CHAPTER 1

Introduction and a Brief Review of the History

During the last thirty years the theory of stability of travelling wave solutions for nonlinear evolution equations has grown into a large field that attracts the attention of both mathematicians and physicists in view of its applications to real-world nonlinear models and of the novelty of the problems. The qualitative theory of nonlinear equations includes, in particular, investigations on the well-posedness of various problems for these equations, the behavior of solutions such as blowing-up, the existence and stability of solitary wave and periodic travelling wave solutions as well as properties of the dynamical system generated by these equations.

The purpose of this book is to give a self-contained presentation of some basic and detailed results concerning the existence and stability of travelling wave solutions. It is intended to be a new source for modern research dealing with nonlinear phenomena of dispersive type. The selection of the material is mainly related to the author's scientific interest. There are four main topics: the existence of solitary wave and periodic travelling wave solutions, the problems of the stability of these special kinds of solutions, the applicability of the *Concentration-Compactness Principle* in the study of the stability of solitary wave solutions of nonlinear dispersive equations, and the instability of solitary waves. A detailed and clear explanation of every concept and method introduced is given. The exposition is accompanied by a careful selection of modern examples. The book provides information that puts the reader at the forefront of current research.

Nonlinear evolution equations for modelling waves take into account both nonlinearity and dispersion effects. Their birth was the discovery of the solitary wave, or great wave of translation, observed on the Edinburgh-Glasgow canal in 1834 by J. Scott Russell. The story of the first encounter of Russell with the solitary wave was reported by him to the British Association in 1844 with the name *Report on Waves* [**246**]. Fascinated with this long water wave without a change in shape, which he called the "great wave of translation, or solitary wave", Russell made some laboratory experiments on this phenomenon, generating solitary waves by dropping a weight at one end of a water channel. He deduced empirically that the volume of water in the wave is equal to the volume displaced by the weight and that the speed c of the solitary wave and its maximum amplitude a above the free surface liquid of finite depth h satisfy the relation

$$c^2 = g(h + a),$$

where g is the acceleration due to gravity. His description of solitary waves contradicted the theories of water waves according to G. B. Airy and G. G. Stokes; they raised questions on the existence of Russell's solitary waves and conjectured that such waves cannot propagate in a liquid medium without a change of form. Despite the mathematical theory, the experimental evidence in favor of solitary waves was convincing. It was not until the 1870s that Russell's prediction was finally and independently confirmed by both J. Boussinesq (1871, [**70**]) and Lord Rayleigh (1876, [**240**]). Assuming that a solitary wave has a length much greater than the depth of the water, they derived from the equations of motion for an inviscid incompressible liquid that the wave height above the mean level h, $z = \psi(x,t)$, is given by

$$\psi(x,t) = a\mathrm{sech}^2[\beta(x-ct)], \tag{1.1}$$

where $\beta^2 = 3a/[4h^2(h+a)]$ for any positive amplitude $a > 0$. Although these authors found the sech^2 solution, they did not write any equation for η that produces (1.1) as a solution. However, Boussinesq did a lot more ([**71**]) and discovered that if a water wave propagates along a flat-bottomed channel of undisturbed depth h and has large wavelength and small amplitude relative to h, then the elevation η of the water surface considered as a function of the coordinate x along the channel and the time t will approximately satisfy the nonlinear evolution equation

$$\eta_{tt} - gh\eta_{xx} - gh\Big[\frac{3}{2h}\eta^2 + \frac{h^2}{3}\eta_{xx}\Big]_{xx} = 0, \tag{1.2}$$

where g is the gravitational acceleration and $\sqrt{gh}$ is the speed of the shallow water waves. This equation is known as the *Boussinesq (bidirectional) equation.* Using this equation, he obtained an explicit representation of solitary waves travelling in both positive and negative x-directions, namely,

$$\eta_{\mathrm{sol}}(x,t) = a\ \mathrm{sech}^2\Big[\sqrt{\frac{3a}{h^3}}(x \pm ct)\Big]. \tag{1.3}$$

As we will see below, Boussinesq's inquiry about the behavior of the solutions of (1.2), with initial data being a slight perturbation of a solitary wave (1.3), was the genesis of the theory nowadays called the theory of stability of solitary wave solutions.

