INTRODUCTION

The roots of *Descriptive Set Theory* go back to the work of Borel, Baire and Lebesgue around the turn of the 20th century, when the young French analysts were trying to come to grips with the abstract notion of a *function* introduced by Dirichlet and Riemann. A function was to be an arbitrary correspondence between objects, with no regard for any method or procedure by which this correspondence could be established. They had some doubts whether so general a concept should be accepted; in any case, it was obvious that all the specific functions which were studied in practice were determined by simple *analytic expressions*, explicit formulas, infinite series and the like. The problem was to delineate the functions which could be defined by such accepted methods and search for their characteristic properties, presumably nice properties not shared by all functions.

Baire was first to introduce in his Thesis [1899] what we now call *Baire functions* (of several real variables), the smallest set which contains all continuous functions and is closed under the taking of (pointwise) limits. He gave an inductive definition: the continuous functions are *of class* 0 and for each countable ordinal ξ, a function is *of class* ξ if it is the limit of a sequence of functions of smaller classes and is not itself of lower class. Baire, however, concentrated on a detailed study of the functions of class 1 and 2 and he said little about the general notion beyond the definition.

The first systematic study of definable functions was Lebesgue's [1905], *Sur les fonctions représentables analytiquement*. This beautiful and seminal paper truly started the subject of descriptive set theory.

Lebesgue defined the collection of *analytically representable functions* as *the smallest set which contains all constants and projections* $(x_1, x_2, \dots, x_n) \mapsto x_i$ *and which is closed under sums, products and the taking of limits*. It is easy to verify that these are precisely the Baire functions. Lebesgue then showed that there exist Baire functions of every countable class and that there exist definable functions which are not analytically representable. He also defined the *Borel measurable* functions and showed that they too coincide with the Baire functions. In fact he proved a much stronger theorem along these lines which relates the *hierarchy* of Baire functions with a natural hierarchy of the Borel measurable sets at each level.

Today we recognize Lebesgue [1905] as a classic work in the *theory of definability*. It introduced and studied systematically several natural notions of definable functions and sets and it established the first important hierarchy theorems and structure results for collections of definable objects. In it we can find the origins of many standard tools and techniques that we use today, for example *universal sets* and applications of the Cantor *diagonal method* to questions of definability.

One of Lebesgue's results in [1905] identified the *implicitly analytically definable* functions with the Baire functions. To take a simple case, suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is analytically representable and for each x, the equation

$$f(x, y) = 0$$

has exactly one solution in y. This equation then defines y implicitly as a function of x; Lebesgue showed that it is an analytically representable function of x, by an argument which was "simple, short but false." The wrong step in the proof was hidden in a lemma taken as (basically) trivial, that a set in the line which is the projection of a Borel measurable set in the plane is itself Borel measurable.

Ten years later the error was spotted by Suslin, then a young student of Lusin at the University of Moscow, who rushed to tell his professor in a scene charmingly described in Sierpinski [1950].

Suslin called the projections of Borel sets *analytic* and showed that indeed there are analytic sets which are not Borel measurable. Together with Lusin they quickly established most of the basic properties of analytic sets and they announced their results in two short notes in the Comptes Rendus, Suslin [1917] and Lusin [1917].

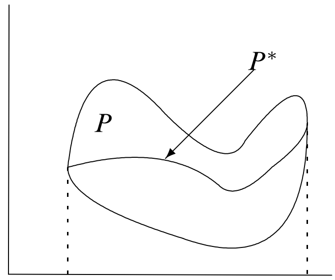
The class of analytic sets is rich and complicated but the sets in it are nice. They are measurable in the sense of Lebesgue, they have the property of Baire and they satisfy the Continuum Hypothesis, i.e., every uncountable analytic set is equinumerous with the set of all real numbers. The best result in Suslin [1917] is a characterization of the Borel measurable sets as precisely those analytic sets which have analytic complements. Lusin [1917] announced another basic theorem which implied that Lebesgue's contention about implicitly analytically definable functions is true, despite the error in the original proof.

Suslin died in 1919 and the study of analytic sets was continued mostly by Lusin and his students in Moscow and by Sierpinski in Warsaw. Because of what Lusin delicately called "difficulties of international communication" those years, they were isolated from each other and from the wider mathematical community, and there were very few publications in western journals in the early twenties.

