further, that by the flow property, if $c : I \to G/H$ is a distinguished curve, then for any $t_0 \in I$ there are $g \in G$ and $X \in \mathfrak{n}$ such that $c(t) = p(g \exp((t - t_0)X))$.

PROPOSITION 1.4.11. *Let G be a Lie group, $H \subset G$ a closed subgroup and $\mathfrak{n} \subset \mathfrak{g}$ a linear subspace complementary to $\mathfrak{h}$. Then we have:*

(1) *If $c : I \to M$ is a distinguished curve and $g \in G$ is any element, then $\ell_g \circ c : I \to M$ is a distinguished curve, too.*
(2) *For any point $x \in G/H$ and any tangent vector $\xi \in T_xG/H$ there is at least one distinguished curve $c : \mathbb{R} \to G/H$ such that $c(0) = x$ and $c'(0) = \xi$.*
(3) *If the complement $\mathfrak{n}$ is H–invariant, then the curve c in (2) is uniquely determined. It coincides with the geodesic of the linear connection on $T(G/H)$ induced by $\mathfrak{n}$.*
(4) *For any $g \in G$, the mapping $\mathfrak{n} \to G/H$, $X \mapsto p(g \exp X)$ defines local coordinates around $p(g)$ in which the straight lines through the origin in $\mathfrak{n}$ map to distinguished curves through $p(g)$.*

PROOF. (1) By definition, there are elements $t_0 \in I$, $g_0 \in G$ and $X \in \mathfrak{n}$ such that $c(t) = p(g_0 \exp((t - t_0)X))$. But then by definition of the action ℓ_g, we get $\ell_g(c(t)) = p(gg_0 \exp((t - t_0)X))$, so $\ell_g \circ c$ is distinguished, too.

(2) By (1) it suffices to consider the case $x = o$. Then for $\xi \in T_oG/H$ there is a unique element $X \in \mathfrak{n}$ such that $\xi = T_ep \cdot X$. Thus, $p(\exp(tX))$ is a distinguished curve as required.

(3) Again, we may confine ourselves to the case $x = o$. As above, take $X \in \mathfrak{n}$ such that $T_ep \cdot X = \xi$. Since $p^{-1}(o) = H$, any other distinguished curve through o in direction ξ is of the form $p(h \exp(tY))$ for h in H, with $Y \in \mathfrak{n}$ the unique element such that $T_hp \cdot L_Y(h) = \xi$. But then

$$\xi = T_hp \circ T_e\lambda_h \cdot Y = T_ep \circ T_h\rho^{h^{-1}} \circ T_e\lambda_h \cdot Y = T_ep \cdot \mathrm{Ad}(h)Y.$$

Thus, $X - \mathrm{Ad}(h)Y \in \mathfrak{h}$, but since $\mathfrak{n}$ is H–invariant, we have $\mathrm{Ad}(h)Y \in \mathfrak{n}$ and thus $Y = \mathrm{Ad}(h^{-1})X$. But then $h \exp(t \mathrm{Ad}(h^{-1})X) = hh^{-1} \exp(tX)h$, so $p(h \exp(tY)) = p(\exp(tX))$.

(4) is obvious from the definitions. □

REMARK 1.4.11. We may equivalently define the distinguished curves by H–invariant data at the origin $o \in G/H$. The distinguished curves $c(t)$ with $c(0) = o$ form an H–invariant set, and the entire set of the distinguished curves is obtained from them by the left shifts.

More generally, we may fix any H-invariant subset A of curves $\alpha(t)$, $\alpha(0) = o$ and to define the A–distinguished curves as all curves of the form $\ell_g \circ \alpha$ for $g \in G$ and $\alpha \in A$. In particular, each choice of an H–invariant subspace $\mathfrak{a} \subset \mathfrak{n}$ in the complement $\mathfrak{n}$ to $\mathfrak{h}$ with respect to the induced adjoint action leads to a subclass of distinguished curves emanating in directions contained in the distribution $\mathcal{A} \subset T(G/H)$ determined by the subspace $\mathfrak{a}$.

1.5. Cartan connections

Having the necessary background at hand, we can now start to investigate Cartan geometries. Throughout this section we will take Cartan geometries as a given input and develop basic tools for the analysis of such structures. We will look for simpler structures underlying a Cartan geometry, but we will not touch the question to what extent these structures determine the Cartan geometry. This question will be taken up in the next section. It should, however, be kept in mind that in many cases of interest Cartan geometries are equivalent to more conventional geometric structures and hence the tools developed here provide additional approaches to the study of those structures.

Any Cartan geometry is derived from a homogeneous space, called the homogeneous model of the geometry. The interplay between this homogeneous model and general Cartan geometries of the given type is one of the main general features of Cartan geometries and an important topic for this section.

1.5.1. Basic concepts. Let $H \subset G$ be a Lie subgroup in a Lie group G, and let $\mathfrak{g}$ be the Lie algebra of G. A *Cartan geometry* of type (G, H) on a manifold M is a principal fiber bundle $p : \mathcal{P} \to M$ with structure group H, which is endowed with a $\mathfrak{g}$–valued one–form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the *Cartan connection*. We require that ω is H–equivariant, reproduces the generators of fundamental vector fields, and defines an absolute parallelism. More formally, this means that

$$(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega \text{ for all } h \in H, \tag{1.12}$$

$$\omega(\zeta_X(u)) = X \text{ for each } X \in \mathfrak{h}, \tag{1.13}$$

$$\omega(u) : T_u\mathcal{P} \to \mathfrak{g} \text{ is a linear isomorphism for all } u \in \mathcal{P}. \tag{1.14}$$

The *homogeneous model* for Cartan geometries of type (G, H) is the canonical bundle $p : G \to G/H$ endowed with the left Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$; see 1.2.4. In the terminology of 1.4.1, the homogeneous model for Cartan geometries of type (G, H) is the Klein geometry of that type.

Given a Cartan geometry $(\mathcal{P} \to M, \omega)$, there are the *constant vector fields* $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ defined for all $X \in \mathfrak{g}$ by $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{P}$. From equivariancy of ω we get

$$\omega^{-1}(X)(u \cdot h) = Tr^h \cdot \omega^{-1}(\mathrm{Ad}(h) \cdot X)(u) \tag{1.15}$$

for all $h \in H$. In the case of the homogeneous model, the constant vector field $\omega^{-1}(X)$ is the left invariant field L_X by definition of the Maurer–Cartan form.

The *curvature form* $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{P} \to M, \omega)$ is defined by the structure equation

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]. \tag{1.16}$$

Notice that the Maurer–Cartan equation implies that the Maurer–Cartan form on $G \to G/H$ always has zero curvature. Therefore, the homogeneous model is often also referred to as the flat model, but we avoid this terminology since it is sometimes confusing.

Since the Cartan connection ω trivializes $T\mathcal{P}$, any differential form on $\mathcal{P}$ is determined by its values on the constant vector fields $\omega^{-1}(X)$. Thus, the complete information about K is contained in the *curvature function* $\kappa : \mathcal{P} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ defined by $\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u))$, and so the standard formula for the exterior derivative d yields

$$\kappa(u)(X, Y) = [X, Y] - \omega\big([\omega^{-1}(X), \omega^{-1}(Y)](u)\big). \tag{1.17}$$

LEMMA 1.5.1. *The curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ is horizontal, so the curvature function may be viewed as $\kappa : \mathcal{P} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. Moreover, for all $h \in H$, we get*

$$(r^h)^* K = \mathrm{Ad}(h^{-1}) \circ K, \tag{1.18}$$

$$\kappa \circ r^h = \lambda(h^{-1}) \circ \kappa, \tag{1.19}$$

where λ is the tensor product of the actions $\Lambda^2 \underline{\mathrm{Ad}}^*$ on $\Lambda^2(\mathfrak{g}/\mathfrak{h})^*$ and Ad *on $\mathfrak{g}$.*

PROOF. We just have to imitate the computation from 1.3.3. By definition of a Cartan connection, if $X \in \mathfrak{h}$, then $\omega^{-1}(X) = \zeta_X$, the fundamental vector field. Equivariancy of ω immediately implies that $\mathcal{L}_{\zeta_X}\omega = i_{\zeta_X} d\omega = -\operatorname{ad}(X) \circ \omega$. But this gives

$$d\omega(\omega^{-1}(X), \eta) + [X, \omega(\eta)] = -\operatorname{ad}(X)(\omega(\eta)) + [X, \omega(\eta)] = 0$$

for all X in $\mathfrak{h}$ and all η. Since the fundamental fields span the vertical bundle, we conclude that K is horizontal, and that each $\kappa(u)$ factors to $\Lambda^2(\mathfrak{g}/\mathfrak{h})$.

The equivariancy property (1.18) of K follows directly from the definition and the compatibility of the pullback with the exterior differential d. To prove (1.19), we have to compute $\kappa(u \cdot h)(X, Y)$. By definition, we get

$$\begin{aligned} K(\omega_{u\cdot h}^{-1}(X), \omega_{u\cdot h}^{-1}(Y)) &= K(Tr^h \cdot \omega_u^{-1}(\mathrm{Ad}(h) \cdot X), Tr^h \cdot \omega^{-1}(\mathrm{Ad}(h) \cdot Y)) \\ &= (r^h)^* K(u)(\omega^{-1}(\mathrm{Ad}(h) \cdot X), \omega^{-1}(\mathrm{Ad}(h) \cdot Y)) \\ &= \mathrm{Ad}(h^{-1}) \cdot \kappa(u)(\mathrm{Ad}(h) \cdot X, \mathrm{Ad}(h) \cdot Y). \end{aligned}$$

Passing from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$ the two occurrences of $\mathrm{Ad}(h)$ inside of $\kappa(u)$ get replaced by $\underline{\mathrm{Ad}}(h)$, and we obtain the required formula. □

EXAMPLE 1.5.1. (i) Let $A(m, \mathbb{R})$ be the affine group in dimension m. In 1.3.5 we have seen that a Cartan geometry of type $(A(m, \mathbb{R}), GL(m, \mathbb{R}))$ on an m–dimensional manifold M is equivalent to a linear connection on the tangent bundle TM. Moreover, the curvature K as defined above exactly encodes the curvature and torsion of this linear connection.

(ii) For a Lie group H and an infinitesimally injective homomorphism $H \to GL(m, \mathbb{R})$ consider the affine extension $B = \mathbb{R}^m \rtimes H$. In 1.3.6 we have seen that a Cartan geometry of type (B, H) is equivalent to a first order G–structure with structure group H endowed with a connection. The curvature of the Cartan connection again can be interpreted as curvature and torsion of the induced linear connection on the tangent bundle.

(iii) More specifically, let us consider $H = O(m) \subset GL(m, \mathbb{R})$. Then the affine extension $\mathbb{R}^m \rtimes H$ is the Euclidean group $\mathrm{Euc}(m)$ as used in 1.1.2. By Example (1) of 1.3.6 an $O(m)$–structure on an m–dimensional smooth manifold M is equivalent to a Riemannian metric g on M. From (ii) we thus conclude that a Cartan geometry of type $(\mathrm{Euc}(m), O(m))$ is equivalent to a connection on the orthonormal frame bundle for g and hence to a metric linear connection on TM. It is a classical result that there is a unique metric linear connection on TM, which, in addition, is torsion free, namely the Levi–Civita connection. This shows that on each Riemannian manifold of dimension m, we can actually obtain a canonical Cartan geometry of type $(\mathrm{Euc}(m), O(m))$. The curvature in this case coincides with the usual Riemann curvature.

This is a prototypical example for a Cartan geometry which is determined by an underlying structure. We will analyze this case more systematically in 1.6.1.

The interpretation of Riemannian structures as Cartan geometries is one of the motivating examples for the concept. Interesting applications of this point of view can be found in the book [**Sh97**]. We will often use this case as an illustration in this chapter. One has to keep in mind, however, that the geometric structures that we will be ultimately interested in are much more complicated than Riemannian structures.

(iv) By 1.1.1 and 1.1.2, there is no difference between Cartan geometries of the types $(O(m+1), O(m))$, $(\mathrm{Euc}(m), O(m))$, and $(O(m,1), O(m))$. This is because $\mathfrak{o}(m+1)$, $\mathfrak{euc}(m+1)$, and $\mathfrak{o}(m,1)$ are all isomorphic as $O(m)$–modules. However, the notion of curvature is different for these three types of geometries, since the homogeneous models S^m and $\mathcal{H}^m$ of the first and last types have (nonzero) constant curvature for the second type. This is an example of model mutation; see [**Sh97**, 4, §3].

1.5.2. Categories of Cartan geometries. A *morphism* between two Cartan geometries $(\mathcal{P} \to M, \omega)$ and $(\mathcal{P}' \to M', \omega')$ of type (G,H) is a principal bundle morphism $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi^*\omega' = \omega$. Notice that compatibility with the Cartan connections implies that any tangent map of ϕ is a linear isomorphism, so ϕ and its base map are local diffeomorphisms. With this definition of morphisms, Cartan geometries of type (G,H) form a category $\mathcal{C}_{(G,H)}$.

LEMMA 1.5.2. *Let $\phi : \mathcal{P} \to \mathcal{P}'$ be a morphism of principal fiber bundles which is a local diffeomorphism. If ω' is a Cartan connection on $\mathcal{P}'$, then $\omega = \phi^*\omega'$ is a Cartan connection on $\mathcal{P}$. If ω' and ω are fixed Cartan connections on $\mathcal{P}$ and $\mathcal{P}'$, then ϕ is a morphism $(\mathcal{P},\omega) \to (\mathcal{P}',\omega')$ if and only if ϕ preserves the constant vector fields, i.e. $T\phi \circ \omega^{-1}(X) = {\omega'}^{-1}(X) \circ \phi$. In this case the curvature forms K and K' are ϕ–related, and the curvature functions satisfy $\kappa = \kappa' \circ \phi$.*

PROOF. The fundamental vector fields are given by $\zeta_X(u) = \frac{d}{dt}|_0 u \cdot \exp tX$, so equivariancy of ϕ implies that $\phi^*\omega'$ reproduces the generators of fundamental vector fields. Similarly, equivariancy of ϕ and ω' implies equivariancy of $\phi^*\omega'$. Since ϕ is assumed to be a local diffeomorphism, $\phi^*\omega' = \omega' \circ T\phi$ restricts to a linear isomorphism on each tangent space, so we have verified that $\phi^*\omega'$ is a Cartan connection.

The pullback $\phi^*\omega'$ evaluates on a constant field as

$$\phi^*\omega'(\omega^{-1}(X)(u)) = \omega'(\phi(u))(T_u\phi \cdot \omega^{-1}(X)(u))$$

and the right–hand side equals X if and only if $T\phi \circ \omega^{-1}(X)(u) = {\omega'}^{-1}(X)(\phi(u))$. Thus, morphisms are characterized by the fact that they preserve the constant fields. The relatedness of the curvature forms follows immediately from their definition via the structure equation. Finally, the relation between K and K' implies

$$\begin{aligned}\kappa(u)(X,Y) = K(u)(\omega^{-1}(X), \omega^{-1}(Y)) &= K'(\phi(u))({\omega'}^{-1}(X), {\omega'}^{-1}(Y)) \\ &= \kappa'(\phi(u))(X,Y)\end{aligned}$$

for all $u \in \mathcal{P}$, $X, Y \in \mathfrak{g}$. □

Various interesting and useful subcategories in $\mathcal{C}_{(G,H)}$ can be defined by restrictions on curvatures. Such restrictions are usually necessary to characterize Cartan geometries that are equivalent to simpler structures. The simplest way to restrict curvatures is by requiring the curvature function κ to have values in a fixed subspace $\mathfrak{N} \subset \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. The simple transformation law $\kappa = \kappa' \circ \phi$ immediately implies that this specifies a full subcategory in $\mathcal{C}_{(G,H)}$. However, as we have seen in 1.5.1, $\kappa(u\cdot g)=g^{-1}\cdot\kappa(u)$, so the values of κ always span an H–invariant subset in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathfrak{g}$. Thus, it is natural to require that $\mathfrak{N}$ is an H–submodule. Having chosen a submodule $\mathfrak{N}$, we obtain the full subcategory $\mathcal{C}^{\mathfrak{N}}_{(G,H)}$ of objects whose curvature functions have values in $\mathfrak{N}$.

The appropriate choice of a normalization condition $\mathfrak{N}$ is often a crucial and difficult step in describing geometric structures as Cartan geometries. However, for any pair (G,H) there are two obvious choices available, namely $\mathfrak{N}=0$ and $\mathfrak{N}=\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathfrak{h}$. In the first case we call the geometries *locally flat* while in the other case we talk about *torsion free* Cartan geometries. The choice of the name "torsion free" should be clear from the examples treated in 1.3.5 and 1.3.6, where it amounts to torsion freeness of the induced linear connection on TM. The name "locally flat" is explained by the first part of the proposition below.

Notice that for any Cartan geometry $(\mathcal{P}\to M,\omega)$ and an open subset $U\subset M$ there is a canonical Cartan geometry $(p^{-1}(U)\to U,\omega|_{p^{-1}(U)})$ on U, so one may restrict Cartan geometries to open subsets.

PROPOSITION 1.5.2. *(1) The curvature of a Cartan geometry $(\mathcal{P}\to M,\omega)$ vanishes identically if and only if any point $x\in M$ has an open neighborhood U such that the restriction $(p^{-1}(U)\to U,\omega)$ is isomorphic to the restriction of the homogeneous model $(G\to G/H,\omega_G)$ to an open neighborhood of o.*

(2) If G/H is connected, then the automorphisms of the Cartan geometry $(G\to G/H,\omega_G)$ are exactly the left multiplications by elements of G.

(3) (Liouville theorem) Suppose that G/H is connected. Then any isomorphism between two restrictions of $(G\to G/H,\omega_G)$ to connected open subsets of G/H uniquely globalizes to an automorphism of the homogeneous model.

PROOF. (1) Assume that the curvature vanishes identically. Then Theorem 1.2.4 implies that for each $u\in\mathcal{P}$, there is a neighborhood V of u in $\mathcal{P}$ and a unique mapping $\phi:V\to G$ such that $\phi(u)=e$ and $\phi^*(\omega_G)=\omega$. In particular, ϕ respects the constant fields restricted to V. This implies that for each $v\in V$, $\phi(v\cdot\exp X)=\phi(v)\cdot\exp X$ on a neighborhood of $0\in\mathfrak{g}$, and so ϕ can be extended uniquely to a principal bundle morphism over a neighborhood U of $p(u)$. By equivariancy we still have $\phi^*\omega_G=\omega$ on the whole domain of ϕ. The other implication is obvious.

(2) Again, by Theorem 1.2.4, a smooth map $f:G\to G$ satisfies $f^*\omega_G=\omega_G$ if and only if it is left multiplication by an element of G. Conversely, since left and right multiplications commute, any left multiplication is an automorphism of the principal bundle $G\to G/H$.

(3) Let $p:G\to G/H$ be the projection and consider connected open subsets U and V in G/H and a principal bundle automorphism $\phi:p^{-1}(U)\to p^{-1}(V)$ such that $\phi^*\omega_G=\omega_G$. Viewing ϕ as a map $p^{-1}(U)\to G$, the uniqueness part of Theorem 1.2.4 tells us that ϕ differs from the inclusion by a left multiplication with a fixed element of G, which implies the result. □

REMARK 1.5.2. (1) Part (2) of this proposition shows that the Cartan geometry of type (G,H) on the homogeneous space G/H is a geometric structure which has precisely G as its automorphism group, thus justifying the point of view we took in Section 1.4.

(2) While the proof of the Liouville theorem in part (3) of the proposition is very simple, this is a rather impressive general result. It becomes particularly powerful

for Cartan geometries determined by some underlying structure. A simple example is the case of Euclidean space. In this case the Cartan geometry is determined by the Riemannian structure, and we obtain the result that any isometry between open subsets of Euclidean space is the restriction of a unique Euclidean motion. The classical Liouville theorem is the version of this result for the conformal sphere from 1.1.5; see 1.6.9. Of course, to deduce this from the proposition above, one needs the result that conformal structures are equivalent to a Cartan geometry, which we will prove in Section 1.6 below.

(3) Parts (1) and (3) of the proposition can be used to obtain an alternative description of locally flat Cartan geometries of type (G, H). If $(p : \mathcal{P} \to M)$ is such a geometry, we can use part (1) to obtain a covering of M by open subsets U_i and isomorphisms from $p^{-1}(U_i) \to U_i$ onto restrictions of $G \to G/H$. The base maps of these isomorphisms are diffeomorphisms ϕ_i from the U_i onto open subsets of G/H. Viewing $\{(U_i, \phi_i)\}$ as an atlas, the transition functions are the restrictions of left actions of elements of G by part (3) of Proposition 1.5.2.

Conversely, suppose we have given an atlas for a manifold M such that the images of the charts are open subsets in G/H and the transition functions are restrictions of left actions of elements of G. Then we can pull back the appropriate restrictions of $G \to G/H$ to the domains of the charts and glue them via the isomorphism provided by left translations to a principal H–bundle over M. The pullbacks of the Maurer–Cartan form to these pieces can be glued together to a Cartan connection on this H–bundle. The resulting Cartan geometry on M is by construction locally isomorphic to $G \to G/H$ and hence locally flat.

This construction is particularly transparent in the case that the Cartan geometry is actually equivalent to some underlying structure. For example, an atlas on M with images in open subsets of $\mathbb{R}^n$ such that the transition functions are conformal isometries for the flat metric on $\mathbb{R}^n$ evidently gives rise to a locally flat conformal structure on M.

1.5.3. Rigidity of morphisms. By definition, morphisms between Cartan geometries are special principal bundle homomorphisms, and it is natural to ask to what extent they are determined by the underlying maps between the base manifolds. In the case of the homogeneous model we have seen above that automorphisms are exactly given by left multiplications by elements of G and left multiplication by g corresponds to the base map $\ell_g : G/H \to G/H$. Thus, the automorphisms covering the identity map are the actions of the elements of the kernel K of the Klein geometry introduced in 1.4.1. There we saw that K is the maximal normal subgroup of G that is contained in H and its Lie algebra $\mathfrak{k}$ is the maximal ideal of $\mathfrak{g}$ which is contained in $\mathfrak{h}$.

Recall also from 1.4.1 that the Klein geometry of type (G, H) is called effective if its kernel K is trivial and infinitesimally effective if K is discrete. Surprisingly, the kernel K also determines the maximal number of morphisms covering a fixed base map in the case of general Cartan geometries of type (G, H). In particular, if (G, H) is effective, then any morphism is uniquely determined by its base map. We shall prove slightly more than this following [**Sh97**, Chapter 5]:

PROPOSITION 1.5.3. *Let G be a Lie group, $H \subset G$ a closed subgroup such that G/H is connected, and let $K \subset H$ be the kernel of the Klein geometry (G, H). Let ϕ_1 and ϕ_2 be two morphisms between two Cartan geometries $(\mathcal{P} \to M, \omega)$ and $(\mathcal{P}' \to M', \omega')$ of type (G, H) which cover the same base mapping $f : M \to M'$.*

Then there is a smooth map $\psi : \mathcal{P} \to K$ such that $\phi_2(u) = \phi_1(u) \cdot \psi(u)$ for all $u \in \mathcal{P}$.

In particular, if (G,H) is effective, then $\phi_1 = \phi_2$, and if (G,H) is infinitesimally effective, then ψ is constant on connected components of M.

Proof. Since the statement is local, we may assume that both ϕ_1 and ϕ_2 are diffeomorphisms, and then by assumption $\phi = \phi_2^{-1} \circ \phi_1$ covers the identity mapping on M and $\phi^*\omega = \omega$. Thus, we may, without loss of generality, assume that $\mathcal{P} = \mathcal{P}'$, $\phi := \phi_1$ covers the identity id_M, and $\phi_2 = \mathrm{id}_{\mathcal{P}}$. These assumptions imply that there is a smooth map $\psi : \mathcal{P} \to H$ such that $\phi(u) = u \cdot \psi(u)$, and to prove the result we have to show that $\psi(u) \in K$ for all u.

Let us compute $\phi^*\omega$ in terms of ψ. If c is a smooth curve in $\mathcal{P}$ with $c(0) = u$ and $c'(0) = \xi$, then $\phi^*\omega(\xi)$ is obtained as the evaluation of ω on the vector defined by the curve

$$t \mapsto r(c(t), \psi(c(t))) = r\big(c(t), \psi(u) \cdot (\psi(u)^{-1}\psi(c(t)))\big),$$

where r is the principal right action of H on $\mathcal{P}$. Thus,

$$\phi^*\omega(\xi) = \omega(Tr^{\psi(u)} \cdot \xi) + \omega(\zeta_Z(u \cdot \psi(u))),$$

where $Z = \frac{d}{dt}|_0(\psi(u)^{-1} \cdot \psi(c(t))) \in \mathfrak{h}$. Since the Cartan connection reproduces the generators of fundamental vector fields, this shows that the whole second summand equals $\psi^*\omega_H$, where ω_H is the Maurer–Cartan form on H. Altogether we have proved

$$\phi^*\omega(u) = \mathrm{Ad}_{\psi(u)^{-1}} \circ \omega(u) + \psi^*\omega_H(u).$$

By our assumptions, $\phi^*\omega = \omega$ and so we conclude

$$(\mathrm{Ad}_{\psi(u)^{-1}} - \mathrm{id}_{\mathfrak{g}})(X) = -\psi^*\omega_H(\omega^{-1}(X)(u)) \tag{1.20}$$

for all $X \in \mathfrak{g}$.

For any Lie subalgebra $\mathfrak{q} \subset \mathfrak{h}$ we write

$$K_{\mathfrak{q}} = \{h \in H : \mathrm{Ad}(h^{-1})(X) - X \in \mathfrak{q} \text{ for all } X \in \mathfrak{g}\}.$$

Since

$$\mathrm{Ad}((hg)^{-1})(X) - X = (\mathrm{Ad}(g^{-1})(\mathrm{Ad}(h^{-1})(X)) - \mathrm{Ad}(h^{-1})(X)) + (\mathrm{Ad}(h^{-1})(X) - X)$$

is in $\mathfrak{q}$ whenever both g and h belong to $K_{\mathfrak{q}}$, the subset $K_{\mathfrak{q}}$ is a closed subgroup and thus a Lie subgroup of H. Now we define a series of Lie subgroups of H by $K_0 = H$ and inductively by $K_i = K_{\mathfrak{k}_{i-1}}$, where $\mathfrak{k}_{i-1}$ is the Lie algebra of K_{i-1}, for all $i = 1, 2, \dots$.

Assume that K_i is normal in H for some i. Then $\mathrm{Ad}(h)(\mathfrak{k}_i) \subset \mathfrak{k}_i$ for all $h \in H$ and so

$$\mathrm{Ad}(h^{-1}gh)(X) - X = \mathrm{Ad}(h^{-1})\big(\mathrm{Ad}(g)(\mathrm{Ad}(h)(X)) - \mathrm{Ad}(h)(X)\big) \in \mathrm{Ad}(h^{-1})(\mathfrak{k}_i) \subset \mathfrak{k}_i$$

for all $g \in K_{i+1}$, $h \in H$. Thus, K_{i+1} is normal in H, too. Since $K_0 = H$ is normal, this implies that all K_i, $i = 1, 2, \dots$ are normal Lie subgroups in H.

But now assume that our function ψ has values in a subgroup $Q \subset H$. Then visibly the right–hand side of equation (1.20) lies in the Lie algebra $\mathfrak{q}$ of Q. But for the left–hand side of (1.20), lying in $\mathfrak{q}$ exactly means that $\psi(u) \in K_{\mathfrak{q}}$. Starting from the fact that ψ has values in $K_0 = H$ we conclude that ψ has values in $K_{\mathfrak{h}} = K_1$, thus in K_2, and so on. Therefore, ψ has values in the intersection $K_\infty = \bigcap_{i=0}^\infty K_i$. Moreover, we clearly have $K_i \supset K_{i+1}$ for all i and so the chain of subalgebras

$\mathfrak{h} = \mathfrak{k}_0 \supset \mathfrak{k}_1 \supset \dots$ has to stabilize at some finite i. Hence, K_∞ coincides with some K_i and therefore is a normal Lie subgroup of H. Let us write $\mathfrak{k}_\infty$ for its Lie algebra and notice

$$K_\infty = \{h \in H : \mathrm{Ad}(h)(X) - X \in \mathfrak{k}_\infty \text{ for all } X \in \mathfrak{g}\}.$$

For $Y \in \mathfrak{k}_\infty$ we have $\mathrm{Ad}(\exp tY)(X) - X \in \mathfrak{k}_\infty$ for all $X \in \mathfrak{g}$. Differentiating this at $t = 0$ shows that $[Y, X] \in \mathfrak{k}_\infty$ for all $X \in \mathfrak{g}$, so $\mathfrak{k}_\infty$ is an ideal in $\mathfrak{g}$. This implies that $\mathrm{Ad}(\exp(X))(\mathfrak{k}_\infty) \subset \mathfrak{k}_\infty$ for all $X \in \mathfrak{g}$. As above, one shows that $\exp(X)K_\infty \exp(-X) = K_\infty$. Since G/H is connected, elements of this form together with elements of H generate G. Since we already know that K_∞ is normal in H, we conclude that K_∞ is normal in G, which completes the proof. □

Remark 1.5.3. It is instructive to look at this result in the case of affine connections on first order G–structures, where a simpler proof is available. Consider a homomorphism $j : H \to GL(m, \mathbb{R})$ such that j' is injective, and let $B = \mathbb{R}^m \rtimes H$ be the corresponding affine extension. The kernel of the Klein geometry (B, H) is simply the kernel of j, so this Klein geometry is always infinitesimally effective and it is effective if H is actually a virtual Lie subgroup of $GL(m, \mathbb{R})$. Given a principal H–bundle $p : \mathcal{P} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{b})$, we get a homomorphism $\mathcal{P} \to \mathcal{P}^1 M$ to the first order frame bundle of M. This homomorphism is characterized by the fact that the pullback of the soldering form on $\mathcal{P}^1 M$ is the form $\pi \circ \omega \in \Omega^1(\mathcal{P}, \mathbb{R}^m)$, where $\pi : \mathfrak{b} \to \mathfrak{b}/\mathfrak{h} \cong \mathbb{R}^m$ is the canonical surjection.

If $(\mathcal{P} \to M, \omega)$ and $(\mathcal{P}' \to M', \omega')$ are two such structures, and $\Phi : \mathcal{P} \to \mathcal{P}'$ is a morphism, then equivariancy of Φ implies that one gets an induced morphism between the images in $\mathcal{P}^1 M$ and $\mathcal{P}^1 M'$, which can be uniquely extended to a morphism $\tilde{\Phi} : \mathcal{P}^1 M \to \mathcal{P}^1 M'$. From $\Phi^* \omega' = \omega$ one immediately concludes $\tilde{\Phi}$ pulls back the soldering form θ' on $\mathcal{P}^1 M'$ to the soldering form θ on $\mathcal{P}^1 M$. But by definition of the soldering form, this means that $\tilde{\Phi}$ is induced by composition with the tangent map Tf of the base map $f : M \to M'$. This shows that $\tilde{\Phi}$ is uniquely determined by f, and hence Φ is determined up to a smooth function with values in the kernel K.

1.5.4. Local description of Cartan connections. To make contact to the classical literature on Cartan connections, let us describe them in a local picture. Given a Cartan geometry $(p : \mathcal{P} \to M, \omega)$ the point about this approach is to build up and/or study the pullback of the Cartan connection ω along local smooth sections of $\mathcal{P}$. In the classical literature, the bundle $\mathcal{P}$ is often not spelled out explicitly. Rather than that one starts with a certain class of local frames or coframes of the tangent bundle of M or of some auxiliary bundle. The class of frames is usually defined by pointwise conditions, and then admissible frames are parametrized by a Lie group H. This group has to be independent of the point. This exactly means that the admissible frames or coframes form a principal subbundle with structure group H in some frame bundle.

To begin, let us assume that $p : \mathcal{P} \to M$ and $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ are given. For an open subset $U \subset M$ and a local smooth section $\sigma : U \to \mathcal{P}$, consider $\sigma^*\omega \in \Omega^1(U, \mathfrak{g})$. In many classical treatments of Cartan connections, $\mathfrak{g}$ is realized as a Lie algebra of matrices, and $\sigma^*\omega$ is viewed as a matrix of real–valued one–forms rather than a one–form with values in a matrix algebra. Any other section over U is of the form $\hat{\sigma}(x) = \sigma(x) \cdot \psi(x)$ for a smooth function $\psi : U \to H$. As in the proof of Proposition

1.5.3, one shows that for $x \in U$ and $\xi \in T_xM$ one has

$$\hat{\sigma}^*\omega(\xi) = \operatorname{Ad}(\psi(x)^{-1})(\sigma^*\omega(\xi)) + \delta\psi(\xi), \tag{1.21}$$

where $\delta\psi = \psi^*\omega_H \in \Omega^1(U, \mathfrak{h})$ is the left logarithmic derivative of $\psi : U \to H$.

Likewise, we can pull back the curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ along σ to obtain $\sigma^*K \in \Omega^2(U, \mathfrak{g})$. By definition, we have

$$\sigma^*K(\xi, \eta) = d\sigma^*\omega(\xi, \eta) + [\sigma^*\omega(\xi), \sigma^*\omega(\eta)],$$

so we obtain the usual structure equation. Changing from σ to $\hat{\sigma}$, the transformation law for the curvature is much easier than the one for the connection. Indeed, since $\delta\psi = \psi^*\omega_H$, it satisfies the Maurer–Cartan equation, and thus this part of the transformation law for the connection does not contribute to the change of curvature. Using that the adjoint action is by Lie algebra homomorphisms, one gets

$$\hat{\sigma}^*K(\xi, \eta) = \operatorname{Ad}(\psi(x)^{-1})(\sigma^*K(\xi, \eta))$$

if $\hat{\sigma}(x) = \sigma(x) \cdot \psi(x)$.

This picture can also be used to construct Cartan connections. One simply tries to associate to an admissible frame (i.e., a local section σ of $\mathcal{P}$) a one–form $\omega_\sigma \in \Omega^1(U, \mathfrak{g})$ (respectively an appropriate matrix of one–forms). Now $U \times H \cong p^{-1}(U)$ via $(x, h) \mapsto \sigma(x) \cdot h$. Using this, it is easy to see that there is a unique one–form $\omega \in \Omega^1(p^{-1}(U), \mathfrak{g})$ which is H–equivariant, reproduces the generators of fundamental vector fields and satisfies $\sigma^*\omega = \omega_\sigma$. Given two local sections with domains overlapping in $V \subset M$, the associated forms induce the same Cartan connection over V, if and only if they transform according to (1.21) for the function ψ relating the two frames.

Associating appropriate matrices of one–forms with the right transformation law to admissible local sections is therefore a way to construct Cartan connections. Having done this, the Cartan curvature can be computed locally as described above.

Finally, one can also analyze morphisms in this picture. One starts with a local diffeomorphism $f : M \to \tilde{M}$ between the bases which pulls back admissible frames to admissible frames. Starting with a local admissible frame $\tilde{\sigma}$ on $\tilde{M}$, one can pull back the corresponding form $\tilde{\omega}_{\tilde{\sigma}}$ along f. Denoting by σ the frame obtained by pulling back $\tilde{\sigma}$ along f, one then has to check whether there is a smooth function ψ for which ω_σ is related to $f^*\tilde{\omega}_{\tilde{\sigma}}$ according to (1.21). If this always works, then f defines a morphism of Cartan geometries.

Doing this in practice often leads to an efficient calculus. Let us sketch this in the case of Riemannian structures. Here one starts with a local orthonormal coframe $\{\sigma^1, \dots, \sigma^n\}$ for an n–dimensional Riemannian manifold M. To construct the Cartan connection with values in $\mathfrak{euc}(n) \cong \mathbb{R}^n \oplus \mathfrak{o}(n)$, one takes the σ^i as the $\mathbb{R}^n$–component. The $\mathfrak{o}(n)$–component then can be written as a family $\omega^i{}_j$ of one–forms such that $\omega^j{}_i = -\omega^i{}_j$.

The Lie bracket in $\mathfrak{euc}(n)$ reads as $[(v, A), (w, B)] = (Aw - Bv, AB - BA)$ for $v, w \in \mathbb{R}^n$ and $A, B \in \mathfrak{o}(n)$. Considering the form $(\sigma^i, \omega^i{}_j)$, the resulting expression for the curvature applied to ξ and η has $\mathbb{R}^n$–component

$$d\sigma^i(\xi, \eta) + \textstyle\sum_j (\omega^i{}_j(\xi)\sigma^j(\eta) - \omega^i{}_j(\eta)\sigma^j(\xi))$$

and $\mathfrak{o}(n)$ component

$$d\omega^i{}_j(\xi, \eta) + \textstyle\sum_k (\omega^i{}_k(\xi)\omega^k{}_j(\eta) - \omega^i{}_k(\eta)\omega^k{}_j(\xi)).$$

Using an analog of the Einstein sum convention, this is usually phrased by saying that the torsion and curvature associated to ${\omega^i}_j$ are represented by the forms $d\sigma^i + {\omega^i}_j \wedge \sigma^j$, respectively, $d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j$.