In 1895, D. J. Korteweg and G. de Vries [**168**] formulated a mathematical model which provided an explanation of the phenomenon observed by Russell (they were apparently unaware of the work of Boussinesq). They derived the now-famous equation for the propagation of waves in one direction on the surface of water of density ρ in the form

$$v_t = \frac{\sqrt{gh}}{h}\Big[\Big(\varepsilon + \frac{3}{2}v\Big)v_x + \frac{1}{2}\sigma v_{xxx}\Big] \tag{1.4}$$

where $v = v(x,t)$, x is a coordinate chosen to be moving with the wave, ε is a small parameter, and

$$\sigma = h\Big(\frac{h^2}{3} - \frac{T}{g\rho}\Big) \sim \frac{1}{3}h^3,$$

when the surface tension $T(\ll \frac{1}{3}g\rho h^2)$ is negligible. This is essentially the original form of the *Korteweg-de Vries equation.* We shall call it the KdV equation. We note that in the approximation used to derive this equation one considers long wave propagating in the direction of increasing x. Equation (1.4) is one of the simplest and most useful nonlinear model equations for solitary waves, and it represents the long-time evolution of wave phenomena in which *the steepening effect of the nonlinear term is counterbalanced by dispersion.*

In the first half of the twentieth century, solitary waves and related evolution equations were not a major topic of scientific conversation. Modern developments in the theory and applications of the KdV solitary waves began with the seminal work published as a Los Alamos Scientific Laboratory Report in 1955 by Enrico Fermi, John Pasta, and Stanislaw Ulam [**106**] on a numerical model of a discrete nonlinear mass-spring system. In 1914, Debye suggested that the finite thermal conductivity of an anharmonic lattice is due to the nonlinear forces in the springs. This suggestion led Fermi, Pasta, and Ulam to believe that a smooth initial state would eventually relax to an equipartition of energy among all modes because of nonlinearity. But their studies led to the remarkable conclusion that there is no equipartition of energy among the modes. What they found did not correspond well to heat conduction; it seems this simple mass-and-spring system features near recurrence of initial states, and not the kind of thermalization that one expects. Although all the energy was initially in the lowest modes, after flowing back and forth among various low-order modes, it eventually returns to the lowest mode, and the end states are a series of recurring states. This remarkable fact has become known as the Fermi-Pasta-Ulam (FPU) recurrence phenomenon. A Los Alamos report was duly constructed and the issue then lay dormant.

A few years later Gardner and Morikawa [**117**] studied the stability of a cold collisionless plasma as it arose in a putative description of nuclear fusion. Starting from the full magneto-hydrodynamic equations and making assumptions about the motion of the plasma, they derived the same equations as had Boussinesq and Korteweg-de Vries, although the physical context was different.

Afterwards, Fermi, Pasta, and Ulam and Gardner and Morikawa inspired Kruskal and Zabusky [**170**] to formulate a continuum model for the nonlinear mass-spring system to understand why recurrence occurred. The system of ordinary differential equations went over to a partial differential equation in this limit, and the equation in question was the Korteweg-de Vries equation. In fact, they considered the initial value problem for the KdV equation

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{1.5}$$

where $\delta > 0$, with the initial condition

$$u(x,0) = \cos \pi x, \qquad 0 \le x \le 2,$$

and the periodic boundary conditions with period 2, so that $u(x,t) = u(x+2,t)$ for all t. Their numerical studies with $\sqrt{\delta} = 0.022$ produced a lot of new and interesting results. In fact, at later times the solutions develop a series of *eight* well-defined waves, each like sech^2 functions with the taller (faster) waves ever catching up and overtaking the shorter (slower) waves. These waves undergo nonlinear interaction, according to the KdV equation, and then emerge from the interaction without a change of form and amplitude, but with only a small change in their phases. So the most remarkable feature is that *these waves retain their identities after the nonlinear interaction.* Another surprising fact is that the initial profile reappears, very close to the FPU recurrence phenomenon. In view of their preservation of shape and their resemblance to the particle-like character of these waves, Kruskal and Zabusky called these solitary waves *solitons* (like photon, proton, etc.). Historically, Kruskal and Zabusky's paper in 1965 marked the birth of the new concept of the *soliton*, a name intended to signify particle-like quantities. Lax in 1968 [**179**] gave rigorous analytical proof that two distinct solitons are preserved through the nonlinear interaction governed by the KdV equation. These discoveries have led, in turn, to extensive theoretical, experimental, and computational studies over the last thirthy-five years.