The next significant step was the introduction of *projective sets* by Lusin and Sierpinski in 1925: a set is projective if it can be constructed starting with Borel measurable sets and iterating the operations of projection and complementation. Using later terminology, let us call analytic sets *A* sets, analytic complements *CA* sets, projections of *CA* sets *PCA* sets, complements of these *CPCA* sets, etc. Lusin in his [1925a], [1925b], [1925c] and Sierpinski [1925] showed that these classes of sets are all distinct and they established their elementary properties. But it was clear from the very beginning that the theory of projective sets was not easy. There was no obvious way to extend to these more complicated sets the regularity properties of Borel and analytic sets; for example it was an open problem whether analytic complements satisfy the Continuum Hypothesis or whether *PCA* sets are Lebesgue measurable.

Another fundamental and difficult problem was posed in Lusin [1930a]. Suppose P is a subset of the plane; a subset P^* of P *uniformizes* P if P^* is the graph of a function and it has the same projection on the line as P, as in the figure on the opposite page. The natural question is whether definable sets admit definable uniformizations and it comes up often, for example when we seek "canonical" solutions for y in terms of x in an equation

$$f(x, y) = 0.$$



Lusin and Sierpinski showed that Borel sets can be uniformized by analytic complements and Lusin also verified that analytic sets can be projectively uniformized. In a fundamental advance in the subject, Kondo [1938] completed earlier work of Novikov and proved that analytic complements and *PCA* sets can be uniformized by sets in the same classes. Again, there was no clear method for extending the known techniques to solve the uniformization problem for the higher projective classes.

As it turned out, the "difficulties of the theory of projective sets" which bothered Lusin from his very first publication in the subject could not be overcome by ingenuity alone. There was an insurmountable technical obstruction to answering the central open questions in the field, since *all of them were independent of the axioms of classical set theory*. It goes without saying that the researchers in descriptive set theory were formulating and trying to prove their assertions within axiomatic Zermelo-Fraenkel set theory (with choice), as all mathematicians still do, consciously or not.

The first independence results were proved by Gödel, in fact they were by-products of his famous consistency proof of the Continuum Hypothesis. He announced in his [1938] that in the model L of constructible sets there is a *PCA* set which is not Lebesgue measurable: it follows that one cannot establish in Zermelo-Fraenkel set theory (with the Axiom of Choice and even if one assumes the Continuum Hypothesis) that all *PCA* sets are Lebesgue measurable. His results were followed up by some people, notably Mostowski and Kuratowski, but that was another period of "difficulties of international communication" and nothing was published until the late forties. Addison [1959b] gave the first exposition in print of the consistency and independence results that are obtained by analyzing Gödel's L.

The independence of the Continuum Hypothesis was proved by Cohen [1963b], whose powerful method of *forcing* was soon after applied to independence questions in descriptive set theory. One of the most significant papers in forcing was Solovay [1970], where it is shown (among other things) that one can consistently assume the axioms of Zermelo-Fraenkel set theory (with choice and even the Continuum Hypothesis) together with the proposition that all projective sets are Lebesgue measurable; from this and Gödel's work it follows that *in classical set theory we can neither prove nor disprove the Lebesgue measurability of PCA sets*.

Similar consistency and independence results were obtained about all the central problems left open in the classical period of descriptive set theory, say up to 1940. It says something about the power of the mathematicians working in the field those years, that in almost every instance they obtained the best theorems that could be proved from the axioms they were assuming.

So the logicians entered the picture in their usual style, as spoilers. There was, however, another parallel development which brought them in more substantially and in a friendlier role. Before going into that, let us make a few remarks about the appropriate context for studying problems of definability of functions and sets.

We have been recounting the development of descriptive set theory on the real numbers, but it is obvious that the basic notions are topological in nature and can be formulated in the context of more general topological spaces. All the important results can be extended easily to *complete, separable, metric spaces*. In fact, it was noticed early on that the theory assumes a particularly simple form on *Baire space*

$$\mathcal{N} = {}^{\omega}\omega,$$

the set of all infinite sequences of natural numbers, topologized with the product topology (taking ω discrete). The key fact about $\mathcal{N}$ is that it is homeomorphic with its own square $\mathcal{N} \times \mathcal{N}$, so that irrelevant problems of dimension do not come up. Results in the theory are often proved just for $\mathcal{N}$, with the (suitable) generalizations to other spaces and the reals in particular left for the reader or simply stated without proof.

