Preface

The roots of the project to write this book originated in the early nineteen–nineties, when several streams of mathematical ideas met after being developed more or less separately for several decades. This resulted in an amazing interaction between active groups of mathematicians working in several directions. One of these directions was the study of conformally invariant differential operators related, on the one hand, to Penrose's twistor program (M.G. Eastwood, T.N. Bailey, R.J. Baston, C.R. Graham, A.R. Gover, and others) and, on the other hand, to hypercomplex analysis (V. Souček, F. Sommen, and others). It turned out that, via the canonical Cartan connection or equivalent data, conformal geometry is just one instance of a much more general picture. Over the years we noticed that the foundations for this general picture had already been developed in the pioneering work of N. Tanaka. His work was set in the language of the equivalence problem and of differential systems, but independent of the much better disseminated developments of the theory of differential systems linked to names like S.S. Chern, R. Bryant, and M. Kuranishi. While Tanaka's work did not become widely known, it was further developed, in particular, by K. Yamaguchi and T. Morimoto, who put it in the setting of filtered manifolds and applied it to the geometric study of systems of PDE's. A lot of input and stimulus also came from Ch. Fefferman's work in complex analysis and geometric function theory, in particular, his parabolic invariant theory program and the relation between CR–structures and conformal structures. Finally, via the homogeneous models, all of these studies have close relations to various parts of representation theory of semisimple Lie groups and Lie algebras, developed for example by T. Branson and B. Ørsted.

Enjoying the opportunities offered by the newly emerging International Erwin Schrödinger Institute for Mathematical Physics (ESI) in Vienna as well as the long lasting tradition of the international Winter Schools "Geometry and Physics" held every year in Srní, Czech Republic, the authors of this book started a long and fruitful collaboration with most of the above mentioned people. Step by step, all of the general concepts and problems were traced back to old masters like Schouten, Veblen, Thomas, and Cartan, and a broad research program led to a conceptual understanding of the common background of the various approaches developed more recently. In the late nineties, the general version of the invariant calculus for a vast class of geometrical structures extended the tools for geometric analysis on homogeneous vector bundles and the direct applications of representation theory expanded in this way to situations involving curvatures. In particular, the celebrated Bernstein–Gelfand–Gelfand resolutions were recovered in the realm of general parabolic geometries by V. Souček and the authors and the cohomological substance of all these constructions was clarified.

The book was written with two goals in mind. On one hand, we want to provide a (relatively) gentle introduction into this fascinating world which blends algebra and geometry, Lie theory and geometric analysis, geometric intuition and categorical thinking. At the same time, preparing the first treatment of the general theory and collection of the main results on the subject, gave us the ambition to provide the standard reference for the experts in the area. These two goals are reflected in the overall structure of the book which finally developed into two volumes. The second volume will be called "Parabolic Geometries II: Invariant Differential Operators and Applications".

Contents of the book. The basic theme of this book is canonical Cartan connections associated to certain types of geometric structures, which immediately causes peculiarities.

Cartan connections and, more generally, various types of absolute parallelisms certainly played a central role in Cartan's work on differential geometry and they still belong to the basic tools in several geometric approaches to differential equations. However, in the efforts in the second half of the last century to put Cartan's ideas into the conceptual framework of fiber bundles, the main part was taken over by principal connections. The isolated treatments in the books by Kobayashi and Sharpe stopped at alternative presentations of the quite well–known conformal Riemannian and projective structures. As a consequence, even basic facts on Cartan connections are neither well represented in the standard literature on differential geometry nor well known among many people working in the field. For this book, we have collected some general facts about Cartan connections on principal fiber bundles in Section 1.5. While this is located in the "Background" part of the book because of its nature, quite a bit of this material will probably be new to most readers.

The other peculiar consequence of the approach is that the geometric structures we study display strong similarities in the picture of Cartan connections, while they are extremely diverse in their original descriptions. Therefore, in large parts of the book we will take the point of view that the Cartan picture is the "true" description of the structures in question, while the original description is obtained as an underlying structure. This point of view is justified by the results on equivalences of categories between Cartan geometries and underlying structures in Section 3.1, which are among the main goals of this volume. This point of view will also be taken in the second volume, in which the Cartan connection (or some equivalent data) will be simply considered as an input.

Let us describe the contents of the first volume in more detail. The technical core of the book is Chapter 3. In Section 3.1 we develop the basic theory of parabolic geometries as Cartan geometries and prove the equivalence to underlying structures in the categorical sense. This is done in the setting of $|k|$–gradings of semisimple Lie algebras, thus avoiding the use of structure theory and representation theory. The structure theory is brought into play in Section 3.2 to get more detailed information on the applicability of the methods developed before. Section 3.3 contains an exposition and a complete proof of Kostant's version of the Bott–Borel–Weil Theorem, which is needed to verify the cohomological conditions that occur in several places in the theory, and proves to be extremely useful later, too.

