphic at z_0 and vanishes at this point.

Here, $\phi(z)$ is called a *root* function of $A(z)$ associated with the characteristic value z_0. The vector $\phi_0 = \phi(z_0)$ is called an eigenvector. The closure of the linear set of eigenvectors corresponding to z_0 is denoted by $\mathrm{Ker}A(z_0)$.

Suppose that z_0 is a characteristic value of the function $A(z)$ and $\phi(z)$ is an associated root function. Then there exists a number $m(\phi) \geq 1$ and a vector-valued function $\psi(z)$ with values in $\mathcal{B}'$, holomorphic at z_0, such that

$$A(z)\phi(z) = (z - z_0)^{m(\phi)}\psi(z), \quad \psi(z_0) \neq 0.$$

The number $m(\phi)$ is called the *multiplicity* of the root function $\phi(z)$.

For $\phi_0 \in \mathrm{Ker}A(z_0)$, we define the rank of ϕ_0, denoted by $\mathrm{rank}(\phi_0)$, to be the maximum of the multiplicities of all root functions $\phi(z)$ with $\phi(z_0) = \phi_0$.

Suppose that $n = \dim \mathrm{Ker}A(z_0) < +\infty$ and that the ranks of all vectors in $\mathrm{Ker}A(z_0)$ are finite. A system of eigenvectors ϕ_0^j, $j = 1, \ldots, n$, is called a *canonical system of eigenvectors* of $A(z)$ associated to z_0 if their ranks possess the following property: for $j = 1, \ldots, n$, $\mathrm{rank}(\phi_0^j)$ is the maximum of the ranks of all eigenvectors in the direct complement in $\mathrm{Ker}A(z_0)$ of the linear span of the vectors $\phi_0^1, \ldots, \phi_0^{j-1}$. We call

$$N(A(z_0)) := \sum_{j=1}^{n} \mathrm{rank}(\phi_0^j)$$

the *null multiplicity* of the characteristic value z_0 of $A(z)$. If z_0 is not a characteristic value of $A(z)$, we put $N(A(z_0)) = 0$.

Suppose that $A^{-1}(z)$ exists and is holomorphic in some neighborhood of z_0, except possibly at z_0. Then the number

$$M(A(z_0)) = N(A(z_0)) - N(A^{-1}(z_0))$$

is called the *multiplicity* of z_0. If z_0 is a characteristic value and not a pole of $A(z)$, then $M(A(z_0)) = N(A(z_0))$ while $M(A(z_0)) = -N(A^{-1}(z_0))$ if z_0 is a pole and not a characteristic value of $A(z)$.

1.1.4. Normal Points. Suppose that z_0 is a pole of the operator-valued function $A(z)$ and the Laurent series expansion of $A(z)$ at z_0 is given by

$$A(z) = \sum_{j \geq -s} (z - z_0)^j A_j. \tag{1.3}$$

If in (1.3) the operators A_{-j}, $j = 1, \ldots, s$, have finite-dimensional ranges, then $A(z)$ is called *finitely meromorphic* at z_0.

The operator-valued function $A(z)$ is said to be of *Fredholm type* (of index zero) at the point z_0 if the operator A_0 in (1.3) is Fredholm (of index zero).

If $A(z)$ is holomorphic and invertible at z_0, then z_0 is called a *regular point* of $A(z)$. The point z_0 is called a *normal point* of $A(z)$ if $A(z)$ is finitely meromorphic, of Fredholm type at z_0, and regular in a neighborhood of z_0 except at z_0 itself.

1.1.5. Trace. Let A be a finite-dimensional operator acting from $\mathcal{B}$ into itself. There exists a finite-dimensional invariant subspace $\mathcal{C}$ of A such that A annihilates some direct complement of $\mathcal{C}$ in $\mathcal{B}$. We define the trace of A to be that of $A|_{\mathcal{C}}$, which is given in the usual way. It is desirable to recall some results about the trace operator.

PROPOSITION 1.6. *The following results hold:*

(i) $\operatorname{tr} A$ *is independent of the choice of* $\mathcal{C}$*, so that it is well-defined.*
(ii) tr *is linear.*
(iii) *If* B *is a finite-dimensional operator from* $\mathcal{B}$ *to itself, then*

$$\operatorname{tr} AB = \operatorname{tr} BA.$$

(iv) *If* M *is a finite-dimensional operator from* $\mathcal{B} \times \mathcal{B}'$ *to itself, given by*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then $\operatorname{tr} M = \operatorname{tr} A + \operatorname{tr} D$.