For instance, we have the ingenious method for finding the exact solution of the KdV equation formulated by Gardner, Greene, Kruskal, and Miura in 1974 [**116**], which is known as the *Method of Inverse Scattering* or the *Inverse Scattering Transform.* This novel method has been generalized to solve several other nonlinear equations (see [**2**] and [**3**]).

Solitary waves in water were first observed scientifically by Russell [**246**], and in his study of these wave forms, he found that individually they appear to be stable states of motion. Propagating along a uniform canal, a solitary wave displays a remarkable property of permanence, such as to give an observer immediate confidence in its stability. Afterward, in the seminal work of Boussinesq, [**71**], after Boussinesq had deduced Russell's empirical great wave of translation (1.3), a new question arose with regard to the stability of these solitary wave solutions by the flow generated by equation (1.2). In fact, Boussinesq proposed to show that the solitary waves (1.3) are stable in the sense that a slight perturbation of a solitary wave will continue to resemble a solitary wave all of the time, rather than evolving into some other wave form. Such a result in fact can be an explanation of why solitary waves are so easily produced and observed in experiments.

Boussinesq obtained three invariant physical quantities, E, F, and V, related to equation (1.2), defined by

$$E(\eta) = \int_{-\infty}^{\infty} \left[\eta_x^2(x) - \frac{3}{h}\eta^3(x)\right] dx, \qquad F(\eta) = \int_{-\infty}^{\infty} \eta^2(x)\, dx, \qquad V(\eta) = \int_{-\infty}^{\infty} \eta(x)\, dx. \tag{1.6}$$

If $\eta = \eta(x,t)$ is a suitable smooth solution of the nonlinear equation (1.2), then $E(\eta(x,t))$, $F(\eta(x,t))$, and $V(\eta(x,t))$ are independent of the temporal variable t. Evidently, F and V represent the energy and the volume, respectively. The quantity E was named by Boussinesq as the *moment of instability.* Boussinesq noticed that the constraint variational problem

$$\delta E = 0 \qquad \text{for } F \text{ fixed} \tag{1.7}$$

is solved by the solitary wave solution (1.3) with a specific value of a. Boussinesq also realized that the extremal property of solitary waves is related to stability, and so he tentatively explained the permanence of these waves. He asserted that, within the class of wave forms whose energy has a given value, those profiles which correspond to the greatest moments of instability will differ the most from solitary wave profiles, while the minimum value of the moment of instability within this class is attained at a solitary wave profile. So by considering η to be close to η_{sol}, *Boussinesq assumed* that $E(\eta_{\text{sol}})$ is the absolute minimum for a given $F(u) = F(\eta_{\text{sol}})$, so that

$$\Delta E(\eta, \eta_{\text{sol}}) \equiv E(\eta) - E(\eta_{\text{sol}}) \geqq 0.$$

But $F(\eta)$ and $\Delta E(\eta, \eta_{\text{sol}})$ are both independent of time if η is a solution. So, if an initial wave form is initially close to η_{sol}, by continuity, its evolution is subject to the constraint that $\Delta E(\eta(t), \eta_{\text{sol}})$ keeps the same small value as in the beginning. Boussinesq then suggested that any wave initially close to a solitary wave remains so for all time. However, while he was on the right track, by modern standards his overall argument for the stability of solitary waves contains some gaps: first, he only proved that $E(\eta_{\text{sol}})$ is stationary, not that it is a minimum. Second, even though $\Delta E(\eta(t), \eta_{\rm sol})$ was nonnegative and small, it is not evident that this is an effective measure of the difference between the functions $\eta(t)$ and $\eta_{\rm sol}$ (see Chapter 6).