Let us now go back to a discussion of the impact of logic and logicians on descriptive set theory.

The fundamental work of Gödel [1931] on incompleteness phenomena in formal systems suggested that it should be profitable to delineate and study those functions (of several variables) on the set ω of natural numbers which are *effectively computable*. A great deal of work was done on this problem in the nineteen thirties by Church, Kleene, Turing, Post and Gödel among others, from which emerged a coherent and beautiful theory of *computability* or *recursion*. The class of *recursive functions* (of several variables) *on* ω was characterized as the smallest set which contains all the constants, the successor and the projections $(x_1, x_2, \dots, x_n) \mapsto x_i$ and which is closed under composition, a form of simple definition by induction (primitive recursion) and minimalization, where g is defined from f by the equation

$$g(x_1, x_2, \dots, x_n) = \text{least } w \text{ such that } f(x_1, x_2, \dots, x_n, w) = 0,$$

assuming that for each $x_1, \dots, x_n$ there is a root to the equation

$$f(x_1, \dots, x_n, w) = 0.$$

Church [1936] and independently Turing [1936] proposed the *Church-Turing Thesis* (hypothesis) that *all number theoretic functions which can be computed effectively by some algorithm are in fact recursive*, and to this date no serious evidence has been presented to dispute this.

Kleene [1952a], [1952b] extended the theory of recursion to functions

$$f : \omega^n \times \mathcal{N}^k \to \omega$$

with domain some finite cartesian product of copies of the natural numbers and Baire space. For example, a function $f : \omega \times \mathcal{N} \to \omega$ is recursive (by the natural extension of the Church-Turing Thesis) if there is an algorithm which will compute $f(n, \alpha)$ given n and a sufficiently long initial segment of the infinite sequence α.

A set $A \subseteq \omega^n \times \mathcal{N}^k$ is recursive if its characteristic function is recursive. By the Church-Turing Thesis again, these are the *decidable* sets for which we have (at least in principle) an algorithm for testing membership.

Using recursion theory as his main tool, Kleene developed a rich and intricate theory of definability on the natural numbers in the sequence of papers [1943], [1955a], [1955b], [1955c].

The class of *arithmetical* sets is the smallest family which contains all recursive sets and is closed under complementation and projection on ω. The *analytical* sets are defined similarly, starting with the arithmetical sets and iterating any finite number of times the operations of complementation, projection on ω and projection on $\mathcal{N}$. Both these classes are naturally ramified into subclasses, much like the subclasses A, CA, PCA, ... of projective sets of reals. Notice that the definitions make sense for subsets of an arbitrary product space of the form $\omega^n \times \mathcal{N}^k$. Kleene, however, was interested in classifying definable sets of natural numbers and he stated his ultimate results just for them. The more complicated product spaces were brought in only so projection on $\mathcal{N}$ could be utilized to define complicated subsets of ω.

Kleene studied a third notion (discovered independently by Davis [1950] and Mostowski [1951]) which is substantially more complicated. The class of *hyperarithmetical* sets of natural numbers is the smallest family of subsets of ω which contains the recursive sets and is closed under complementation and "recursive" countable union, suitably defined. The precise definition is quite intricate and the proofs of the main results are subtle, often depending on delicate estimates of the complexity of explicit and inductive definitions.

Using later terminology, let us call Σ^1_1 the simplest analytical sets of numbers, those which are projections to ω of arithmetical subsets of $\omega \times \mathcal{N}$. The most significant result of Kleene [1955c] (and the whole theory for that matter) was a characterization of the hyperarithmetical sets as precisely those Σ^1_1 sets which have Σ^1_1 complements.

Now this is clearly reminiscent of Suslin's characterization of the Borel sets. A closer look at specific results reveals a deep resemblance between these two fundamental theorems and suggests the following *analogy* between the classical theory and Kleene's definability theory for subsets of ω:

$\mathbb{R}$ or $\mathcal{N}$	ω
continuous functions	recursive functions
Borel sets	hyperarithmetical sets
analytic sets	Σ^1_1 sets
projective sets	analytical sets.

In fact, the theories of the corresponding classes of objects in this table are so similar, that one naturally conjectures that Kleene was consciously trying to create an "effective analog" on the space ω of classical descriptive set theory.