Recall that if an operator-valued function $C(z)$ is finitely meromorphic in the neighborhood V of z_0, which contains no poles of $C(z)$ except possibly z_0, then $\int_{\partial V} C(z)\, dz$ is a finite-dimensional operator. The following identity will also be used frequently.

PROPOSITION 1.7. *Let* $A(z)$ *and* $B(z)$ *be two operator-valued functions which are finitely meromorphic in the neighborhood* $\overline{V}$ *of* z_0*, which contains no poles of* $A(z)$ *and* $B(z)$ *other than* z_0*. Then we have*

$$\operatorname{tr} \int_{\partial V} A(z)B(z)\, dz = \operatorname{tr} \int_{\partial V} B(z)A(z)\, dz. \tag{1.4}$$

1.2. Factorization of Operators

We say that $A(z) \in \mathfrak{U}(z_0)$ admits a factorization at z_0 if $A(z)$ can be written as

$$A(z) = E(z)D(z)F(z), \tag{1.5}$$

where $E(z), F(z)$ are regular at z_0 and

$$D(z) = P_0 + \sum_{j=1}^{n} (z - z_0)^{k_j} P_j. \tag{1.6}$$

Here, P_j's are mutually disjoint projections, $P_1, \ldots, P_n$ are one-dimensional operators, and $I - \sum_{j=0}^{n} P_j$ is a finite-dimensional operator.

THEOREM 1.8. *$A(z) \in \mathfrak{U}(z_0)$ admits a factorization at z_0 if and only if $A(z)$ is finitely meromorphic and of Fredholm type of index zero at z_0.*

PROOF. Suppose that $A(z)$ is finitely meromorphic and of Fredholm type of index zero at z_0. We shall construct E, F, and D such that (1.5) holds. Write the Laurent series expansion of $A(z)$ as follows:

$$A(z) = \sum_{j=-\nu}^{+\infty} (z - z_0)^j A_j$$

in some neighborhood U of z_0. Since $\operatorname{ind} A_0 = 0$, then by the Fredholm alternative $B_0 := A_0 + K_0$ is invertible for some finite-dimensional operator K_0. Consequently,

$$B(z) := K_0 + \sum_{j=0}^{+\infty} (z - z_0)^j A_j$$

is invertible in some neighborhood U_1 of z_0 and

$$A(z) = C(z) + B(z) = B(z)[I + B^{-1}(z)C(z)], \tag{1.7}$$

where

$$C(z) = \sum_{j=-\nu}^{-1} (z - z_0)^j A_j - K_0.$$

Since $K(z) := B^{-1}(z)C(z)$ is finitely meromorphic, we can write $K(z)$ in the form

$$K(z) = \sum_{j=1}^{\nu} (z - z_0)^{-j} K_j + T_1(z),$$

where K_j, $j = 1, \ldots, \nu$, are finite-dimensional and T_1 is holomorphic.

Since the operators A_j and K_j are finite-dimensional, there exists a subspace $\mathfrak{N}$ of $\mathcal{B}$ of finite codimension such that

$$\begin{cases} \mathfrak{N} \subset \operatorname{Ker} A_j, \ j = -\nu, \ldots, -1, \\ \mathfrak{N} \subset \operatorname{Ker} K_j, \ j = 0, \ldots, \nu, \\ \mathfrak{N} \bigcap \operatorname{Im} K_j = \{0\}, \ j = 1, \ldots, \nu. \end{cases}$$

Let $\mathfrak{C}$ be a direct finite-dimensional complement of $\mathfrak{N}$ in $\mathcal{B}$ and let P be the projection onto $\mathfrak{C}$ satisfying $P(I-P)=0$. Set $P_0 := I - P$. We have

$$\begin{aligned} I + K(z) &= I + PK(z)P + P_0K(z)P \\ &= I + PK(z)P + P_0T_1(z)P, \end{aligned}$$

and therefore,

$$I + K(z) = (I + PK(z)P)(I + P_0T_1(z)P). \tag{1.8}$$

Since $P(I + K(z))P$ can be viewed as an operator from $\mathfrak{C}$ into itself and $\mathfrak{C}$ is finite-dimensional, it follows from Gaussian elimination that