One then proves that, given a local orthonormal coframe $\{\sigma^1, \dots, \sigma^n\}$, there are unique one–forms ${\omega^i}_j$ such that ${\omega^j}_i = -{\omega^i}_j$ and $d\sigma^i + {\omega^i}_j \wedge \sigma^j = 0$. From this it already follows that the forms ${\omega^i}_j$ transform appropriately: For a second local orthonormal coframe $\{\hat\sigma^1, \dots, \hat\sigma^n\}$, there is an orthogonal matrix ${A^i}_j$ of smooth functions such that $\hat\sigma^i = {A^i}_j\sigma^j$. Denoting by $({B^i}_j)$ the inverse matrix to $({A^i}_j)$ we compute

$$\begin{aligned} d\hat\sigma^i &= d{A^i}_j \wedge \sigma^j + {A^i}_j d\sigma^j \\ &= d{A^i}_j \wedge {B^j}_k\hat\sigma^k - {A^i}_j{\omega^j}_k \wedge \sigma^k \\ &= ({B^j}_k d{A^i}_j - {A^i}_j{\omega^j}_\ell {B^\ell}_k) \wedge \hat\sigma^k. \end{aligned}$$

To interpret this, observe that ${A^i}_j\sigma^j$ is the $\mathbb{R}^n$–component of $\mathrm{Ad}({A^i}_j)(\sigma^i, {\omega^i}_j)$. Consequently, the frame $\hat\sigma$ is obtained from σ by the right action of ${B^i}_j = ({A^i}_j)^{-1}$. Moreover, since ${B^j}_k{A^i}_j = \delta^i_k$ we see that ${B^j}_k d{A^i}_j = -{A^i}_j d{B^j}_k$. Apart from the sign, this is exactly the well–known expression $B^{-1}dB$ for the left logarithmic derivative of B. Then the above equation indeed reads as $\hat\omega = \delta B + \mathrm{Ad}(B^{-1}) \circ \omega$ as required.

Having the forms ${\omega^i}_j$, the Riemannian curvature in the coframe σ can be computed as $d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j \in \Omega^2(M, \mathfrak{o}(TM))$.

1.5.5. Natural bundles. Let us fix a Klein geometry (G, H) and consider the category $\mathcal{C}_{(G,H)}$ of Cartan geometries of type (G, H). The most general definition of a natural bundle in this setting is as a functor which associates to each object $(\mathcal{P} \to M, \omega)$ a fiber bundle $FM \to M$ and to any morphism Φ from $(\mathcal{P} \to M, \omega)$ to $(\mathcal{P}' \to M', \omega')$ covering $f : M \to M'$ a fiber bundle morphism $F\Phi : FM \to FM'$ covering f. This is the concept of *gauge natural bundles* as studied in [**KMS**, Chapter XII].

In the case of Cartan geometries a much simpler concept of natural bundles is sufficient for most purposes. Suppose that F is a natural bundle in the above sense and consider the value $F(G/H)$ on the homogeneous model $(G \to G/H, \omega_G)$. From 1.5.2 we know that the left multiplications by elements $g \in G$ are exactly the automorphisms of this Cartan geometry. Applying F we obtain an action of G on $F(G/H)$ by bundle automorphisms, which lifts the canonical action on G/H. This makes $F(G/H)$ into a homogeneous bundle in the sense of 1.4.2. By Proposition 1.4.3 we get an H–action on the standard fiber $S := F_o(G/H)$ which determines the homogeneous bundle $F(G/H)$ up to isomorphism.

Given the left action of H on S, we can map each Cartan geometry $(\mathcal{P} \to M, \omega)$ of type (G, H) to the associated bundle $\mathcal{P} \times_H S$. This defines a natural bundle on $\mathcal{C}_{(G,H)}$ by the functorial properties of the associated bundle construction; see 1.2.7. Such natural bundles will be called *natural bundles associated to the Cartan bundle*. If not explicitly stated otherwise, in the sequel we will only consider natural bundles of that type.

Formally, natural bundles associated to the Cartan bundle depend only on the principal bundle $\mathcal{P} \to M$ and not the Cartan connection ω. The Cartan connection is, however, necessary to identify natural bundles associated to the Cartan bundle with more traditional geometric objects, for example, with tensor bundles. From

1.4.3 we know that in the case of the homogeneous model $(G \to G/H, \omega_G)$ the tangent bundle is the associated bundle $G \times_H (\mathfrak{g}/\mathfrak{h})$ (and the Maurer–Cartan form is used to identify the two bundles).

The same identification works for a general Cartan geometry $(\mathcal{P} \to M, \omega)$. Consider the mapping $\mathcal{P} \times \mathfrak{g} \to TM$ defined by $(u, X) \mapsto T_u p \cdot \omega_u^{-1}(X)$. For $X \in \mathfrak{h}$, the field $\omega^{-1}(X)$ is the fundamental field ζ_X and thus vertical, so this map factors to $\mathcal{P} \times (\mathfrak{g}/\mathfrak{h})$ and fixing u, one gets a linear isomorphism $\mathfrak{g}/\mathfrak{h} \to T_{p(u)}M$. Equivariancy of ω immediately implies that this factors to a bundle map $\mathcal{P} \times_H (\mathfrak{g}/\mathfrak{h}) \to TM$, where the H–action $\underline{\mathrm{Ad}}$ on $\mathfrak{g}/\mathfrak{h}$ is induced by the restriction of the adjoint action of G. This map induces a linear isomorphism in each fiber and covers the identity on M, and thus is an isomorphism of vector bundles. The cotangent bundle T^*M may then be identified with the natural bundle corresponding to $(\mathfrak{g}/\mathfrak{h})^*$ and similarly for arbitrary tensor bundles.

As in the case of the homogeneous model, there is a way to view this as an underlying G–structure. Consider the action $\underline{\mathrm{Ad}} : H \to GL(\mathfrak{g}/\mathfrak{h})$ from above and let $H^1 \subset H$ be the kernel of this homomorphism, which is a closed normal subgroup of H. In particular, we can form the quotient group $H_0 := H/H^1$ and the space $\mathcal{P}_0 := \mathcal{P}/H^1$ is naturally a principal bundle over M with structure group H_0. There is an evident projection $p_0 : \mathcal{P} \to \mathcal{P}_0$. Now define $\theta \in \Omega^1(\mathcal{P}_0, \mathfrak{g}/\mathfrak{h})$ as follows: for $u_0 \in \mathcal{P}_0$ and $\xi \in T_{u_0}\mathcal{P}_0$ choose a point $u \in \mathcal{P}$ over u_0 and a tangent vector $\tilde{\xi} \in T_u\mathcal{P}$ such that $T_{u_0}p_0 \cdot \tilde{\xi} = \xi$, and put $\theta(\xi) := \omega(\tilde{\xi}) + \mathfrak{h}$.

Two choices for $\tilde{\xi}$ differ by a vector which is vertical for $\mathcal{P} \to M$, so this choice plays no role. Any other choice for the point u is of the form $u \cdot h$ with $h \in H^1$ and we may choose $Tr^h \cdot \tilde{\xi}$ as the lift of ξ. Equivariancy of ω then implies that $\omega(Tr^h \cdot \tilde{\xi}) = \mathrm{Ad}(h^{-1})(\omega(\tilde{\xi}))$ and since $h \in H^1$ we conclude that θ is well defined and strictly horizontal. Similarly, one verifies that θ is H_0–equivariant and using a local smooth section of $\mathcal{P} \to \mathcal{P}_0$ one shows that θ is smooth. Thus, $(\mathcal{P}_0 \to M, \theta)$ is a first order G–structure with structure group H_0; see 1.3.6. Left actions of H_0 are the same thing as left actions of H such that H^1 acts trivially, and the former correspond to natural bundles for G–structures with structure group H_0. Via the Cartan connection ω one can view any such bundle as a natural bundle for Cartan geometries of type (G, H).

The relation to the homogeneous model continues to work in the question of natural sections. Given a natural bundle F on the category $\mathcal{C}_{(G,H)}$ a *natural section* is a family of smooth sections $\sigma_M : M \to FM$ of the values such that for any morphism $\Phi : (\mathcal{P} \to M, \omega) \to (\mathcal{P}' \to M', \omega')$ covering $f : M \to M'$ we have $F\Phi \circ \sigma_M = \sigma_{M'} \circ f$. Looking at automorphisms of the homogeneous model, we see that for any natural section, the section $\sigma_{G/H}$ must be a G–invariant section of the homogeneous bundle $F(G/H)$ as introduced in 1.4.4. By Theorem 1.4.4 such sections are in bijective correspondence with H–invariant elements in the standard fiber S.

But for any H–invariant element $s_0 \in S$ and any Cartan geometry $(p : \mathcal{P} \to M, \omega)$ of type (G, H) we may define $\sigma_M : M \to \mathcal{P} \times_H S$ by $\sigma_M(x) := [\![u, s_0]\!]$ where $u \in \mathcal{P}$ is any point such that $p(u) = x$. Invariance of s_0 implies that this is well defined and using local smooth sections of $\mathcal{P}$ we immediately see that σ_M is smooth. Clearly, the family σ_M defines a natural section. For example, there exists a natural Riemannian metric on Cartan geometries of type (G, H) if and only if there is a

G–invariant Riemannian metric on G/H and any choice of a G–invariant metric on G/H canonically extends to a natural Riemannian metric.

In 1.4.5 we have discussed the existence of G–invariant principal connections on the bundle $G \to G/H$. We have shown that such a connection is equivalent to an H–invariant subspace $\mathfrak{n} \subset \mathfrak{g}$, which is complementary to $\mathfrak{h}$. In particular, such a connection exists if and only if the Klein geometry (G, H) is reductive. Suppose that we have chosen an H–invariant decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. Then any Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ splits as $\omega = \omega_{\mathfrak{n}} + \omega_{\mathfrak{h}}$ and it follows immediately from the definitions that $\omega_{\mathfrak{h}}$ is a principal connection on $\mathcal{P}$. Forming induced connection, we see that there is a natural connection on any natural bundle associated to the Cartan bundle.

1.5.6. Natural connections. We have just seen that an invariant principal connection on $G \to G/H$ gives rise to a principal connection on the principal H–bundle $\mathcal{P}$ for any Cartan geometry $(p : \mathcal{P} \to M, \omega)$ of type (G, H). Via induced connections, one obtains natural (linear) connections on all natural (vector) bundles associated to the Cartan bundle. We next extend this to general invariant principal connections on homogeneous principal bundles as discussed in 1.4.5.

By Lemma 1.4.5 any homogeneous principal K–bundle over G/H is of the form $G \times_i K$ for a homomorphism $i : H \to K$. This is the associated bundle with respect to the action of H on K defined by $h \cdot k := i(h)k$. From 1.5.5 we know that this extends to a natural bundle on the category of Cartan geometries of type (G, H). Given a Cartan geometry $(\mathcal{P} \to M, \omega)$, one simply has to form the associated bundle $\mathcal{P} \times_i K \to M$. This is again a K–principal bundle with principal action induced by multiplication from the right.

In Theorem 1.4.5 we have shown that invariant principal connections on $G \times_i K$ are in bijective correspondence with linear maps $\alpha : \mathfrak{g} \to \mathfrak{k}$ which satisfy two conditions. First, one has to assume that $\alpha|_{\mathfrak{h}} = i'$, the derivative of i. Second, α has to be equivariant, i.e. $\alpha \circ \mathrm{Ad}(h) = \mathrm{Ad}(i(h)) \circ \alpha$ for all $h \in H$. Given such a map, we can define a natural principal connection on the category of Cartan geometries of type (G, H). Observe first that there is an obvious map $j : \mathcal{P} \to \mathcal{P} \times_i K$ induced by mapping $u \in \mathcal{P}$ to the class of (u, e).

THEOREM 1.5.6. *Let G and K be Lie groups, $H \subset G$ a closed subgroup, $i : H \to K$ a homomorphism and $\alpha : \mathfrak{g} \to \mathfrak{k}$ a linear map satisfying conditions (i) and (ii) from Theorem 1.4.5.*

(1) For any Cartan geometry $(p : \mathcal{P} \to M, \omega)$ of type (G, H), there is a unique principal connection γ_α on $\mathcal{P} \times_i K$ such that $j^*\gamma_\alpha = \alpha \circ \omega \in \Omega^1(\mathcal{P}, \mathfrak{k})$.

(2) The assignment from (1) is functorial, i.e. any morphism of Cartan geometries induces a morphism of principal bundles which is compatible with the principal connections.

PROOF. (1) Let us write π for the projection $\mathcal{P} \times_i K \to M$. Since $\pi \circ j = p$ we see that for a point $u \in \mathcal{P}$ the tangent space $T_{j(u)}(\mathcal{P} \times_i K)$ is spanned by vertical vectors and elements of $T_u j(T_u \mathcal{P})$. Hence, there is only one possible definition for $\gamma_\alpha(j(u))$:

$$\gamma_\alpha(j(u))(T_u j \cdot \xi + \zeta_A(j(u))) := \alpha(\omega(u)(\xi)) + A \tag{1.22}$$

for $\xi \in T_u \mathcal{P}$ and $A \in \mathfrak{k}$. We have to show that this is well defined. If $T_u j \cdot \xi$ is vertical, then $T\pi \circ Tj \cdot \xi = Tp \cdot \xi = 0$, so $\xi = \zeta_X(u)$ for some $X \in \mathfrak{h}$. By definition of j, we have $j(u\cdot h)=j(u)\cdot i(h)$. Putting $h=\exp(tX)$ and differentiating at $t=0$ we see that $T_uj\cdot\zeta_X(u)=\zeta_{i'(X)}(j(u))$. Since $\alpha(X)=i'(X)$ for $X\in\mathfrak{h}$ by property (i), we see that (1.22) uniquely defines a linear map $T_{j(u)}(\mathcal{P}\times_i K)\to\mathfrak{k}$.

From the definition, we see that $\gamma_\alpha(j(u))$ reproduces the generators of fundamental vector fields. To ensure equivariancy, we next have to define

$$\gamma_\alpha(j(u)\cdot k)(\eta)=\operatorname{Ad}(k^{-1})(\gamma_\alpha(j(u))(Tr^{k^{-1}}\cdot\eta)). \tag{1.23}$$

To verify that this is well defined, suppose that $j(u)\cdot k=j(\hat{u})\cdot\hat{k}$. Projecting to G/H, we see that $\hat{u}=u\cdot h$ for some $h\in H$. Then $j(\hat{u})\cdot\hat{k}=j(u)\cdot(i(h)\hat{k})$, so $\hat{k}=i(h^{-1})k$. Writing the right–hand side of (1.23) in terms of $\hat{u}$ and $\hat{k}$, we get

$$\operatorname{Ad}(k^{-1})\operatorname{Ad}(i(h))\left(\gamma_\alpha(j(u\cdot h))(Tr^{i(h)}\cdot Tr^{k^{-1}}\cdot\eta)\right).$$

To see that γ_α is well defined, we only have to verify that for all η in $T_{j(u)}(\mathcal{P}\times_i K)$ we have

$$\gamma_\alpha(j(u\cdot h))(Tr^{i(h)}\cdot\eta)=\operatorname{Ad}(i(h)^{-1})(\omega(j(u))(\eta)).$$

If $\eta=\zeta_A(j(u))$ for some $A\in\mathfrak{k}$, then this immediately follows from equivariancy of the fundamental vector fields. On the other hand, if $\eta=T_uj\cdot\xi$ for some $\xi\in T_u\mathcal{G}$, then $T_{j(u)}r^{i(h)}\cdot T_uj\cdot\xi=T_{u\cdot h}j\cdot T_ur^h\cdot\xi$, and $\omega(u\cdot h)(T_ur^h\cdot\xi)=\operatorname{Ad}(h^{-1})(\omega(u)(\xi))$, and the result follows from the equivariancy property (ii) of α.

Hence, we have constructed $\gamma_\alpha\in\Omega^1(\mathcal{P}\times_i K,\mathfrak{k})$. By construction, $j^*\gamma_\alpha=\alpha\circ\omega$ and, as a principal connection form, γ_α is uniquely determined by this property. From the definition in (1.23) it follows immediately that $(r^k)^*\gamma_\alpha=\operatorname{Ad}(k^{-1})\circ\gamma_\alpha$. Again by definition, $\gamma_\alpha(j(u))$ reproduces generators of fundamental vector fields, so by equivariancy, this holds on all of $\mathcal{P}\times_i K$.

(2) Let $\Phi:(\mathcal{P}\to M,\omega)\to(\tilde{\mathcal{P}}\to\tilde{M},\tilde{\omega})$ be a morphism of Cartan geometries. Then by definition $\Phi:\mathcal{P}\to\tilde{\mathcal{P}}$ is a principal bundle map. Hence, $\Phi\times\mathrm{id}_K$: $\mathcal{P}\times K\to\tilde{\mathcal{P}}\times K$ induces a principal bundle map $F(\Phi):\mathcal{P}\times_i K\to\tilde{\mathcal{P}}\times_i K$. Denoting by $\tilde{j}:\tilde{\mathcal{P}}\to\tilde{\mathcal{P}}\times_i K$ the natural map, by definition we get $F(\Phi)\circ j=\tilde{j}\circ\Phi$. Denoting by $\tilde{\gamma}_\alpha$ the connection on $\tilde{\mathcal{P}}\times_i K$ constructed according to (1), we can form the pullback $F(\Phi)^*\tilde{\gamma}_\alpha$. Since $F(\Phi)$ is a principal bundle homomorphism, this is a principal connection on $\tilde{\mathcal{P}}\times_i K$. Now we compute

$$j^*F(\Phi)^*\tilde{\gamma}_\alpha=\Phi^*\tilde{j}^*\tilde{\gamma}_\alpha=\Phi^*(\alpha\circ\tilde{\omega})=\alpha\circ\Phi^*\tilde{\omega}=\alpha\circ\omega.$$

But by part (1), γ_α is the unique principal connection which is pulled back to $\alpha\circ\omega$ along j, so $F(\Phi)^*\tilde{\gamma}_\alpha=\gamma_\alpha$. □

Via associated bundles and induced connections we conclude that any invariant connection on a homogeneous bundle over G/H extends to a natural connection on the corresponding natural bundle on Cartan geometries of type (G,H). Let us describe this for vector bundles and linear connections. Given a homogeneous vector bundle $E\to G/H$, let E_o be the fiber over $o=eH$. Then by 1.4.3 we obtain a representation ρ of H on E_o and $E\cong G\times_H E_o$. In 1.4.7 we have seen that the frame bundle of E is $G\times_\rho GL(E_o)$, and homogeneous linear connections on E are equivalent to homogeneous principal connections on the frame bundle. For a Cartan geometry $(p:\mathcal{P}\to M,\omega)$ of type (G,H), we evidently have $(\mathcal{P}\times_\rho GL(E_o))\times_{GL(E_o)}E_o=\mathcal{P}\times_H E_o$. Starting from an invariant linear connection on E, the theorem gives us a natural principal connection on $\mathcal{P}\times_\rho GL(E_o)$ and hence a linear connection on $\mathcal{P} \times_H E$. Naturality of this connection follows from functoriality of the construction of associated bundles.

1.5.7. Tractor bundles. Starting from an arbitrary Klein geometry (G, H), there is always a class of homogeneous bundles which do admit canonical invariant connections. By 1.5.6 this leads to natural connections on the corresponding bundles on Cartan geometries of type (G, H). We have not studied these bundles in 1.4 since they are canonically trivial on G/H, so the existence of a G–invariant connection is obvious. However, on general Cartan geometries these bundles are nontrivial and the canonical connections on them become a highly interesting and fundamental tool.

The idea here is very simple. In the setting of Theorem 1.4.5, we can simply use the inclusion $i : H \to G$ and $\alpha = \mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$. This gives rise to a homogeneous principal connection on the principal bundle $G \times_H G \to G/H$. This extended principal bundle is canonically trivial: The map $G \times G \to (G/H) \times G$ defined by $(g, g') \mapsto (gH, gg')$ factors to an isomorphism $G \times_H G \to (G/H) \times G$ of homogeneous principal bundles. The natural connection on $G \times_H G$ is the pullback along this isomorphism of the canonical flat connection on the product bundle.

Theorem 1.5.6 then implies that for any Cartan geometry $(\mathcal{P} \to M, \omega)$ there is a natural principal connection on the *extended principal bundle* $\tilde{\mathcal{P}} := \mathcal{P} \times_H G \to M$, which is a principal G–bundle. Correspondingly, there are natural connections on all natural bundles associated to the Cartan bundle with respect to an action of H which is the restriction of an action of G. These bundles are not trivial in general, so the existence of natural connections is not evident.

A particularly important special case is natural vector bundles corresponding to the restriction of a representation of G to the subgroup H. These are called *tractor bundles* and from above we know that they carry canonical linear connections, called *tractor connections*

Among all tractor bundles, the *adjoint tractor bundle* is of fundamental importance in the study of Cartan geometries. This is the tractor bundle $\mathcal{A}$ corresponding to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. For a Cartan geometry $(\mathcal{P} \to M, \omega)$ of type (G, H), we have $\mathcal{A}M = \mathcal{P} \times_H \mathfrak{g}$, where H acts on $\mathfrak{g}$ by the restriction of the adjoint action. The short exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of H–modules gives rise to the short exact sequence

$$0 \to \mathcal{P} \times_H \mathfrak{h} \to \mathcal{A}M \to TM \to 0.$$

In particular, there is a natural surjective bundle map $\Pi : \mathcal{A}M \to TM$, and we may view the adjoint tractor bundle as an extension of the tangent bundle. Let us explore further important properties of the adjoint tractor bundle.

PROPOSITION 1.5.7. *Let $(\mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H), $\mathcal{A}M \to M$ its adjoint tractor bundle and $\Pi : \mathcal{A}M \to TM$ the natural projection. Let $\mathcal{V}M$ be the tractor bundle corresponding to a representation of G on V.*

(1) The Cartan curvature κ of ω can be naturally interpreted as a two–form κ on M with values in $\mathcal{A}M$.

(2) There is a natural bundle map $\{\ ,\ \} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$, which makes each fiber $\mathcal{A}_xM$ into a Lie algebra isomorphic to $\mathfrak{g}$.

(3) There is an isomorphism between the space $\Gamma(\mathcal{A}M)$ of smooth sections of $\mathcal{A}M$ and the space $\mathfrak{X}(\mathcal{P})^H$ of vector fields on $\mathcal{P}$ which are invariant under the principal right action of H. This induces a Lie bracket $[\ ,\ ]$ on $\Gamma(\mathcal{A}M)$. For

$s_1, s_2 \in \Gamma(\mathcal{A}M)$, one has $\Pi([s_1, s_2]) = [\Pi(s_1), \Pi(s_2)]$, where in the right–hand side we use the Lie bracket of vector fields.

(4) There is a natural bundle map $\bullet : \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$. For each point $x \in M$, this makes the fiber $\mathcal{V}_x M$ into a module over the Lie algebra $\mathcal{A}_x M$. In particular, for sections $s_1, s_2 \in \Gamma(\mathcal{A}M)$ and $t \in \Gamma(\mathcal{V}M)$, we get

$$\{s_1, s_2\} \bullet t = s_1 \bullet (s_2 \bullet t) - s_2 \bullet (s_1 \bullet t).$$

(5) The operations introduced in (2) and (4) are parallel for the canonical tractor connections. Denoting them by $\nabla^{\mathcal{A}}$ and $\nabla^{\mathcal{V}}$ we get

$$\nabla^{\mathcal{A}}_\xi \{s_1, s_2\} = \{\nabla^{\mathcal{A}}_\xi s_1, s_2\} + \{s_1, \nabla^{\mathcal{A}}_\xi s_2\},$$
$$\nabla^{\mathcal{V}}_\xi (s \bullet t) = (\nabla^{\mathcal{A}}_\xi s) \bullet t + s \bullet (\nabla^{\mathcal{V}}_\xi t)$$

for sections $s_1, s_2 \in \Gamma(\mathcal{A}M)$ and $t \in \Gamma(\mathcal{V}M)$ and all vector fields $\xi \in \mathfrak{X}(M)$.

Proof. (1) The curvature function $\kappa : \mathcal{P} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ was shown to be H–equivariant in Lemma 1.5.1. Hence, it corresponds to a smooth section of the associated bundle, which by definition is $\Lambda^2 T^*M \otimes \mathcal{A}M$.

(2) $\{\ ,\ \}$ is simply the morphism between associated bundles induced by the H–equivariant map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

(3) Since $\mathcal{A}M = \mathcal{P} \times_H \mathfrak{g}$, sections of the adjoint tractor bundle are in bijective correspondence with smooth functions $f : \mathcal{P} \to \mathfrak{g}$ such that $f(u \cdot h) = \operatorname{Ad}(h^{-1})(f(u))$. On the other hand, since ω trivializes $T\mathcal{P}$, the map $\xi \mapsto \omega(\xi)$ defines a linear isomorphism between $\mathfrak{X}(\mathcal{P})$ and $C^\infty(\mathcal{P}, \mathfrak{g})$. Now ξ corresponds to an equivariant function if and only if

$$\omega(\xi(u \cdot h)) = \operatorname{Ad}(h^{-1})(\omega(\xi(u))) = \omega(u \cdot h)(Tr^h \cdot \xi(u)),$$

for all $h \in H$, where in the last equality we use equivariancy of ω. Since the values of ω are linear isomorphisms, this is equivalent to $\xi(u \cdot h) = Tr^h \cdot \xi(u)$ and hence $(r^h)^*\xi = \xi$ for all $h \in H$.

Naturality of the Lie bracket implies that $(r^h)^*([\xi, \eta]) = [(r^h)^*\xi, (r^h)^*\eta]$. Therefore, $\mathfrak{X}(\mathcal{P})^H$ is a Lie subalgebra in $\mathfrak{X}(\mathcal{P})$, and we can pull back the Lie bracket via the isomorphism to $\Gamma(\mathcal{A}M)$. Right invariant vector fields on $\mathcal{P}$ are projectable. From the identification $TM \cong \mathcal{P} \times_H (\mathfrak{g}/\mathfrak{h})$ constructed in 1.5.5 we see that $\Pi : \mathcal{A}M \to TM$ corresponds to projecting right invariant vector fields. Thus, $\Pi([s_1, s_2]) = [\Pi(s_1), \Pi(s_2)]$ follows from naturality of the Lie bracket.

(4) Denoting by $\rho : G \to GL(V)$ the representation inducing the tractor bundle, consider its derivative $\rho' : \mathfrak{g} \to L(V, V)$. For $g \in G$ and $X \in \mathfrak{g}$, we have $\exp(t \operatorname{Ad}(g)(X)) = g \exp(tX) g^{-1}$. Applying the representation ρ and differentiating at $t = 0$, we see that

$$\rho'(\operatorname{Ad}(g)(X))(\rho(g)(v)) = \rho(g)(\rho'(X)(v)).$$

This means that the bilinear map $\mathfrak{g} \times V \to V$ induced by ρ' is G–equivariant and hence H–equivariant, and thus induces a natural map $\bullet : \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$ on associated bundles.

(5) The operations in (2) and (4) are actually induced by G–equivariant maps on the corresponding representations. Hence, we can also view them as being induced on bundles associated to the extended principal bundle $\mathcal{P} \times_H G$. But then the tractor connections are all induced from a fixed principal connection on that bundle. Maps between associated bundles coming from equivariant maps between

the inducing representations are clearly parallel for these connections. Expanding this leads to the claimed formulae. □

The bracket $\{\ ,\ \}$ on $\mathcal{A}M$ from part (2) is called the *algebraic bracket* on adjoint tractors, while the bracket $[\ ,\ ]$ from part (3) is called the *Lie bracket* on adjoint tractors. Note that part (3), in particular, says that $\Pi : \mathcal{A}M \to TM$ makes $(\mathcal{A}M, [\ ,\ ])$ into a Lie algebroid over M.

Composing the curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ with the projection $\Pi : \mathcal{A}M \to TM$, we obtain a form $T := \Pi \circ \kappa \in \Omega^2(M, TM)$, which is called the *torsion* of the Cartan connection ω. By construction, this torsion vanishes if and only if the Cartan geometry in question is torsion free in the sense introduced in 1.5.1 and then its Cartan curvature is a two–form with values in the bundle $\mathcal{P} \times_H \mathfrak{h}$.

Using the interpretation of κ as a two–form with values in $\mathcal{A}M$, we can give a general description of the curvatures of natural principal connections. For a Cartan geometry $(p : \mathcal{P} \to M, \omega)$ of type (G, H) and a homomorphism $i : H \to K$ consider the bundle $\mathcal{P} \times_i K$. The map $\alpha : \mathfrak{g} \to \mathfrak{k}$ used to determine a natural principal connection on this bundle, in particular, is H–equivariant. Hence, it induces a bundle map

$$\mathcal{A}M = \mathcal{P} \times_H \mathfrak{g} \to \mathcal{P} \times_H \mathfrak{k} \cong (\mathcal{P} \times_i K) \times_K \mathfrak{k},$$

which we also denote by α. On the other hand, by Proposition 1.4.6, the map $(X, Y) \mapsto [\alpha(X), \alpha(Y)] - \alpha([X, Y])$ descends to an H–equivariant map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{k}$. In view of 1.5.5 this gives rise to a natural section of the associated natural bundle, i.e. a natural element $R^0_\alpha \in \Omega^2(M, \mathcal{P} \times_H \mathfrak{k})$.

Corollary 1.5.7. *Let $\gamma_\alpha \in \Omega^1(\mathcal{P} \times_i K, \mathfrak{k})$ be the natural principal connection associated to $(\mathcal{P} \to M, \omega)$ via the map $\alpha : \mathfrak{g} \to \mathfrak{k}$ as in Theorem 1.5.6. Then the curvature $R_\gamma \in \Omega^2(M, \mathcal{P} \times_H \mathfrak{k})$ is given by*

$$R_\gamma = \alpha \circ \kappa + R^0_\alpha,$$

where $\kappa \in \Omega^2(M, \mathcal{A}M)$ is the Cartan curvature of ω.

In particular, if $\mathcal{V}M \to M$ is a tractor bundle, then the curvature $R^{\mathcal{V}}$ of the canonical tractor connection $\nabla^{\mathcal{V}}$ is given by

$$R^{\mathcal{V}}(\xi, \eta)(t) = \kappa(\xi, \eta) \bullet t$$

for $\xi, \eta \in \mathfrak{X}(M)$ and $t \in \Gamma(\mathcal{V}M)$

Proof. Let $j : \mathcal{P} \to \mathcal{P} \times_i K$ be the natural map used in 1.5.6, so the connection γ_α is characterized by $j^*\gamma_\alpha = \alpha \circ \omega$. In particular,

$$[j^*\gamma_\alpha(\xi), j^*\gamma_\alpha(\eta)] = [\alpha(\omega(\xi)), \alpha(\omega(\eta))].$$

On the other hand, we also get $j^*d\gamma_\alpha = \alpha \circ d\omega$, and inserting the definition of the Cartan curvature, we obtain

$$j^*d\gamma_\alpha(\xi, \eta) = \alpha(\kappa(\omega(\xi), \omega(\eta)) - [\omega(\xi), \omega(\eta)]).$$

Summing with the expression above, by definition we obtain the pullback of the curvature of γ_α interpreted as an element of $\Omega^2(\mathcal{P} \times_i K, \mathfrak{k})$; see 1.3.3. The isomorphism $\mathcal{P} \times_H \mathfrak{k} \cong (\mathcal{P} \times_i K) \times_K \mathfrak{k}$ is induced by $[\![u, A]\!] \mapsto [\![j(u), A]\!]$, which shows that this pullback exactly represents R_γ, and the claimed formula follows.

In the special case of a tractor bundle $\alpha = \mathrm{id}_{\mathfrak{g}}$, so $R^0_\alpha = 0$, and we are left with $j^*R_\alpha = \kappa$. To convert to the curvature of the induced connection, we have to interpret the values of κ as acting on $\mathcal{V}M$ via the infinitesimal representation; see 1.3.4. But this was exactly the definition of $\bullet$ in the proposition. □

1.5.8. The fundamental derivative. The next important property of the adjoint tractor bundle is the existence of a basic family of natural differential operators on arbitrary natural bundles. The idea to obtain these operators is simply that differentiating H–equivariant smooth functions with respect to H–invariant vector fields, one obtains again H–equivariant functions. We restrict to natural vector bundles here, the case of general bundles is briefly sketched below.

Let E be a natural vector bundle associated to the Cartan bundle with respect to a representation $\rho : H \to GL(V)$. We define the *fundamental derivative* D : $\Gamma(\mathcal{A}M) \times \Gamma(EM) \to \Gamma(EM)$, which we write as $(s, \sigma) \mapsto D_s\sigma$ as follows: The section $s \in \Gamma(\mathcal{A}M)$ corresponds to an H–invariant vector field $\xi \in \mathfrak{X}(\mathcal{P})^H$, while the section σ corresponds to a smooth equivariant function $\phi : \mathcal{P} \to V$. Now for the function $\xi \cdot \phi : \mathcal{P} \to V$ we compute

$$\xi(u \cdot h) \cdot \phi = (Tr^h \cdot \xi(u)) \cdot \phi = \xi(u) \cdot (\phi \circ r^h) = \rho(h^{-1})(\xi(u) \cdot \phi).$$

Hence, $\xi \cdot \phi$ is H–equivariant, so it corresponds to a smooth section $D_s\sigma$ of EM. By construction, this operator is bilinear, and tensorial and thus linear over $C^\infty(M, \mathbb{R})$ in s.

We next establish some basic properties of the fundamental derivative. Consider the derivative $\rho' : \mathfrak{h} \to L(V, V)$ of the representation ρ inducing E. In the proof of part (4) of Proposition 1.5.7 we have seen that the corresponding bilinear map $\mathfrak{h} \times V \to V$ is H–equivariant. Hence, we obtain a natural bundle map $\bullet : (\mathcal{P} \times_H \mathfrak{h}) \times E \to E$. In the case of a tractor bundle, this is just the restriction of the bundle map from part (4) of Proposition 1.5.7.

PROPOSITION 1.5.8. *(1) For a smooth function $f : M \to \mathbb{R}$ and $s \in \Gamma(\mathcal{A}M)$ we get $D_s f = \Pi(s) \cdot f$.*

(2) If s is a section of the subbundle $\mathcal{P} \times_H \mathfrak{h} \subset \mathcal{A}M$, then $D_s\sigma = -s \bullet \sigma$ for any $\sigma \in \Gamma(E)$.

(3) The fundamental derivative is compatible with all natural bundle maps coming from H–equivariant maps between the inducing representations. In particular, for natural vector bundles E and F, the dual E^* of E, and sections $\sigma \in \Gamma(E)$, $\tau \in \Gamma(F)$ and $\beta \in \Gamma(E^*)$ we get

$$D_s(f\sigma) = (\Pi(s) \cdot f)\sigma + fD_s\sigma,$$
$$D_s(\sigma \otimes \tau) = (D_s\sigma) \otimes \tau + \sigma \otimes D_s\tau,$$
$$\Pi(s) \cdot (\beta(\sigma)) = (D_s\beta)(\sigma) + \beta(D_s\sigma).$$

PROOF. (1) Writing $p : \mathcal{P} \to M$ for the projection, the equivariant function $\mathcal{P} \to \mathbb{R}$ corresponding to f is simply $f \circ p$. But then for $\xi \in \mathfrak{X}(\mathcal{P})$ we get $\xi \cdot (f \circ p) = (Tp \cdot \xi) \cdot f$, and the result follows.

(2) If s is a section of the subbundle $\mathcal{P} \times_H \mathfrak{h}$, then the corresponding vector field ξ has the property that $\omega(\xi)$ has values in $\mathfrak{h}$. Thus, $\xi(u) = \zeta_{\omega(\xi)(u)}(u)$. Let $\phi : \mathcal{P} \to V$ be the equivariant function corresponding to s. Applying $\phi(u \cdot h) = \rho(h)(\phi(u))$ for $h = \exp(tA)$ for $A \in \mathfrak{h}$ and differentiating at $t = 0$, we get $\zeta_A \cdot \phi(u) = -\rho'(A)(\phi(u))$, and the claim follows.