As it happened, Kleene did not know the classical theory, since he was a logician by trade and at the time that was considered part of topology. Mostowski knew it, being Polish, and he first used classical methods in his [1946], where he obtained independently many of the results of Kleene [1943]. More significantly, Mostowski introduced the hyperarithmetical sets following closely the classical approach to Borel sets, as opposed to Kleene's initial rather different definition in his [1955b].

First to establish firmly the analogies in the table above was Addison, in his Ph.D. Thesis [1954] and later in his [1959a]. Over the years and with the work of many people, what was first conceived as "analogies" developed into a general theory which yields in a unified manner both the classical results and the theorems of the recursion theorists; more precisely, this effective theory yields *refinements* of the classical results and *extensions* of the theorems of the recursion theorists.

It is this extended, *effective descriptive set theory* which concerns us here.

Powerful as they are, the methods from logic and recursion theory cannot solve the "difficulties of the theory of projective sets," since they too are restricted by the limitations of Zermelo-Fraenkel set theory. The natural next step was taken in the fundamental paper Solovay [1969], where for the first time strong set theoretic hypotheses were shown to imply significant results about projective sets.

Solovay proved that if there exist measurable cardinals, then *PCA* sets are Lebesgue measurable, they have the property of Baire and they satisfy the Continuum Hypothesis. Later, he and Martin proved a difficult uniformization theorem about *CPCA* sets in their joint [1969], and Martin [1971] established several deep properties of *CPCA* sets, all under the same hypothesis, that there exist measurable cardinals.

For our purposes here, it is not important to know exactly what measurable cardinals are. Suffice it to say that their existence cannot be shown in Zermelo-Fraenkel set theory and that if they exist, they are terribly large sets: bigger than the continuum, bigger than the first strongly inaccessible cardinal, bigger than the first Mahlo cardinal, etc. It is also fair to add that few people are willing to buy their existence after a casual look at their definition. Nevertheless, no one has shown that they do not exist, and it was known from previous work of Scott, Gaifman, Rowbottom and Silver that the existence of measurable cardinals implies new and interesting propositions about sets, even about real numbers. These, however, were metamathematical results, the kind that only logicians can love. Solovay's chief contribution was that he used this new and strange hypothesis to solve natural, mathematical problems posed by Lusin more than forty years earlier.

Unfortunately, measurable cardinals were not a panacea. Soon after Solovay's original work it was shown by himself, Martin and Silver among others that they do not resolve the open questions about projective sets beyond the *CPCA* class, except for some isolated results about *PCPCA* sets.

The next step was quite unexpected, even by those actively searching for strong hypotheses to settle the old open problems. Blackwell [1967] published a new, short and elegant proof of an old result of Lusin's about analytic sets, using the determinacy of open games.

Briefly, an *infinite game* (of perfect information) *on* ω is described by an arbitrary subset $A \subseteq \mathcal{N}$ of Baire space. We imagine two players I and II successively choosing natural numbers, with I choosing k_0, then II choosing k_1, then I choosing k_2, etc.; after an infinite sequence

$$\alpha = (k_0, k_1, \dots)$$

has been specified in this manner, we say that I wins if $\alpha \in A$, II wins if $\alpha \notin A$. The game (or the set A which describes it) is *determined*, if one of the two players has a *winning strategy*, a method of playing against arbitrary moves of his opponent which will always produce a sequence winning for him.

It was known that open games are determined and Blackwell's proof hinged on that fact. It was also known that one could prove the existence of non-determined games using the Axiom of Choice, but no definable non-determined game on ω had ever been produced.

Working independently, Addison and Martin realized that Blackwell's proof could be lifted to yield new results about the third class of projective sets, if only one *assumed* the hypothesis that enough projective sets are determined. Soon after, Martin and Moschovakis again independently used the *hypothesis of projective determinacy* to settle a whole slew of old questions about all levels of the projective hierarchy, see Addison and Moschovakis [1968] and Martin [1968]. Three years later the uniformization problem was solved on the same hypothesis in Moschovakis [1971a] and the methods introduced there led quickly to an almost complete structure theory for the classes of projective sets, see especially Kechris [1973], [1974], [1975], Martin [1971] and Moschovakis [1973], [1974c].

This is where matters stand today.