$$P(I + K(z))P = E_1(z)D_1(z)F_1(z),$$

where $D_1(z)$ is diagonal and $E_1(z)$ and $F_1(z)$ are holomorphic and invertible. In view of (1.8), this implies that

$$\begin{aligned} A(z) &= B(z)(P_0 + P(I + K(z))P)(I + P_0T_1(z)P) \\ &= B(z)(P_0 + E_1(z)D_1(z)F_1(z))(I + P_0T_1(z)P) \\ &= B(z)(P_0 + E_1(z))(P_0 + D_1(z))(P_0 + F_1(z))(I + P_0T_1(z)P). \end{aligned}$$

Here $I + P_0T_1(z)P$ is holomorphic and invertible with inverse $I - P_0T_1(z)P$. Thus, taking

$$E(z) := B(z)(P_0 + E_1(z)), \qquad F(z) := (P_0 + F_1(z))(I + P_0T_1(z)P)$$

yields the desired factorization for A since $E(z)$ and $F(z)$, given by the above formulas, are holomorphic and invertible at z_0.

The converse result, that $A(z) = E(z)D(z)F(z)$ with $E(z), F(z)$ regular at z_0 and $D(z)$ satisfying (1.6) is finitely meromorphic and of Fredholm type of index zero at z_0, is easy. □

Corollary 1.9. *$A(z)$ is normal at z_0 if and only if $A(z)$ admits a factorization such that $I = \sum_{j=0}^{n} P_j$ in (1.6). Moreover, we have*

$$M(A(z_0)) = k_1 + \cdots + k_n$$

for $k_1, \cdots, k_n$, given by (1.6).

Corollary 1.10. *Every normal point of $A(z)$ is a normal point of $A^{-1}(z)$.*

1.3. Main Results of the Gohberg and Sigal Theory

We now tackle our main goal of this chapter, which is to generalize the argument principle and Rouché's theorem to operator-valued functions.

1.3.1. Argument Principle. Let V be a simply connected bounded domain with rectifiable boundary ∂V. An operator-valued function $A(z)$ which is finitely meromorphic and of Fredholm type in V and continuous on ∂V is called *normal* with respect to ∂V if the operator $A(z)$ is invertible in $\overline{V}$, except for a finite number of points of V which are normal points of $A(z)$.

Lemma 1.11. *An operator-valued function $A(z)$ is normal with respect to ∂V if it is finitely meromorphic and of Fredholm type in V, continuous on ∂V, and invertible for all $z \in \partial V$.*

PROOF. To prove that A is normal with respect to ∂V, it suffices to prove that $A(z)$ is invertible except at a finite number of points in V. To this end choose a connected open set U with $\overline{U} \subset V$ so that $A(z)$ is invertible in $V \setminus U$. Then, for each $\xi \in U$, there exists a neighborhood U_ξ of ξ in which the factorization (1.5) holds. In U_ξ, the kernel of $A(z)$ has a constant dimension except at ξ. Since $\overline{U}$ is compact, we can find a finite covering of $\overline{U}$, *i.e.*,

$$\overline{U} \subset U_{\xi_1} \cup \cdots \cup U_{\xi_k},$$

for some points $\xi_1, \ldots, \xi_k \in U$. Therefore, $\dim \operatorname{Ker} A(z)$ is constant in $V \setminus \{\xi_1, \ldots, \xi_k\}$, and so $A(z)$ is invertible in $\overline{V} \setminus \{\xi_1, \ldots, \xi_k\}$. □

Now, if $A(z)$ is normal with respect to the contour ∂V and z_i, $i = 1, \ldots, \sigma$, are all its characteristic values and poles lying in V, we put

$$\mathcal{M}(A(z); \partial V) = \sum_{i=1}^{\sigma} M(A(z_i)). \tag{1.9}$$

The full multiplicity $\mathcal{M}(A(z); \partial V)$ of $A(z)$ in V is the number of characteristic values of $A(z)$ in V, counted with their multiplicities, minus the number of poles of $A(z)$ in V, counted with their multiplicities.