(3) In the picture of equivariant functions, all the operations act only on the values of functions. The natural bundle maps are given by applying linear and

multilinear maps to the values of the functions. Of course, this is compatible in the appropriate sense with differentiation. The three claimed formulae are evident examples of this situation. □

Except for the fact that the tangent bundle has been replaced by the adjoint tractor bundle, the fundamental derivative looks very similar to the family of covariant derivatives by the Levi–Civita connection on a Riemannian manifold. The naturality properties of the fundamental derivative justify the use of the same symbol D to denote all fundamental derivatives.

Of course, we may also leave the algebraic slot of D free, and view $s \mapsto D_s\sigma$ as a section $D\sigma$ of $\mathcal{A}^*M \otimes EM$, and thus the fundamental derivative as a differential operator $\Gamma(EM) \to \Gamma(\mathcal{A}^*M \otimes EM)$. In this version, the fundamental derivative can be iterated, i.e. for $\sigma \in \Gamma(EM)$ and $k \in \mathbb{N}$ we obtain $D^k\sigma \in \Gamma(\otimes^k \mathcal{A}^*M \otimes EM)$.

Now we can use the fundamental derivative to derive a formula for an arbitrary natural linear connection. This generalizes the formula for homogeneous connections from Proposition 1.4.7. As above, let E be the natural vector bundle corresponding to a representation $\rho : H \to GL(V)$. By Theorem 1.4.7, a homogeneous linear connection on $E(G/H) \to G/H$ is induced by a linear map $\alpha : \mathfrak{g} \to L(V, V)$ such that

(i) $\alpha|_{\mathfrak{h}} = \rho'$, the derivative of the representation ρ,
(ii) $\alpha(\mathrm{Ad}(h)(X)) = \rho(h) \circ \alpha(X) \circ \rho(h^{-1})$ for all $X \in \mathfrak{g}$ and $h \in H$.

In 1.5.6 we have noted that, via principal and induced connections, one obtains from α a natural linear connection on E. Now property (ii) says that α, and hence the corresponding bilinear map $\mathfrak{g} \times V \to V$ is H–equivariant. Thus, it induces a natural bundle map $\mathcal{A}M \times EM \to \mathcal{A}M$ which we also denote by α.

Theorem 1.5.8. *Consider the operation* $\Gamma(\mathcal{A}M) \times \Gamma(EM) \to \Gamma(EM)$ *defined by* $(s, \sigma) \mapsto D_s\sigma + \alpha(s, \sigma)$*. This vanishes identically if* s *is a section of the subbundle* $\mathcal{P} \times_H \mathfrak{h} \subset \mathcal{A}M$*. Hence, it descends to an operator* $\mathfrak{X}(M) \times \Gamma(EM) \to \Gamma(EM)$ *which is exactly the covariant derivative with respect to the natural linear connection induced by* α*. In particular, if* E *is a tractor bundle* $\mathcal{V}$*, then the tractor connection* $\nabla^{\mathcal{V}}$ *is given by*

$$\nabla_{\Pi(s)} t = D_s t + s \bullet t$$

for $s \in \Gamma(\mathcal{A}M)$ *and* $t \in \Gamma(\mathcal{V}M)$*.*

Proof. Put $K := GL(V)$, let $\mathcal{P} \times_\rho K$ be the frame bundle of E, and let $j : \mathcal{P} \to \mathcal{P} \times_\rho K$ be the natural map. Then the natural principal connection γ_α on the frame bundle corresponding to the linear map α is characterized by $j^*\gamma_\alpha = \alpha \circ \omega$. Now take a tangent vector $\xi \in T_u\mathcal{P}$. Then the horizontal lift of $T_u p \cdot \xi$ in the point $j(u)$ by definition is $T_u j \cdot \xi - \zeta_{\alpha(\omega(\xi))}(u)$. If $\phi : \mathcal{P} \times_\rho K \to V$ is the equivariant map corresponding to $\sigma \in \Gamma(EM)$, then its derivative with respect to this horizontal lift is

$$(T_u j \cdot \xi) \cdot \phi + \rho'(\alpha(\omega(\xi)))(\phi(j(u))).$$

The first summand can be written as $\xi \cdot (\phi \circ j)$. Viewing EM as $\mathcal{P} \times_H V$, the equivariant map corresponding to σ is $\phi \circ j$. Taking ξ to be the right invariant vector field corresponding to s, the formula follows.

In the case of a tractor bundle, we start with a representation $\rho : G \to GL(V)$, and the map $\alpha : \mathfrak{g} \to L(V, V)$ simply becomes the derivative ρ'. □

We can now use these results to compute the Lie bracket on adjoint tractor fields.

COROLLARY 1.5.8. *For $s_1, s_2 \in \Gamma(\mathcal{A}M)$, the Lie bracket is given by*

$$\begin{aligned}[s_1, s_2] &= D_{s_1} s_2 - D_{s_2} s_1 - \kappa(\Pi(s_1), \Pi(s_2)) + \{s_1, s_2\} \\ &= \nabla^{\mathcal{A}}_{\Pi(s_1)} s_2 - \nabla^{\mathcal{A}}_{\Pi(s_2)} s_1 - \{s_1, s_2\} - \kappa(\Pi(s_1), \Pi(s_2)).\end{aligned}$$

PROOF. For $i = 1, 2$ let $\xi_i \in \mathfrak{X}(\mathcal{P})^H$ be the vector field corresponding to s_i. By definition, the function $\mathcal{P} \to \mathfrak{g}$ corresponding to the Lie bracket $[s_1, s_2]$ is $\omega([\xi_1, \xi_2])$. Inserting the definition of the exterior derivative and of the curvature form, this reads as

$$\xi_1 \cdot \omega(\xi_2) - \xi_2 \cdot \omega(\xi_1) - K(\xi_1, \xi_2) + [\omega(\xi_1), \omega(\xi_2)].$$

Now inserting the definitions of D, κ, and $\{\ ,\ \}$ this is exactly the first claimed formula. To get the second formula, we just have to note that for the adjoint tractor bundle, the formula for the tractor connection from the theorem reads as $\nabla_{\Pi(s_1)} s_2 = D_{s_1} s_2 + \{s_1, s_2\}$ and likewise for the other term. □

REMARK 1.5.8. There also is a nonlinear analog of the fundamental derivative. Since we will use this only rarely, we are brief about it. Let S be a smooth manifold with a left H–action and let F be the corresponding natural bundle. As above, smooth sections of FM may be identified with smooth H–equivariant functions $\mathcal{P} \to S$, but hitting this with a vector field ξ the resulting function $\xi \cdot f$ now has values in TS and is a lift of f. Nevertheless, if $\xi \in \mathfrak{X}(\mathcal{P})^H \cong \Gamma(\mathcal{A}M)$, then the same argument as above shows that $\xi \cdot f$ is equivariant, and defines a smooth section of $\mathcal{P} \times_H TS$ which may be identified with the vertical tangent bundle VFM. Hence, we get a fundamental derivative $D : \Gamma(\mathcal{A}M) \times \Gamma(FM) \to \Gamma(VFM)$, which is linear and tensorial in the first argument. Moreover, for $s \in \Gamma(\mathcal{A}M)$ and $\sigma \in \Gamma(FM)$ the section $D_s\sigma$ of VFM is a lift of σ.

1.5.9. Bianchi and Ricci identities. We next derive the basic differential identities for the curvature and describe iterated fundamental derivatives. In 1.5.8 we saw that for any natural vector bundle E there is a sequence of differential operators $D^k : \Gamma(EM) \to \Gamma(\otimes^k \mathcal{A}^* M \otimes EM)$. Moreover, there is a natural projection $\Pi : \mathcal{A}M \to TM$, so any adjoint tractor field has an underlying vector field. Dually, we get an inclusion $T^*M \to \mathcal{A}^*M$ which means that any differential form canonically extends to taking adjoint tractor fields as an input. Otherwise put, we insert adjoint tractor fields into differential forms by first projecting to the underlying vector fields. We will often suppress the projection Π and simply insert adjoint tractor fields as arguments into differential forms.

PROPOSITION 1.5.9. *Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H) with curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$, let $\nabla^{\mathcal{A}}$ be the adjoint tractor connection and $\{\ ,\ \}$ the algebraic bracket on $\mathcal{A}M$.*

(1) (Bianchi–identity) The curvature κ satisfies

$$\sum_{cycl} \Big(\nabla^{\mathcal{A}}_{\xi_1}(\kappa(\xi_2, \xi_3)) - \kappa([\xi_1, \xi_2], \xi_3) \Big) = 0 \tag{1.24}$$

for all vector fields $\xi_i \in \mathfrak{X}(M)$ or equivalently

$$\sum_{cycl} \Big(\{s_1, \kappa(s_2, s_3)\} - \kappa(\{s_1, s_2\}, s_3) + \kappa(\kappa(s_1, s_2), s_3) + (D_{s_1}\kappa)(s_2, s_3) \Big) = 0 \tag{1.25}$$

for all $s_i \in \Gamma(\mathcal{A}M)$, where the sums are over all cyclic permutations of the arguments.

(2) (Ricci–identity) For any natural vector bundle E and any section $\sigma \in \Gamma(EM)$ the alternation of the square of the fundamental derivative is given by

$$(D^2\sigma)(s_1, s_2) - (D^2\sigma)(s_2, s_1) = -D_{\kappa(s_1,s_2)}\sigma + D_{\{s_1,s_2\}}\sigma.$$

Proof. (1) Let us first prove the equivalence of (1.24) and (1.25). In view of the fact that $\Pi([s_1, s_2]) = [\Pi(s_1), \Pi(s_2)]$, we may equivalently replace the vector fields ξ_i in (1.24) by adjoint tractor fields s_i. Then the formula for the adjoint tractor connection from Theorem 1.5.8 shows that

$$\nabla^{\mathcal{A}}_{s_1}(\kappa(s_2, s_3)) = D_{s_1}(\kappa(s_2, s_3)) + \{s_1, \kappa(s_2, s_3)\},$$

while the formula for the Lie bracket of adjoint tractors from Corollary 1.5.8 gives

$$-\kappa([s_1, s_2], s_3) = -\kappa(D_{s_1}s_2, s_3) + \kappa(D_{s_2}s_1, s_3) + \kappa(\kappa(s_1, s_2), s_3) - \kappa(\{s_1, s_2\}, s_3).$$

On the other hand, naturality of the fundamental derivative implies

$$(D_{s_1}\kappa)(s_2, s_3) = D_{s_1}(\kappa(s_2, s_3)) - \kappa(D_{s_1}s_2, s_3) - \kappa(s_2, D_{s_1}s_3).$$

Inserting this, we see that, replacing the ξ_i by s_i in (1.24) we obtain the cyclic sum of

$$\begin{aligned}\{s_1, \kappa(s_2, s_3)\} + (D_{s_1}\kappa)(s_2, s_3) + \kappa(\kappa(s_1, s_2), s_3)& \\ - \kappa(\{s_1, s_2\}, s_3) + \kappa(s_2, D_{s_1}s_3) + \kappa(D_{s_2}s_1, s_3)&.\end{aligned}$$

Forming the cyclic sum, the last two terms cancel by skew symmetry of κ, and we obtain (1.25).

Now (1.25) is visibly linear over smooth functions in all arguments s_i, so it can be verified in a point. We may view κ as the curvature function $\mathcal{P} \to L(\Lambda^2\mathfrak{g}, \mathfrak{g})$ (recalling that the result vanishes if one entry is from $\mathfrak{h}$). In these terms, the claimed identity has the form

$$0 = \sum_{\text{cycl}}\Big([\kappa(X, Y), Z] + \kappa([X, Y], Z) \\ - \kappa\big(\kappa(X, Y), Z\big) - \omega^{-1}(Z) \cdot \kappa(X, Y)\Big) \tag{1.26}$$

for all X, Y, $Z \in \mathfrak{g}$. (Observe that since evaluation in (X, Y) is a linear map, there is no difference between $(\omega^{-1}(Z) \cdot \kappa)(X, Y)$ and $\omega^{-1}(Z) \cdot (\kappa(X, Y))$.) Let us evaluate the structure equation on the vector fields $[\tilde{X}, \tilde{Y}]$ and $\tilde{Z}$, where $\tilde{X} = \omega^{-1}(X)$, $\tilde{Y} = \omega^{-1}(Y)$, and $\tilde{Z} = \omega^{-1}(Z)$. Remember, in particular, $\kappa(X, Y) = K(\tilde{X}, \tilde{Y})$ and $\omega([\tilde{X}, \tilde{Y}]) = -\kappa(X, Y) + [X, Y]$. Thus,

$$\begin{aligned}K([\tilde{X}, \tilde{Y}], \tilde{Z}) &= -\tilde{Z} \cdot \omega([\tilde{X}, \tilde{Y}]) - \omega\big([[\tilde{X}, \tilde{Y}], \tilde{Z}]\big) + \big[\omega([\tilde{X}, \tilde{Y}]), Z\big] \\ &= \tilde{Z} \cdot (\kappa(X, Y)) - \omega\big([[\tilde{X}, \tilde{Y}], \tilde{Z}]\big) + [[X, Y], Z] - [\kappa(X, Y), Z]\end{aligned}$$

and the left–hand side equals $\kappa([X, Y], Z) - \kappa(\kappa(X, Y), Z)$. Now, let us perform the cyclic permutation over X, Y, and Z. Leaving out the two terms which disappear by virtue of the Jacobi identity for vector fields and the Lie algebra $\mathfrak{g}$, while collecting all remaining terms on one side of the equality, we obtain exactly the required identity.

(2) To prove the Ricci identity, note first that for sections $s_i \in \Gamma(\mathcal{A}M)$ corresponding to $\xi_i \in \mathfrak{X}(\mathcal{P})^H$ the definition of the fundamental derivative immediately

implies that $D_{s_1}(D_{s_2}\sigma) - D_{s_2}(D_{s_1}\sigma) = D_{[s_1,s_2]}\sigma$, where on the right–hand side we have the Lie bracket of adjoint tractor fields. Naturality of the fundamental derivative implies $(D^2\sigma)(s_1, s_2) = D_{s_1}(D_{s_2}\sigma) - D_{D_{s_1}s_2}\sigma$. Alternating this, we see that

$$(D^2\sigma)(s_1, s_2) - (D^2\sigma)(s_2, s_1) = D_{[s_1,s_2]-D_{s_1}s_2+D_{s_2}s_1}\sigma,$$

and the Ricci identity immediately follows from the formula for $[s_1, s_2]$ in Corollary 1.5.8. □

Remark 1.5.9. (i) The first form of the Bianchi identity matches up nicely with the classical Bianchi identity for a linear connection on a vector bundle which says that the covariant exterior derivative of the curvature vanishes, and in fact this leads to an alternative proof of the identity.

(ii) Since κ does not depend on adjoint tractor fields inserted but only on the underlying vector fields, it follows that the term $\kappa(\kappa(s_1, s_2), s_3)$ depends only on the torsion T, the projection to TM of the values of κ; see 1.5.7. In particular, this term vanishes for torsion–free Cartan geometries.

To get the classical versions of the Bianchi and Ricci identities, let us specialize to Cartan geometries corresponding to a reductive Klein geometry (G, H). Thus, we have to assume that there is a distinguished H–invariant subspace $\mathfrak{n} \subset \mathfrak{g}$ which is complementary to the Lie subalgebra $\mathfrak{h}$. Since we will be mainly interested in the non–reductive case in the sequel, we only give a rough treatment of this case, leaving some details to the reader.

Corollary 1.5.9. *Let $(\mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H), where (G, H) is a reductive Klein geometry with H–invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$. Let R be the curvature of the natural principal connection on $\mathcal{P}$ and let T be the torsion of the induced connection on the tangent bundle TM. Denoting by ∇ the natural covariant derivatives we have:*

(i) *The algebraic Bianchi identity for $\xi_i \in \mathfrak{X}(M)$:*

$$\sum_{cycl}\Big(-R(\xi_1, \xi_2)(\xi_3) + T(T(\xi_1, \xi_2), \xi_3) + (\nabla_{\xi_1}T)(\xi_2, \xi_3)\Big) = 0.$$

(ii) *The differential Bianchi identity*

$$\sum_{cycl}\Big((\nabla_{\xi_1}R)(\xi_2, \xi_3) + R(T(\xi_1, \xi_2), \xi_3)\Big) = 0.$$

(iii) *The Ricci identity for a section σ of an arbitrary natural vector bundle and $\xi, \eta \in \mathfrak{X}(M)$,*

$$(\nabla^2\sigma)(\xi, \eta) - (\nabla^2\sigma)(\eta, \xi) = R(\xi, \eta)(\sigma) - \nabla_{T(\xi,\eta)}\sigma.$$

Proof. For any Cartan geometry $(\mathcal{P} \to M, \omega)$ of type (G, H), the H–invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ induces a splitting $\mathcal{A}M = (\mathcal{P} \times_H \mathfrak{h}) \oplus (\mathcal{P} \times_H \mathfrak{n})$, and the second summand is isomorphic to TM. According to this, we will write sections of $\mathcal{A}M$ as column vectors with a vector field as lower row and a section of $\mathcal{P} \times_H \mathfrak{h}$ as upper row.

In the setting or Theorem 1.5.8, we have to set $\alpha(s, \sigma) = \pi(s) \bullet \sigma$, where $\pi : \mathfrak{g} \to \mathfrak{h}$ is the projection along $\mathfrak{n}$. Hence, viewing $\xi \in \mathfrak{X}(M)$ as an adjoint tractor, $D_\xi\sigma = \nabla_\xi\sigma$ by Theorem 1.5.8. Via the splitting of $\mathcal{A}M$, also the Bianchi and Ricci identities split into components.

The assumptions on $\mathfrak{h}$ and $\mathfrak{n}$ imply that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{n}] \subset \mathfrak{n}$, and we may split the remaining bracket $\Lambda^2\mathfrak{n} \to \mathfrak{g}$ according to the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. From Theorem 1.4.7 we know that the $\mathfrak{n}$–component of the bracket corresponds to $-T_0$, where T_0 is the torsion of the invariant connection on $T(G/H)$ induced by $\mathfrak{n}$, while the $\mathfrak{h}$–component corresponds to $-R_0$, where R_0 is the curvature of the invariant principal connection on $G \to G/H$.

Taking into account that $\alpha : \mathfrak{g} \to \mathfrak{h}$ is the projection along $\mathfrak{n}$, Corollary 1.5.7 shows that the Cartan curvature is given by

$$\kappa(\xi,\eta) = \begin{pmatrix} (R-R_0)(\xi,\eta) \\ (T-T_0)(\xi,\eta) \end{pmatrix}$$

for all $\xi,\eta \in \mathfrak{X}(M)$. We can also split the algebraic bracket on $\mathcal{A}M$ according to the decomposition of the bracket on $\mathfrak{g}$. To obtain the two Bianchi identities (i) and (ii), we shall apply formula (1.25) from Proposition 1.5.9 to $\xi_1,\xi_2,\xi_3 \in \mathfrak{X}(M) \subset \Gamma(\mathcal{A}M)$. For the first summand $\{\xi_1,\kappa(\xi_2,\xi_3)\}$ we get

$$\left\{\begin{pmatrix} 0 \\ \xi_1 \end{pmatrix}, \begin{pmatrix} (R-R_0)(\xi_2,\xi_3) \\ (T-T_0)(\xi_2,\xi_3) \end{pmatrix}\right\} = \begin{pmatrix} -R_0(\xi_1,(T-T_0)(\xi_2,\xi_3)) \\ -(R-R_0)(\xi_2,\xi_3)(\xi_1) - T_0(\xi_1,(T-T_0)(\xi_2,\xi_3)) \end{pmatrix},$$

and after passing to the cyclic sum, the right–hand side can be replaced by

$$\begin{pmatrix} R_0((T-T_0)(\xi_1,\xi_2),\xi_3) \\ -(R-R_0)(\xi_2,\xi_3)(\xi_1) + T_0((T-T_0)(\xi_1,\xi_2),\xi_3) \end{pmatrix}.$$

For the next two summands $-\kappa(\{\xi_1,\xi_2\},\xi_3) + \kappa(\kappa(\xi_1,\xi_2),\xi_3)$, we use that by construction the TM–component of $\kappa(\xi_1,\xi_2) - \{\xi_1,\xi_2\}$ is given by $T(\xi_1,\xi_2)$ to conclude that these two summands contribute

$$\begin{pmatrix} (R-R_0)(T(\xi_1,\xi_2),\xi_3) \\ (T-T_0)(T(\xi_1,\xi_2),\xi_3) \end{pmatrix}.$$

For the last summand, we directly get the contribution

$$\begin{pmatrix} (\nabla_{\xi_1}(R-R_0))(\xi_2,\xi_3) \\ (\nabla_{\xi_1}(T-T_0))(\xi_2,\xi_3) \end{pmatrix}.$$

Collecting the $\mathcal{P} \times_H \mathfrak{h}$–components, we see that the terms containing R_0 and T cancel, and we conclude that vanishing of this component is equivalent to

$$\sum_{\text{cycl}}\Big((\nabla_{\xi_1}R)(\xi_2,\xi_3) + R(T(\xi_1,\xi_2),\xi_3)\Big) = \sum_{\text{cycl}}\Big((\nabla_{\xi_1}R_0)(\xi_2,\xi_3) + R_0(T_0(\xi_1,\xi_2),\xi_3)\Big).$$

Similarly, collecting the TM–components we see that the terms mixing T and T_0 cancel and vanishing of the TM–component is equivalent to

$$\sum_{\text{cycl}}\Big(-R(\xi_1,\xi_2)(\xi_3) + T(T(\xi_1,\xi_2),\xi_3) + (\nabla_{\xi_1}T)(\xi_2,\xi_3)\Big) = \sum_{\text{cycl}}\Big(-R_0(\xi_1,\xi_2)(\xi_3) + T_0(T_0(\xi_1,\xi_2),\xi_3) + (\nabla_{\xi_1}T_0)(\xi_2,\xi_3)\Big)$$

To complete the proof, it thus suffices to show that the right–hand sides of these two equations vanish automatically. But this can be easily concluded as follows: Define a new Lie algebra structure on $\mathfrak{h} \oplus \mathfrak{n}$ by keeping the brackets $\mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ and

$\mathfrak{h} \times \mathfrak{n} \to \mathfrak{n}$ and declaring the bracket to be zero on $\mathfrak{n} \times \mathfrak{n}$. One immediately verifies that this defines a Lie algebra $\tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{g}} \cong \mathfrak{g}$ as an $\mathfrak{h}$–module. Taking as a Lie group $\tilde{G}$ with this Lie algebra the affine extension of H, we may view $(G \to G/H, \omega_G)$ as a (non–flat) Cartan geometry of type $(\tilde{G}, H)$. Now the curvature and torsion of this geometry is by construction given by R_0 and T_0 from before, while for the new geometry the canonical connection on the homogeneous model is torsion free and flat, so the result follows by using the above formulae in this case.

(iii) Here we just have to observe that for $\xi_1, \xi_2 \in \mathfrak{X}(M) \subset \Gamma(\mathcal{A}M)$, the expression $\kappa(\xi_1, \xi_2) - \{\xi_1, \xi_2\}$ has components $R(\xi_1, \xi_2)$ and $T(\xi_1, \xi_2)$. The claimed identity now immediately follows from part (2) of Proposition 1.5.9 applied to $\xi_i \in \mathfrak{X}(M) \subset \Gamma(\mathcal{A}M)$. □

1.5.10. Fundamental derivative and jet prolongations. Tractor connections and fundamental derivatives as discussed in the last few subsections are examples of invariant differential operators defined for all Cartan geometries. By construction, they are intrinsic to the given geometric structure, and this is the property one tries to capture in the general notion of natural or invariant differential operators for Cartan geometries. It turns out that this leads to very deep and interesting problems and results, which will be the main topic of volume two, so we will discuss the technicalities there and only present an outline here.

For the case of a Klein geometry (G, H), we have discussed the basics on invariant differential operators in 1.4.9 and 1.4.10. We first observe that the situation becomes very simple in the presence of invariant connections. Assume that we have given a homogeneous vector bundle $E \to G/H$ which admits an invariant linear connection and that there also is an invariant linear connection on $T(G/H)$. Then Proposition 1.4.9 gives a complete description of invariant differential operators which map sections of E to sections of an arbitrary homogeneous bundle F. This description is in terms of invariant bundle maps $S^kT^*(G/H) \otimes E \to F$ for $k \geq 0$.

Now all the ingredients generalize to arbitrary Cartan geometries of type (G, H). The bundles E and F give rise to natural bundles on the category $\mathcal{C}_{(G,H)}$, and from 1.5.6 we know that there are natural connections on E and on the tangent bundle. An invariant bundle map $\Phi : S^kT^*(G/H) \otimes E \to F$ is induced by a G–equivaraint map between the inducing representation and hence extends to a natural bundle map on $\mathcal{C}_{(G,H)}$. Applying this bundle map to symmetrized iterated covariant derivatives as in the proof of Proposition 1.4.9 we obtain a natural differential operator, whose symbol is given by the natural bundle map from above. Thus, we see

Observation 1.5.10. Suppose that (G, H) is a Klein geometry such that there is a G–invariant linear connection on $T(G/H)$. Suppose further that $E \to G/H$ is a homogeneous vector bundle which admits a G–invariant linear connection. Then any invariant linear differential operator mapping $\Gamma(E)$ to sections of some homogeneous vector bundle F canonically extends to a natural differential operator on the category $\mathcal{C}_{(G,H)}$.

In the case of a general homogeneous vector bundle $E \to G/H$, we started by considering a jet prolongation J^rE. This is again homogeneous and hence induced by a representation of H on the standard fiber J^rE_o. For another homogeneous vector bundle F, invariant differential operators of order $\leq r$ are then equivalent to H–module homomorphisms $J^rE_o \to F_o$. Now, again, the representations give rise to natural vector bundles on $\mathcal{C}_{(G,H)}$ and the H–module homomorphism gives rise to a natural bundle map. Hence, it may seem as if invariant differential operators still would automatically extend to natural operators on $\mathcal{C}_{(G,H)}$.

This is not true, however. The problem is that while for the homogeneous model $J^r(G\times_H E_o)$ is isomorphic to $G\times_H J^r E_o$, it is not true for a general Cartan geometry $(\mathcal{P}\to M,\omega)$ of type (G,H) that $\mathcal{P}\times_H J^r E_o$ is naturally isomorphic to $J^r(\mathcal{P}\times_H E_0)$. Already for conformal geometry (which is a rather simple example of a parabolic geometry) there are examples of invariant differential operators on the homogeneous model, which do not extend to the category of all conformal structures; see [**Gr92**] and [**GoHi04**]. These results also prove that in these cases the bundle $\mathcal{P}\times_H J^r E_o$ is really different from $J^r(\mathcal{P}\times_H E_0)$. (The simplest case in which this occurs is $r=6$ and a one–dimensional representation E_o.) It is this partial breakdown of the correspondence between a Klein geometry and the associated Cartan geometries that makes the theory of invariant differential operators difficult and interesting.

In spite of these difficulties, jet prolongations and the associated representations are an essential ingredient in the theory of natural operators for Cartan geometries. As a first basic result, we show that the fundamental derivative can be used to encode arbitrarily high jets of sections of any bundle associated to the Cartan bundle into a section of a natural bundle. Fix a Klein geometry (G,H) and consider a natural bundle E corresponding to a representation V of H. This means that for any Cartan geometry $(p:\mathcal{G}\to M,\omega)$ of type (G,H) we have $EM=\mathcal{P}\times_H V$. Given a section $\sigma\in\Gamma(EM)$, we can form the fundamental derivative $D\sigma\in\Gamma(\mathcal{A}^*M\otimes EM)$ and iterating we get for any r the operator $D^r\sigma\in\Gamma(\otimes^r\mathcal{A}^*M\otimes EM)$. Formally, we define $D^1\sigma=D\sigma$ and inductively $D^r\sigma=D(D^{r-1}\sigma)$. By construction, D is a first order operator, so we may also view $\sigma\mapsto D^r\sigma$ as a vector bundle map from the rth jet prolongation J^rEM to $\otimes^r\mathcal{A}^*M\otimes EM$. To come closer to the usual jets, the obvious idea is to replace $D^r\sigma$ by its complete symmetrization $\operatorname{Symm}(D^r\sigma)$ defined by

$$\operatorname{Symm}(D^r\sigma)(s_1,\dots,s_r)=\tfrac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}(D^r\sigma)(s_{\tau(1)},\dots,s_{\tau(r)}),$$

for all $s_j\in\Gamma(\mathcal{A}M)$, where $\mathfrak{S}_r$ denotes the permutation group. This may then be viewed as a section of $S^r\mathcal{A}^*M\otimes EM$.

Proposition 1.5.10. *For any $r\in\mathbb{N}$, the operator*

$$\sigma\mapsto(\sigma,D\sigma,\operatorname{Symm}(D^2\sigma),\dots,\operatorname{Symm}(D^r\sigma))$$

induces an injective bundle map $J^rEM\to\bigoplus_{j=0}^r(S^j\mathcal{A}^*M\otimes EM)$.

Proof. We first claim that $j^{r-1}_x\sigma=0$ implies that

$$D^r\sigma(x)(s_1,\dots,s_r)=D_{s_1}D_{s_2}\dots D_{s_r}\sigma(x)$$

for all sections $s_i\in\Gamma(\mathcal{A}M)$. Naturality of the fundamental derivative implies that for any i and arbitrary sections $s_1,\dots,s_r$ of $\mathcal{A}M$ we have

$$D^i\sigma(s_1,\dots,s_i)=D_{s_1}(D^{i-1}\sigma(s_2,\dots,s_i))-\sum_\ell D^{i-1}\sigma(s_2,\dots,D_{s_1}s_\ell,\dots,s_i).$$

Since D is first order, $j^{r-1}_x\sigma=0$ implies $j^{r-i}_x(D^{i-1}\sigma)=0$, and thus

$$j^{r-i}_x(D^i\sigma(s_1,\dots,s_i))=j^{r-i}_x(D_{s_1}(D^{i-1}\sigma(s_2,\dots,s_i))).$$

Applied to $i=r$, this shows that $D^r\sigma(s_1,\dots,s_r)(x)=D_{s_1}(D^{r-1}\sigma(s_2,\dots,s_r))(x)$. But then we apply the same fact for $i=r-1$ to conclude that the one–jet in x

of $D^{r-1}\sigma(s_2,\dots,s_r)$ coincides with the one–jet of $D_{s_2}(D^{r-2}(s_3,\dots,s_r))$, and the claim follows by induction.

Still assuming $j^{r-1}_x\sigma = 0$, we next claim that $(D_{s_1}D_{s_2}\dots D_{s_r}\sigma)(x)$ is completely symmetric in the s_j. The Ricci identity from part (2) of Proposition 1.5.9 implies that for any section ϕ of a natural bundle one may compute $(D_{s_1}D_{s_2}\phi)(x) - (D_{s_2}D_{s_1}\phi)(x)$ from $D\phi(x)$. In particular, if $j^\ell_x\phi = 0$, then $j^{\ell-1}_x D\phi = 0$, and thus $j^{\ell-1}_x(D_{s_1}D_{s_2}\phi) = j^{\ell-1}_x(D_{s_2}D_{s_1}\phi)$. Now consider an index ℓ such that $1 \le \ell \le r-1$. Then $j^{r-1}_x\sigma = 0$ implies that $D_{s_{\ell+2}}\dots D_{s_r}\sigma$ has vanishing ℓ–jet in x and thus

$$j^{\ell-1}_x D_{s_\ell}D_{s_{\ell+1}}D_{s_{\ell+2}}\dots D_{s_r}\sigma = j^{\ell-1}_x D_{s_{\ell+1}}D_{s_\ell}D_{s_{\ell+2}}\dots D_{s_r}\sigma.$$

Hence, the value of $D_{s_1}\dots D_{s_r}\sigma(x)$ does not change if one exchanges s_i and s_{i+1}, which implies the claim.

To complete the proof, we only have to verify injectivity, which we do by induction on r. Hence, let us first assume that $\sigma \in \Gamma(EM)$ satisfies $\sigma(x) = 0$ and $D\sigma(x) = 0$ for some $x \in M$. Denoting by $f : \mathcal{P} \to V$ the equivariant function corresponding to σ, let us take any point $u \in \mathcal{P}$ such that $p(u) = x$ and any tangent vector $\xi \in T_u\mathcal{P}$. Then $\xi \cdot f(u) = 0$, since this equals $D_s\sigma(x)$, where $s \in \Gamma(\mathcal{A}M)$ corresponds to any extension of ξ to an H–invariant vector field on $\mathcal{P}$. Thus, $j^1_u f = 0$ and hence $j^1_x\sigma = 0$.

So let us assume that $r > 1$ and $\sigma \in \Gamma(\mathcal{V}M)$ is such that $\sigma(x) = 0$, $D\sigma(x) = 0$ and $\operatorname{Symm}(D^i\sigma)(x) = 0$ for $i \le r$. By induction, this implies $j^{r-1}_x\sigma = 0$ and using the two claims above we conclude that $\operatorname{Symm}(D^r\sigma)(x) = D^r\sigma(x)$, so we conclude that $D^r\sigma$ vanishes in x. By the above considerations, this implies that $D^{r-1}\sigma$ and thus $j^{r-1}\sigma$ has vanishing one–jet in x, whence $j^r_x\sigma = 0$. □

Even in the case $r = 1$ (where there is no issue of symmetrization) the bundle $EM \oplus \mathcal{A}^*M \otimes EM$ is much bigger than the first jet prolongation J^1EM. It is also easy to see directly that the pair $(\sigma, D\sigma)$ for $\sigma \in \Gamma(EM)$ contains redundant information. By part (2) of Proposition 1.5.8, for a section s of the subbundle $\mathcal{P} \times_H \mathfrak{h} \subset \mathcal{A}M$ and any section σ of a natural bundle, $D_s\sigma$ is just the negative of the natural algebraic action of s on σ.

These observations suggest a way to get a better hold on the first jet prolongation. Suppose that V is the representation of H which induces the natural bundle E. Then we define a subspace $J^1V \subset V \oplus L(\mathfrak{g}, V)$ as the space of all (v, ϕ) such that $\phi(A) = -A \cdot v$ for all $A \in \mathfrak{h} \subset \mathfrak{g}$.

THEOREM 1.5.10. *The subspace $J^1V \subset V \oplus L(\mathfrak{g}, V)$ is H–invariant and hence gives rise to a natural subbundle in $EM \oplus \mathcal{A}^*M \otimes EM$. The operator $\sigma \mapsto (\sigma, D\sigma)$ always has values in this subbundle. For any Cartan geometry $(\mathcal{P} \to M, \omega)$, this gives rise to a natural isomorphism $J^1EM \to \mathcal{P} \times_H (J^1V)$.*

In particular, any first order invariant linear differential operator between homogeneous bundles on G/H canonically extends to a natural operator on $\mathcal{C}_{(G,H)}$.

PROOF. The natural H–action on $V \oplus L(\mathfrak{g}, V)$ is given by

$$h \cdot (v, \phi) = (h \cdot v, A \mapsto h \cdot \phi(\operatorname{Ad}(h^{-1})(A))).$$

Assuming that (v, ϕ) lies in the subspace J^1V and that $A \in \mathfrak{h}$, the second component of this is given by

$$A \mapsto -h \cdot (\operatorname{Ad}(h^{-1})(A) \cdot v) = -h \cdot (h^{-1} \cdot A \cdot h \cdot v) = -A \cdot h \cdot v,$$

which implies that $h \cdot (v, \phi) \in J^1V$, so this is an H–submodule. We have noted above that if s is a section of the subbundle $\mathcal{P} \times_H \mathfrak{h} \subset \mathcal{A}M$, then $D_s\sigma$ is given by the negative of the algebraic action on σ, which implies that $(\sigma, D\sigma)$ has values in the subbundle $\mathcal{P} \times_H (J^1V)$. From the proposition we obtain an injective bundle map $J^1EM \to \mathcal{P} \times_H (J^1V)$. Projecting on the first factor induces a surjection $J^1V \to V$, whose kernel by definition is isomorphic to $L(\mathfrak{g}/\mathfrak{h}, V)$. Hence, we see that the dimension of J^1V equals the rank of the bundle J^1EM, so the bundle map $J^1EM \to \mathcal{P} \times_H (J^1V)$ has to be an isomorphism.