The first rigorous proof of stability of solitary wave solutions appeared only a century later, in Benjamin's article [**42**] on solitary wave solutions of the KdV equation (1.5) with $\delta = 1$ (obviously this normalization does not affect in any sense the theory). If we consider a wave of the form $u(x,t) = \phi_c(x - ct)$ and substitute it in (1.5) with the boundary condition that $\phi_c(\xi) \to 0$ as $|\xi| \to \infty$, we see that the wave profile ϕ_c satisfies the nonlinear differential equation

$$-\phi_c''(\xi) + c\phi_c(\xi) - \frac{1}{2}\phi_c^2(\xi) = 0 \tag{1.8}$$

where $\xi = x - ct$. It is well known that, via integration, the wave profile ϕ_c is given by

$$\phi_c(\xi) = 3c\,\mathrm{sech}^2\Big(\frac{1}{2}\sqrt{c}\,\xi\Big). \tag{1.9}$$

Making an elaboration of Boussinesq's original ideas, Benjamin showed that the solitary wave solutions (1.9) are stable by the flow generated by the KdV equation for every $c > 0$. The arguments given by Benjamin were also based on the functionals E $(h = 1)$ and F, which are also invariants of motion for the KdV equation. In a simple analysis, we cannot expect to prove stability of solitary wave solutions with respect to the metric in $H^s(\mathbb{R})$, $s \geqq 0$ (see Chapter 5). So, one way of avoiding this difficulty is to give up knowing where the solution is in exchange for knowing its *shape* very well. This discrimination leads to a new measure of distance and a new notion of stability, which Benjamin called *stability in shape* (or orbital stability). This notion in a few words tell us that ϕ is stable whenever an initial wave form ψ is sufficiently near to ϕ in the $H^1(\mathbb{R})$-norm and then for each instant t there is a translation, $\gamma(t)$, such that the shape of the function $x \to u(x + \gamma(t), t)$ will resemble and remain close to ϕ in the $H^1(\mathbb{R})$-norm. The heart of Benjamin's analysis was to show that if $\psi \in H^1(\mathbb{R})$ is sufficiently close to ϕ_c in $H^1(\mathbb{R})$-norm and $F(\psi) = F(\phi_c)$, then

$$E(\psi) - E(\phi_c) \geq A \inf_{y \in \mathbb{R}} \|\psi - \phi_c(\cdot + y)\|_1^2 \tag{1.10}$$

where A denotes a positive constant which is independent of ψ. Note that inequality (1.10) shows that ϕ_c is a local minimizer of the functional E, but it does not show Boussinesq's assertion (in the case of equation (1.5)) that ϕ_c is a minimizer of E over the set of all admissible functions ψ satisfying $F(\psi) = F(\phi_c)$. As we will see later, this assertion was only established years later as a consequence of the *Concentration-Compactness Method* (Lions [**193**], [**194**]) and from the pioneering work of Cazenave and Lions [**82**] about the orbital stability of standing waves for some nonlinear Schrödinger equations.

The local analysis given in [**42**] is centered on the quadratic form $\langle \mathcal{L}\varphi, \varphi\rangle$ generated by the closed, unbounded, self-adjoint operator on $L^2(\mathbb{R})$

$$\mathcal{L} = -\frac{d^2}{dx^2} + c - \phi_c. \tag{1.11}$$

Upper and lower bounds on the quadratic term are the key ingredients to obtain inequality (1.10) and so establish stability. Upper bounds are straightforward in this case, and the crux of the matter is effective lower bounds. At this point, a careful spectral theory for the operator $\mathcal{L}$ arises as a crucial piece of information for completing the stability. The basic properties about $\mathcal{L}$ are

(1.12) *$\mathcal{L}$ has a single negative eigenvalue which is simple, the zero eigenvalue is simple with eigenfunction ϕ_c', and the remainder of the spectrum is positive and bounded away from zero.*

So, since ϕ_c in (1.9) has a single maximum and is an even function monotonically decreasing to zero at infinity, the Sturm-Liouville theory immediately implies the required conditions in (1.12). So, inequality (1.10) and other ingredients imply that by suitably translating the solitary wave profile ϕ_c, one can nearly match it to the solution u and so the assertion that the solitary wave is stable in shape is a consequence.

For several years after these initial advances there were no new developments in the stability theory of solitary wave solutions for nonlinear evolution equations. This is owing, in part, to the overall complexity of Benjamin's theory and, in part, to the difficulty of establishing the crucial information (1.12).