THEOREM 1.12 (Generalized argument principle). *Suppose that the operator-valued function $A(z)$ is normal with respect to ∂V. Then we have*

$$\mathcal{M}(A(z); \partial V) = \frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V} A^{-1}(z) \frac{d}{dz} A(z) dz. \tag{1.10}$$

PROOF. Let z_j, $j = 1, \ldots, \sigma$, denote all the characteristic values and all the poles of A lying in V. The key of the proof lies in using the factorization (1.5) in each of the neighborhoods of the points z_j. We have

$$\frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V} A^{-1}(z) \frac{d}{dz} A(z) dz = \sum_{j=1}^{\sigma} \frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V_j} A^{-1}(z) \frac{d}{dz} A(z) dz, \tag{1.11}$$

where, for each j, V_j is a neighborhood of z_j. Moreover, in each V_j, the following factorization of A holds:

$$A(z) = E^{(j)}(z) D^{(j)}(z) F^{(j)}(z), \qquad D^{(j)}(z) = P_0^{(j)} + \sum_{i=1}^{n_j} (z - z_j)^{k_{ij}} P_i^{(j)}.$$

As for the matrix-valued case at the beginning of this chapter, it is readily verified that

$$\begin{aligned} \frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V_j} A^{-1}(z) \frac{d}{dz} A(z) dz &= \frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V_j} (D^{(j)}(z))^{-1} \frac{d}{dz} D^{(j)}(z) dz \\ &= \sum_{i=1}^{n_j} k_{ij} = M(A(z_j)). \end{aligned}$$

Now, (1.10) follows by using (1.11). □

The following is an immediate consequence of Lemma 1.11, identity (1.10), and (1.4).

COROLLARY 1.13. *If the operator-valued functions $A(z)$ and $B(z)$ are normal with respect to ∂V, then $C(z) := A(z)B(z)$ is also normal with respect to ∂V, and*

$$\mathcal{M}(C(z); \partial V) = \mathcal{M}(A(z); \partial V) + \mathcal{M}(B(z); \partial V).$$

The following general form of the argument principle will be useful. It can be proven by the same argument as the one in Theorem 1.12.

THEOREM 1.14. *Suppose that $A(z)$ is an operator-valued function which is normal with respect to ∂V. Let $f(z)$ be a scalar function which is analytic in V and continuous in $\overline{V}$. Then*

$$\frac{1}{2\sqrt{-1}\pi} \operatorname{tr} \int_{\partial V} f(z) A^{-1}(z) \frac{d}{dz} A(z) dz = \sum_{j=1}^{\sigma} M(A(z_j)) f(z_j),$$

where z_j, $j = 1, \ldots, \sigma$, are all the points in V which are either poles or characteristic values of $A(z)$.

1.3.2. Generalization of Rouché's Theorem. A generalization of Rouché's theorem to operator-valued functions is stated below.

THEOREM 1.15 (Generalized Rouché's theorem). *Let $A(z)$ be an operator-valued function which is normal with respect to ∂V. If an operator-valued function $S(z)$ which is finitely meromorphic in V and continuous on ∂V satisfies the condition*

$$\|A^{-1}(z)S(z)\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} < 1, \quad z \in \partial V,$$

then $A(z) + S(z)$ is also normal with respect to ∂V and

$$\mathcal{M}(A(z); \partial V) = \mathcal{M}(A(z) + S(z); \partial V).$$

PROOF. Let $C(z) := A^{-1}(z)S(z)$. By Corollary 1.10, $C(z)$ is finitely meromorphic in V. Suppose that $z_1, z_2, \ldots, z_n$, are all of the poles of $C(z)$ in V and that $C(z)$ has the following Laurent series expansion in some neighborhood of each z_j:

$$C(z) = \sum_{k=-\nu_j}^{+\infty} (z - z_j)^k C_k^{(j)}.$$

Let $\mathfrak{N}$ be the intersection of the kernels $\operatorname{Ker} C_k^{(j)}$ for $j = 1, \ldots, n$ and $k = 1, \ldots, \nu_j$. Then, $\dim \mathcal{B}/\mathfrak{N} < +\infty$ and the restriction $C(z)|_{\mathfrak{N}}$ of $C(z)$ to $\mathfrak{N}$ is holomorphic in V.