Applying this to $E \to G/H$, we see that J^1V is the representation inducing the homogeneous bundle J^1E. Hence, any first order linear invariant operator defined on sections of E is induced by an H–homomorphism from J^1V to some other representation. This homomorphism induces a natural bundle map on J^1EM and hence a natural differential operator on $\mathcal{C}_{(G,H)}$. □

Again, it may seem that this result can be extended to higher orders, but as we have seen already this cannot be true in general. The problem is that the behavior of $\mathrm{Symm}(D^r\sigma)$ under insertion of one section of the subbundle $\mathcal{P} \times_H \mathfrak{h}$ can be computed from $D^i\sigma$ with $i < r$, but it is not sufficent to know $\mathrm{Symm}(D^i\sigma)$ for $i < r$. As we shall see in volume two, there is an analog of the theorem for $r = 2$. This means that the second jet prolongation of any natural bundle associated to the Cartan bundle is again associated to the Cartan bundle. Moreover, any second order invariant operator on G/H canonically extends to a natural operator on $\mathcal{C}_{(G,H)}$.

For higher orders, one can start by looking at *non–holonomic jet prolongations*, i.e. iterated first jet prolongations. For a natural vector bundle bundle EM, we can use the theorem to identify $J^1(J^1EM)$ with the associated bundle corresponding to $J^1(J^1V)$, and similary for higher orders. This can be improved by passing to *semi–holonomic jet prolongations*, which can be constructed from the non–holonomic ones using only functorial properites. For any $r > 0$ this leads to a natural bundle $\bar{J}^rEM$ which is associated to the Cartan bundle with respect to a representation $\bar{J}^rV$. The rth jet prolongation J^rEM naturally includes into $\bar{J}^rEM$. Therefore, H–equivariant maps from $\bar{J}^rV$ to other representations give rise to natural differential operators. However, nonzero maps may lead to the zero operator and not all natural operators are obtained in this way. This circle of ideas will be one of the main topics of volume two.

1.5.11. Automorphisms of Cartan geometries. We next switch to another nice general feature of Cartan geometries. We have seen in 1.5.2 that the automorphisms of the homogeneous model $(G \to G/H, \omega_G)$ are exactly given by left multiplications by elements of G. The aim of this subsection is to show that the group of automorphisms of any Cartan geometry $(\mathcal{P} \to M, \omega)$ is a Lie group (which may have uncountably many connected components) and the dimension of this group at most equals the dimension of G. The first ingredient we need is usually referred to as Lie's second fundamental theorem. This generalizes the fact that Lie algebra homomorphisms integrate to local group homomorphisms (see 1.2.4) to the case of a diffeomorphism group, i.e. it is an analogous result for group actions.

The infinitesimal data for an action of a group G on M is provided by the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$, which is a Lie algebra homomorphism for right actions. Now a *local right action* of the group G on M is given by an open neighborhood U of $M \times \{e\}$ in $M \times G$ and a smooth map $r : U \to M$, such that $r(x, e) = x$ and if $x \in M$ and $g, h \in G$ are such that (x, g), (x, gh) and $(r(x, g), h)$ all are in U, then $r(r(x, g), h) = r(x, gh)$. Given a local action $r : U \to M$, then for $x \in M$ and $X \in \mathfrak{g}$ we get a curve $t \mapsto r(x, \exp(tX))$ defined locally around zero, so the derivative $\frac{d}{dt}|_0 r(x, \exp(tX))$ is always well defined.

LEMMA 1.5.11 (Lie's second fundamental theorem). *Let G be a Lie group with Lie algebra $\mathfrak{g}$, let M be a smooth manifold, and let $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ be a homomorphism of Lie algebras. Then there exists a local right action $r : U \to M$ such that $\phi(X)(x) = \frac{d}{dt}|_0 r(x, \exp(tX))$ for all $x \in M$ and $X \in \mathfrak{g}$.*

This is a classical result, which basically goes back to Sophus Lie. A proof of the version above in modern language can be found in [**Pa57**]. The basic idea of the proof is to consider the distribution on $M \times G$ formed by all pairs $(\phi(X)(x), L_X(g))$, where L_X denotes the left invariant vector field corresponding to X. Since ϕ is assumed to be a Lie algebra homomorphism, this distribution is involutive and thus integrable by the Frobenius theorem. By construction the second projection induces a local diffeomorphism from any leaf to G. For $x \in M$ one can then look at the leaf through (x, e) and inverting the second projection, one obtains a smooth map from an open neighborhood of e in G to M. This can be used to define the action on x which visibly produces the correct fundamental vector fields in x. A careful analysis of the equivariancy properties of this foliation shows that this leads to a local action defined on a neighborhood of $M \times \{e\}$ in $M \times G$.

The next step is to prove a sufficient condition for a transformation group to be a Lie group which is due to [**Pa57**].

PROPOSITION 1.5.11. *Let G be a group of diffeomorphisms of a smooth manifold M and let $S \subset \mathfrak{X}(M)$ be the subset of all vector fields ξ whose flows Fl^{ξ}_t are defined and lie in G for all $t \in \mathbb{R}$. If the Lie subalgebra of $\mathfrak{X}(M)$ generated by S is finite–dimensional, then G is a Lie group of transformations of M and S is the Lie algebra of G.*

PROOF. We present a short proof following [**Ko72**, Theorem 3.1]. Let us write $\mathfrak{g} \subset \mathfrak{X}(M)$ for the Lie algebra of vector fields generated by S and consider the connected and simply connected Lie group $\tilde{G}$ with the Lie algebra $\mathfrak{g}$. By Lie's second fundamental theorem the inclusion $\mathfrak{g} \to \mathfrak{X}(M)$ integrates to a local group action, i.e. there is an open neighborhood U of $M \times \{e\}$ in $M \times \tilde{G}$ and a local action $r : U \to M$ such that $X(x) = \frac{d}{dt}|_0 r(x, \exp(tX))$ for all $X \in \mathfrak{g}$. Of course, this implies that $r(x, \exp tX) = \mathrm{Fl}^X_t(x)$ whenever $(x, \exp tX) \in U$.

First we claim that S generates $\mathfrak{g}$ as a vector space. Denoting by $V \subset \mathfrak{g}$ the vector space generated by S it suffices to prove $[V, V] \subset V$. Consider $X, Y \in S$ and put $Z = \mathrm{Ad}(\exp X)Y \in \mathfrak{g}$. Then $\exp tZ = \exp X \exp tY \exp(-X)$. Now for $x \in M$ and s small enough the action property implies that locally around x one has $r(y, \exp(sX)g\exp(-sX)) = \mathrm{Fl}^X_s(r(\mathrm{Fl}^X_{-s}(y), g))$ and using $\exp(X) = (\exp(\frac{1}{N}X))^N$ for sufficiently large N, we conclude that $r(x, \exp(tZ)) = (\mathrm{Fl}^X_1 \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_{-1})(x)$ for $|t|$ small enough. But again for $|t|$ small enough we have $\mathrm{Fl}^Z_t = r(x, \exp(tZ))$ and hence

$$\mathrm{Fl}^Z_t = \mathrm{Fl}^X_1 \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_{-1}\,.$$

The right–hand side of this equation is defined for all t and is a one–parameter group of diffeomorphisms, so we conclude that Fl^Z_t is defined for all t and thus

$Z \in S$. For $X \in S$ and $t \in \mathbb{R}$ we have $tX \in S$, so our argument shows that $\mathrm{Ad}(\exp(tX))(Y) \in S \subset V$ for all $X, Y \in S$. But differentiating this smooth curve at $t = 0$, the resulting element must also lie in V, so we obtain $[S, S] \subset V$, and since the bracket is bilinear this implies $[V, V] \subset V$.

Now we claim that $S = \mathfrak{g}$. Choose a basis $\{X_1, \dots, X_k\}$ of the vector space $\mathfrak{g}$ consisting of elements of S, and consider the map $\mathfrak{g} \to \tilde{G}$ defined by

$$\sum c^i X_i \mapsto \exp(c^1 X_1) \dots \exp(c^k X_k).$$

This restricts to a diffeomorphism from an open neighborhood of zero in $\mathfrak{g}$ onto an open neighborhood of the unit $e \in \tilde{G}$. For $Y \in \mathfrak{g}$ we thus get smooth functions $c^1, \dots, c^k$, defined for sufficiently small t, such that

$$\exp tY = \exp(c^1(t) X_1) \dots \exp(c^k(t) X_k).$$

Similarly, as above, we next conclude that for each point $x \in M$ we find a neighborhood in M and a bound on $|t|$ up to which we have

$$r(y, \exp tY) = (\mathrm{Fl}^{X_1}_{c^1(t)} \circ \dots \circ \mathrm{Fl}^{X_k}_{c^k(t)})(y)$$

for all y in the neighborhood. Again, the left–hand side coincides with $Fl^Y_t(y)$, while the right–hand side is a one–parameter subgroup defined for all t for which the c^i are defined. Hence, we conclude that the formula for Fl^Y_t is valid for all such t, so for those t the flow Fl^Y_t is defined globally on M. This implies that Fl^Y_t is defined for all $t \in \mathbb{R}$, and hence $Y \in S$.

Now, we know $S = \mathfrak{g}$ so, in particular, S is a Lie algebra and thus $G_0 := \{\mathrm{Fl}^X_t : X \in S, t \in \mathbb{R}\}$ is a subgroup of G. Moreover, $X \mapsto \mathrm{Fl}^X_1$ can be used to define a local chart from an open neighborhood of zero in $\mathfrak{g}$ onto a neighborhood of id in G_0. Transporting this chart around using left multiplications, we obtain an atlas making G_0 into a Lie group. By construction G_0 is connected and since conjugating a one–parameter group of diffeomorphisms by a fixed diffeomorphism gives rise to a one–parameter group, we conclude that G_0 is a normal subgroup of G. Further, it is easy to see that for any $\phi \in G$, conjugation by ϕ defines a continuous homomorphism from G_0 to itself. Thus, we may transport the topology from G_0 to G using either left or right multiplications. This makes G into a topological group which contains G_0 as an open normal subgroup, so G_0 must also be closed and thus the connected component of the identity. Now we can carry over the smooth structure from G_0 to the other connected components, thus making G into a Lie group acting smoothly on M. □

Let us remark, that the topology of the group G from Proposition 1.5.11 is not necessarily second countable since G may have uncountably many connected components. We do not know examples in which this actually occurs.

Theorem 1.5.11. *Let $(\mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H) over a connected manifold M. Then the group* $\mathrm{Aut}(\mathcal{P}, \omega)$ *of all automorphisms of $(\mathcal{P} \to M, \omega)$ is a Lie group of dimension at most* $\dim(G)$.

Proof. By definition, an automorphism Φ of $(\mathcal{P} \to M, \omega)$ is a principal bundle automorphism of $\mathcal{P}$ such that $\Phi^*\omega = \omega$. Equivalently, Φ is a diffeomorphisms of $\mathcal{P}$ such that $\Phi^*\omega = \omega$ and $\Phi \circ r^h = r^h \circ \Phi$ for all $h \in H$. For a vector field $\xi \in \mathfrak{X}(\mathcal{P})$ such that Fl^ξ_t is defined for all t, the condition that each Fl^ξ_t is an automorphism is equivalent to $\mathcal{L}_\xi \omega = 0$ and $(r^h)^*\xi = \xi$. Let us denote by $\mathfrak{a}$ the space of all *infinitesimal automorphisms*, i.e. the space of all vector fields ξ satisfying these two conditions (without assuming existence of the flow for all times). From the conditions it is obvious that $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{X}(P)$. For $\xi \in \mathfrak{X}(\mathcal{P})$ and $X \in \mathfrak{g}$ we get $(\mathcal{L}_\xi \omega)(\omega^{-1}(X)) = 0 - \omega([\xi, \omega^{-1}(X)])$. Thus, $\xi \in \mathfrak{a}$ if and only if ξ commutes with each of the fields $\omega^{-1}(X)$ for $X \in \mathfrak{g}$.

For $\xi \in \mathfrak{a}$, the condition that $[\xi, \omega^{-1}(X)] = 0$ implies that the flows of the two fields commute, and thus, in particular, $\xi(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)) = T\,\mathrm{Fl}_t^{\omega^{-1}(X)} \cdot \xi(u)$ whenever the flow is defined. But for $u \in \mathcal{P}$ the map $X \mapsto \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$ defines a diffeomorphism from an open neighborhood of zero in $\mathfrak{g}$ onto an open neighborhood of u in $\mathcal{P}$. This shows that the value of ξ in u determines ξ locally around u, which together with H–invariance of ξ and connectedness of M implies that ξ is uniquely determined by $\xi(u)$ globally. In particular, we have proved that the evaluation mapping $\mathfrak{a} \to T_u\mathcal{P}$ is injective and hence $\dim(\mathfrak{a}) \leq \dim(\mathfrak{g})$.

Next, let $S \subset \mathfrak{a}$ be the subset of those infinitesimal automorphisms whose flow is defined for all $t \in \mathbb{R}$. By construction, the set S and the group $\mathrm{Aut}(\mathcal{P}, \omega)$ of diffeomorphisms of $\mathcal{P}$ satisfy the assumptions of the proposition above and so $\mathrm{Aut}(\mathcal{P}, \omega)$ is a Lie group with Lie algebra S. Of course, we have $\dim(S) \leq \dim(\mathfrak{a}) \leq \dim(\mathfrak{g})$. □

1.5.12. Infinitesimal automorphisms. In the last subsection, we have introduced the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of a Cartan geometry of type (G, H) as the Lie algebra of all $\xi \in \mathfrak{X}(\mathcal{P})$ such that $(r^h)^*\xi = \xi$ and $\mathcal{L}_\xi \omega = 0$. Notice, in particular, that $\mathfrak{a} \subset \mathfrak{X}(\mathcal{P})^H \cong \Gamma(\mathcal{A}M)$, so infinitesimal automorphisms can be naturally viewed as sections of the adjoint tractor bundle. In the proof of Theorem 1.5.11, we have seen that the Lie algebra of the automorphism group $\mathrm{Aut}(\mathcal{P}, \omega)$ consists exactly of those $\xi \in \mathfrak{a}$ whose flow is defined for all $t \in \mathbb{R}$. Now we can easily characterize $\mathfrak{a} \subset \Gamma(\mathcal{A}M)$ and at the same time determine the Lie algebra structure on $\mathfrak{a}$ in this picture. By definition, $\mathcal{L}_\xi \omega = 0$ is equivalent to $0 = \xi \cdot (\omega(\eta)) - \omega([\xi, \eta])$ for all $\eta \in \mathfrak{X}(\mathcal{P})$. Of course, it suffices to have this property for $\eta \in \mathfrak{X}(\mathcal{P})^H \cong \Gamma(\mathcal{A}M)$. Hence, we see that $s \in \Gamma(\mathcal{A}M)$ corresponds to an infinitesimal automorphism if and only if $0 = D_s t - [s, t]$ for all $t \in \Gamma(\mathcal{A}M)$. Using the formulae derived in 1.5.8 this equation can be equivalently rewritten as

$$\begin{aligned} 0 &= D_s t - (D_s t - D_t s + \{s, t\} - \kappa(s, t)) \\ &= D_t s + \{t, s\} + \kappa(s, t) \\ &= \nabla^{\mathcal{A}}_{\Pi(t)} s + \kappa(s, t), \end{aligned}$$

where $\nabla^{\mathcal{A}}$ is the adjoint tractor connection. Thus, we have proved

LEMMA 1.5.12. *Let $(\mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H). Then for an adjoint tractor field $s \in \Gamma(\mathcal{A}M)$ the following four conditions are equivalent*

(1) *The vector field $\xi \in \mathfrak{X}(\mathcal{P})^H$ corresponding to s is an infinitesimal automorphism.*
(2) *$D_s t = [s, t]$ for all $t \in \Gamma(\mathcal{A}M)$.*
(3) *$D_t s = -\{t, s\} + \kappa(t, s)$ for all $t \in \Gamma(\mathcal{A}M)$.*
(4) *$\nabla^{\mathcal{A}} s = -i_{\Pi(s)}\kappa$.*

Viewing formula (4) as a differential equation defining infinitesimal automorphisms, one observes that this actually means that the infinitesimal automorphisms are exactly the parallel sections for the connection $\hat{\nabla}$ on the vector bundle $\mathcal{A}M$ which is defined as $\hat{\nabla}_\xi s = \nabla^{\mathcal{A}}_\xi s - \kappa(\xi, \Pi(s))$.

Corollary 1.5.12. *The space of infinitesimal automorphisms of a Cartan geometry* $(\mathcal{P} \to M, \omega)$ *of type* (G, H) *is isomorphic to the space of smooth sections of the adjoint tractor bundle* $\mathcal{A}M$*, which are parallel for the linear connection* $\hat{\nabla}$.

This provides an alternative proof for the fact that any infinitesimal automorphism is determined by its value in a single point, which implies the bound on the dimension of the automorphism group.

The lemma also describes the Lie algebra structure on $\mathfrak{a}$, which is by definition induced by the Lie bracket of vector fields on $\mathcal{P}$ and thus by the Lie bracket of adjoint tractors. Indeed, using (2) and (3) we see that for $s, t \in \mathfrak{a} \subset \Gamma(\mathcal{A}M)$ we have $[s, t] = \kappa(s, t) - \{s, t\}$. Notice that $\Pi(s)(u) = 0$ exactly means that the corresponding vector field ξ is vertical in u, so the underlying point $x \in M$ is a fixed point of the base map of the infinitesimal automorphism. If one of the two infinitesimal automorphisms involved has this property, then the bracket is simply the negative of the algebraic bracket on $\mathcal{A}M$, while in general one gets a curvature correction.

1.5.13. Correspondence spaces. Let G be a Lie group and let $K \subset H \subset G$ be closed subgroups. Then there is an obvious G–equivariant projection $G/K \to G/H$. The fiber of this projection over $o = eH \in G/H$ is simply H/K. Left multiplication by elements of G evidently makes $G/K \to G/H$ into a homogeneous bundle, and from 1.4.3 we conclude that $G/K \cong G \times_H (H/K)$. Let us rephrase this in terms of Klein geometries as introduced in 1.4.1. Starting from the Klein geometry (G, H) and a closed subgroup $K \subset H$, we conclude that the total space of the homogeneous fiber bundle $G \times_H (H/K)$ carries the Klein geometry (G, K).

This simple observation carries over to Cartan geometries, leading to a general construction for natural geometries on the total spaces of certain natural bundles. These are then referred to as *correspondence spaces*, since the concept was first formalized in the context of twistor correspondences. We will take up this concept in the realm of parabolic geometries in Section 4.4, but the basic constructions make sense for arbitrary Cartan geometries.

Definition 1.5.13. Let G be a Lie group and $K \subset H \subset G$ be closed subgroups. Let $(p : \mathcal{P} \to N, \omega)$ be a Cartan geometry of type (G, H). Then we define the *correspondence space* $\mathcal{C}N$ of N for $K \subset H$ to be the quotient space $\mathcal{P}/K$.

On the level of Lie algebras, we of course have $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$. In particular, there is an obvious projection $\mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{h}$.

Proposition 1.5.13. *For a closed subgroup* $K \subset H$*, let* $\mathcal{C}N$ *be the correspondence space of a Cartan geometry* $(p : \mathcal{P} \to N, \omega)$ *of type* (G, H)*. Then we have:*

(1) $\mathcal{C}N$ *is the total space of a natural fiber bundle over* N *with fiber the homogeneous space* H/K*, and it carries a canonical Cartan geometry* $(\pi : \mathcal{P} \to \mathcal{C}N, \omega)$ *of type* (G, K).

(2) The curvature functions κ^N *of* $(p : \mathcal{P} \to N, \omega)$ *and* $\kappa^{\mathcal{C}N}$ *of* $(\pi : \mathcal{P} \to \mathcal{C}N, \omega)$ *are related as follows. For* $u \in \mathcal{P}$ *and* $X, Y \in \mathfrak{g}$ *we have*

$$\kappa^{\mathcal{C}N}(u)(X + \mathfrak{k}, Y + \mathfrak{k}) = \kappa^N(u)(X + \mathfrak{h}, Y + \mathfrak{h}),$$

so κ^N *and* $\kappa^{\mathcal{C}N}$ *are induced by the same function* $\mathcal{G} \to L(\Lambda^2\mathfrak{g}, \mathfrak{g})$. *In particular,* $(\pi : \mathcal{P} \to \mathcal{C}N, \omega)$ *is locally flat if and only if* $(p : \mathcal{P} \to N, \omega)$ *is locally flat.*

(3) The subspace $\mathfrak{h}/\mathfrak{k} \subset \mathfrak{g}/\mathfrak{k}$ *is a* K*–submodule, which gives rise to a distribution* $V\mathcal{C}N \subset T\mathcal{C}N$. *This is exactly the vertical subbundle of the projection* $\mathcal{C}N \to N$. *The Cartan curvature* $\kappa^{\mathcal{C}N}$ *of* $\mathcal{C}N$ *has the property that* $i_\xi\kappa^{\mathcal{C}N} = 0$ *for any* $\xi \in \Gamma(V\mathcal{C}N) \subset \mathfrak{X}(\mathcal{C}N)$.

(4) The construction of correspondence spaces defines a functor from the category of Cartan geometries of type (G, H) *to the category of Cartan geometries of type* (G, K). *If the homogeneous space* H/K *is connected, then this is an equivalence onto a subcategory, i.e. any morphism between two correspondence spaces comes from a morphism of the original geometries.*

Proof. (1) Evidently, $\mathcal{C}N = \mathcal{P}/K \cong \mathcal{P} \times_H (H/K)$, so the first claim follows. Since H acts freely on $\mathcal{P}$, the same is true for the subgroup K. Thus, the natural projection $\pi : \mathcal{P} \to \mathcal{C}N$ is a principal fiber bundle with structure group K. By definition the Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is a trivialization of the tangent bundle $T\mathcal{P}$, it is H–equivariant, and $\omega(\zeta_A) = A$ for $A \in \mathfrak{h}$. But then, of course, ω is K–equivariant and $\omega(\zeta_A) = A$ for $A \in \mathfrak{k}$, and hence defines a Cartan connection on $\pi : \mathcal{P} \to \mathcal{C}N$.

(2) The curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ by construction is the same for both geometries. Then the claim about the curvature functions follows from the definition. Since local flatness is equivalent to vanishing of the curvature function, the last statement is evident, too.

(3) Since K is a subgroup of H, we get $\operatorname{Ad}(k)(\mathfrak{h}) \subset \mathfrak{h}$ for all $k \in K$. Hence, $\mathfrak{h}/\mathfrak{k} \subset \mathfrak{g}/\mathfrak{k}$ is K–invariant and $\mathcal{P}\times_K(\mathfrak{h}/\mathfrak{k})$ is a smooth subbundle of $\mathcal{P}\times_K(\mathfrak{g}/\mathfrak{k})$. Since ω defines a Cartan connection on $\pi : \mathcal{P} \to \mathcal{C}N$, we know from 1.5.5 that the latter bundle is isomorphic to $T\mathcal{C}N$. Explicitly, the isomorphism is induced from the map $\mathcal{P} \times \mathfrak{g}/\mathfrak{k} \to T\mathcal{C}N$ defined by $(u, X + \mathfrak{k}) \mapsto T_u\pi \cdot \omega_u^{-1}(X)$. Since the identification of TN with $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$ is also obtained using ω, we see that the tangent map of the projection $\mathcal{C}N \to N$ corresponds to the canonical projection $\mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{h}$. Hence, the vertical subbundle of $\mathcal{C}N \to N$ corresponds to the kernel $\mathfrak{h}/\mathfrak{k}$ of $\mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{h}$. The statement on the curvatures follows immediately from the description of the curvature functions in part (2).

(4) A morphism $(\mathcal{P} \to N, \omega) \to (\tilde{\mathcal{P}} \to \tilde{N}, \tilde{\omega})$ is by definition a principal bundle map $\Phi : \mathcal{P} \to \tilde{\mathcal{P}}$ such that $\Phi^*\tilde{\omega} = \omega$. Since Φ is H–equivariant, it is also K–equivariant. Hence, it induces a smooth map $\mathcal{C}N = \mathcal{P}/K \to \tilde{\mathcal{P}}/K = \mathcal{C}\tilde{N}$ and defines a principal bundle map from $\mathcal{P} \to \mathcal{C}N$ to $\tilde{\mathcal{P}} \to \mathcal{C}\tilde{N}$. Since $\Phi^*\tilde{\omega} = \omega$ by assumption, it is a morphism of Cartan geometries of type (G, K).

Conversely, a morphism $\Phi : (\mathcal{P} \to \mathcal{C}N, \omega) \to (\tilde{\mathcal{P}} \to \mathcal{C}\tilde{N}, \tilde{\omega})$ is a K–equivariant map $\mathcal{P} \to \tilde{\mathcal{P}}$ which is compatible with the Cartan connections. This also defines a morphism $(\mathcal{P} \to N, \omega) \to (\tilde{\mathcal{P}} \to \tilde{N}, \omega)$ if and only if Φ is even H–equivariant. Now observe that since $\tilde{\omega}$ is a Cartan connection on $\tilde{\mathcal{P}} \to \tilde{N}$, the fundamental vector field generated by $A \in \mathfrak{h}$ equals $\tilde{\omega}^{-1}(A)$. Compatibility of Φ with the Cartan connections thus implies that Φ pulls back fundamental vector fields on $\tilde{\mathcal{P}} \to \tilde{N}$ to fundamental vector fields with the same generator on $\mathcal{P} \to N$.

Therefore, Φ commutes with the flows of fundamental vector fields. But the flow of ζ_A is by the principal right action by $\exp(tA)$, so Φ commutes with the principal right actions of elements of the form $\exp(A)$ for $A \in \mathfrak{h}$. Connectedness of H/K implies that K meets each connected component of H, so elements of K together with elements of the form $\exp(A)$ generate the group H. Under this assumption Φ is therefore automatically H–equivariant, which completes the proof. □

Example 1.5.13. Consider a Riemannian manifold N of dimension n. By example (iii) of 1.5.1, N carries a canonical Cartan geometry of type $(\mathrm{Euc}(n), O(n))$. Now take the subgroup $O(n-1) \subset O(n)$ and consider the associated correspondence space $\mathcal{C}N$. We can realize $O(n-1)$ as the stabilizer of a unit vector in $\mathbb{R}^n$ and, as in 1.1.1, identify the homogeneous space $O(n)/O(n-1)$ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Now the associated bundle $\mathcal{P} \times_{O(n)} \mathbb{R}^n$ is the tangent bundle TN. Hence, $\mathcal{C}N = \mathcal{P} \times_{O(n)} S^{n-1}$ can be identified with the *unit sphere bundle* $SN \subset TN$ of all tangent vectors of length one.

By the proposition, we obtain a natural Cartan geometry $\mathcal{P} \to SN$ of type $(\mathrm{Euc}(n), O(n-1))$ on the unit sphere bundle. Realizing $\mathrm{Euc}(n)$ as a matrix group as in 1.1.2 and $O(n-1) \subset O(n)$ as the stabilizer of the first standard basis vector in $\mathbb{R}^n$, we get

$$\mathrm{Euc}(n) = \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} : v \in \mathbb{R}^n, A \in O(n) \right\},$$

and the subgroup $O(n-1)$ corresponds to the matrices in which $v = 0$ and A is of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)$. Looking at the Lie algebras, we see that, as an $O(n-1)$–module, we have $\mathfrak{euc}(n) = \mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \oplus \mathfrak{o}(n-1)$, with $\mathfrak{o}(n)$ corresponding to the last two summands. The first three summands provide us with an $O(n-1)$–invariant complement $\mathfrak{n}$ to $\mathfrak{o}(n-1) \subset \mathfrak{euc}(n)$.

Using the standard inner products on these three summands, we obtain an $O(n-1)$–invariant inner product on $\mathfrak{n}$, which induces a canonical Riemannian metric on SN. By construction, TSN decomposes into the orthogonal direct sum of three subbundles. By part (3) of the proposition, the last summand is the vertical subbundle of $SN \to N$. The other two summands constitute the horizontal subbundle for the Levi–Civita connection (with the lifted metric). This decomposes further into the line subbundle formed by multiples of the footpoint and its orthogonal complement.

1.5.14. Characterization of correspondence spaces. We continue working in the setting of a Lie group G with closed subgroups $K \subset H \subset G$. In 1.5.13 we have shown how to associate to a Cartan geometry of type (G, H) a Cartan geometry of type (G, K) on the correspondence space. Now we want to characterize Cartan geometries of type (G, K) which are locally isomorphic to correspondence spaces.

Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, K). As in part (3) of Proposition 1.5.13, the K–invariant subspace $\mathfrak{h}/\mathfrak{k} \subset \mathfrak{g}/\mathfrak{k}$ determines a smooth subbundle $VM \subset TM$. In the case of a correspondence space $\mathcal{C}N$, this bundle becomes the vertical subbundle of the projection $\mathcal{C}N \to N$, so, in particular, it must be involutive. If M is locally isomorphic to a correspondence space, then this isomorphism is compatible with the subbundles, so VM must be involutive, too. We can characterize involutivity in terms of the torsion of the Cartan connection ω. Recall from 1.5.7 that the torsion $T \in \Omega^2(M, TM)$ of ω is obtained by applying the projection $\Pi : \mathcal{A}M \to TM$ to the values of the Cartan curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$.

Lemma 1.5.14. *Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, K) with torsion $T \in \Omega^2(M, TM)$, and let $VM \subset TM$ be the subbundle corresponding to $\mathfrak{h}/\mathfrak{k} \subset \mathfrak{g}/\mathfrak{k}$. The VM is integrable if an only if $T(VM, VM) \subset VM$.*

PROOF. Let ξ and η be local sections of $VM \subset TM$, and choose local lifts $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{P})$. Then $[\tilde{\xi}, \tilde{\eta}]$ is a local lift of the Lie bracket $[\xi, \eta]$. Thus, we have to check whether $Tp \cdot [\tilde{\xi}, \tilde{\eta}]$ lies in $VM \subset TM$. Since the identification of TM with $\mathcal{P} \times_K (\mathfrak{g}/\mathfrak{k})$ is obtained from $(u, X + \mathfrak{k}) \mapsto T_u p \cdot \omega(u)^{-1}(X)$, this is the case if and only if $\omega([\tilde{\xi}, \tilde{\eta}])$ has values in $\mathfrak{h} \subset \mathfrak{g}$.

The assumptions that ξ and η are sections of VM likewise is equivalent to the fact that $\omega(\tilde{\xi})$ and $\omega(\tilde{\eta})$ have values in $\mathfrak{h} \subset \mathfrak{g}$. In this case, also $\tilde{\xi} \cdot \omega(\tilde{\eta})$ and $\tilde{\eta} \cdot \omega(\tilde{\xi})$ have values in $\mathfrak{h}$. Hence, we see that $[\xi, \eta] \in \Gamma(VM)$ is equivalent to $d\omega(\tilde{\xi}, \tilde{\eta})$ having values in $\mathfrak{h}$. Since $\mathfrak{h}$ is a Lie subalgebra, also $[\omega(\tilde{\xi}), \omega(\tilde{\eta})]$ automatically has values in $\mathfrak{h}$, so we can equivalently replace $d\omega(\tilde{\xi}, \tilde{\eta})$ by $K(\tilde{\xi}, \tilde{\eta})$. But this having values in $\mathfrak{h}$ is equivalent to $T(\xi, \eta)$ having values in the subbundle of TM corresponding to $\mathfrak{h}/\mathfrak{k}$. □

Suppose that the geometry $(p : \mathcal{P} \to M, \omega)$ of type (G, K) satisfies this necessary condition for being locally isomorphic to a correspondence space. Then we can actually construct a candidate for a space N such that M may be locally isomorphic to $\mathcal{C}N$. Namely, in the case of a correspondence space, N is simply the (global) space of leaves of the foliation corresponding to the subbundle $V\mathcal{C}N$. Returning to M, we have to consider spaces which locally parametrize the leaves of the foliation defined by $VM \subset TM$. Due to the origins of this whole circle of ideas in twistor theory, such spaces are called local twistor spaces for M.

DEFINITION 1.5.14. Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, K) such that the subbundle $VM \subset TM$ is integrable. Then a (local) *twistor space* for M is a local leaf space for the foliation defined by VM, i.e. a smooth manifold N together with an open subset $U \subset M$ and a surjective submersion $\psi : U \to N$ such that $\ker(T_x\psi) = V_xM$ for all $x \in U$.

Existence of local twistor spaces follows immediately from the local version of the Frobenius theorem (see [**KMS**, Theorem 3.22]) by projecting onto one factor of an adapted chart. Note that for two local twistor spaces $\psi_i : U_i \to N_i$ there is a unique diffeomorphism $\phi : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$ such that $\phi \circ \psi_1 = \psi_2$.

We know already that correspondence spaces satisfy a much stronger curvature condition than the one from the lemma, since by part (3) of Proposition 1.5.13 we must have $i_\xi \kappa = 0$ for any section ξ of $VM \subset TM$. Surprisingly, this curvature condition is actually equivalent to local isomorphism to a correspondence space:

THEOREM 1.5.14. *Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, K) with curvature κ. Suppose that $i_\xi \kappa = 0$ for all $\xi \in \Gamma(VM)$.*

Then for any sufficiently small local twistor space $\psi : U \to N$ of M, one obtains a Cartan geometry of type (G, H) on N such that $(p^{-1}(U), \omega|_{p^{-1}(U)})$ is isomorphic to an open subspace in the correspondence space $\mathcal{C}N$. If H/K is connected, then this Cartan geometry is uniquely determined.

PROOF. The composition $\psi \circ p : p^{-1}(U) \to N$ is a surjective submersion, so it admits local smooth sections. Choosing U sufficiently small, we may therefore assume that there is a global smooth section $\sigma : N \to p^{-1}(U)$ of $\psi \circ p$. In terms of the curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$, the condition on κ implies $0 = K(\omega^{-1}(A), \omega^{-1}(B))$ for all $A, B \in \mathfrak{h} \subset \mathfrak{g}$. This can be written as $0 = -\omega([\omega^{-1}(A), \omega^{-1}(B)]) + [A, B]$, which means that $A \mapsto \omega^{-1}(A)$ defines a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{X}(\mathcal{P})$, i.e. an action of $\mathfrak{h}$ on $\mathcal{P}$. By Lie's second fundamental theorem (Lemma 1.5.11), this

Lie algebra action integrates to a local group action. There is an open neighborhood W of $\mathcal{P} \times \{e\}$ in $\mathcal{P} \times H$ and a smooth map $F : W \to \mathcal{P}$ such that

- $F(u, e) = u$ and $\frac{d}{dt}|_{t=0} F(u, \exp(tA)) = \omega^{-1}(A)(u)$ for all $u \in \mathcal{P}$ and all $A \in \mathfrak{h}$.
- $F(F(u, g), h) = F(u, gh)$ provided that (u, g), (u, gh) and $(F(u, g), h)$ all lie in W.

Possibly shrinking the leaf space further, we find an open neighborhood $\tilde{V}$ of e in H such that $(\sigma(x), g) \in W$ and $(F(\sigma(x), g), e) \in W$ for all $x \in N$ and all $g \in \tilde{V}$. Then we define $\Phi : N \times \tilde{V} \to \mathcal{P}$ by $\Phi(x, g) := F(\sigma(x), g)$. For $x \in N$ the tangent map $T_{(x,e)}\Phi : T_xN \times \mathfrak{h} \to T_{\sigma(x)}\mathcal{G}$ is evidently given by $(\xi, A) \mapsto T_x\sigma \cdot \xi + \omega^{-1}(A)(\sigma(x))$, so it is a linear isomorphism. Possibly shrinking U and $\tilde{V}$, we may assume that Φ is a diffeomorphism onto an open subset $\tilde{U} \subset \mathcal{P}$, and we arrive at the following picture:

$$\begin{array}{ccccc} N \times V & \xrightarrow{\Phi} & p^{-1}(U) & \hookrightarrow & \mathcal{P} \\ \downarrow \mathrm{pr}_1 & \nearrow \sigma & \downarrow p & & \downarrow p \\ N & \xleftarrow{\psi} & U & \hookrightarrow & M \end{array}$$

Possibly shrinking $\tilde{V}$ further, we may assume that it is of special form: Choose a linear subspace $\mathfrak{n} \subset \mathfrak{h}$ which is complementary to $\mathfrak{k}$. Choose an open ball V_1 around 0 in $\mathfrak{n}$ which is so small that $(X, k) \mapsto \exp(X)k$ is a diffeomorphism from $V_1 \times K$ onto an open neighborhood V of K in H. Then assume that there is an open ball V_2 around 0 in $\mathfrak{k}$ such that $(X, B) \mapsto \exp(X)\exp(B)$ is a diffeomorphism $V_1 \times V_2 \to \tilde{V}$.