In Chapter 4, the general results of Chapter 3 are turned into explicit descriptions of a wide variety of examples of geometries covered by our methods.

In particular, we thoroughly discuss the geometries corresponding to $|1|$–gradings (which can be described as classical first order G–structures) and the parabolic contact geometries, which have an underlying contact structure. In Sections 4.4 and 4.5, we discuss two general constructions relating geometries of different types, the construction of correspondence spaces and twistor spaces, and analogs of the Fefferman construction.

The developments in Chapter 5 admit two interpretations. On the one hand, via the notion of Weyl structures, we associate to any parabolic geometry a class of distinguished connections and we define classes of distinguished curves. On the other hand, the data associated to a Weyl structure offer an equivalent description of the canonical Cartan connection in terms of objects associated to the underlying structure. In this way, one also obtains a more explicit description of the canonical Cartan connections. Throughout Sections 5.2 and 5.3 we also discuss various applications of the theory developed in the book.

The first part of the book (Chapters 1 and 2) provides necessary background and motivation. Chapter 1 is general and rather elementary and should be digestible and enjoyable even for newcomers. Here Cartan's concept of "curved analogs" of Klein's homogeneous spaces and also the related general calculi are explained using the effective general language of Lie groups and Lie algebras but no structure theory. As mentioned before, some of the material presented in this discussion is not easy to find in the literature. Section 1.6 contains an explicit and elementary treatment of conformal (pseudo)–Riemannian structures. Apart from motivating further developments, this also indicates clearly that a deep understanding of the algebraic structure of the algebras and groups of symmetries in question is the key to further progress. This naturally leads to Chapter 2, which contains background material on semisimple Lie algebras and Lie groups. While the material we cover in this chapter is certainly available in book form in many places, there are some unusual aspects. The main point is that, apart from the complex theory, we also discuss the structure theory and representation theory of real semisimple Lie algebras. The real theory is typically scattered in the textbooks among the advanced topics and hence rather difficult to learn quickly elsewhere. In this way, the first part of the book makes the whole project more or less self–contained. In addition, it should be of separate interest as well. As an important counterpart to the theory developed in Chapter 2 we provide tables containing the central structural information on semisimple Lie algebras in Appendix B.

The second volume will be devoted to invariant differential operators for parabolic geometries, in particular, the technique of BGG–sequences, and several applications. While the links of the Cartan geometry to the more easily visible and understandable underlying structures are among the main targets of the first volume, the second one will treat the Cartan connections as given abstract data. This will further underline the algebraic and cohomological character of the available tools and methods.

Suggestions for reading. We have tried to design the book in a way which allows fruitful reading for people with different interests. Readers interested in one or a few specific examples of the geometries covered by the general theory could start reading the parts of the fourth and fifth chapters devoted to the structures in question, and return to the earlier chapters to get background or general results and concepts as needed. Apart from the well–known conformal and projective structures, the book contains extensive material on almost Grassmannian structures, almost quaternionic geometries, CR–geometries and Lagrangean contact geometries, quaternionic contact geometries, low–dimensional distributions, path geometries, and many others. This includes part of their twistor theory, correspondence spaces and further functorial constructions.

Readers familiar with differential geometry and Lie theory, who are interested in the general approach to parabolic geometries might prefer to look at the "Prologue" 1.1 in order to get a sense for the typical examples of the structures in question, inspect briefly the generalities on Cartan connections in 1.5 and then begin a serious reading straight from Chapter 3. If necessary, it should be possible to find background material, concepts and technicalities quickly in Chapters 1 and 2.

Finally, both Chapters 1 and 2 are also intended to be useful as a broader introduction to the subject and have been used successfully as underlying material for various graduate courses, both in Vienna and Brno. Thus we also believe that readers at the graduate student level may enjoy reading the book in the order it is written, with possible glimpses backward and forward for illustrations of the general considerations as they go.

Acknowledgement. Although we heard that delays are common with mathematical books, the work on this one certainly has taken much longer than anyone expected, and we are grateful to the publisher for his patience and support during all of these years. We would also like to mention here many of our collaborators and colleagues who contributed countless suggestions and comments during the ten years of the development of this project. Our particular thanks are due to those colleagues and students of ours who attended our seminars in Vienna and Brno, where most of the contents of this first volume of our book was read, presented and discussed in detail — the project could hardly have been accomplished without their efforts.

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