Let $q := \max_{z \in \partial V} \|C(z)\|$, which by assumption is less than 1. Since

$$\Delta_z \|C(z)|_{\mathfrak{N}}\|^2 = 4 \|\frac{\partial}{\partial z} C(z)|_{\mathfrak{N}}\|^2,$$

then $\|C(z)|_{\mathfrak{N}}\|$ is subharmonic in V, and hence we have from the maximum principle

$$\max_{z \in V} \|C(z)|_{\mathfrak{N}}\| \leq q.$$

It then follows that

$$\|(I + C(z))x\| \geq (1 - q)\|x\|, \qquad x \in \mathfrak{N}, \ z \in V.$$

This implies that $(I + C(z))|_{\mathfrak{N}}$ has a closed range and $\mathrm{Ker}(I + C(z))|_{\mathfrak{N}} = 0$. Therefore, $I + C(z)$ has a closed range and a kernel of finite dimension for $z \in V \setminus \{z_1, \dots, z_n\}$. By a slight extension of Proposition 1.5 [**241**], $\mathcal{I}(z)$ defined by

$$\mathcal{I}(z) = \mathrm{dimKer}(I + C(z)) - \mathrm{codim}\ \mathrm{Im}(I + C(z))$$

is continuous for $z \in \overline{V} \setminus \{z_1, \dots, z_n\}$. Thus,

$$\mathrm{ind}(I + C(z)) = 0 \quad \text{for } z \in \overline{V} \setminus \{z_1, \dots, z_n\}.$$

Moreover, since the Laurent series expansion of $(I + C(z))|_{\mathfrak{N}}$ in a neighborhood of z_j is given by

$$(I + C(z))|_{\mathfrak{N}} = I|_{\mathfrak{N}} + \sum_{k=0}^{+\infty} (z - z_j)^k C_k^{(j)}|_{\mathfrak{N}}, \tag{1.12}$$

it follows that $(I+C_0^{(j)})|_{\mathfrak{N}}$ has a closed range and a trivial kernel. Using Propositions 1.4 and 1.5, we have

$$\mathrm{ind}(I + C_0^{(j)}) = \mathrm{ind}(I + \sum_{k=0}^{+\infty} (z - z_j)^k C_k^{(j)}) = \mathrm{ind}(I + C(z)) = 0.$$

Thus, $(I + C_0^{(j)})$ is Fredholm. By Lemma 1.11, we deduce that $I + C(z)$ is normal with respect to ∂V.

Now we claim that $\mathcal{M}(I + C(z); \partial V) = 0$. To see this, we note that $I + tC(z)$ is normal with respect to ∂V for $0 \le t \le 1$. Let

$$f(t) := \mathcal{M}(I + tC(z); \partial V).$$

Then $f(t)$ attains integers as its values. On the other hand, since

$$f(t) = \frac{1}{2\sqrt{-1}\pi} \mathrm{tr} \int_{\partial V} t(I + tC(z))^{-1} \frac{d}{dz} C(z)\, dz \tag{1.13}$$

and $(I + tC(z))^{-1}$ is continuous in $[0, 1]$ in operator norm uniformly in $z \in \partial V$, $f(t)$ is continuous in $[0, 1]$. Thus, $f(1) = f(0) = 0$.

Finally, with the help of Corollary 1.13, we can conclude that the theorem holds. □

1.3.3. Generalization of Steinberg's theorem. Steinberg's theorem asserts that if $K(z)$ is a compact operator on a Banach space, which is analytic in z for z in a region V in the complex plane, then $I + K(z)$ is meromorphic in V. See [**238**]. A generalization of this theorem to finitely meromorphic operators was first given by Gohberg and Sigal in [**114**]. The following important result holds.

THEOREM 1.16 (Generalized Steinberg's theorem). *Suppose that $A(z)$ is an operator-valued function which is finitely meromorphic and of Fredholm type in the domain V. If the operator $A(z)$ is invertible at one point of V, then $A(z)$ has a bounded inverse for all $z \in V$, except possibly for certain isolated points.*

1.4. Concluding Remarks

In this chapter, we have reviewed the main results in the theory of Gohberg and Sigal on meromorphic operator-valued functions. These results concern the generalization of the argument principle and the Rouché theorem to meromorphic operator-valued functions. Some of these results have been extended to very general operator-valued functions in [**46, 170**] and with other types of spectrum than isolated eigenvalues in [**174**].

Throughout this book, the theory of Gohberg and Sigal will be applied to perturbation theory of eigenvalues. Other interesting applications include the investigation of scattering resonances and scattering poles [**118, 57**] and the study of the regularity of the solutions of elliptic boundary value problems near conical points [**154**].