For a fixed point $x \in N$, any vector tangent to $\{x\} \times \tilde{V}$ can be written as $\zeta_A(x, g) = \frac{d}{dt}|_{t=0}(x, g\exp(tA))$ for some $g \in \tilde{V}$ and $A \in \mathfrak{h}$. For sufficiently small t, by construction we have $\Phi(x, g\exp(tA)) = F(\Phi(x, g), \exp(tA))$, and thus

$$T_{(x,g)}\Phi \cdot \zeta_A(x, g) = \omega^{-1}(A)(\Phi(x, g)).$$

Thus, we see that $T\Phi \circ \zeta_A = \omega^{-1}(A) \circ \Phi$ for all $A \in \mathfrak{h}$. Moreover, $T\Phi \circ \zeta_A$ always lies in $\omega^{-1}(\mathfrak{h}) \subset T\mathcal{P}$, which implies that $\Phi(\{x\} \times \tilde{V})$ is contained in one leaf of the foliation corresponding to the integrable distribution $\omega^{-1}(\mathfrak{h}) \subset T\mathcal{P}$. Hence, the map $\psi \circ \pi \circ \Phi$ is constant on $\{x\} \times \tilde{V}$, and since $\Phi(x, e) = \sigma(x)$, we conclude that $\psi \circ \pi \circ \Phi = \mathrm{pr}_1 : N \times \tilde{V} \to N$.

For $X \in V_1$ and $B \in V_2$ we have $\exp(X)\exp(tB) \in \tilde{V}$ for all $t \in [0, 1]$. Since $\omega^{-1}(B) \in \mathfrak{X}(\mathcal{P})$ is the fundamental vector field generated by $B \in \mathfrak{k}$, the infinitesimal condition $T\Phi \circ \zeta_B = \omega^{-1}(B) \circ \Phi$ implies

$$\Phi(x, \exp(X)\exp(B)) = \Phi(x, \exp(X)) \cdot \exp(B),$$

where in the right–hand side we use the principal right action on $\mathcal{P}$. Since K acts freely both on $N \times V$ and on $\mathcal{P}$ we can uniquely extend Φ to a K–equivariant diffeomorphism from $N \times V$ to the K–invariant open subset

$$\{u \cdot g : u \in \tilde{U}, g \in K\} = p^{-1}(p(\tilde{U})) \subset \mathcal{P}.$$

Since the family of fundamental vector fields on $N \times P$ and the family of the vector fields $\omega^{-1}(A)$ on $\mathcal{P}$ have the same equivariancy property, this extension still satisfies $T\Phi \circ \zeta_A = \omega^{-1}(A) \circ \Phi$ for all $A \in \mathfrak{h}$.

Next, $\Phi^*\omega \in \Omega^1(N \times V, \mathfrak{g})$ restricts to a linear isomorphism on each tangent space. We can consider the restriction of this form to $N \times \{e\}$ and extend it equivariantly to $\tilde{\omega} \in \Omega^1(N \times H, \mathfrak{g})$ by defining

$$\tilde{\omega}(x,h) := \mathrm{Ad}(h^{-1}) \circ (\Phi^*\omega)(x,e) \circ Tr^{h^{-1}},$$

where r denotes the principal right action of H on $N \times H$. By construction, $\tilde{\omega}$ is smooth, it restricts to a linear isomorphism on each tangent space and satisfies $(r^h)^*\tilde{\omega} = \mathrm{Ad}(h^{-1}) \circ \tilde{\omega}$ for all $h \in H$. In points of the form (x,e) we have $\tilde{\omega} = \Phi^*\omega$, which together with $T\Phi \circ \zeta_A = \omega^{-1}(A) \circ \Phi$ implies that $\tilde{\omega}$ reproduces the generators of all fundamental vector fields in such points. By equivariancy, this holds globally, and thus $\tilde{\omega}$ is a Cartan connection on the principal H–bundle $N \times H \to N$. Hence, we have obtained a Cartan geometry of type (G,H) on N.

Since the vector fields ζ_A and $\omega^{-1}(A)$ are Φ–related, their flows are Φ–related, so $\Phi \circ r^{\exp(tA)} = \mathrm{Fl}_t^{\omega^{-1}(A)} \circ \Phi$ whenever defined. In terms of the curvature form K, the condition $i_\xi \kappa = 0$ for all $\xi \in \Gamma(VM)$ reads as $0 = i_{\omega^{-1}(A)} K$ for all $A \in \mathfrak{h}$. Since $i_{\omega^{-1}(A)}\omega$ is constant, this implies $\mathcal{L}_{\omega^{-1}(A)}\omega = -\,\mathrm{ad}(A) \circ \omega$, and in turn

$$\tfrac{d}{dt}\left(\mathrm{Fl}_t^{\omega^{-1}(A)}\right)^* \omega = -\,\mathrm{ad}(A) \circ \left(\mathrm{Fl}_t^{\omega^{-1}(A)}\right)^* \omega.$$

Solving this differential equation, we obtain $\left(\mathrm{Fl}_t^{\omega^{-1}(A)}\right)^* \omega = \mathrm{Ad}(\exp(-tA)) \circ \omega$ whenever the flow is defined. Now we compute

$$\begin{aligned}(r^{\exp(tA)})^*\Phi^*\omega =& (\Phi \circ r^{\exp(tA)})^*\omega = (\mathrm{Fl}_t^{\omega^{-1}(A)} \circ \Phi)^*\omega \\ =& \Phi^*(\mathrm{Fl}_t^{\omega^{-1}(A)})^*\omega = \mathrm{Ad}(\exp(-tA)) \circ \Phi^*\omega.\end{aligned}$$

Using for A an element X in the open subset V_1, this makes sense for all $t \in [0,1]$, and we conclude that $\Phi^*\omega$ and $\tilde{\omega}$ coincide on all elements of the form $(x, \exp(X))$ for $x \in V_1$. But then K–equivariancy of Φ shows that $\Phi^*\omega = \tilde{\omega}$ on all of $N \times V$. But this exactly means that Φ defines an isomorphism of Cartan geometries from the open subset $N \times (V/H) \subset \mathcal{C}N = N \times (H/K)$ to the open subset $p(\tilde{U}) \subset M$. Now we can simply replace the leaf space $\psi : U \to N$ by $\psi : p(\tilde{U}) \to N$ to obtain the first part of the claim.

In part (4) of Proposition 1.5.13 we have verified that isomorphism of correspondence space implies isomorphism of the underlying geometries provided that H/K is connected, so the uniqueness statement follows. □

The results on correspondence spaces and their characterizations become particularly interesting if the Cartan geometries are determined by some underlying structures. We will take up these issues in the realm of parabolic geometries in Section 4.4.

1.5.15. Invariant Cartan connections and extension functors. Now we move to a second way to relate Cartan geometries of a different type. This nicely illustrates the interplay between the homogeneous model and the associated Cartan geometries. We consider Lie groups G and L and closed subgroups $H \subset G$ and $K \subset L$. We first describe G–invariant Cartan geometries of type (L,K) on G/H. For this to make sense, we must of course assume that $\dim(G/H) = \dim(L/K)$. Then we show that any such geometry gives rise to a functor mapping Cartan geometries of type (G,H) to Cartan geometries of type (L,K). An exposition of the theory of invariant Cartan geometries and substantial examples can be found in [**Ha06**]; see also [**Ha07**].

Recall first the description of G–homogeneous principal bundles over G/H with structure group K from 1.4.5. Any such bundle is of the form $G \times_i K$, where $i : H \to K$ is a smooth homomorphism, and one forms the associated bundle with respect to the left action $h \cdot k = i(h)k$. Two homomorphisms i and $\hat{i}$ give rise to isomorphic homogeneous principal bundles if an only if there is an element $k \in K$ such that $\hat{i}(h) = ki(h)k^{-1}$ for all $h \in H$.

Multiplication in G induces a left action of G on $G \times_i K$ by principal bundle automorphisms. We write ℓ_g for the action of $g \in G$. Then a Cartan connection ω on $G \times_i K$ is called *invariant* if $(\ell_g)^*\omega = \omega$ for all $g \in G$. This means that G acts by automorphisms on the Cartan geometry $(G \times_i K \to G/H, \omega)$.

PROPOSITION 1.5.15. *Let G and L be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{l}$, $H \subset G$ and $K \subset L$ closed subgroups such that* $\dim(G/H) = \dim(L/K)$*, and let $i : H \to K$ be a homomorphism. Then the set of G–invariant Cartan connections on the principal K–bundle $G \times_i K$ is in bijective correspondence with the set of linear maps $\alpha : \mathfrak{g} \to \mathfrak{l}$ such that:*

(i) $\alpha \circ \mathrm{Ad}(h) = \mathrm{Ad}(i(h)) \circ \alpha$ *for all $h \in H$.*
(ii) $\alpha|_{\mathfrak{h}} = i' : \mathfrak{h} \to \mathfrak{k} \subset \mathfrak{l}$*, where i' denotes the derivative of i.*
(iii) *The map $\underline{\alpha} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{l}/\mathfrak{k}$ induced by α is a linear isomorphism.*

For $k_0 \in K$ put $\hat{i}(h) = k_0 i(h) k_0^{-1}$. Then the pullback of the Cartan connection determined by $\alpha : \mathfrak{g} \to \mathfrak{l}$ under the isomorphism $G \times_{\hat{i}} K \to G \times_i K$ corresponds to the map $\hat{\alpha} = \mathrm{Ad}(k_0) \circ \alpha$.

PROOF. This is closely parallel to the description of homogeneous principal connections in Theorem 1.4.5. Let $q : G \times K \to G \times_i K$ be the canonical projection and put $u_0 = q(e, e)$. Given an invariant Cartan connection ω, define $\alpha : \mathfrak{g} \to \mathfrak{l}$ by $\alpha(X) := \omega(u_0)(T_{(e,e)}q \cdot (X, 0))$. In the proof of Theorem 1.4.5, we saw that $T_{(e,e)}q \cdot (0, A) = \zeta_A(u_0)$. As in that proof, this implies that $\omega(u_0)$ is determined by α, and by K–equivariance and G–invariance, this in turn determines ω. Still as in the proof of Theorem 1.4.5 we obtain properties (i) and (ii) for α, and conversely having given α with (i) and (ii) we can construct $\omega \in \Omega^1(G \times_i K, \mathfrak{l})$ which is K–equivariant, G–invariant and reproduces the generators of fundamental vector fields.

Now by K–equivariance and G–invariance, the values of ω all are linear isomorphisms if and only if $\omega(u_0)$ is a linear isomorphism. By dimensional reasons, it suffices to show that $\omega(u_0)$ is surjective. Since $\omega(u_0)$ evidently maps onto $\mathfrak{k}$, this is equivalent to the composition with the projection to $\mathfrak{l}/\mathfrak{k}$ being surjective. This in turn is equivalent to surjectivity of $X \mapsto \alpha(X) + \mathfrak{k}$, and hence by condition (ii) and dimensional reasons to (iii). The behavior of α under the change from i to $\hat{i}$ is proved exactly as in Theorem 1.4.5. □

A homogeneous Cartan geometry of type (L, K) on G/H determines a homomorphism $i : H \to K$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{l}$ which satisfies the conditions (i)–(iii) from the proposition. Suppose now that $(p : \mathcal{P} \to M, \omega)$ is an arbitrary Cartan geometry of type (G, H). As for the homogeneous model, we can of course define a principal K–bundle as $\mathcal{P} \times_i K$. Mapping $u \in \mathcal{P}$ to the class of (u, e) induces a bundle map $j : \mathcal{P} \to \mathcal{P} \times_i K$ which is equivariant over $i : H \to K$.

LEMMA 1.5.15. *In this situation, there is a uniquely determined Cartan connection $\omega_\alpha \in \Omega^1(\mathcal{P} \times_i K, \mathfrak{l})$ such that $j^*\omega_\alpha = \alpha \circ \omega \in \Omega^1(\mathcal{P}, \mathfrak{l})$.*

Proof. This is closely parallel to the proof of part (1) of Theorem 1.5.6. First for $u \in \mathcal{P}$, the tangent space $T_{j(u)}(\mathcal{P} \times_i K)$ is spanned by vertical vectors and elements of $T_u j(T_u\mathcal{P})$. Then one defines

$$\omega_\alpha(j(u))(T_u j \cdot \xi + \zeta_A(j(u))) := \alpha(\omega(u)(\xi)) + A$$

for $A \in \mathfrak{k}$, and verifies that it is well defined using property (i) of α. By definition, we see that $\omega_\alpha(j(u))$ reproduces the generators of fundamental vector fields. Suppose that $\omega_\alpha(j(u))(T_u j \cdot \xi + \zeta_A(j(u))) = 0$. Projecting to $\mathfrak{l}/\mathfrak{k}$, we see that

$$0 = \alpha(\omega(u)(\xi)) + \mathfrak{k} = \underline{\alpha}(\omega(u)(\xi) + \mathfrak{h}).$$

By condition (iii) on α, this implies that $\omega(u)(\xi) =: X \in \mathfrak{h}$, i.e. $\xi = \zeta_X(u)$. But this means that our original vector was vertical, and by construction $\omega_\alpha(j(u))$ is injective on vertical vectors. Thus, $\omega_\alpha(j(u))$ is a linear isomorphism.

As in the proof of Theorem 1.5.6 we then define

$$\tilde{\omega}(j(u) \cdot k)(\eta) = \mathrm{Ad}(k^{-1})(\tilde{\omega}(j(u))(Tr^{k^{-1}} \cdot \eta))$$

and verify that this is well defined using property (ii) of α. Having defined $\omega_\alpha \in \Omega^1(\mathcal{P} \times_i K, \mathfrak{l})$, we see that by construction the value in each point is a linear isomorphism, $j^*\omega_\alpha = \alpha \circ \omega$ and ω_α is uniquely determined by this property. Equivariancy of ω_α immediately follows from the construction. Finally, the fact that ω_α reproduces generators of fundamental vector fields follows by equivariancy from the fact that $\omega_\alpha(j(u))$ has this property. □

Fixing the data (i, α), which are equivalent to a G–invariant Cartan geometry of type (L, K) on G/H, we can now associate to each Cartan geometry of type (G, H) a Cartan geometry of type (L, K). In fact, this construction is functorial:

Theorem 1.5.15. *Let G and L be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{l}$, and let $H \subset G$ and $K \subset L$ be closed subgroups. Fix a homomorphism $i : H \to K$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{l}$ with properties (i)–(iii) from the proposition.*

Then mapping $(\mathcal{P} \to M, \omega)$ to $(\mathcal{P} \times_i K \to M, \omega_\alpha)$ defines a functor from Cartan geometries of type (G, H) to Cartan geometries of type (L, K). Passing from (i, α) to $(\hat{i}, \hat{\alpha})$ as described in the proposition, one obtains a naturally isomorphic functor.

Proof. As in the proof of part (2) of Theorem 1.5.6, a morphism $\Phi : (\mathcal{P} \to M, \omega) \to (\tilde{\mathcal{P}} \to \tilde{M}, \tilde{\omega})$ induces a principal bundle map $F(\Phi) : \mathcal{P} \times_i K \to \tilde{\mathcal{P}} \times_i K$. Since Φ is a local diffeomorphism, the same is true for $F(\Phi)$ and hence $F(\Phi)^*\tilde{\omega}_\alpha$ is a Cartan connection on $\mathcal{G} \times_i K$. Then the fact that $F(\Phi)^*\tilde{\omega}_\alpha = \omega_\alpha$ follows as in Theorem 1.5.6, and functoriality is obvious.

Suppose that $\hat{i}(h) = k_0 i(h) k_0^{-1}$ and $\hat{\alpha} = \mathrm{Ad}(k_0) \circ \alpha$. For a fixed geometry $(\mathcal{P} \to M, \omega)$ of type (G, H), the map $\mathcal{P} \times K \to \mathcal{P} \times K$ defined by $(u, k) \mapsto (u, k_0 k)$ induces an isomorphism $\phi_{\mathcal{P}} : \mathcal{P} \times_i K \to \mathcal{P} \times_{\hat{i}} K$. Denoting by j and $\hat{j}$ the two inclusions, this satisfies $r^{k_0} \circ \hat{j} = \phi_{\mathcal{P}} \circ j$. Using this, we compute

$$j^*(\phi_{\mathcal{P}})^*\omega_{\hat{\alpha}} = \hat{j}^*(r^{k_0})^*\omega_{\hat{\alpha}} = \hat{j}^*(\mathrm{Ad}(k_0^{-1})\omega_{\hat{\alpha}}) = \mathrm{Ad}(k_0^{-1}) \circ \hat{\alpha} \circ \omega = \alpha \circ \omega.$$

By uniqueness in the lemma, this implies that $\phi_{\mathcal{P}}$ defines an isomorphism between the Cartan geometries $(\mathcal{G} \times_i K, \omega_\alpha)$ and $(\mathcal{P} \times_{\hat{i}} K, \omega_{\hat{\alpha}})$. From the constructions it is evident that this defines a natural transformation between the two extension functors. □

Notice that by construction, applying the extension functor determined by (i,α) to the homogeneous model, we exactly recover the invariant Cartan connection on $G\times_i K$ constructed in the proposition.

1.5.16. Extension functors and curvature. Our next task is to compute the curvature of geometries obtained via the functors we have just constructed. Consider a linar map $\alpha:\mathfrak{g}\to\mathfrak{l}$ with the properties required in Proposition 1.5.15. Let us start from $(\mathcal{P}\to M,\omega)$ and consider the adjoint tractor bundles $\mathcal{A}M:=\mathcal{P}\times_H\mathfrak{g}$ and $\tilde{\mathcal{A}}M:=(\mathcal{G}\times_i K)\times_K\mathfrak{l}\cong\mathcal{P}\times_H\mathfrak{l}$. Here $h\in H$ acts on $\mathfrak{l}$ by $\mathrm{Ad}(i(h))$. Now by assumption α is H–equivariant, and hence induces a natural bundle map $\alpha:\mathcal{A}M\to\tilde{\mathcal{A}}M$. Moreover, from Proposition 1.4.6, we know that conditions (i) and (ii) on α imply that

$$(X+\mathfrak{h},Y+\mathfrak{h})\mapsto[\alpha(X),\alpha(Y)]-\alpha([X,Y])$$

defines an H–invariant element of $\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathfrak{l}$. This gives rise to a natural section $\Phi_\alpha\in\Omega^2(M,\tilde{\mathcal{A}}M)$.

Proposition 1.5.16. *In the setting of Proposition 1.5.15, the Cartan curvature $\kappa_\alpha\in\Omega^2(M,\tilde{\mathcal{A}}M)$ of the Cartan geometry $(\mathcal{P}\times_i K,\omega_\alpha)$ is given by*

$$\kappa_\alpha=\alpha\circ\kappa+\Phi_\alpha,$$

where $\kappa\in\Omega^2(M,\mathcal{A}M)$ is the Cartan curvature of $(\mathcal{P}\to M,\omega)$ and $\Phi_\alpha\in\Omega^2(M,\tilde{\mathcal{A}}M)$ is the natural section constructed above.

Proof. By definition of the extension functor $j^*\omega_\alpha=\alpha\circ\omega$. From this, the result follows exactly as in the proof of Corollary 1.5.7. □

Applying this result to the homogeneous model, we see that Φ_α is the natural section corresponding to the Cartan curvature of the invariant Cartan geometry of type (L,K) on G/H obtained in Proposition 1.5.15. In particular, we see that this invariant Cartan geometry on G/H is locally flat if and only if $\alpha:\mathfrak{g}\to\mathfrak{l}$ is a Lie algebra homomorphism.

Example 1.5.16. Here we will only look at a very simple class of examples, a much more substantial one will be discussed in 5.3.13 and 5.3.14. However, in spite of its simple origin, this class of examples leads to quite deep mathematics in specific situations.

The simplest source for the data (i,α) we need is given by homomorphisms $\phi:G\to L$ which have the property that $\phi(H)\subset K$. If this is the case, then ϕ induces a smooth map $\underline{\phi}:G/H\to L/K$. To have a chance for getting an appropriate α, we had to require that G/H and L/K have the same dimension. Therefore, it makes sense to assume that $\underline{\phi}$ is a local diffeomorphism. (Indeed, if $\underline{\phi}$ is a diffeomorphism locally around $o=eH$, then it is a local diffeomorphism everywhere by homogeneity.) If this is the case, then we can put $i:=\phi|_H:H\to K$ and $\alpha:=\phi':\mathfrak{g}\to\mathfrak{l}$. Then $\alpha|_{\mathfrak{h}}=i'$ holds by construction, and $\underline{\alpha}$ can be identified with $T_o\underline{\phi}:\mathfrak{g}/\mathfrak{h}\to\mathfrak{l}/\mathfrak{k}$, which was assumed to be a linear isomorphism. Equivariancy of α then follows from naturality of the adjoint action, i.e. the standard fact that $\mathrm{Ad}(\phi(h))(\phi'(X))=\phi'(\mathrm{Ad}(h)(X))$.

Even simpler, we can start with the closed subgroup $K\subset L$, and consider another closed subgroup $G\subset L$. Then G naturally acts on L/K and we have to assume that the G–orbit of eK is open. If this is the case, then putting $H=G\cap K$, the inclusion of G into L induces an open embedding of G/H into L/K, so we may use this inclusion as the homomorphism ϕ.

Applying this simplest version in the case that both G and L are semisimple and K is an appropriate parabolic subgroup of L leads, in combination with the construction of correspondence spaces, to a number of deep relations between parabolic geometries of different type; see Section 4.5. The most prominent among these is Fefferman's construction of a natural conformal structure on the total space of a circle bundle over a manifold endowed with a CR–structure; see 4.5.1 and 4.5.2.

1.5.17. Cartan's space $\mathcal{S}$ and development of curves. To conclude this section, we will discuss the generalization of the concept of distinguished curves on homogeneous spaces to general Cartan geometries. A nice conceptual approach to this is based on Cartan's space $\mathcal{S}$. This also provides an interesting example of an associated (nonlinear) fiber bundle corresponding to the restriction of an action of G, which carries a canonical general connection; see 1.5.7. The natural bundle $\mathcal{S}$ is defined as the associated bundle with fiber G/H, with H acting on G/H by the restriction of the natural G–action.

The first remarkable fact about $\mathcal{S}$ is that $x \mapsto O(x) = [\![u, o]\!] \in \mathcal{P} \times_H G/H$ defines a natural section O of $\mathcal{S}M \to M$ for every Cartan geometry $(\mathcal{P} \to M, \omega)$ of type (G, H). Moreover, along the image of O, the vertical subbundle of the projection $\mathcal{S}M \to M$ is the associated bundle $\mathcal{P} \times_H T_o(G/H)$. Since $T_o(G/H)$ is canonically isomorphic with $\mathfrak{g}/\mathfrak{h}$, this can be naturally identified with the tangent bundle TM. Thus, we may view Cartan's space $\mathcal{S}$ as a nonlinear version of the tangent bundle in which the geometry in question is encoded by means of the parallel transport of the induced connection. This point of view goes back to Cartan (cf. [**Car37**]) and it was developed further in an abstract way in the second half of the twentieth century (see e.g. [**Kol71**]).

Let us illustrate the power of this parallelism on the classical concept of the development of curves. Fix a point $x \in M$ and consider a smooth curve $c : I \to M$ defined on some open interval $I \subset \mathbb{R}$ containing zero such that $c(0) = x$. The development dev_c of c around $x = c(0)$ is then a smooth curve into the fiber $\mathcal{S}_x M$, which is defined locally around zero and maps 0 to $O(x)$. The idea of the definition is simple: To obtain $\mathrm{dev}_c(t)$, follow the curve c up to time t. Then consider the unique parallel curve in $\mathcal{S}M$ which lies over c and goes through the point $O(c(t))$. Follow this curve back to $t = 0$, to obtain a unique point $\mathrm{dev}_c(t) \in \mathcal{S}_{c(0)}M = \mathcal{S}_x M$.

More formally, given c, define smooth curves c_t by $c_t(s) := c(t + s)$. As in [**KMS**, 9.8], let us denote by $s \mapsto \mathrm{Pt}_\gamma(u, s)$ the parallel curve lying over $s \mapsto \gamma(s)$ starting at the point $u \in \mathcal{S}_{\gamma(0)}$. Then we have $\mathrm{dev}_c(t) = \mathrm{Pt}_{c_t}(O(c(t)), -t)$. By Theorem 9.8 of [**KMS**], this defines a smooth curve for $t \in \mathbb{R}$ with $|t|$ small enough.

Fixing a point $u \in \mathcal{P}_x$, there is a unique curve $\alpha(t)$ in G/H mapping $0 \in \mathbb{R}$ to $o \in G/H$ such that $\mathrm{dev}_c(t) = [\![u, \alpha(t)]\!]$. Choosing a different frame $u \cdot h \in \mathcal{P}_x$, the curve changes to $\ell_{h^{-1}} \circ \alpha$. Hence, if we fix a family of smooth curves through the origin $o = eH$ in G/H which is H–invariant, then the requirement that a local smooth curve through x develops to a member of this family is unambiguously defined.

To obtain a more explicit description of the development, we consider the extended principal bundle $\tilde{\mathcal{P}} = \mathcal{P} \times_H G$. Then $\mathcal{S}M = \mathcal{P} \times_H (G/H) \cong \tilde{\mathcal{P}} \times_G (G/H)$, and we write $q : \mathcal{P} \times (G/H) \to \mathcal{S}M$ and $\tilde{q} : \tilde{\mathcal{P}} \times (G/H) \to \mathcal{S}M$ for the corresponding projections. They are related by $q = \tilde{q} \circ (j \times \mathrm{id})$, where $j : \mathcal{P} \to \tilde{\mathcal{P}}$ is the natural inclusion. From 1.5.7 and Theorem 1.5.6 we know that the Cartan connection ω on $\mathcal{P}$ induces a principal connection $\tilde{\omega}$ on $\tilde{\mathcal{P}}$ which is characterized by $j^*\tilde{\omega} = \omega$.

Now suppose that $c : I \to M$ is a smooth curve with $c(0) = x$. Choose a point $u \in \mathcal{P}_x$ and consider the horizontal lift c^{hor} of c to $\tilde{\mathcal{P}}$ with initial point $j(u)$, which is defined locally around 0. Choosing an arbitrary lift $\bar{c} : I \to \mathcal{P}$ of c, the curve c^{hor} can be written as $j(\bar{c}(t)) \cdot g(t)$ for some smooth G–valued function g. Any other choice of lift can be written as $\bar{c}(t) \cdot h(t)$ for some smooth H–valued function h, and then the function g gets replaced by $h(t)^{-1}g(t)$. Using this we now formulate:

THEOREM 1.5.17. *Let $c : I \to M$ be a smooth curve with $c(0) = x$, let $u \in \mathcal{P}_x$ be a point, and let g be a smooth G–valued function defined locally around zero such that $g(0) = e$.*

(1) Locally around zero, we have $\text{dev}_c(t) = q(u, g(t)^{-1} \cdot o)$ *if and only if there is a lift $\bar{c} : I \to \mathcal{P}$ of c with $\bar{c}(0) = u$ such that the curve $j(\bar{c}(t)) \cdot g(t)$ in $\tilde{\mathcal{P}}$ is horizontal locally around zero.*

(2) Mapping c to dev_c *defines a bijection between germs of smooth curves through x in M and germs of smooth curves through o in G/H. This bijection is compatible with having contact to some order, i.e. two curves c_1 and c_2 have contact of order r in x if and only if their developments have contact of order r in o.*

PROOF. (1) Assume first that we have chosen a lift $\bar{c} : I \to \mathcal{P}$ such that $j(\bar{c}(t)) \cdot g(t)$ is a horizontal curve in $\tilde{\mathcal{P}}$, and we denote by $J \subset \mathbb{R}$ its domain of definition. If we fix some point $y \in G/H$, then by definition of an induced connection, the curve $\alpha(t) := \tilde{q}(j(\bar{c}(t)) \cdot g(t), y)$ is horizontal in $\mathcal{S}M$. Of course, it is a lift of $c : J \to M$. Now for some $t_0 \in J$ put $y = g(t_0)^{-1} \cdot o$. Then

$$\alpha(t_0) = \tilde{q}(j(\bar{c}(t_0)) \cdot g(t_0), g(t_0)^{-1} \cdot o) = \tilde{q}(j(\bar{c}(t_0)), o) = O(c(t_0)).$$

Reading this backwards in time, we see that $\text{Pt}_{c_{t_0}}(O(c(t_0)), s) = \alpha(t_0 - s)$ for all s such that $t_0 - s \in J$. In particular, $\text{dev}_c(t_0) = \alpha(0) = \tilde{q}(j(\bar{c}(0)), g(t_0)^{-1} \cdot o)$. Putting $u = \bar{c}(0)$, we get $\text{dev}_c(t) = q(u, g(t)^{-1} \cdot o)$ as required.

Conversely, assume that $\text{dev}_c(t) = q(u, g(t)^{-1} \cdot o)$ holds locally around zero. Define a smooth function ϕ with values in $\mathfrak{g}$ by $\phi(t) := -\text{Ad}(g(t))(\delta g(t))$, where δ denotes the left logarithmic derivative. Now $\omega(\bar{c}'(t)) = \phi(t)$ is a time dependent first order ODE on $\mathcal{P}$, so locally around 0 we find a unique solution $\bar{c}$ such that $\bar{c}(0) = u$. Now consider $\tilde{\omega}(\frac{d}{dt}(j(\bar{c}(t)) \cdot g(t)))$. As in the proof of Proposition 1.5.3 we see that this is given by

$$\tilde{\omega}\left(Tr^{g(t)} \cdot Tj \cdot \bar{c}'(t) + \zeta_{\delta g(t)}\right) = \text{Ad}(g(t)^{-1})(\omega(\bar{c}'(t))) + \delta g(t),$$

which vanishes by construction. Hence, locally around zero we have constructed a horizontal lift of c as required.

(2) The second part of the proof of (1) actually gives a construction of a curve with given development, at least on the level of germs. Suppose we have given a curve γ through $O(x)$ in $\mathcal{S}_xM$. Fixing $u \in \mathcal{P}$, we can find a smooth G–valued function g such that $\gamma(t) = q(u, g(t)^{-1} \cdot o)$. As in the proof of part (1), we construct a local curve $\bar{c}$ in $\mathcal{P}$, and then we can project this curve down to a curve c through x in M. By construction, $j(\bar{c}(t)) \cdot g(t)$ is the horizontal lift of c starting in u, so $\gamma = \text{dev}_c$ locally around zero.

To prove bijectivity, it therefore remains to show that the curve c is independent of the choices made in its construction. Keeping the point u fixed, any other choice for the function g has the form $h(t)g(t)$ for a smooth H–valued function h. Now the product rule for the left logarithmic derivative reads as

$$\delta(h(t)g(t)) = \mathrm{Ad}(g(t)^{-1})(\delta h(t)) + \delta g(t),$$

compare with the proof of Theorem 1.2.4. Consequently, replacing $g(t)$ by $h(t)g(t)$, the function $\phi(t) = -\mathrm{Ad}(g(t))(\delta g(t))$ gets replaced by

$$-\mathrm{Ad}(h(t)^{-1})(\delta h(t) - \phi(t)) = \mathrm{Ad}((\nu \circ h)(t))(\phi(t)) + \delta(\nu \circ h)(t).$$

Here, ν denotes the inversion on H, and we have used $\delta(\nu\circ h)(t) = -\mathrm{Ad}(h(t))(\delta h(t))$, compare again with 1.2.4. But now suppose that $\bar{c}$ is a local solution of the differential equation $\omega(\bar{c}'(t)) = \phi(t)$. Then as above, we compute that

$$\omega(\tfrac{d}{dt}(\bar{c}(t)\cdot h(t)^{-1}) = \mathrm{Ad}(h(t)^{-1})(\phi(t)) + \delta(\nu \circ h)(t),$$

so we have found the solution for the modified function. But this evidently projects onto the same curve c.

Replacing u by $u{\cdot}h$ for some fixed $h \in H$, we can replace $g(t)$ by $h^{-1}g(t)h$ (which also maps 0 to e). Then the left logarithmic derivative is given by $\delta(h^{-1}g(t)h) = \mathrm{Ad}(h^{-1})(\delta g(t))$, so our function $\phi(t)$ gets replaced by $\mathrm{Ad}(h^{-1})(\phi(t))$. But if $\bar{c}$ is the solution of $\omega(\bar{c}'(t)) = \phi(t)$ with initial value u, then evidently $\bar{c}(t)\cdot h$ is the solution for $\mathrm{Ad}(h^{-1})(\phi(t))$ with initial value $u \cdot h$. Again, this projects to the same curve as $\bar{c}$, so we have established bijectivity of the development.

Finally, suppose that c_1 and c_2 are smooth local curves, which have kth order contact in x. Fixing a point $u \in \mathcal{P}$, the horizontal lifts of c_1 and c_2 to $\tilde{\mathcal{P}}$ with initial value $j(u)$ have kth order contact in $j(u)$. Further, we can choose lifts $\bar{c}_1$ and $\bar{c}_2$ of the curves through u in $\mathcal{P}$, which have kth order contact in u. But then writing the horizontal lifts as $j(\bar{c}_i(t)) \cdot g_i(t)$, we see that the resulting functions g_1 and g_2 have kth order contact in e. Hence, also the developments $q(u, g_1(t)^{-1} \cdot o)$ and $q(u, g_2(t)^{-1} \cdot o)$ have kth order contact in o.

Conversely, if the developments have kth order contact in o, then we can realize them as $q(u, g_i(t)^{-1} \cdot o)$ for G–valued functions g_1 and g_2 which have kth order contact in u. Then the associated $\mathfrak{g}$–valued functions ϕ_i are given in terms of a left logarithmic derivative, so they have contact of order $(k-1)$ in 0. But then the solutions of the associated first order ODEs with initial value u have kth order contact in u, and projecting to M preserves kth order contact. □

1.5.18. Canonical curves. We have discussed canonical curves on a homogeneous space G/H in 1.4.11. Using the development introduced in 1.5.17 above, we can extend any reasonable family of distinguished curves on G/H to a natural family of locally defined curves on each Cartan geometry of type (G, H).

Definition 1.5.18. (1) A family $\mathcal{C}$ of smooth curves through o in G/H is called *admissible* if and only if for each $\gamma \in \mathcal{C}$, each t_0 in the domain of γ and each element $g \in G$ such that $\gamma(t_0) = g^{-1} \cdot o$, the curve $t \mapsto g \cdot \gamma(t + t_0)$ also lies in $\mathcal{C}$.

(2) Let $(\mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H), $x \in M$ a point, and let $c : I \to \mathcal{S}_x M$ be a smooth curve defined on an interval containing zero such that $c(0) = O(x)$. Let γ be a smooth curve in G/H. Then we say that c *is represented by* γ *on* I if and only if γ is defined on I and for some $u \in \mathcal{P}_x$ we have $c(t) = [\![u, \gamma(t)]\!]$ for all $t \in I$.

(3) Fix an admissible family $\mathcal{C}$ of curves and consider a Cartan geometry $(\mathcal{P} \to M, \omega)$ of type (G, H). Then a smooth curve $c : I \to M$ is said to be a *canonical*

curve of type $\mathcal{C}$ if and only if for each $t_0 \in I$, the development $\operatorname{dev}_{c_{t_0}}$ of the curve $c_{t_0}(t) = c(t + t_0)$ is represented by an element of $\mathcal{C}$ on some neighborhood of zero.

The defining property for a canonical curve in (3) can be loosely phrased as "c develops to an element of $\mathcal{C}$ locally around each point". Note that applying the definition of admissibility in part (1) to $t_0 = 0$, we obtain that for $\gamma \in \mathcal{C}$ and $h \in H$ we also have $\ell_h \circ \gamma \in \mathcal{C}$. This property, however, is not sufficient for admissibility. The motivation for the notion of admissibility is the following result, which makes the notion of a canonical curve much more manageable.

Proposition 1.5.18. *Let* $\mathcal{C}$ *be an admissible family of curves in* G/H*, let* $(\mathcal{P} \to M, \omega)$ *be any Cartan geometry of type* (G, H)*, and let* $c : I \to M$ *be a smooth curve.*

Suppose that there is a point $t_0 \in I$ *such that the development of* c_{t_0} *is defined and represented by an element of* $\mathcal{C}$ *on the whole interval* $J = \{t : t + t_0 \in I\}$*. If, in addition,* c *admits a horizontal lift to* $\tilde{\mathcal{P}}$ *which is defined on all of* I *and maps* t_0 *to* $j(\mathcal{P}) \subset \tilde{\mathcal{P}}$*, then* c *is a canonical curve of type* $\mathcal{C}$*.*

In particular, if an arbitrary curve develops to an element of $\mathcal{C}$ *in one point, then its restriction to a sufficiently small interval around that point is a canonical curve.*

Proof. It suffices to consider $t_0 = 0$ and hence $J = I$. Choose some lift $\bar{c} : I \to \mathcal{P}$ of c. By assumption, we have a horizontal lift mapping 0 to $j(u)$ for some $u \in \mathcal{P}$. Choosing some lift $\bar{c} : I \to \mathcal{P}$ of c such that $\bar{c}(0) = u$, the horizontal lift can be written as $t \mapsto j(\bar{c}(t)) \cdot g(t)$ for some smooth function $g : I \to G$. From the proof of Theorem 1.5.17 we see that $\operatorname{dev}_c(t) = [\![u, g(t)^{-1} \cdot o]\!]$ for all $t \in I$. In particular, the curve $t \mapsto g(t)^{-1} \cdot o$ belongs to $\mathcal{C}$.