The next advance appears to have been made by Bennet *et al.* [**49**] in their approach to the stability of the Benjamin-Ono equation's solitary wave (see (3.5) and (3.6)). Their theory still required a complete spectral analysis. Weinstein in 1986–1987 ([**274**], [**276**]) simplified the overall argument demonstrating stability, clarifying and sharpening what was required from the spectral analysis, and very considerably extending the range of the theory's applicability. A large number of the extensive collection of cases that fall within the theory produced the development of sharper conditions for the stability and instability of general travelling wave solutions for nonlinear evolution equations, such as the general stability theory established by Grillakis, Shatah, and Strauss in [**123**] and [**124**]. This theory was set for abstract Hamiltonian equations (or systems) of the form

$$u_t = JE'(u(t)) \tag{1.13}$$

where J should be an onto skew-symmetric linear operator. In [**123**] a sufficient condition of stability was established, one which will imply the crucial inequality (1.10). In the case of the KdV equation, this condition is given by the strict convexity property of the real-valued function

$$d(c) = E(\phi_c) + cF(\phi_c) \tag{1.14}$$

for $c \in (0, \infty)$. This condition is the same as the one given by Shatah in 1983 ([**251**]) and by Weinstein in 1987 ([**274**]) in their analyses of the stability of ground states for certain nonlinear evolution equations.

Instability conditions were also considered in the Grillakis *et al.* abstract results [**123**], in which the assumption on J is crucial. So, this result does not apply directly when J is a differential operator such as ∂_x. In this situation an improvement of the theory was given by Bona, Souganidis, and Strauss [**64**] and Souganidis and Strauss [**255**], by taking into consideration general one-dimensional evolution equations of the forms

$$u_t + f(u)_x - Mu_x = 0 \quad \text{and} \quad M_0 u_t + f(u)_x = 0,$$

where M and M_0 are pseudo-differential operators of order $\mu \geqq 1$. So, in the presence of the right spectral information (1.12), a solitary wave that is not proved to be stable by the theory is in fact unstable. Namely, the basic function d satisfies the property that $d''(c) < 0$.

Condition (1.12) is a crucial piece of information to be verified in order to obtain a stability or instability theory. In general, it is not easy to prove this condition. In particular the fact that zero is a simple eigenvalue is a delicate business. A more accurate study about this point in the theory has been developed by Albert in [**6**] and Albert and Bona in [**10**]. In their works novel sufficient conditions for obtaining the needed spectral information were established (see Chapter 7).

We have seen that the stability of solitary wave solutions, in a general form, is based on conditions that prove an inequality of the form (1.10). This means that we must show that the solitary wave solution is a local constrained minimizer of a Hamiltonian functional, and the procedure for this is carried out basically studying specific spectral properties of a linear operator obtained by linearizing the solitary wave equation. In practice this spectral analysis is particularly difficult to carry out (see Chapter 7). To avoid these difficulties, an alternative method of proving stability of solitary waves, which does not rely on local analysis, was developed by Cazenave and Lions in [**82**] using Lions's method of *Concentration-Compactness*. In this method, instead of starting with a given solitary wave and trying to prove that it reaches a local minimum for a constrained variational problem, one starts with the constrained variational problem and looks for global minimizers. When the method works and the functionals involved in the variational analysis are conserved quantities for the equation in question, one shows not only the existence of global minimizers which are solitary wave solutions, but also that the set of global minimizers is a stable set for the flow generated by the evolution equation under discussion. This means that a solution which starts near the set of minimizers will remain near it for all time (see Chapters 8 and 9).

In the last couple of decades, a series of applications and variants of the Concentration-Compactness Method was obtained for a great range of dispersive evolution equations. We cite for instance the works of Weinstein [**276**], Albert, Bona, and Saut [**14**], Kuznetsov [**172**], Kuznetsov, Rubenchik, and Zakharov [**173**], Albert [**8**], de Bouard and Saut [**95**], Kichenassamy [**164**], Lopes [**195**], [**197**], [**198**], Angulo [**21**], and Albert and Angulo [**9**].