Now take $t_1 \in I$, and consider the curve $t \mapsto j(\bar{c}(t + t_1)) \cdot g(t + t_1)g(t_1)^{-1}$ in $\tilde{\mathcal{P}}$. This is obtained by the principal right action of a fixed element of G on a horizontal curve, so it is horizontal, too. Its value in $t = 0$ is $j(\bar{c}(t_1)) \in j(\mathcal{P})$ and it lifts the curve c_{t_1}. By Theorem 1.5.17 the development of c_{t_1} is, locally around zero, represented by $t \mapsto [\![\bar{c}(t_1), g(t_1)g(t+t_1)^{-1} \cdot o]\!]$. But since $g(t)^{-1} \cdot o$ lies in $\mathcal{C}$, by admissibility $g(t_1)g(t + t_1)^{-1} \cdot o$ also lies in $\mathcal{C}$. Since $t_1 \in I$ is arbitrary, the result follows. $\square$

Together with Theorem 1.5.17 we see that the structure of the local canonical curves of type $\mathcal{C}$ through any point x in any Cartan geometry looks exactly as the structure of local curves through o in G/H which are in $\mathcal{C}$. This means that many questions about canonical curves (e.g. how many derivatives in the point x are needed to uniquely specify a canonical curve of type $\mathcal{C}$) can be reduced to looking at $o \in G/H$. Questions of this type in the realm of parabolic geometries will be studied in Section 5.3. Notice further, that the proof of Theorem 1.5.17 also shows how canonical curves of type $\mathcal{C}$ can be constructed as projections of solutions of appropriated ODEs.

As a simple example let us look at the case of exponential curves. In 1.4.11 we have started from a subspace $\mathfrak{n} \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$ and we have considered the curves $t \mapsto g \exp((t - t_0)X)H$ for $X \in \mathfrak{n}$ and $g \in G$. If this maps 0 to o, then $g \exp(-t_0X) =: h \in H$, and our curve is given by $t \mapsto h \exp(tX) \cdot o = \exp(t \operatorname{Ad}(h)(X)) \cdot o$. More generally, for $X \in \mathfrak{g}$, consider $c^X(t) := \exp(tX) \cdot o$. If $g^{-1} \cdot o = \exp(t_0X) \cdot o$, then as above we see that $g \cdot c^X(t+t_0) = c^{\operatorname{Ad}(h)(X)}(t)$. Hence, if we suppose that $A \subset \mathfrak{g}$ is any subset such that $\mathrm{Ad}(h)(A) \subset A$ for all $h \in H$, then the family $\mathcal{C}_A := \{c^X : X \in A\}$ of curves through o is admissible. Hence, we have the notion of canonical curves of type $\mathcal{C}_A$ on arbitrary Cartan geometries of type (G, H). In this case, we can describe the canonical curves explicitly:

COROLLARY 1.5.18. *Let $A \subset \mathfrak{g}$ be a subset which is invariant under $\mathrm{Ad}(h)$ for all $h \in H$. Let $(\mathcal{P} \to M, \omega)$ be any Cartan geometry. Then a curve $c : I \to M$ is canonical of type $\mathcal{C}_A$ if an only if locally it coincides up to a constant shift in parameter with the projection of a flow line of a vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ for some $X \in A$.*

PROOF. Put $\bar{c}(t) = \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ for some $u \in \mathcal{P}$ and some $X \in A$. From the definitions and using that $\mathrm{Ad}(\exp(tX))(X) = X$, one immediately verifies that the curve $j(\bar{c}(t)) \cdot \exp(-tX)$ is horizontal in $\tilde{\mathcal{P}}$. By Theorem 1.5.17, the development of the projection to M is represented by $[\![\bar{c}(0), \exp(tX) \cdot o]\!]$, so by the proposition it is canonical of type $\mathcal{C}_A$. Therefore, any curve which locally coincides with such flow lines (up to a constant shift in parameter) is canonical, too.

Conversely, if c is canonical, then around each point, it develops to an element of $\mathcal{C}_A$. Shifting the parameter, the development is given by $[\![u, \exp(tX) \cdot o]\!]$. Now we can locally reconstruct c as in the proof of Theorem 1.5.17. But the resulting data for the ODE is $\phi(t) = -\mathrm{Ad}(\exp(-tX))(-X) = X$, so the solution of the ODE is a flow line of $\omega^{-1}(X)$. □

The simplest possible situation arises if one starts from a reductive Klein geometry (G, H). Then for A one can take a fixed H–invariant complement $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$. The flow lines of constant vector fields corresponding to elements of $\mathfrak{n}$ are preserved by the principal right action on $\mathcal{P}$. Projecting them down to M, one exactly obtains the geodesics of the natural connection on the tangent bundle induced by $\mathfrak{n}$; compare with 1.5.6 and 1.4.11.

In general, fixing some complementary linear subspace $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{g}$, one can also construct families of distinguished coordinates. Choosing $u \in \mathcal{P}$, one can take a neighborhood U of $0 \in \mathfrak{n}$ and define a map $U \to M$ by $X \mapsto p(\mathrm{Fl}_1^{\omega^{-1}(X)}(u))$, which is well defined for small U. In the reductive case, this exactly recovers the usual normal coordinates, while in general one obtains larger families of distinguished coordinates. This will be explored in the realm of parabolic geometries in more detail in 5.1 and 5.3.

1.6. Conformal Riemannian structures

In this section, we finally switch to the question of constructing Cartan geometries from underlying structures. Taking pseudo–Riemannian structures as a motivation, we first discuss the case of affine connections on G–structures. While this case is rather easy, since one has already given the bundle, it nicely illustrates the algebraic nature of the problems involved. The main part of the section is devoted to conformal structures, which are the best known example of a parabolic geometry. Many of the general features of parabolic geometries show up in this case in rather simple form, so conformal structures will be a guideline throughout the book. At the same time, this completes the discussion of the example of the conformal sphere from 1.1.5.

We will treat conformal structures in an explicit and classical style, and only try to indicate the Lie algebraic background that will be used systematically in more

Steps (D) and (E): There is only one element of length two in the subset $W^{\mathfrak{p}} \subset W$, namely $s_n \circ s_{n-2}$. The corresponding cohomology component sits in homogeneity -1 (this uses $n > 4$) and it is the highest weight subspace in $\Lambda^2(\Lambda^2\mathbb{R}^{n*}) \otimes \Lambda^2\mathbb{R}^n$. This highest weight subspaces can be characterized as the kernel of the unique (up to multiples) nonzero contraction

$$\Lambda^2(\Lambda^2\mathbb{R}^{n*}) \otimes \Lambda^2\mathbb{R}^n \to \otimes^3\mathbb{R}^{n*} \otimes \mathbb{R}^n.$$

Geometrically, this means that the full harmonic curvature will be given as the tracefree part of the torsion of any connection on the G_0–structure. The connections on TM coming from the G_0–structure are exactly those, which are induced from linear connections on E via the isomorphism $\Lambda^2 E \cong TM$.

Step (F): The standard tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{2n}$ comes equipped by the canonical split signature scalar product inherited from the defining one on $\mathbb{R}^{2n}$. Moreover, it comes with a natural subbundle $\mathcal{T}^1 \subset \mathcal{T}$ which is isomorphic to E^* and such that $\mathcal{T}/\mathcal{T}^1 \cong E$. This gives rise to filtrations on tensor products, and hence on more general tractor bundles. Enlarging G to the spin group $Spin(n, n)$ also the spin representations give rise to tractor bundles, but we do not go into details here.

4.2. Parabolic contact structures

These are parabolic geometries for which the underlying geometric structure is a contact structure with some additional structure on the contact subbundle. The most important example is provided by CR–structures, for which the additional structure is a complex structure on the contact subbundle. We start by recalling some background on contact structures.

4.2.1. Contact structures and contact connections. Recall from linear algebra that nondegenerate skew symmetric bilinear forms exist only on vector spaces of even dimension, and there they are uniquely determined up to isomorphism. Looking at an even dimensional smooth manifold M, a smooth family of skew symmetric bilinear forms on the tangent spaces of M is just a two–form τ on M. A form $\tau \in \Omega^2(M)$ such that $\tau_x : T_xM \times T_xM \to \mathbb{R}$ is nondegenerate for each $x \in M$, is called an *almost symplectic form* on M, and τ is called a *symplectic form* if, in addition, $d\tau = 0$. In this case (M, τ) is called a *symplectic* manifold. Note that the nondegeneracy condition can also be expresses as the fact that $\tau \wedge \cdots \wedge \tau$ (with half the dimension many factors) is a volume form on M. Since there is an obvious symplectic form on $\mathbb{R}^{2n}$, they exist locally on any manifold of even dimension. The question of global existence of symplectic structures is surprisingly difficult with lots of recent progress, for example, via the Seiberg–Witten equations; see e.g. [**Tau94**].

The standard example of a symplectic structure is provided by the cotangent bundle T^*N of an arbitrary smooth manifold N. This carries a tautological one form $\alpha \in \Omega^1(T^*N)$ defined as follows. Let $\pi : T^*N \to N$ be the projection and $T\pi : TT^*N \to TN$ its tangent map. Then for $\phi \in T_x^*N$ and $\xi \in T_\phi T^*N$ one defines $\alpha(\phi)(\xi) := \phi(T\pi \cdot \xi)$. Choosing local coordinates q^i on M and using the induced coordinates (q^i, p_i) on T^*N one gets $\alpha = \sum_i p_i dq^i$ and hence $d\alpha = \sum dp_i \wedge dq^i$. This is immediately seen to be nondegenerate and thus defines a symplectic structure. Putting $N = \mathbb{R}^n$, we obtain the (constant) standard linear symplectic structure on

$\mathbb{R}^{2n}$, which usually is written in terms of local coordinates as

$$((x_1, \dots, x_{2n}), (y_1, \dots, y_{2n})) \mapsto \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i).$$

A basic result in symplectic geometry is the Darboux theorem, which states that for any symplectic form τ on M and around each point $x \in M$, there exist local coordinates (q^i, p_i) for M such that $\tau = \sum dp_i \wedge dq^i$. In particular, symplectic structures do not have any local invariants.

Looking for an odd–dimensional analog of this concept, one is led to the notion of a *contact form* $\alpha \in \Omega^1(M)$ on a smooth manifold M of dimension $2n+1$, i.e. a form such that $\alpha \wedge (d\alpha)^n$ is a volume form on M. Again there are simple examples of such forms on $\mathbb{R}^{2n+1}$ and hence locally on any manifold of odd dimension. Namely, using coordinates (t, q_i, p_i), one defines $\alpha = dt + \sum p_i dq^i$, which is immediately seen to be a contact form. There is a contact version of the Darboux theorem which says that given any contact form, one may locally choose coordinates in which it is given by the above expression. So contact forms do not have local invariants either.

If α is a contact form on M, then by definition $\alpha(x) \neq 0$ for all $x \in M$, so the pointwise kernels of α form a codimension one subbundle H of the tangent bundle TM, called the *contact subbundle*. Again by construction, the restriction of $d\alpha$ to $H \times H$ is a nondegenerate skew symmetric bilinear form. The quotient bundle $Q := TM/H$ is a real line bundle, which is trivialized by α.

Defining $T^{-2}M = TM$ and $T^{-1}M := H$, we obtain a filtration of the tangent bundle of M with associated graded $\text{gr}(TM) = Q \oplus H$. The condition on compatibility with the Lie bracket from Definition 3.1.7 is vacuous in this case, so this makes M into a filtered manifold, and we can look at the Levi bracket $\mathcal{L} : H \times H \to Q$ induced by this filtration. For sections ξ and η of $H = \ker(\alpha)$ the definition of the exterior derivative implies that $d\alpha(\xi, \eta) = -\alpha([\xi, \eta])$, so we see that, viewing α as a trivialization of Q, we have $\alpha \circ \mathcal{L} = -d\alpha$. Hence, $\mathcal{L}$ is nondegenerate, and the symbol algebra in each point is $\mathbb{R} \oplus \mathbb{R}^{2n}$ with the bracket $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ being nondegenerate. This graded Lie algebra is called the real *Heisenberg algebra* $\mathfrak{h}_{2n+1}$.

Now one defines a *contact structure* on a smooth manifold M of dimension $2n+1$ as a smooth subbundle $H \subset TM$ of rank $2n$ such that, putting $Q = TM/H$, the Levi bracket $\mathcal{L} : H \times H \to Q$ is nondegenerate in each point. More elegantly, one can say that a contact manifold is a filtered manifold for which each symbol algebra is a Heisenberg algebra.

As we have observed in 3.1.7, the last statement implies that for any contact structure $H \subset TM$, the associated graded to the tangent bundle, $\text{gr}(TM) = Q \oplus H$ has a natural frame bundle with structure group $\text{Aut}_{\text{gr}}(\mathfrak{h}_{2n+1})$, the group of automorphisms of the graded Lie algebra $\mathfrak{h}_{2n+1}$. The fiber of this bundle over $x \in M$ is just the space of isomorphisms $\mathfrak{h}_{2n+1} \to (Q_x \oplus H_x, \mathcal{L})$ of graded Lie algebras. Let us first determine the group $\text{Aut}_{\text{gr}}(\mathfrak{h}_{2n+1})$.

LEMMA 4.2.1. *Any automorphism of $\mathfrak{h}_{2n+1}$ is uniquely determined by its restriction $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Viewed as a subgroup of $GL(2n, \mathbb{R})$, the group $\text{Aut}_{\text{gr}}(\mathfrak{h}_{2n+1})$ is generated by $Sp(2n, \mathbb{R})$, multiples of the identity, and the diagonal matrix $\mathbb{I}_{n,n}$ with the first n entries equal to 1 and the other n equal to -1.*

The subgroup of those automorphisms, which, in addition, preserve an orientation on $\mathbb{R} \subset \mathfrak{h}_{2n+1}$ is the conformal symplectic group $CSp(2n, \mathbb{R})$ generated by $Sp(2n, \mathbb{R})$ and multiples of the identity.

PROOF. As before let $[\ ,\ ]:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$ be the bracket in $\mathfrak{h}_{2n+1}$ given by the standard symplectic form on $\mathbb{R}^n$. Surjectivity of this bracket immediately implies the first statement. For $A\in Sp(2n,\mathbb{R})$ we by definition have $[Ax,Ay]=[x,y]$, so this gives rise to an automorphism. Likewise, for $a\in\mathbb{R}\setminus 0$, we get $[ax,ay]=a^2[x,y]$, so multiples of the identity are in $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{h}_{2n+1})$, too. Finally, by definition of $[\ ,\ ]$, we see that $[\mathbb{I}_{n,n}x,\mathbb{I}_{n,n}y]=-[x,y]$.

Conversely, suppose that $A\in GL(2n,\mathbb{R})$ and $a\in\mathbb{R}\setminus 0$ are such that $[Ax,Ay]=a[x,y]$. If $a<0$, then replace A by $\mathbb{I}_{n,n}A$ to get an element for which $a>0$. Dividing then by $\sqrt{a}$, we obtain an element which preserves $[\ ,\ ]$ and thus lies in $Sp(2n,\mathbb{R})$. The last statement is then obvious. □

The first statement implies that any isomorphism $\mathfrak{h}_{2n+1}\to(H_x\oplus Q_x,\mathcal{L}_x)$ is uniquely determined by the component mapping $\mathbb{R}^{2n}$ to H_x. Hence, the natural frame bundle for $H\oplus Q$ can be viewed as a subbundle of the frame bundle of H. If we choose an orientation on Q (for example the one induced by a contact form), then the structure group of the natural frame bundle is reduced to $CSp(2n,\mathbb{R})$.

We collect some important properties of contact structures in the following:

PROPOSITION 4.2.1. *Let $H\subset TM$ be a contact structure on a smooth manifold M of dimension $2n+1$ with quotient bundle $Q=TM/H$. Let $p:E\to M$ be the natural frame bundle for $H\oplus Q$ with structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{h}_{2n+1})$. Let $\mathcal{L}:\Lambda^2H\to Q$ be the Levi bracket and let $\Lambda^2_0H\subset\Lambda^2H$ be the kernel of $\mathcal{L}$.*

(1) Locally, there exists a contact form α which has H as its contact subbundle, and this form is unique up to multiplication by a nowhere vanishing function. In particular, contact structures have no local invariants. There exists a global contact form α for H if and only if the quotient bundle Q is orientable and hence trivial.

(2) Any principal connection on E is completely determined by the induced linear connection on the vector bundle H. A linear connections on H arises in this way if and only if the induced connection on Λ^2H preserves the subbundle Λ^2_0H.

(3) If $\alpha\in\Omega^1(M)$ is a contact form with contact subbundle H, then there is a unique vector field r on M such that $\alpha(r)=1$ and $i_rd\alpha=0$. In particular, α induces an isomorphism $TM\cong H\oplus\mathbb{R}$.

(4) Given α as in (3), there is a linear connection ∇ on TM such that ∇ preserves the subbundle H, $\nabla\alpha=0$, $\nabla d\alpha=0$, and $\nabla r=0$, and such that the restriction to H is induced by a principal connection on E as in (2).

PROOF. (1) The bundle $Q=TM/H$ is locally trivial, and a local trivialization α of this bundle can be viewed as a local one–form on M whose kernel in each point is the fiber of H. In this picture, $d\alpha|_{\Lambda^2H}=-\alpha\circ\mathcal{L}$, which implies that α is a contact form. Conversely, a local contact form with contact subbundle H factors to a local trivialization of Q, so we get a bijective correspondence. From this, (1) follows immediately.

(2) Since E can be viewed as a subbundle of the frame bundle of H, a principal connection on E is uniquely determined by the induced linear connection on H. On the other hand, by definition, a linear map $A\in GL(2n,\mathbb{R})$ extends to an automorphism of $\mathfrak{h}_{2n+1}$ if and only if the induced map on $\Lambda^2\mathbb{R}^{2n*}$ preserves the line generated by $[\ ,\ ]$. By duality this is equivalent to the fact that the induced map on $\Lambda^2\mathbb{R}^{2n}$ preserves the kernel of $[\ ,\ ]$. From this, the description of the induced connections follows immediately.

(3) Since α is nowhere vanishing, we can locally find a vector field ξ such that $\alpha(\xi)$ is nowhere vanishing, and mutliplying by an appropriate function we may assume $\alpha(\xi) = 1$. Then we can look at the restriction of $i_\xi d\alpha$ to H, which defines a section of H^*. By nondegeneracy, there is a section η of H such that $i_\xi d\alpha = i_\eta d\alpha$, and $r := \xi - \eta$ has the required properties. If $\hat{r}$ has the same properties, then $\alpha(\hat{r} - r) = 0$, so $\hat{r} - r \in \Gamma(H)$. But then $i_{\hat{r}-r} d\alpha = 0$ implies $\hat{r} = r$ by nondegeneracy of $d\alpha$ on H, and uniqueness follows. The isomorphism $TM \cong H \oplus \mathbb{R}$ is then given by $\xi \mapsto (\xi - \alpha(\xi)r, \alpha(\xi))$.

(4) The choice of contact form α reduces the structure group of the frame bundle E from part (2) further to $Sp(2n, \mathbb{R})$. Explicitly, the fiber over $x \in M$ of this reduction is given by all linear isomorphisms $\phi : \mathbb{R}^{2n} \to H_x$ for which $d\alpha(\phi(v), \phi(w)) = [v, w]$. This bundle admits a principal connection, and we take the induced linear connection on H and extend it by the trival connection to a connection on $H \oplus \mathbb{R}$. Via the isomorphism from (3), this gives a linear connection on TM which preserves the subbundle H. By construction of the frame bundle, this linear connection satisfies $\nabla d\alpha = 0$ and by the trival extension we have $\nabla r = 0$. Since H is preserved, this easily implies $\nabla \alpha = 0$. □

The connections described in part (2) are called *contact connections* for the contact structure H. Given a choice α of contact form, the vector field r from (3) is called the *Reeb vector field* for α. Linear connections as in (4) are called contact connections adapted to the contact form α. There are evident analogs of (2) and (4) for partial connections (see 1.3.7) which leads to the notion of partial contact connections.

There is a contact analog of the canonical symplectic structure on a cotangent bundle. Namely, let M be a smooth manifold of dimension $n+1$ and let $\mathcal{P}T^*M$ be the *projectivized cotangent bundle*. This is the $\mathbb{R}P^n$–bundle over M whose fiber over x is the space of all lines in T^*_xM. If ℓ is such a line, we define a hyperplane $H_\ell \subset T_\ell \mathcal{P}T^*M$ as the space of those tangent vectors, whose projection to T_xM is annihilated by ℓ. This can be viewed as the image of the kernel of the tautological one–form α on T^*M under the tangent map of the projection from $T^*M \setminus M$ (the complement of the zero section) to $\mathcal{P}T^*M$. Choosing a local section σ of this projection, the subbundle H is realized as the kernel of $\sigma^*\alpha$. Now in a point ϕ of T^*M, the restriction of $d\alpha(\phi)$ to $\ker(\alpha(\phi))$ is degenerate with null space given by those vertical tangent vectors which are multiples of the foot point ϕ. This immediately implies that $\sigma^*\alpha$ defines a local contact form for H. By part (1) of the proposition, any contact structure in dimension $2n+1$ is locally isomorphic to $\mathcal{P}T^*M$.

4.2.2. Generalities on parabolic contact structures. Recall from 3.2.4 that a contact grading is a $|2|$–grading such that $\mathfrak{g}_{-2}$ is one–dimensional and the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate. This exactly means that the Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra. In 3.2.4 and 3.2.10 we have seen that contact gradings exist only on simple Lie algebras and obtained a complete classification of both complex and real contact gradings.

Given a real contact grading and corresponding groups $P \subset G$, the first ingredient for an infinitesimal flag structure of type (G, P) is a filtration $TM = T^{-2}M \supset T^{-1}M$, where $\dim(M) = \dim(\mathfrak{g}_-)$ and $H := T^{-1}M$ has corank one. Regularity of the infinitesimal flag structure in particular requires that each symbol algebra of this filtration is isomorphic to $\mathfrak{g}_-$, i.e. that H defines a contact structure on M. The subgroup $G_0 \subset P$ acts on $\mathfrak{g}_-$ by Lie algebra automorphism, so it can be viewed as a subgroup of $GL(\mathfrak{g}_{-1})$, and the additional ingredient of a regular infinitesimal flag structure is a reduction of structure group of H to this subgroup.

If $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in homogeneous degrees ≤ 0, then regular normal parabolic geometries are equivalent to regular infinitesimal flag structures and hence to contact structures with an additional reduction of structure group to $\mathrm{Ad}(G_0) \subset \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. There is only one parabolic contact structure for which this condition is not satisfied, namely contact projective structures. These will be discussed separately in 4.2.6 below.

Next, let us move towards a description of harmonic curvature components. We will be less detailed here than in the case of $|1|$–gradings. A general approach to the description of harmonic curvature of arbitrary parabolic geometry will be developed using Weyl structures in Chapter 5. First we can prove a general fact on the cohomology for contact gradings, which heavily restricts the possibilities for harmonic curvatures of torsion type.

LEMMA 4.2.2. *Consider a contact grading* $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_{-2}$, *and the Kostant Laplacian* $\square$ *on* $\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}$. *Then*

$$\ker(\square) \cap (\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}_-) \subset (\Lambda_0^2\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_-.$$

PROOF. Since the statement of the lemma is invariant under complexification, we may assume that we deal with a complex contact grading. The key point here is that Kostant's version of the Bott–Borel–Weyl theorem asserts that the highest weight of any irreducible component of $\ker(\square)$ occurs with multiplity one in $\Lambda^*\mathfrak{g}_-^* \otimes \mathfrak{g}$; see part (2) of Theorem 3.3.5.

Now as a $\mathfrak{g}_0$–module $\Lambda^2\mathfrak{g}_-^*$ decomposes as $\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \oplus \Lambda^2\mathfrak{g}_{-1}^*$. Via the decomposition $\Lambda^2\mathfrak{g}_{-1} = \Lambda_0^2\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ induced by the bracket, the second summand decomposes further as the sum of $(\Lambda_0^2\mathfrak{g}_{-1})^*$ with $\mathfrak{g}_{-2}^* \cong \mathfrak{g}_2$. Tensoring with $\mathfrak{g}$ we again obtain a decomposition into three summands. The last of these is (isomorphic to) $\mathfrak{g}_2 \otimes \mathfrak{g}$. Since the same module is also contained in $\mathfrak{g}_-^* \otimes \mathfrak{g}$, none of its weights can occur with multiplicity one inside $\Lambda^*\mathfrak{g}_-^* \otimes \mathfrak{g}$.

For the first summand, we can apply similar arguments in a more restricted situation. Since $\mathfrak{g}_{-2}$ is one–dimensional, the representation $\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$ is trivial, so $\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2} \cong \mathfrak{g}_1$, which also sits in $\mathfrak{g} = \Lambda^0\mathfrak{g}_-^* \otimes \mathfrak{g}$. Finally, consider the bracket $\mathfrak{g}_2 \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_1$. Recall that the Killing form B induces dualities between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ and between $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$. Now for $X, Y \in \mathfrak{g}_{-1}$ and $0 \neq \beta \in \mathfrak{g}_2$ we get $B([\beta, X], Y) = B(\beta, [X, Y])$. Nondegeneracy of the bracket on $\mathfrak{g}_{-1}$ shows that $\mathrm{ad}(\beta) : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is injective and thus an isomorphism, so $\mathfrak{g}_2 \otimes \mathfrak{g}_{-1} \cong \mathfrak{g}_1$. Hence, we conclude that

$$\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \cong \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1,$$

which also sits in $\mathfrak{g}_-^* \otimes \mathfrak{g}$.

Altogether we see that for any weight of $\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}_-$ that occurs with multiplicity one in $\Lambda^*\mathfrak{g}_-^* \otimes \mathfrak{g}_-$ the weight space has to be contained in $(\Lambda_0^2\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_-$, so the same is true for the $\mathfrak{g}_0$–submodule generated by this weight space. $\square$

Let us analyze the consequences of this proposition for the possible locations of harmonic curvature components. Components of $\ker(\square)$ contained in homogeneity zero are irrelevant for harmonic curvature. For homogeneity one, there is only one possibility, namely components of $(\Lambda^2_0\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}$. For homogeneity two, the lemma does not leave any room in torsion types. Therefore, the only possibility is curvatures coming form $\Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$, and from the proof we again see that these actually have to be contained in $(\Lambda^2_0\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0$.

To describe the harmonic curvatures in homogeneity one and two, we have to interpret the procedure for constructing a normal Cartan connection from an infinitesimal flag structure similarly as in the proof of Theorem 4.1.1. Suppose that (E, θ) is a regular infinitesimal flag structure of some type (G, P) corresponding to a contact grading. The idea of the prolongation procedure in Section 3.1 was to first make some choices in order to construct a Cartan connection $\tilde{\omega}$ on $\mathcal{G} := E \times P_+$ and then modify this to a normal Cartan connection ω. To obtain $\tilde{\omega}$, one has to choose a principal connection γ on E, as well as a projection π from TM onto the subbundle H. Having made these choices, one constructs from θ a Cartan connection $\tilde{\theta}$ on E, which then can be trivially extended to a regular Cartan connection $\tilde{\omega}$ on $\mathcal{G}$. Now we can interpret this in terms of linear connections.

Since both H and $Q = TM/H$ are associated bundles to E, the principal connection γ induces linear connections ∇^H on H and ∇^Q on Q. By construction, these have the property that $\mathcal{L} : H \times H \to Q$ is parallel, which can be used as a characterization of ∇^Q. Since E is a subbundle of the natural frame bundle of $Q \oplus H$, we see from 4.2.1 that ∇^H is a contact connection. Conversely, a contact connection ∇^H which is compatible with the additional structure induced by E can be used to define γ, and then ∇^Q is obtained from $\nabla\mathcal{L} = 0$. A choice of a projection $\pi : TM \to H$ induces an isomorphism $TM \cong H \oplus Q$ via $\xi \mapsto (\pi(\xi), \xi + H)$. Using this isomorphism, the connections ∇^H and ∇^Q induce a linear connection ∇ on TM. Using this, we can now formulate the basic result on the interpretation of harmonic curvature components of parabolic contact structures.

Theorem 4.2.2. *Let $(E \to M, \theta)$ be an infinitesimal flag structure of type (G, P) corresponding to a contact grading such that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in homogeneities ≤ 0. Let $H \subset TM$ be the contact subbundle, $Q = TM/H$ the quotient bundle, and $q : TM \to Q$ the natural quotient map. Let κ_H be the harmonic curvature of a regular normal parabolic geometry of type (G, P) with underlying infinitesimal flag structure (E, θ). For a principal connection γ and a projection $\pi : TM \to H$ let ∇^H, ∇^Q and ∇ be the induced linear connections on H, Q, and TM.*

(1) For any choice of γ and π, the component $(\kappa_H)_1$ in homogeneity 1 is represented by the component in $\ker(\Box) \subset \Lambda^2H^* \otimes H$ of the tensor $\tau \in \Gamma(\Lambda^2H^* \otimes H)$ defined by

$$\tau(\xi, \eta) := \nabla_\xi\eta - \nabla_\eta\xi - \pi([\xi, \eta]), \tag{4.5}$$

for $\xi, \eta \in \Gamma(H)$. The tensor τ depends only on ∇^H in H–directions.

(2) For any choice of γ, there is a unique π such that component $H \otimes Q \to Q$ of the torsion of ∇ vanishes. The projection π is characterized by

$$\mathcal{L}(\pi(\eta), \xi) = \nabla^Q_\xi q(\eta) - q([\xi, \eta]) \tag{4.6}$$

for all $\xi \in \Gamma(H)$ and all $\eta \in \mathfrak{X}(M)$. In particular, if ∇ is compatible with a contact form α, then this is equivalent to $\pi(\eta) = \eta - \alpha(\eta)r$, where r is the Reeb vector field.

(3) Suppose that γ and π are chosen in such a way that the homogeneous component of degree one of the torsion of ∇ has values in $\ker(\Box)$ (which in particular

implies that π *is the projection from (2)). Then the component* $(\kappa_H)_2$ *in homogeneity* 2 *is represented by the component in* $\ker(\Box) \subset \Lambda_0^2 H^* \otimes (E \times_{G_0} \mathfrak{g}_0)$ *of the curvature* R *of* ∇*.*

Proof. (1) As indicated above, we put $\mathcal{G} = E \times P_+$, and use γ and π to define $\tilde{\omega}$. Denoting by $i : E \to \mathcal{G}$ the inclusion and by $\tilde{\kappa}$ the curvature of $\tilde{\omega}$, the pullback $i^*\tilde{\kappa}$ is given by the torsion and the curvature of ∇. Now normalizing $\tilde{\omega}$, we see that the homogeneous components of degree one $\text{gr}_1(\tilde{\kappa})$ of $\tilde{\kappa}$ and $\text{gr}_1(\kappa)$ differ by an element in the image of ∂. Normality of ω implies that $\text{gr}_1(\kappa)$ lies in $\ker(\Box)$. As in the proof of Theorem 4.1.1, the harmonic curvature component $(\kappa_H)_1$ is represented by $\text{gr}_1(\kappa)$, which coincides with the $\ker(\Box)$–component of the $\text{gr}_1(\tilde{\kappa})$. It follows directly from the definition of torsion that the component $H \times H \to H$ of the torsion of ∇ is given by (4.5). The last claim is obvious from the formula.

(3) Now if we manage to choose γ and π in such a way that $\text{gr}_1(\tilde{\kappa})$ actually is a section of $\ker(\Box)$, then the homogeneous component in degree two of $\tilde{\kappa}$ differs from the one of κ only by elements in the image of ∂. Hence, the $\ker(\Box)$–components in homogeneity two coincide and by the lemma we know that $\ker(\Box) \subset L(\Lambda^2 TM, E \times_{G_0} \mathfrak{g}_0)$. Then the description follows immediately.

(2) First look at the right–hand side of (4.6) for fixed η and variable ξ. Since $q(\xi) = 0$, this is linear over smooth functions, and hence defines (still for fixed η) a bundle map $H \to Q$. By nondegeneracy, there is a unique section $\pi(\eta)$ such that (4.6) holds. But by the Leibniz rule, the right–hand side is also linear over smooth functions in η, so we actually get a bundle map $\pi : TM \to H$ in this way. Finally, if $\eta \in \Gamma(H)$, then $q(\eta) = 0$ and $q([\xi, \eta]) = \mathcal{L}(\xi, \eta)$ and hence $\pi(\eta) = \eta$. Therefore, (4.6) uniquely defines a projection $\pi : TM \to H$.

To get the component $Q \otimes H \to Q$ of the torsion, we have to proceed as follows. For a vector field η consider $\eta - \pi(\eta)$ (which represents the section $q(\eta)$ of Q), take a section ξ of H and consider

$$q\left(\nabla_\xi(\eta - \pi(\eta)) - \nabla_{\eta-\pi(\eta)}\xi - [\xi, \eta - \pi(\eta)]\right) = q(\nabla_\xi \eta) - q([\xi, \eta]) + \mathcal{L}(\xi, \pi(\eta)),$$

where we have used that ∇ preserves the subbundle H. But by construction of ∇ we have $q(\nabla_\xi \eta) = \nabla_\xi^Q q(\eta)$, so the characterization of π follows.

In the case that ∇ is compatible with a contact form α, we can rewrite the characterizing equation as

$$-d\alpha(\pi(\eta), \xi) = \xi \cdot \alpha(\eta) - \alpha([\xi, \eta]).$$

Since $\alpha(\xi) = 0$, the right–hand side equals $d\alpha(\xi, \eta)$, which shows that $\eta - \pi(\eta)$ must be a multiple of the Reeb field. But then $\alpha(\pi(\eta)) = 0$ implies that the factor must equal $\alpha(\eta)$. □

This result is not sufficient to describe all harmonic curvatures in homogeneity one and two. While it tells us how to choose π for given γ, it does not show how to choose γ in order to satisfy the condition in (3). This depends on the concrete choice of structure, and we will indicate it in some cases below. Finally, let us remark that a complete description of the harmonic curvature will be obtained using Weyl structures in Chapter 5.

4.2.3. Lagrangean contact structures. We start the discussion of the individual parabolic contact structures with the A_n–series. From 3.2.10 we know that in this series the algebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{su}(p, q)$ with $p, q > 0$ admit contact gradings.

Lagrangean contact structures correspond to the split real form $\mathfrak{sl}(n,\mathbb{R})$. The name of this structures goes back to M. Takeuchi; see [**Tak94**]. It is derived from the fact that maximal isotropic subspaces in symplectic vector spaces are called *Lagrangean subspaces*. In contact geometry, maximal isotropic subbundles of the contact bundle are often called *Legendrean*, so the name Legendrean contact strucutures would also be appropriate.

For $n \geq 1$ consider the Lie algebra $\mathfrak{g} := \mathfrak{sl}(n+2,\mathbb{R})$. The contact grading on this algebra comes from decomposing into blocks of size 1, n, and 1, so

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & \gamma \\ X & A & W \\ \beta & Y & b \end{pmatrix} : a, b, \beta, \gamma \in \mathbb{R}; X, W \in \mathbb{R}^n; Z, Y \in \mathbb{R}^{n*}; a + b + \operatorname{tr}(A) = 0 \right\}.$$

The grading components are indicated by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^E & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1^F \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^F & \mathfrak{g}_0 \end{pmatrix},$$

where we have indicated the splittings $\mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm 1}^E \oplus \mathfrak{g}_{\pm 1}^F$ for later use. The following facts are easily seen from this block form. The trace form (and hence any invariant form on $\mathfrak{g}$) induces a duality between $\mathfrak{g}_{-1}^E$ and $\mathfrak{g}_1^E$ and between $\mathfrak{g}_{-1}^F$ and $\mathfrak{g}_1^F$. The splittings of $\mathfrak{g}_{\pm 1}$ are invariant under the adjoint action of $\mathfrak{g}_0$ which induces a surjection $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1}^E)$ with one–dimensional kernel. The Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is trivial on $\mathfrak{g}_{-1}^E \times \mathfrak{g}_{-1}^E$ as well as on $\mathfrak{g}_{-1}^F \times \mathfrak{g}_{-1}^F$ and its restriction to $\mathfrak{g}_{-1}^E \times \mathfrak{g}_{-1}^F$ induces an isomorphism $\mathfrak{g}_{-1}^F \cong L(\mathfrak{g}_{-1}^E, \mathfrak{g}_{-2})$. In particular, this bracket is nondegenerate, so we really have found a contact grading. Viewing this bracket as a symplectic form on $\mathfrak{g}_{-1}$, the subspaces $\mathfrak{g}_{-1}^E$ and $\mathfrak{g}_{-1}^F$ of $\mathfrak{g}_{-1}$ are Lagrangean.

As a group G with Lie algebra $\mathfrak{g}$ we take $PGL(n+2,\mathbb{R})$. We can either realize this group as the quotient of $GL(n+2,\mathbb{R})$ by scalar multiples of the identity or by taking the subgroup of matrices whose determinant has absolute value one and identifying each matrix with its negative. In any case, we will work with representative matrices. For the parabolic subgroup $P \subset G$ we take the subgroup of matrices which are block upper triangular with blocks of sizes 1, n, and 1. The resulting Levi subgroup $G_0 \subset P$ consists of the block diagonal matrices with these block sizes.

The homogeneous model G/P is the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$ of lines in hyperplanes in $\mathbb{R}^{n+2}$. Mapping such a flag to its line makes $F_{1,n+1}$ into a fiber bundle over $\mathbb{R}P^{n+1}$. The fiber of this bundle is the space of all hyperplanes containing a fixed line, which can be identified with hyperlanes in the quotient by that line. Since a hyperplane in a vector space is equivalent to a line in its dual, the fiber is $\mathbb{R}P^{n*}$. Evidently, the vertical bundle of this fibration corresponds to $\mathfrak{g}_{-1}^F$. Likewise, projecting to the hyperplane shows that $F_{1,n+1}$ is a fiber bundle over $\mathbb{R}P^{(n+1)*}$ with fiber $\mathbb{R}P^n$, and the vertical bundle of this fibration corresponds to $\mathfrak{g}_{-1}^E$. To complete the interpretation of the structure, one shows that the projection to $\mathbb{R}P^{n+1}$ actually identifies $F_{1,n+1}$ with the projectivized cotangent bundle $\mathcal{P}T^*\mathbb{R}P^{n+1}$. The subspace spanned by the two vertical bundles (which are transversal) thereby gets identified with the tautological subbundle, so it defines the canonical contact structure on $\mathcal{P}T^*\mathbb{R}P^{n+1}$.

By Proposition 3.3.7, $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in homogeneous degrees ≤ 0, so we only have to understand regular infinitesimal flag structures of type (G, P). Let us denote elements of $\mathfrak{g}_-$ as triples (β, X, Y). Likewise, we denote a block diagonal matrix $\left(\begin{smallmatrix} c & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & e \end{smallmatrix}\right)$ with $c, e \in \mathbb{R} \setminus 0$, $C \in GL(n, \mathbb{R})$ as (c, C, e). In this language, the adjoint action is given by $(c, C, e) \cdot (\beta, X, Y) = (\frac{e}{c}\beta, c^{-1}CX, eYC^{-1})$. Observe that this is unchanged if we replace (c, C, e) by a nonzero multiple. Taking the representative $(1, c^{-1}C, \frac{e}{c})$ we see that the second component represents the action on $\mathfrak{g}^E_{-1}$ and the last component the one on $\mathfrak{g}_{-2}$, and this completely determines the element of G_0. In particular, the action (as expected) preserves the bracket and the decomposition of $\mathfrak{g}_{-1}$. Conversely, suppose that we take an automorphism of the graded Lie algebra $\mathfrak{g}_-$, which preserves the decomposition of $\mathfrak{g}_{-1}$. If $C \in GL(\mathfrak{g}^E_{-1})$ denotes the restriction of this automorphism, then compatibility with the bracket implies that the automorphism must be given by $(\beta, X, Y) \mapsto (e\beta, CX, eYC^{-1})$ for some nonzero number e. Since this is the action of $(1, C, e)$, we conclude that the adjoint action identifies G_0 with the subgroup of those autmorphisms of the graded Lie algebra $\mathfrak{g}_-$, which, in addition, preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$.

From the discussion in 4.2.2 we know that an infinitesimal flag structure of type (G, P) on a smooth manifold M of dimension $2n + 1$ is given by a contact structure $H \subset TM$ together with a redcution of structure group corresponding to $G_0 \subset \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. From the description of G_0 above it is clear that such a reduction is equivalent to a decomposition $H = E \oplus F$ of the contact subbundle as a direct sum of two Legendrean subbundles. This means that each of the subbundles has rank n, and the restriction of $\mathcal{L}$ to $E \times E$ and $F \times F$ vanishes identically. Note that this implies that $\mathcal{L}$ identifies F with the bundle $L(E, Q)$ of linear maps. A contact structure with an additonal decomposition $H = E \oplus F$ into the direct sum of two Legendrean subbundles is called a *Lagrangean contact structure*.

To get an overview of the basic completely reducible natural bundles for these structures we have to look at representations of G_0. Now this has a two–dimensional center, so there is a two–parameter family of one–dimensional representations and correspondingly a two–parameter family of natural real line bundles. We do not go into detail of how these are best parametrized, but just observe that Q, $\Lambda^n E$ and $\Lambda^n F \cong \Lambda^n E^* \otimes Q$ are typical examples. The semisimple part of G_0 is $SL(n, \mathbb{R})$ with the standard representation $\mathfrak{g}^E_{-1}$ corresponding to the bundle E. Hence, all natural bundles can be obtained from tensor bundles of E and natural line bundles.

Let us next compute the cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$. This is completely different for $n = 1$ and $n > 1$, and we consider the case $n = 1$ first. If $n = 1$, then we actually deal with the Borel subalgebra in A_2, i.e. the Dynkin diagram ×——×. There are two elements of length two in the Weyl group, namely the two possible compositions of the two simple reflections. The corresponding sets Φ_w are $\{\alpha_1, \alpha_1 + \alpha_2\}$, respectively, $\{\alpha_2, \alpha_1 + \alpha_2\}$. On the other hand, the highest root $\alpha_1 + \alpha_2$ is mapped by these two Weyl group elements to $-\alpha_1$, respectively, $-\alpha_2$. By Theorem 3.3.5 (and dualization to get from cohomology of $\mathfrak{p}_+$ to cohomology of $\mathfrak{g}_-$), we conclude that the two irreducible components in the cohomology are represented by the one–dimensional representations consisting of maps $\mathfrak{g}_{-2} \times \mathfrak{g}^E_{-1} \to \mathfrak{g}^E_1$, respectively, $\mathfrak{g}_{-2} \times \mathfrak{g}^F_{-1} \to \mathfrak{g}^F_1$. The corresponding harmonic curvatures are represented by Cotton–York type tensors mapping $Q \otimes E$ to E^* and $Q \otimes F \to F^*$. These can be determined more explicity using Weyl structures.

For $n > 1$, the Hasse diagram contains three elements of length two. Denoting by $\alpha_1, \dots, \alpha_{n+1}$ the simple roots and by σ_i the simple reflection corresponding to

α_i, these three elements are given as $\sigma_1 \circ \sigma_2$, $\sigma_{n+1} \circ \sigma_n$, and $\sigma_1 \circ \sigma_{n+1} = \sigma_{n+1} \circ \sigma_1$. The corresponding sets Φ_w evidently are $\{\alpha_1, \alpha_1 + \alpha_2\}$, $\{\alpha_{n+1}, \alpha_n + \alpha_{n+1}\}$, and $\{\alpha_1, \alpha_{n+1}\}$. The images of the highest root $\alpha_1 + \cdots + \alpha_{n+1}$ under these three elements are $\alpha_2 + \cdots + \alpha_{n+1}$, $\alpha_1 + \cdots + \alpha_n$, and $\alpha_2 + \cdots + \alpha_n$, respectively. Again using Theorem 3.3.5 and dualization, we conclude that the irreducible components of $\ker(\Box)$ are the highest weight parts in the sets of maps $\Lambda^2 \mathfrak{g}^E_{-1} \to \mathfrak{g}^F_{-1}$, $\Lambda^2 \mathfrak{g}^F_{-1} \to \mathfrak{g}^E_{-1}$, and $\mathfrak{g}^E_{-1} \otimes \mathfrak{g}^F_{-1} \to \mathfrak{g}_0$, respectively. The first two components are torsions in homogeneity 1 and the last one is a curvature in homogeneity 2, so we can use Theorem 4.2.2 to analyze the corresponding harmonic curvature components.

For the two torsions, the interpretation is simple. Suppose that ∇^H is the contact connection induced by a principal connection on the regular infinitesimal flag structure determined by a Lagrangean contact structure $H = E \oplus F \subset TM$. Then of course $\nabla^H = \nabla^E \oplus \nabla^F$ for connections on the subbundles, so, in particular, the subbundles are preserved. Now from 4.2.2 we know that that we have to look at components of

$$\tau(\xi, \eta) := \nabla^H_\xi \eta - \nabla^H_\eta \xi - \pi([\xi, \eta])$$

for $\xi, \eta \in \Gamma(H)$ and a certain projection π from TM onto the subbundle H. To get the first torsion component, we have to take $\xi, \eta \in \Gamma(E)$ and project the result to F. But then the covariant derivatives produce sections of E, and since E is Legendrean the bracket $[\xi, \eta]$ is a section of H, so we can leave out π. Hence, we end up with mapping $\xi, \eta \in \Gamma(H)$ to the F–component of $-[\xi, \eta] \in \Gamma(H)$. Since ξ and η actually have trivial F–components, this is bilinear over smooth functions, and hence defines a tensor $\tau_E \in \Gamma(\Lambda^2 E^* \otimes F)$.

To understand the highest weight component, recall that $F \cong E^* \otimes Q$ via $\mathcal{L}$. Thus, $\Lambda^2 E^* \otimes F \cong \Lambda^2 E^* \otimes E^* \otimes Q$ and the highest weight part in there is the kernel of the alternation map to $\Lambda^3 E^* \otimes Q$. Now viewed as a trilinear map on E with values in Q, the torsion τ_E maps (ξ, η, ζ) to $\mathcal{L}$ applied to the F–component of $[\xi, \eta]$ and ζ. Replacing the F–component by $[\xi, \eta]$ does not change the value of $\mathcal{L}$, so we are left with

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}([\xi, \eta], \zeta) = q([[\xi, \eta], \zeta]).$$

But this has trivial alternation by the Jacobi identity. Hence, we obtain

Proposition 4.2.3. *The two harmonic curvature components in homogeneity one of the regular normal parabolic geometry determined by a Lagrangean contact structure $H = E \oplus F \subset TM$ are represented by the torsions $\tau_E \in \Gamma(\Lambda^2 E^* \otimes F)$ and $\tau_F \in \Gamma(\Lambda^2 F^* \otimes E)$, induced by projecting the negative of the Lie bracket of two sections of one subbundle to the other subbundle. In particular, τ_E vanishes identically if and only if the subbundle $E \subset TM$ is integrable and likewise for τ_F. Vanishing of both τ_E and τ_F is equivalent to torsion freeness of the normal parabolic geometry.*

Proof. Apart from the last claim, everything has been proved already above. For the last claim, recall that the lowest nontrivial homogeneous component of the curvature of a regular normal parabolic geometry is harmonic; see Theorem 3.1.12. Vanishing of τ_E and τ_F implies that this lowest nonzero component has homogeneity at least two. By Lemma 4.2.2 the harmonic part of homogeneity two cannot produce any torsions. Since the same is true for arbitrary maps of homogeneity at least three, the result follows. □

To interpret the remaining harmonic curvature component, one has to choose a contact connection ∇^H and a projection π as above, and then modify it in such a way that the homogeneity one component of the torsion is contained in $\ker(\Box)$. Then one looks at the appropriate part of the curvature of the resulting connection. We will give a detailed description of connections adapted to parabolic contact structure in this sense (as well as in stronger senses) in Section 5.2. At this point, we only give a short sketch how such a connection can be constructed.

Observe that a contact connection ∇^H comes from a principal connection on the infintesimal flag strucutre if and only if it is of the form $\nabla^E \oplus \nabla^F$. Starting with an arbitrary choice of such a connection, the tensor τ from above is a section of $\Lambda^2 H^* \otimes H$. Decomposing this bundle, we obtain

$$(\Lambda^2 E^* \oplus (E \otimes F)^*_0 \oplus Q^* \oplus \Lambda^2 F^*) \otimes (E \oplus F).$$

Here we have denoted by $(E \otimes F)_0$ the kernel of $\mathcal{L} : E \otimes F \to Q$ and identified a complementary subbundle with Q. Note that by definition of τ, the only part of this that depends on π is the part in $Q^* \otimes (E \oplus F)$. Now take the components in $(\Lambda^2 E^* \oplus (E \otimes F)^*_0) \otimes E$, interepret them as a section of $H^* \otimes E^* \otimes E$ and subtract this from ∇^E. Likewise, take the components in $((E \otimes F)^*_0 \oplus \Lambda^2 F^*) \otimes F$, view them as a section of $H^* \otimes F^* \otimes F$ and subtract this from ∇^F. Finally, we can use the component in $Q^* \otimes (E \oplus F)$ to change the projection π.

The resulting pair of connnection and projection by construction has the property that the nonzero components of the tensor τ only lie in $\Lambda^2 E^* \otimes F$ and in $\Lambda^2 F^* \otimes E$, and we know from above that these parts automatically lie in $\ker(\Box)$. We claim that this is already an appropriate connection, i.e. the part of the torsion which maps $Q \otimes H$ to Q has to vanish automatically. To see this, observe that we are dealing with the lowest homogeneous component of the curvature of a regular Cartan connection, so by the Bianchi identity, it is contained in the kernel of ∂; see Theorem 3.1.12. Let us denote the homogeneity one part of the torsion by ψ and for sections $\xi, \eta, \zeta \in \Gamma(H)$ expand the equation $0 = \partial\psi(\xi, \eta, \zeta)$. Using that $\{\ ,\ \}$ coincides with $\mathcal{L}$, this gives

$$\begin{aligned} 0 = &\mathcal{L}(\xi, \psi(\eta, \zeta)) - \mathcal{L}(\eta, \psi(\xi, \zeta)) + \mathcal{L}(\zeta, \psi(\xi, \eta)) \\ &- \psi(\mathcal{L}(\xi, \eta), \zeta) + \psi(\mathcal{L}(\xi, \zeta), \eta) - \psi(\mathcal{L}(\eta, \zeta), \xi). \end{aligned} \tag{4.7}$$

Now assume that $\xi, \eta \in \Gamma(E)$ and $\zeta \in \Gamma(F)$. Then in the first two terms, ψ already gives zero while in the third term ψ has values in F, so this does not contribute either. In the fourth term we get $\mathcal{L}(\xi, \eta) = 0$ since both are sections of E, so (4.7) reduces to $\psi(\mathcal{L}(\xi, \zeta), \eta) = \psi(\mathcal{L}(\eta, \zeta), \xi)$. But now given $\beta \in \Gamma(Q)$ and $\eta \in \Gamma(E)$ we can choose $\xi \in \Gamma(E)$ and $\zeta \in \Gamma(F)$ such that $\mathcal{L}(\xi, \zeta) = \beta$ and $\mathcal{L}(\eta, \zeta) = 0$, and we get

$$\psi(\beta, \eta) = \psi(\mathcal{L}(\xi, \zeta), \eta) = \psi(\mathcal{L}(\eta, \zeta), \xi) = 0.$$

Thus, ψ vanishes on $Q \otimes E$ and likewise one shows that it vanishes on $Q \otimes F$.

4.2.4. Partially integrable almost CR–structures. This is certainly the most important example of a parabolic contact structure, which has often been studied independently. The constructions of canonical Cartan connections by N. Tanaka in [**Tan62**] and by S.S. Chern and J. Moser in [**ChMo76**] (for the subclass of integrable CR–structures) are among the best known results of this kind and were a strong motivation for the development of the general theory.

We have partly discussed this example in 3.1.7, so we will go through the basics rather quickly. For $p+q=n\geq 1$ we consider the real form $\mathfrak{su}(p+1,q+1)$ of $\mathfrak{sl}(n+2,\mathbb{C})$. We choose the Hermitian form on $\mathbb{C}^{n+2}$ which is given by

$$\langle (z_0,\dots,z_{n+1}),(w_0,\dots,w_{n+1})\rangle = z_0\overline{w_{n+1}}+z_{n+1}\overline{w_0}+\sum_{j=1}^{p} z_j\overline{w_j}-\sum_{j=p+1}^{n} z_j\overline{w_j}.$$

Denoting by $\mathbb{I}=\mathbb{I}_{p,q}$ the $n\times n$–diagonal matrix with the first p entries equal to 1 and the remaining entries equal to -1, we can represent the Lie algebra in block form with blocks of sizes 1, n, and 1, similarly to 4.2.3 as

$$\mathfrak{g}=\left\{\begin{pmatrix} a & Z & iz\\ X & A & -\mathbb{I}Z^*\\ ix & -X^*\mathbb{I} & -\bar{a}\end{pmatrix} : \begin{matrix} A\in\mathfrak{u}(n), a\in\mathbb{C}, X\in\mathbb{C}^n, Z\in\mathbb{C}^{n*},\\ x,z\in\mathbb{R};\quad a+\operatorname{tr}(A)-\bar{a}=0\end{matrix}\right\}.$$

The grading components are as for Lagrangean contact structures in 4.2.3 above. Rather than the splitting of $\mathfrak{g}_{\pm 1}$ into two irreducible pieces we have a complex structure on these subspaces. After complexification, the splitting into two components is recovered as the splitting of $\mathfrak{g}_{\pm 1}\otimes\mathbb{C}$ into holomorphic and anti–holomorphic parts. This will also be crucial for the interpretation of cohomologies. The bracket $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-2}$ is given by $[X,Y]=Y^*\mathbb{I}X-X^*\mathbb{I}Y$, so this is twice the imaginary part of the standard Hermitian inner product of signature (p,q). Note that this is compatible with the complex structure in the sense that $[iX,iY]=[X,Y]$.

As a group with Lie algebra $\mathfrak{g}$, we take $G=PSU(p+1,q+1)$. The parabolic subgroup P is then the stabilizer of the isotropic complex line generated by the first basis vector. (This automatically stabilizes also its orthocomplement, which is a hyperplane containing the given line.) The subgroup G_0 again is given by block diagonal matrices, i.e. we have matrices $\left(\begin{smallmatrix} c & 0 & 0\\ 0 & C & 0\\ 0 & 0 & 1/\bar{c}\end{smallmatrix}\right)$ with $c\in\mathbb{C}\setminus 0$ and $C\in U(n)$ such that $c\det(C)/\bar{c}=1$. We have to identify matrices which are multiples of each other, which leaves the freedom of multiplying by an $(n+2)$nd root of unity. Using a notation similar to 4.2.3, the adjoint action is immediately computed to be given by $(c,C)\cdot(ix,X)=(|c|^{-2}ix,c^{-1}CX)$, which is complex linear on $\mathfrak{g}_{-1}$ and orientation preserving on $\mathfrak{g}_{-2}$. Notice that there is a p–dimensional subspace in $\mathfrak{g}_{-1}$ on which $X\mapsto[X,iX]$ is nonzero with all values of the same sign and a q–dimensional subspace for which the same is true for the opposite sign. Hence, if $p\neq q$, then preserving the bracket and the complex structure on $\mathfrak{g}_{-1}$ implies that the orientation on $\mathfrak{g}_{-2}$ is preserved. For $p=q$, this is an additional condition.

Conversely, assume that $A:\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ is a complex linear isomorphism such that $[AX,AY]=\lambda[X,Y]$ for some $\lambda>0$. Since the standard Hermitian form of signature (p,q) is obtained as $1/2(i[X,iY]+[X,Y])$, we conclude that A has the same compatibility with that Hermitian form. In particular, $|\det(A)|^2=\lambda^n$. Now choose $c\in\mathbb{C}$ such that $|c|^2c^{-n-2}=\det(A)$. Then we get $|\det(A)|^2=|c|^{-2n}=\lambda^n$, and since $\lambda>0$ this implies $\lambda=|c|^{-2}$. Hence, cA has the property that $[cAX,cAY]=[X,Y]$ and hence $cA\in U(n)$. But then A is realized by the adjoint action of (c,cA) and $c\det(cA)/\bar{c}=c^{n+2}|c|^{-2}\det(A)=1$ as required. Note that in this procedure c is only unique up to multiplication with an $(n+2)$nd root of unity.

From the discussion in 4.2.2 we conclude that a regular infinitesimal flag structure (and hence a regular normal parabolic geometry) of type (G,P) on a smooth manifold M of dimension $2n+1$ is equivalent to a contact structure $H\subset TM$ together with a complex structure J on H such that $\mathcal{L}(J\xi,J\eta)=\mathcal{L}(\xi,\eta)$ for all

$\xi, \eta \in \Gamma(H)$. If this last condition is satisfied, then identifying the fiber Q_x of Q over $x \in M$ with $\mathbb{R}$, the map $\mathcal{L}$ is the imaginary part of a Hermitian form, and one requires that this form has signature (p, q). If $p = q$, one in addition has to choose an orientation on Q (which requires Q to be trivial). Since as a complex vector bundle H is canonically oriented, this is equivalent to choosing an orientation on M. For $p \neq q$, this orientation is automatically chosen by deciding between signature (p, q) and signature (q, p).

Let us rephrase this in the language of CR geometry. Given a real smooth manifolds M of dimension $2n+1$, a rank n complex subbundle (H, J) in TM is called an *almost CR–structure of hypersurface type*. Correspondingly, there is the notion of a (local) *CR–diffeomorphism*, which requires the tangent map to preserve the CR–subbundle H and the restriction to H being complex linear. The almost CR–structure is called *nondegenerate* if H defines a contact structure on M. Next, we have the condition that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. This condition is not used very often in CR geometry, since it is implied by the integrability condition to be discussed below. One usual terminology (see e.g. [**Miz93**]) for this condition is *partial integrability*. Then $\mathcal{L}$ becomes the imaginary part of a Hermitian form and choosing an orientation on Q the signature of this form is called the signature of (M, H, J). Hence, we conclude that regular normal parabolic geometries of type $(PSU(p+1, q+1), P)$ are equivalent to oriented nondegenerate partially integrable hypersurface type almost CR–structures of signature (p, q).

To understand the terminology "partial integrability" and its relation to the integrability condition, it is best to pass to the complexified setting. Since H is a complex vector bundle, the image $H \otimes \mathbb{C}$ in the complexified tangent bundle $TM \otimes \mathbb{C}$ splits into holomorphic and anti–holomorphic part as $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. Typical sections of $H^{0,1}$ are of the form $\xi + iJ\xi$ for $\xi \in \Gamma(H)$. Applying the complex bilinear extension of $\mathcal{L}$ to two such sections, we obtain

$$\mathcal{L}(\xi + iJ\xi, \eta + iJ\eta) = (\mathcal{L}(\xi, \eta) - \mathcal{L}(J\xi, J\eta)) + i(\mathcal{L}(J\xi, \eta) + \mathcal{L}(\xi, J\eta)).$$

Evidently, partial integrability is equivalent to vanishing of this expression and hence to the fact that the bracket of two sections of $H^{0,1}$ is a section of $H \otimes \mathbb{C}$. Alternatively, this can be phrased as follows. Consider the complex linear extension $q_{\mathbb{C}} : TM \otimes \mathbb{C} \to Q \otimes \mathbb{C}$, and the tensorial map $H^{0,1} \times H^{0,1} \to \mathbb{C}$ induced by an imaginary multiple of $q_{\mathbb{C}}([\xi, \bar{\eta}])$ for sections $\xi, \eta \in \Gamma(H^{0,1})$. Partial integrability is equivalent to this form being Hermitian, thus defining (with appropriate normalization) the classical *Levi form*. The signature of M is then the signature of this form.

A partially integrable almost CR–manifold is called *integrable* or a *CR–manifold* if the bundle $H^{0,1}$ is involutive. In the real picture, this is expressed by vanishing of the *Nijenhuis tensor* $N : \Lambda^2 H \to H$, which is induced by

$$(\xi, \eta) \mapsto [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]).$$

Note that N is of type $(0, 2)$ i.e. conjugate linear in both arguments.

The most important examples for CR–structures come from complex analyis. Let $(\mathcal{M}, J)$ be a complex manifold of complex dimension $n+1$, and let $M \subset \mathcal{M}$ be a smooth real hypersurface. For $x \in M$ define $H_x := T_xM \cap J(T_xM)$, the maximal complex subspace of $T_xM \subset T_x\mathcal{M}$. These subspaces must have complex dimension n and they fit together to define a complex subbundle $H \subset TM$. By definiton $H^{0,1} = TM \otimes \mathbb{C} \cap T^{0,1}\mathcal{M}$ and as the interesection of two involutive subbundles, this is automatically involutive. Generically, the subbundle H will be maximally nondegenerate, and then (M, H, J) is automatically a CR–structure. Note further that a biholomorphism of $\mathcal{M}$ which maps M to itself automatically restricts to a CR–diffeomorphism on M.

This picture can be nicely used to understand the homogeneous model G/P. The subgroup P is the stabilizer of a null line and G acts transitively on the space of all such lines. Hence, G/P can be identified with the projectivized null cone, which is a smooth real hypersurface in $\mathbb{C}P^{n+1}$. Since G acts by biholomorphisms on $\mathbb{C}P^{n+1}$, it acts by CR–diffeomorphisms on G/P. It is easy to verify directly the the CR–structure on G/P is nondegenerate of signature (p, q). Alternatively, one may describe the CR–structure in an elementary way along the lines of Example 1.1.6: It is easy to describe the manifold G/P more explicitly. Writing $\mathbb{C}^{n+2} = V' \oplus V''$ such that the Hermitian form is positive definite on V' and negative definite on V'', the null cone can be written as $\{(z', z'') : |z'| = |z''|\}$ for the Euclidean norm on both factors. Factoring by complex multiples we may first assume that $|z'| = |z''| = 1$, and then the remaining freedom is multiplication by elements of $U(1)$. Hence, G/P is obtained by factoring $S^{2p+1} \times S^{2q+1}$ by the diagonal action of $U(1)$. Note that in the special case $q = 0$, we obtain S^{2p+1} as described in Example 1.1.6.

The further discussion of this geometry is closely parallel to the case of Lagrangean contact structures in 4.2.3. As for the basic completely reducible bundles, we have real and complex line bundles as well as tensor bundles of H (taking into account the complex structure to form subbundles). Concerning harmonic curvature components, the cohomologies here have the same complexifications as the ones in 4.2.3. To interpret these we have to recall that on the complexification, the splitting of the subbundles corresponds to holomorphic and anti–holomorphic parts, so this admits an interpretation in terms of complex linearity and anti–linearity. For $n = 1$, the complexified cohomology corresponds to sections of the complex line bundles $Q^* \otimes \mathbb{C} \otimes (H^{1,0})^* \otimes (H^{1,0})^*$ and $Q^* \otimes \mathbb{C} \otimes (H^{0,1})^* \otimes (H^{0,1})^*$. The sum of these bundles is the complexification of the bundle whose sections are bilinear maps $Q \times H \to H^*$, which are complex linear in the second variable. Hence, there is only one basic curvature invariant for three–dimensional CR–structures.

If $n > 1$, then the two components in homogeneity one are the complexification of a single complex representation. Namely, they correspond to bilinear maps $\Lambda^2 H^{1,0} \to H^{0,1}$ and $\Lambda^2 H^{0,1} \to H^{1,0}$, which is the complexification of maps $\Lambda^2 H \to H$ which are conjugate linear in both variables. So there is just one torsion in this case, which corresponds to bilinear maps of type $(0, 2)$. The other cohomology component corresponds to the complexification of the bundle $\Lambda^{1,1} H^* \otimes \mathfrak{su}(H)$ of forms of type $(1, 1)$, so the last basic curvature has its values there.

The development in 4.2.3 already suggests that the torsion in this case should be the obstruction to involutivity of the bundles $H^{0,1}$ and $H^{1,0}$ (which are conjugate to each other) and hence to integrability of the almost CR–structure. This is indeed true and can be verified nicely as follows. If ∇^H is a contact connection induced by a principal connection on the infinitesimal flag structure, then J is parallel for the induced connection. By 4.2.2 we have to consider the $(0, 2)$–component of the tensor

$$\tau(\xi, \eta) = \nabla^H_\xi \eta - \nabla^H_\eta \xi - \pi([\xi, \eta]),$$

for some chosen projection $\pi : TM \to H$. Up to a nonzero factor, the $(0,2)$–component is given by

$$\tau(\xi,\eta) - \tau(J\xi, J\eta) + J(\tau(J\xi,\eta) + \tau(\xi, J\eta)).$$

Expanding the corresponding expression, one immediately concludes that π can be left out by partial integrability. Moreover, all terms involving ∇^H cancel since J is parallel, and the bracket terms add up to the negative of the Nijenhuis tensor. In particular, we conclude that the category of CR–structures is equivalent to the category of torsion–free normal parabolic geometries of type (G,P).

4.2.5. Lie contact structures. We next switch to the contact structures associated to the orthogonal Lie algebras. Hence, we have to consider real forms of $\mathfrak{so}(n,\mathbb{C})$ for $n \geq 7$. Our convention will be that for $p+q = n \geq 3$, we consider $\mathfrak{g} := \mathfrak{so}(p+2, q+2)$. Let V be the real vector space $\mathbb{R}^{n+4}$. Let $\mathbb{J}$ be the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and let $\mathbb{I}_{p,q}$ be the diagonal matrix with the first p entries equal to 1 and the others equal to -1. On V, we use the inner product associated to the matrix $\left(\begin{smallmatrix} 0 & 0 & \mathbb{J} \\ 0 & \mathbb{I}_{p,q} & 0 \\ \mathbb{J} & 0 & 0 \end{smallmatrix}\right)$. This means that the first two and the last two basis vectors are null, and the only non–trivial inner products among these vectors are between the first and the last and between the second and the last but one. The orthocomplement of these four vectors carries a standard inner product of signature (p,q). In particular, the whole inner product has the right signature $(p+2, q+2)$. A direct computation shows that with respect to this inner product, the Lie algebra $\mathfrak{so}(V)$ has the following form with blocks of size 2, n, and 2, where we write $\mathbb{I}$ for $\mathbb{I}_{p,q}$,

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & Z & z\mathbb{I}_{1,1} \\ X & E & -\mathbb{I}Z^t\mathbb{J} \\ x\mathbb{I}_{1,1} & -\mathbb{J}X^t\mathbb{I} & -\mathbb{J}A^t\mathbb{J} \end{pmatrix} : E \in \mathfrak{so}(p,q), x, z \in \mathbb{R} \right\}.$$

Here A, X, and Z are arbitrary matrices of size 2×2, $n \times 2$, and $2 \times n$, respectively, and we have used that a 2×2–matrix B such that $\mathbb{J}B = -B^t\mathbb{J}$ must be a real multiple of $\mathbb{I}_{1,1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The grading is by blocks as usual, so, in particular, $\mathfrak{g}_{\pm 2}$ has real dimension one. The bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is given by $[X,Y] := \mathbb{J}Y^t\mathbb{I}X - \mathbb{J}X^t\mathbb{I}Y$. Denoting the columns of an element $X \in \mathfrak{g}_{-1}$ by X_1 and X_2, and by $\langle\ ,\ \rangle$ the standard inner product of signature (p,q) on $\mathbb{R}^{p+q}$, we obtain

$$[X,Y] = (\langle X_1, Y_2\rangle - \langle X_2, Y_1\rangle)\mathbb{I}_{1,1}.$$

This is evidently nondegenerate, so we have obtained a contact grading. For an element $A \in GL(p+q,\mathbb{R})$ the composition with X is obtained by applying A to the columns of X, which immediately shows that $[AX, AY] = [X,Y]$ for $A \in O(p,q)$. This uniquely characterizes the bracket as a bilinear form:

Lemma 4.2.5. *Up to real multiples, there is a unique nondegenerate skew symmetric bilinear form on $L(\mathbb{R}^2, \mathbb{R}^{p+q})$ which is invariant under composition by elements of $O(p,q)$.*

Proof. The space $\Lambda^2(\mathbb{R}^{2*} \otimes \mathbb{R}^{p+q})$ decomposes as

$$(\Lambda^2\mathbb{R}^{2*} \otimes S^2\mathbb{R}^{p+q}) \oplus (S^2\mathbb{R}^{2*} \otimes \Lambda^2\mathbb{R}^{p+q}).$$

This decomposition is valid over $GL(2,\mathbb{R}) \times GL(p+q,\mathbb{R})$, but we have to analyze it over $O(p,q)$, with $\mathbb{R}^2$ being viewed as a trivial representation. Then $\Lambda^2\mathbb{R}^{2*}$ is a trivial one–dimensional representation which we can forget. Hence, the first summand is isomorphic to $S^2\mathbb{R}^{p+q} \cong \mathbb{R} \oplus S^2_0\mathbb{R}^{p+q}$. In the second summand, $S^2\mathbb{R}^{2*}$ is a trivial three–dimensional representation, so this summand is isomorphic to the sum of three copies of the irreducible representation $\Lambda^2\mathbb{R}^{p+q}$. In total we conclude that $\Lambda^2(\mathbb{R}^{2*} \otimes \mathbb{R}^{p+q})$ contains a unique one–dimensional subspace on which $O(p,q)$ acts trivially, which completes the proof. □

For the group with Lie algebra $\mathfrak{g}$ we choose $G := O(p+2, q+2)$. For the parabolic subgroup P we take the stabilizer of the isotropic plane spanned by the first two basis vectors, i.e. the subgroup of matrices which are block upper triangular with blocks of size 2, n, and 2. The Levi subgroup $G_0 \subset P$ then consists of all block diagonal matrices with these block sizes contained in $O(p+2, q+2)$. A short computation shows that these are exactly the matrices of the form

$$\begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \mathbb{J}(B^t)^{-1}\mathbb{J} \end{pmatrix} : B \in GL(2,\mathbb{R}), C \in O(p,q).$$

We will denote such a matrix as a pair (B, C) and elements of $\mathfrak{g}_-$ as pairs (x, X). For a 2×2–matrix B one easily computes that $\mathbb{J}B^t\mathbb{J}\mathbb{I}_{1,1}B = \det(B)\mathbb{I}_{1,1}$, and using this one verfies that the adjoint action is given by $(B,C)\cdot(x,X) = (\det(B^{-1})x, CXB^{-1})$. Using the same identity once more, one checks that this is really compatible with the bracket and by definition it is compatible with the identification of $\mathfrak{g}_{-1}$ with $L(\mathbb{R}^2, \mathbb{R}^{p+q})$, including the inner product on $\mathbb{R}^{p+q}$.

Definition 4.2.5. For $p + q = n \geq 3$ a *Lie contact structure* of signature (p,q) on a smooth manifold M of dimension $2n+1$ is given by a contact structure $H \subset TM$, two auxilliary vector bundles $E \to M$ of rank 2 and $F \to M$ of rank n, a bundle metric on F of signature (p,q), and an isomorphism $H \cong L(E,F)$ such that for each $x \in M$ the Levi bracket $\mathcal{L}_x$ is invariant under the resulting action of the orthogonal group $O(F_x)$ on H_x.

A morphism of Lie contact structures from (M, H, E, F) to $(\tilde{M}, \tilde{H}, \tilde{E}, \tilde{F})$ is a local contact diffeomorphism $f : M \to \tilde{M}$ such that for each $x \in M$ the restriction $T_xf : H_x \to \tilde{H}_{f(x)}$ comes, via the fixed isomorphisms from a linear isomorphisms $E_x \to \tilde{E}_{f(x)}$ and an orthogonal isomorphism $F_x \to \tilde{F}_x$.

Proposition 4.2.5. *For $G = O(p+2, q+2)$ and $P \subset G$ the stabilizer of a null plane, the category of regular normal parabolic geometries of type (G,P) is equivalent to the category of Lie contact structures of signature (p,q).*

Proof. By Proposition 3.3.7, the Lie algebra cohomology $H^1(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ is concentrated in homogeneous degrees ≤ 0, so it suffices to prove that regular infinitesimal flag structures of type (G,P) are equivalent to Lie contact structures of signature (p,q).

Consider a regular infinitesimal flag structure $(H := T^{-1}M, \mathcal{G}_0 \to M, \theta)$ of type (G,P). Then $\mathcal{G}_0$ is a principal bundle with structure group $G_0 \cong GL(2,\mathbb{R}) \times O(p,q)$. Define $E := \mathcal{G}_0 \times_{G_0} \mathbb{R}^2$ and $F := \mathcal{G}_0 \times_{G_0} \mathbb{R}^{p+q}$, where we use the defining representations of the two components of G_0. Then F carries a natural bundle metric of signature (p,q). Next, we have the component $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$ of the frame form θ. By definition, $T^{-1}\mathcal{G}_0$ is the preimage of H, and $\mathfrak{g}_{-1} = L(\mathbb{R}^2, \mathbb{R}^{p+q})$. Via θ_{-1}, we get an isomorphism

$$H \cong \mathcal{G}_0 \times_{G_0} L(\mathbb{R}^2, \mathbb{R}^{p+q}) \cong L(E,F);$$

see 3.1.6. On the other hand, the component $\theta_{-2} \in \Omega^2(\mathcal{G}_0, \mathfrak{g}_{-2})$ identifies $Q = TM/H$ with $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-2}$. Regularity of the infinitesimal flag structure now exactly means that via these identifications, the Levi bracket $\mathcal{L} : H \times H \to Q$ is induced by $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. This immediately implies that H is a contact structure and $\mathcal{L}$ is invariant under the action of $O(F)$, so we have obtained a Lie contact structure. In the same way, a morphism of regular infinitesimal flag structures induces a morphism of Lie contact structures.

Conversely, assume we have given a Lie contact structure (H, E, F) on M. Define $p : \mathcal{G}_0 \to M$ to be the fibered product of the linear frame bundle of E and the orthonormal frame bundle of F. Then this is a principal bundle with structure group $GL(2, \mathbb{R}) \times O(p, q) \cong G_0$. A point in $\mathcal{G}_0$ over $x \in M$ by definition is a pair (ϕ, ψ) of a linear isomorphism $\phi : \mathbb{R}^2 \to E_x$ and an orthogonal isomorphism $\psi : \mathbb{R}^{p+q} \to F_x$. For a tangent vector $\xi \in T^{-1}_{(\phi,\psi)}\mathcal{G}_0$ we have $Tp \cdot \xi \in H_x M$, so via the fixed isomorphism $H \cong L(E, F)$, we can view $Tp \cdot \xi$ as a linear map $E_x \to F_x$. Now define

$$\theta_{-1}(\xi) := \psi^{-1} \circ (Tp \cdot \xi) \circ \phi \in L(\mathbb{R}^2, \mathbb{R}^{p+q}) = \mathfrak{g}_{-1}.$$

This evidently defines a smooth section θ_{-1} of the bundle $L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1})$, and since ϕ and ψ are isomorphisms, the kernel of θ_{-1} in each point is the vertical subbundle of $\mathcal{G}_0 \to M$. The principal right action of $(B, C) \in G_0 = GL(2, \mathbb{R}) \times O(p, q)$ maps (ϕ, ψ) to $(\phi \circ B, \psi \circ C)$. Since acting on ξ by the derivative of the principal action leaves the image under Tp unchanged, we conclude that

$$(r^{(B,C)})^*\theta_{-1}(\xi) = C^{-1} \circ \psi^{-1} \circ (Tp \cdot \xi) \circ \phi \circ B = Ad((B.C)^{-1})(\theta_{-1}(\xi)),$$

so we obtain the right equivariancy condition. Fixing the point (ϕ, ψ), the form θ_{-1} induces an isomorphism $H_x \to \mathfrak{g}_{-1}$ which intertwines the action of $O(F_x)$ on H_x with the action of $O(p, q)$ on $\mathfrak{g}_{-1} = L(\mathbb{R}^2, \mathbb{R}^{p+q})$. Pulling back $\mathcal{L}_x$ via the inverse β of this isomorphism, we obtain a nondegenerate skew symmetric bilinear map $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to Q_x$ which is invariant under the action of $O(p, q)$. By the lemma, there is a unique linear isomorphism $\gamma : Q_x \to \mathfrak{g}_{-2}$ such that

$$\gamma \circ \mathcal{L}_x \circ (\beta \times \beta) = [\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}.$$

Now for $\xi \in T_{(\phi,\psi)}\mathcal{G}_0$ we define $\theta_{-2}(\xi) := \gamma(Tp \cdot \xi + H_x) \in \mathfrak{g}_{-2}$. Clearly, this defines $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$, and the kernel of this form in each point u is $T^{-1}_u\mathcal{G}_0$. Now take $(B, C) \in G_0$ and pass from (ϕ, ψ) to $(\phi \circ B, \psi \circ C)$. By construction, this replaces β by $\beta \circ \mathrm{Ad}(B, C)$. Since $\gamma \circ \mathcal{L}_X \circ (\beta \times \beta) = [\ ,\ ]$ is Ad–equivariant, we see that we have to replace γ by $\mathrm{Ad}((B, C)^{-1}) \circ \gamma$. As above, this shows that $(r^{(B,C)})^*\theta_{-2} = \mathrm{Ad}((B, C)^{-1}) \circ \theta_{-2}$, so $(\mathcal{G}_0 \to M, \theta)$ is an infinitesimal flag structure of type (G, P). The construction of θ_{-2} was done in such a way that under the isomorphism $H \oplus Q \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$, the Levi bracket corresponds to $[\ ,\ ]$. This implies regularity of the infinitesimal flag structure.

A morphism of Lie contact structures by definition gives rise to bundle maps between the auxilliary bundles, with the F–part being orthogonal and both covering the given local diffeomorphism. Using the induced maps on the frame bundles, one obtains a principal bundle map between the corresponding infinitesimal flag structures. From the above construction of the frame form θ one easily deduces that this principal bundle map is compatible with the frame forms, and thus a morphism of infinitesimal flag structures. □

Since $G_0 \cong GL(2,\mathbb{R}) \times O(p,q)$, all irreducible bundles in this case can be obtained from natural line bundles (with $Q = TM/H$ as a basic example) and tensor bundles of E, E^*, and F. Tensor bundles constructed from H can be decomposed into sums of these basic bundles similarly as discussed for almost Grassmannian structures of type $(2,n)$ in 4.1.3, but the fact that F is equipped with a bundle metric leads to finer decompositions. It has to be emphasized, however, that the bundles E and F, as well as the metric on F, have almost no intrinsic meaning. For example, a change of the bundle metric on F can be absorbed into a change of the isomorphism $H \to L(E,F)$. The main purpose of these data is, on the one hand, to express a compatibility condition between the Levi bracket and the tensor product decomposition. On the other hand, the finer decompositions of tensor bundles constructed from H are uninfluenced by the freedom in the choices one has.

The interpretation of harmonic curvature components is relatively easy in this case, since they are all contained in homogeneity one. In high dimensions, this is very similar to the discussion for almost Grassmanian structures in 4.1.4 (apart from the fact that the highest root is different). In lower dimensions, one has to discuss the cases of even and odd n (which correspond to D_n or B_n series) separately, but the final outcome is always the same (apart from a small speciality for $n = 4$ to be discussed below). The cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ consists of two irreducible components, which are both contained in homogeneity one. In the language of bundles, they can be described as follows. Via the isomorphism $H \cong E^* \otimes F$, we get the decomposition

$$\Lambda^2 H^* \otimes H \cong (\Lambda^2 E \otimes E^* \otimes S^2 F^* \otimes F) \oplus (S^2 E \otimes E^* \otimes \Lambda^2 F^* \otimes F).$$

The two harmonic curvature components are the highest weight components of the two summands. In the first summand, $\Lambda^2 E \otimes E^*$ is already irreducible and via the bundle metric we can identify F with F^*, and the highest weight component is then given by the totally symmetric and tracefree part $S^3_0 F^*$. For the other summand, the highest weight part in $S^2 E \otimes E^*$ is the tracefree part, while in $\Lambda^2 F^* \otimes F$, we can identify F with F^* and then have to take the tracefree part of the kernel of the alternation. If $n = 4$, then the bundle F has rank four and the middle dimensional exterior power $\Lambda^2 F^*$ splits into self–dual and anti–self–dual parts. Accordingly, we get a splitting of the corresponding harmonic curvature into two components. Hence, for $n = 4$ there are three basic harmonic curvature quantities.

Following the general procedure from 4.2.2, we can determine the two harmonic curvature components from any contact connnection ∇^H, which is induced from a principal connection on the infinitesimal flag structure. The latter condition clearly means the ∇^H is induced from a linear connection ∇^E on E and a linear connection ∇^F on F, which is compatible with the bundle metric. Having chosen such connections and a projection π from TM onto the subbundle H, we can form the tensor $\tau \in \Gamma(\Lambda^2 H^* \otimes H)$ given by

$$\tau(\xi,\eta) = \nabla^H_\xi \eta - \nabla^H_\eta \xi - \pi([\xi,\eta]),$$

and the components of τ in $\ker(\Box)$ represent the harmonic curvatures by Theorem 4.2.2. These components can be determined explicitly along similar lines as for almost Grassmannian structures in 4.1.3.

Let us finally remark that for even n, there is another real form of $\mathfrak{so}(n+2,\mathbb{C})$ for which the contact grading makes sense, namely $\mathfrak{so}^*(\frac{n}{2}+1)$. From the Satake diagram of this algebra we see that the semisimple part of the subalgebra $\mathfrak{g}_0$ for this grading is given by $\mathfrak{sp}(1) \oplus \mathfrak{so}^*(\frac{n}{2})$. In 2.3.11 we have seen that $\mathfrak{so}^*(\frac{n}{2})$ is the Lie algebra of maps compatible with a quaternionic Hermitian form with skew symmetric real part and symmetric imaginary part on $\mathbb{H}^n$, while $\mathfrak{sp}(1)$ can be viewed as acting on $\mathbb{H}^n$ by quaternionic scalar multiplications. Thus, this grading is related to Lie contact structures similarly as the almost quaternionic structures studied in 4.1.8 are related to the almost Grassmannian structures from 4.1.3. This analogy actually goes further, since one may easily show that a parabolic contact structure corresponding to $\mathfrak{so}^*(\frac{n}{2}+1)$ is given by a contact structure H on a manifold of dimension $4n+1$ together with a prequaternionic structure on the contact subbundle H for which the Levi–bracket is Hermitian. This means that there is a rank three subbundle $\mathcal{Q} \subset L(H, H)$, which can be locally spanned by smooth sections I, J, K, such that $I \circ I = J \circ J = -\mathrm{id}$, $I \circ J = -J \circ I = K$, and such that the Levi bracket $\mathcal{L}$ is Hermitian with respect to I, J and K. To our knowledge, the study of these structures has just begun recently; see [**Ž09**].

4.2.6. Contact projective structures. This is the last remaining example of a parabolic contact structure associated to a classical Lie algebra. At the same time, it provides the second basic example of a parabolic geometry which is not determined by the underlying infinitesimal flag structure.

For $n \geq 1$, consider $\mathfrak{g} := \mathfrak{sp}(2n+2, \mathbb{R})$. To obtain a description similar to the other examples, it is better not to use the standard symplectic form on $\mathbb{R}^{2n+2}$, but the form defined by

$$((x_0, \dots, x_{2n+1}), (y_0, \dots, y_{2n+1})) \mapsto x_0 y_{2n+1} - y_0 x_{2n+1} + \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

For this choice, the Lie algebra $\mathfrak{g}$ gets the following form with blocks of size 1, n, n, and 1,

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & W & z \\ X & A & B & W^t \\ Y & C & -A^t & -Z^t \\ x & Y^t & -X^t & -a \end{pmatrix} : B^t = B, C^t = C \right\}.$$

The grading is by the usual block form, but taking the two middle blocks as one, i.e. $\mathfrak{g}_{\pm 2} \cong \mathbb{R}$ via the entries x and z, $\mathfrak{g}_{-1} \cong \mathbb{R}^{2n}$ via the entries X and Y, and $\mathfrak{g}_1 \cong \mathbb{R}^{2n*}$ via the entries Z and W. Finally, $\mathfrak{g}_0$ is the block diagonal part given by the a entry and the central block, which evidently is $\mathfrak{sp}(2n, \mathbb{R})$. The bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is (with obvious notation) given by

$$\left[\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \right] = Y_1^t X_2 - X_1^t Y_2 - Y_2^t X_1 + X_2^t Y_1,$$

which, up to a factor -2, is just the standard symplectic form on $\mathbb{R}^{2n}$. Hence, we have found a contact grading.

As a group with Lie algbra $\mathfrak{g}$, we take $G := PSp(2n+2, \mathbb{R})$ the quotient of $Sp(2n+2, \mathbb{R})$ by its center $\{\pm \mathrm{id}\}$. As usual, we will work with representative matrices taking into account that they are only defined up to sign. As the parabolic subgroup we use the stabilizer of the line generated by the first basis vector. This immediately shows that the homogeneous model G/P is the projective space $\mathbb{R}P^{2n+1}$ as discussed in 1.1.4. If we use $G = Sp(2n+2, \mathbb{R})$ and the connected component of the identity of the stabilizer for the parabolic subgroup P (which means that the entry in the top left corner is positive), then we obtain the sphere S^{2n+1}. The subgroup $G_0 \subset P$ is given by the classes of block diagonal matrices in G. The middle block of such a matrix simply represents the standard action of $Sp(2n,\mathbb{R})$ on $\mathfrak{g}_{-1} = \mathbb{R}^{2n}$, while the rest just gives multiples of the identity. Hence, Ad identifies G_0 with $CSp(\mathfrak{g}_{-1})$. In view of 4.2.2 we conclude that a regular infinitesimal flag structure of type (G,P) is equivalent to a contact structure H together with an orientation of the quotient bundle $Q = TM/H$. Hence, it is clear, that parabolic geometries of type (G,P) cannot be determined by the underlying ininfinitesimal flag structure. We also see that all irreducible bundles come from line bundles and tensor bundles of H.

By part (1) of Proposition 3.3.7, $H^1(\mathfrak{g}_-,\mathfrak{g})$ is concentrated in homogeneous degrees ≤ 1 so by Theorem 3.1.16, regular normal parabolic geometries of type (G,P) are equivalent to regular normal P–frame bundles of degree one. Interpreting these is parallel to the case of projective structures discussed in 4.1.5. Any such P–frame bundle $(\mathcal{G}_1 \to M, \theta)$ has an underlying regular infinitesimal flag structure of type (G,P) and we have seen above that this is equivalent to a contact structure $H \subset TM$ plus an orientation of the quotient bundle $Q = TM/H$. Now $\mathcal{G}_1 \to M$ by definition is a principal bundle with structure group P/P_+^2. The frame form θ consists of two components, namely

$$\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_1, \mathfrak{g}^{-1}/\mathfrak{g}^1)) \qquad \theta_{-2} \in \Omega^1(\mathcal{G}_1, \mathfrak{g}/\mathfrak{p}).$$

Here, $T^{-1}\mathcal{G}_1$ is the preimage of H. The underlying infinitesimal flag structure is $\mathcal{G}_0 = \mathcal{G}_1/(P_+/P_+^2)$ with the frame form induced by θ. Now we follow the discussion in 4.1.5. Suppose that $u_0 \in \mathcal{G}_0$ is a point. Then for $u \in \mathcal{G}_1$ over u_0, we can view the component in $\mathfrak{g}^0/\mathfrak{g}^1 = \mathfrak{g}_0$ of θ_{-1} as a map $T^{-1}_{u_0}\mathcal{G}_0 \to \mathfrak{g}_0$, which reproduces the generators of fundamental vector fields. Choosing a local section of $\mathcal{G}_1 \to M$, one obtains a local partial principal connection on $\mathcal{G}_0$; see 1.3.7. Equivalently, we can view this as a local partial contact connection ∇^σ on H. This means that ∇^σ is an operator $\Gamma(H) \times \Gamma(H) \to \Gamma(H)$ which is linear over smooth functions in the first variable and satisfies a Leibniz rule in the second variable. Morover, the induced partial connection on $\Lambda^2 H$ preserves the subbundle $\Lambda^2_0 H$.

The component θ_{-2} of the frame form can be interpreted as follows. As above, we fix u_0 and u and suppose they lie over $x \in M$. For a tangent vector $\xi \in T_xM$ we can choose a lift $\tilde{\xi} \in T_u\mathcal{G}_1$, and since this is unique up to elements of $T^0\mathcal{G}_1 = \ker(\theta_{-2}(u))$ the value of $\theta_{-2}(\tilde{\xi}) \in \mathfrak{g}/\mathfrak{p}$ depends only on ξ. Recall that we have a natural linear isomorphism $\mathfrak{g}_- \to \mathfrak{g}/\mathfrak{p}$. Then define $\pi^u(\xi) := T_up \cdot \eta \in H_x \subset T_xM$, where $\eta \in T_u\mathcal{G}_1$ has the property that $\theta_{-2}(\eta)$ has vanishing $\mathfrak{g}_{-2}$–component, while the $\mathfrak{g}_{-1}$–component coincides with the one of $\theta_{-2}(\tilde{\xi})$. This is immediately seen to be well defined and by definition $\pi(\xi) = \xi$ for $\xi \in H_x$. Using equivariancy of θ_{-2} one shows similarly as for θ_{-1} that a local section σ of $\mathcal{G}_1$ leads to a locally defined projection π^σ from TM onto the subbundle H.

Next, one has to compute how a change of the section σ changes the data $(\nabla^\sigma, \pi^\sigma)$. The possible change of a section over $U \subset M$ is detemined by functions $g_0 : U \to G_0$ and $Z : U \to \mathfrak{g}_1$, via $\hat{\sigma}(x) = \sigma(x) \cdot g_0(x)\exp(Z(x))P_+^2$. As in 4.1.5 the data remain unchanged if $Z = 0$, so we may restrict to the case that $\hat{\sigma}(x) = \sigma(x) \cdot \exp(Z(x))P_+^2$. Using equivariancy of θ, it follows as in 4.1.5 that the change is determined by $-\operatorname{ad}(Z(x))$. Keeping in mind that we view $\mathfrak{g}_0$ as a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$ via the adjoint action, we compute as follows: Take (with obvious notation) $(Z,W) \in \mathfrak{g}_1$ and $\binom{X_1}{Y_1}, \binom{X_2}{Y_2} \in \mathfrak{g}_{-1}$. Then one computes that $-\left[\left[(Z,W),\binom{X_1}{Y_1}\right],\binom{X_2}{Y_2}\right]$ is given by

$$(ZX_2+WY_2)\begin{pmatrix}X_1\\Y_1\end{pmatrix}+(ZX_1+WY_1)\begin{pmatrix}X_2\\Y_2\end{pmatrix}+(X_1^tY_2-Y_1^tX_2)\begin{pmatrix}W^t\\-Z^t\end{pmatrix}.$$

Taking the traceform on $\mathfrak{g}$ to describe the duality between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, the expression (ZX_2+WY_2) is simply the pairing between (Z,W) and $\binom{X_2}{Y_2}$, so the first two terms are easy to interpret and look exactly as in the projective case. To interpret the last term, we observe that taking its bracket with $\binom{X}{Y}$ gives

$$(ZX+WY)2(Y_1^tX_2-X_1^tY_2)=(ZX+WY)\left[\begin{pmatrix}X_1\\Y_1\end{pmatrix},\begin{pmatrix}X_2\\Y_2\end{pmatrix}\right].$$

Using this we see that changing σ to $\hat{\sigma}$, the partial connection changes as

$$\nabla^{\hat{\sigma}}_\xi\eta=\nabla^\sigma_\xi\eta+\Upsilon(\xi)\eta+\Upsilon(\eta)\xi+\Upsilon^\#(\mathcal{L}(\xi,\eta)) \tag{4.8}$$

for $\xi,\eta\in\Gamma(H)$. Here Υ is the map Z interpreted as a section of H^* and $\Upsilon^\#:Q\to H$ is characterized by $\mathcal{L}(\Upsilon^\#(\beta),\zeta)=\Upsilon(\zeta)\beta$ for $\beta\in Q$ and $\zeta\in H$.

For the projection, the interpretation is easier. For $(Z,W)\in\mathfrak{g}_1$ and $x\in\mathfrak{g}_{-2}$ we obtain $-[(z,W),x]=x\binom{W^t}{-Z^t}$. This shows that

$$\pi^{\hat{\sigma}}(\xi)=\pi^\sigma(\xi)+2\Upsilon^\#(q(\xi)),$$

for $\xi\in\mathfrak{X}(M)$, with $q:TM\to TM/H=Q$ the natural quotient map.

This relation has two nice consequences. On the one hand, consider the tensor τ from 4.2.2. For the data associated to σ this is given by

$$\tau(\xi,\eta)=\nabla^\sigma_\xi\eta-\nabla^\sigma_\eta\xi-\pi^\sigma([\xi,\eta]).$$

Changing to $\hat{\sigma}$, the terms in (4.8) involving Υ (rather than $\Upsilon^\#$) are symmetric in ξ and η and hence do not contribute to the change of τ. Hence, the full contribution to the change of τ from (4.8) is $\Upsilon^\#(\mathcal{L}(\xi,\eta)-\mathcal{L}(\eta,\xi))=2\Upsilon^\#(\mathcal{L}(\xi,\eta))$, which exactly cancels with the contribution from the projection term. Thus, for all choices of sections, we obtain the same torsion tensor τ.

For the other part of homogeneity one in the torsion, we have to use the induced connection on Q, so we have to compute its change first. This connection is characterized by the fact that $\mathcal{L}$ is parallel, so

$$\nabla^{\hat{\sigma}}_\xi\mathcal{L}(\eta,\zeta)=\mathcal{L}(\nabla^{\hat{\sigma}}_\xi\eta,\zeta)+\mathcal{L}(\eta,\nabla^{\hat{\sigma}}_\xi\zeta),$$

for $\xi,\eta,\zeta\in\Gamma(H)$. Expanding the right–hand side, collecting terms and using that $\mathcal{L}$ is parallel for ∇^σ, one obtains $\nabla^\sigma_\xi\mathcal{L}(\eta,\zeta)+2\Upsilon(\xi)\mathcal{L}(\eta,\zeta)$. In the proof of Theorem 4.2.2, we have seen that the homogeneity one part of the torsion for the choice associated to σ is given by

$$\nabla^\sigma_\xi q(\eta)-q([\xi,\eta])+\mathcal{L}(\xi,\pi^\sigma(\eta)),$$

for $\xi\in\Gamma(H)$ and $\eta\in\mathfrak{X}(M)$. Passing to $\hat{\sigma}$, the middle term remains unchanged, while the changes caused by the first and last term evidently cancel. Hence, the whole homogeneity one part of the torsion is independent of the choice of σ.

Definition 4.2.6. Let (M,H) be a contact manifold, and let $\nabla:\Gamma(H)\times\Gamma(H)\to\Gamma(H)$ be a partial contact connection.

(1) The *contact torsion* $\tau : \Lambda^2 H \to H$ of ∇ is the tensor induced by

$$\tau(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \pi([\xi, \eta]),$$

where $\pi : TM \to H$ is the projection associated to ∇ in part (2) of Theorem 4.2.2.

(2) A partial contact connection $\hat{\nabla}$ on H, is said to be *contact projectively equivalent* to ∇ if and only if there is a smooth section $\Upsilon \in \Gamma(H^*)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi + \Upsilon^\#(\mathcal{L}(\xi, \eta)),$$

where $\Upsilon^\# : Q \to H$ is characterized by $\mathcal{L}(\Upsilon^\#(\beta), \zeta) = \Upsilon(\zeta)\beta$ for $\beta \in Q$ and $\zeta \in H$.

The upshot of the above discussion was that a regular P–frame bundle of degree one over M induces a contact structure H on M as well as a contact projective equivalence class of partial contact connections. Parallel to 4.1.5, one establishes the converse. On the other hand, we have seen that contact projectively equivalent partial connections have the same contact torsion, so it make sense to talk about the contact torsion of a projective class.

Thus, it remains to discuss the normality condition, which needs some basic information on $H^2(\mathfrak{g}_-, \mathfrak{g})$. There is only one element w of length two in the Hasse diagram, namely acting with the reflection corresponding to the second simple root and then with the one corresponding to the first simple root, which is crossed. Applying this to the highest root $2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$ we get $2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ (respectively α_2 if $n = 2$). The corresponding root space lies in $\mathfrak{g}_0$. It is still a positive root though, so the root spaces corresponding to elements of Φ_w must be contained in $\mathfrak{g}_1$. This shows that the irreducible representation $H^2(\mathfrak{g}_-, \mathfrak{g})$ sits in $\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$. In particular, there is no cohomology in homogeneity one. Using this, one verifies (again parallel to 4.1.5 and using 4.2.2) that a regular P–frame bundle of degree one is normal if and only if it corresponds to a contact projective class with vanishing contact torsion. The harmonic curvature can then be read off as in part (3) of Theorem 4.2.2 using any connection extending a member of the contact projective equivalence class.

4.2.7. Contact projective structures and geodesics. Similarly, as discussed for projective structures in 4.1.6, there is an interpretation of contact projective structures in terms of unparametrized geodesics. We discuss this only briefly, more details can be found in [**Fox05a**] and [**Fox05b**]. Suppose first that (M, H) is a contact structure, ∇ is a contact connection on TM and $c : I \to M$ is a geodesic for ∇. Let α be a contact form for H with Reeb vector field r, and put $f(t) := \alpha(c'(t))$. Then $c'(t) - f(t)r \in H$ for all t, and since ∇ is a contact connection we get

$$0 = \alpha(\nabla_{c'}(c' - fr)) = -\alpha(\nabla_{c'} fr) = -c' \cdot f - f\alpha(\nabla_{c'} r).$$

This is a linear first order ODE on f, so if f vanishes in one point, then it vanishes identically. Hence, we conclude that any geodesic for a contact connection that is tangent to H in one point is tangent to H everywhere. We call these geodesics the *contact geodesics* of ∇. Evidently, they depend only on the partial connection underlying ∇.

Now given a partial contact connection, consider the set of all partial contact connections which have the same contact torsion and the same geodesics up to parametrization. Here the contact torsion is determined with respect to the projection π associated to the partial contact connection according to part (2) of

Theorem 4.2.2. Similarly, as in 4.1.6, one shows that this recovers the contact projective equivalence class as defined in 4.2.6 above. The family of contact geodesics can then be viewed as a smooth family of unparametrized curves, with exactly one curve through each point in each direction in the contact subbundle. Among such families there are those, which are the geodesics of a (partial) contact connection with vanishing contact torsion.

Suppose that we have contact manifolds (M, H) and $(\tilde{M}, \tilde{H})$ endowed with such families of paths and a contact diffeomorphism $f : M \to \tilde{M}$, which is compatible with these families. Then the families determine contact projective equivalence classes of partial contact connections on M and $\tilde{M}$. Take a representative of the class on $\tilde{M}$, and pull it back to M using f. The result is a partial contact connection with vanishing contact torsion, which by construction has the distinguished paths as contact geodesics. Thus, it lies in the contact projective equivalence class, and f is a morphism of contact projective structures. Hence, we see that the families of paths provide an equivalent description of projective contact structures.

At this point there occurs a subtlety which is not present for classical projective structures. Any linear connection on the tangent bundle of a manifold can be changed into a torsion–free connection. Since the necessary change is given by a skew symmetric tensor, this does not change the set of geodesics. Thus, the family of geodesics of an arbitrary linear connection can always be described by a torsion–free projective structure and hence a regular normal parabolic geometry.

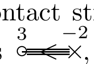
This is no longer true in the contact case. The deformation tensor between two partial contact connections is a section of $H^* \otimes \mathfrak{csp}(H) \subset H^* \otimes H^* \otimes H$. Such a change does not affect the contact geodesics, if at the same time it is contained in $\Lambda^2 H^* \otimes H$, and then it also directly describes the change of torsion. The intersection $(H^* \otimes \mathfrak{csp}(H)) \cap (\Lambda^2 H^* \otimes H)$ turns out to be too small to remove arbitrary torsions. Only those families of contact paths, which can be described by a partial contact connection with vanishing contact torsion, admit an equivalent description as a regular normal parabolic geometry.

It turns out, however, that also contact projective structures with nonvanishing contact torsion admit a canonical regular Cartan connection of type (G, P). This was shown by D.J.F. Fox in [**Fox05a**], where he generalized the normalization condition on the curvature of a Cartan connection. In the case of vanishing contact torsion, the original normalization condition is recovered. Hence, this extends the approach via Cartan geometries to arbitrary families of contact geodesics.

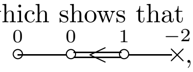
4.2.8. Exotic parabolic contact structures. In this section we briefly indicate what the parabolic contact structures associated to exceptional Lie algebras look like. We also use this to demonstrate that a rough picture of the nature of a parabolic geometry can be obtained with very little effort. To our knowledge, the details have not been worked out yet for any of these geometries. It should, however, be remarked that there are relations between contact gradings on simple Lie algebras and Jordan algebras (see [**Kan73**]), which should be useful for a more detailed study of these geometries.

We will not discuss harmonic curvature components here. Indeed, one shows that in all cases, there is only one irreducible component in $H^2(\mathfrak{g}_-, \mathfrak{g})$ and that component sits in homogeneity one. Hence, the harmonic curvatures can always be read off as appropriate components of the tensor τ associated to any contact connection induced by from a principal connection on the infinitesimal flag structure.

The principles along which we discuss the geometries are fairly easy: The list of complex contact gradings can be found in 3.2.4 and from the Satake diagram of each real form one immediately sees whether the contact grading exists on that real form or not. In that table one also finds the type of $\mathfrak{g}_0$. Using this, one computes the dimension, in which the geometry exists as $1/2(\dim(\mathfrak{g}) - \dim(\mathfrak{g}_0))$. For the exceptional algebras, $\mathfrak{g}_0$ always has one–dimensional center and the Satake diagram of the semisimple part is obtained by removing the crossed node and all edges connecting to it; see [**Kane93**]. The single crossed node represents the simple root for which the corresponding root space is contained in $\mathfrak{g}_1$. But this immediately implies that the negative of this root is the highest weight of $\mathfrak{g}_{-1}$. From the definition of the Dynkin diagram, one may immediately write out this weight, thus finding the nature of the reduction of structure group of a contact structure which is equivalent to the parabolic contact structure.

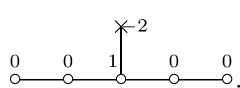
The parabolic contact structure associated to G_2. Here there is just one noncompact noncomplex real form, namely the split real form. From the table in 3.2.4 we see that the semisimple part of $\mathfrak{g}_0$ is $\mathfrak{sl}(2,\mathbb{R})$, so $\mathfrak{g}_0 \cong \mathfrak{gl}(2,\mathbb{R})$. Since G_2 has dimension 14, the associated parabolic contact structure exists on manifolds of dimension 5. The highest weight of $\mathfrak{g}_{-1}$ is , so up to the action of the center, this is the third symmetric power of the standard representation of $\mathfrak{g}_0$. The weights of $S^3\mathbb{R}^2$ are 3, 1, -1, and -3, all with multiplicity one, which shows that the weights of $\Lambda^2(S^3\mathbb{R}^2)$ are 4, 2, -2, and -4 with multiplicity one and 0 with multiplicity two. This shows that $\Lambda^2(S^3\mathbb{R}^2) \cong S^4\mathbb{R}^2 \oplus \mathbb{R}$, so up to scale there is a unique skew symmetric bilinear form on $S^3\mathbb{R}^2$ which is invariant under $\mathfrak{sl}(2,\mathbb{R})$.

Thus, we conclude that a parabolic contact structure associated to G_2 on a smooth manifold M of dimension 5 is given by a contact structure $H \subset TM$ and an auxilliary rank two vector bundle $E \to M$, together with an isomorphism $S^3E \to H$, such that the Levi bracket $\mathcal{L}$ is invariant under the action of $\mathfrak{sl}(E)$. The contact connections on H coming from a principal connection on the associated infinitesimal flag structure are exactly those induced by connections on E.

The parabolic contact structure associated to F_4. Again there is only the split real form to consider. The subalgebra $\mathfrak{g}_0$ has semisimple part $\mathfrak{sp}(6,\mathbb{R})$, which shows that the geometry exists in dimension 15. The highest weight of $\mathfrak{g}_{-1}$ is , so this is the tracefree part $\Lambda^3_0\mathbb{R}^6 \subset \Lambda^3\mathbb{R}^6$. The wedge product $\Lambda^3\mathbb{R}^6 \times \Lambda^3\mathbb{R}^6 \to \Lambda^6\mathbb{R}^6$ defines a nondegenerate skew symmetric bilinear form, which is invariant under $\mathfrak{sp}(6,\mathbb{R})$.

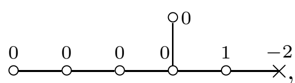
Hence, the parabolic contact structure associated to F_4 on a contact manifold (M,H) of dimension 15 is given by an auxilliary vector bundle E of rank 6 which is endowed with a nondegenerate skew symmetric bilinear form and an isomorphism $\Lambda^3_0E \to H$, such that $\mathcal{L}$ is invariant under $\mathfrak{sp}(E)$. From this it is also clear what the associated contact connections look like.

The parabolic contact structure associated to E_6. Here there are three different real forms to consider. For the semisimple part of $\mathfrak{g}_0$ we obtain $\mathfrak{sl}(6,\mathbb{R})$, $\mathfrak{su}(3,3)$, and $\mathfrak{su}(5,1)$, respectively. Hence, these types of geometries exist in dimension 21. The highest weight of $\mathfrak{g}_{-1}$ is given by



For the split form, we therefore get $\mathfrak{g}_{-1} \cong \Lambda^3\mathbb{R}^6$, and the description is completely parallel to the F_4 case above but without a symplectic form on the auxilliary bundle. For the two $\mathfrak{su}$–algebras, $\mathfrak{g}_{-1}$ is a real subrepresentation in $\Lambda^3\mathbb{C}^6$. Existence of this real subrepresentation comes from the fact that on one hand, the wedge product identifies $\Lambda^3\mathbb{C}^6$ with its dual. On the other hand, we get an induced Hermitian form on this space, which leads to an identification with the conjugate dual. Together with the above, this defines a invariant conjugation, whose fixed points form the real subrepresentation. Hence, in these two cases, one has an auxiliary complex vector bundle E of rank 6 with a Hermitian bundle metric of signature $(3,3)$, respectively, $(5,1)$ and an identification of the contact subbundle with the appropriate real subspace in $\Lambda^3_{\mathbb{C}}E$.

The parabolic contact structure associated to E_7. In this case, we do not know an explicit interpretation of the reduction of structure group one obtains. Again there are three different real forms to consider for which the semisimple parts of $\mathfrak{g}_0$ are isomorphic to $\mathfrak{so}(6,6)$, $\mathfrak{so}(4,8)$, and $\mathfrak{so}^*(12)$. In particular, the geometries exist in dimension 33. For the highest weight of the complexification of $\mathfrak{g}_{-1}$, one obtains



so this corresponds to one of the spin representations. As before it turns out that on this representation there is a unique skew symmetric bilinear form. Hence, the geometry is given by a reduction of the contact subbundle to the appropriate group G_0, which is included in $GL(32,\mathbb{R})$ via a real subrepresentation of the appropriate basic spin representation.

The parabolic contact structure associated to E_8. Here things get really involved, since the semisimple part of $\mathfrak{g}_0$ is a real form of E_7, and two of these real forms actually occur. The geometry exists in dimension 57 and is given by a reduction of structure group of the contact subbundle to the appropriate real form of E_7, which has a unique representation in dimension 56.

4.3. Examples of general parabolic geometries

4.3.1. Geometries determined by filtrations. A general regular infinitesimal flag structure has two ingredients, the filtration of the tangent bundle and the reduction of structure group. In the case of $|1|$–gradings discussed in section 4.1, the filtration was trivial and all the geometry was given by the reduction of structure group. Now we consider the other extreme, in which the geometry is determined by the filtration only. The reduction of structure group corresponds to the homomorphism $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$, where $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ denotes the group of automorphisms of the graded Lie algebra $\mathfrak{g}_-$. If this is an isomorphism, then the whole geometry is determined by the filtration. While it is not evident a priori, whether this condition is satisfied, there is a simple cohomological criterion, which shows that it is even true generically. Recall that for a semisimple Lie algebra $\mathfrak{g}$, the group $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$. Since any derivation of $\mathfrak{g}$ is inner by Corollary 2.1.6, this is isomorphic to $\mathfrak{g}$. Hence, $G := \mathrm{Aut}(\mathfrak{g})$ has Lie algebra $\mathfrak{g}$ and the adjoint action is given by applying automorphisms. Consequently, given a $|k|$–grading on $\mathfrak{g}$ the maximal parabolic subgroup $P \subset \mathrm{Aut}(\mathfrak{g})$ for the given grading is the group $\mathrm{Aut}_f(\mathfrak{g})$ of all automorphisms