The next development in the study of the applicability of the Concentration-Compactness Method in the existence and stability of solitary wave solutions was achieved by Levandosky [**182**]. Here Levandosky studied the stability of a fourth-order wave equation (see Chapter 8, Section 8.3) and observed that the method can still be used in the stability theory if the functionals involved in the variational problem are not conserved quantities.

Overall, the study of the qualitative properties of solitary wave solutions and its influence on the development of the theory of nonlinear evolution equations have produced a large number of papers in the past decade such that it now becomes difficult to trace the developments. In particular, a very substantial contribution relative to the large-time asymptotic behavior of KdV-type equations has been established in the works of Pego and Weinstein [**235**], [**236**], Bona and Soyeur [**63**], Laedke, Blaha, Spatschek, and Kuznetsov [**176**], Merle [**220**], Martel and Merle [**210**], [**211**], [**212**], [**214**], [**216**], [**217**], Martel [**208**], [**209**], and Merle and Vega [**221**].

Now, the situation regarding the existence and stability of periodic travelling wave solutions is very different from that for solitary wave solutions; because of these progressive wave trains have received comparatively little attention. A first study of these wave forms was made by Benjamin in [**45**]. This study focused on the periodic steady solutions called *cnoidal waves*, which were found initially by Korteweg-de Vries in [**168**] for the KdV equation (1.5) ($\delta = 1$). Benjamin put forward an approach to proving the stability of cnoidal waves of the form

$$\varphi(\xi) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k\right), \tag{1.15}$$

but he did not provide a detailed justification of his assertions, and several aspects of his argument seemed problematic. Recently, Angulo, Bona, and Scialom in [**28**] gave a complete theory of the stability of cnoidal waves for the KdV equation (see also [**26**]). To obtain this result, they used the theory of elliptic integrals and the modern theory of stability of Grillakis, Shatah, and Strauss [**123**] which they adapted to the periodic context.

Afterwards, Angulo in [**25**] found new formulas of periodic travelling wave solutions of the focusing nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0$$

based on the Jacobian elliptic function of type *cnoidal and dnoidal* and established a stability and instability study of the wave form of dnoidal type (see also Gallay and Hărăguş [**113**], [**114**]). An similar study was also obtained for the models Modified Korteweg-de Vries and Hirota-Satsuma in [**25**], [**26**], [**27**].

It is remarkable to see that in all these works it was necessary to use the method of quadrature to obtain the explicit profile of the periodic travelling wave solutions, in other words, to put our *differential equation* in the form

$$[\psi']^2 = F(\psi).$$

Moreover, the necessary spectral information for studying stability or instability was obtained via an elaborated spectral theory for the periodic eigenvalue problem associated with the Jacobi form of the *Lamé's equation*, namely,

$$\begin{cases} \dfrac{d^2}{dx^2}\Psi + [\rho - n(n+1)k^2 sn^2(x;k)]\Psi = 0, \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases}$$

when $n \in \mathbb{N}$, $sn(\cdot; k)$ represents the Jacobian elliptic function snoidal, $k \in (0,1)$, and K is the complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

We note that Gardner in [**115**] provided a theory for determining that the large wavelength periodic waves are *linearly unstable* whenever the limiting homoclinic wave (solitary wave) is unstable. He applied his theory to diverse types of nonlinear evolutions equations in one space variable. In the case of the generalized KdV equations

$$u_t + u^p u_x + u_{xxx} = 0,$$

with $p \in \mathbb{N}$, if we assume that this equation produces a family of large wavelength periodic waves U^α such that the period $2T_\alpha$ tends to infinity as α tends to zero, then this family is unstable whenever $p > 4$ and $\alpha > 0$ is sufficiently small.

Recently Angulo and Natali in [**37**] developed a novel theory for studying the existence and stability of periodic travelling waves based on the theory of totally positive operators and the *Poisson Summation Theorem*. Their theory was based on the study of the general equation of KdV-type:

$$u_t + u^p u_x - M u_x = 0,$$

where $p \geq 1$ is an integer and M is a differential or pseudo-differential operator in the framework of periodic functions. In other words, M is defined as a Fourier multiplier operator by

$$\widehat{Mg}(k) = \alpha(k)\widehat{g}(k), \quad k \in \mathbb{Z},$$

where α is assumed to be a measurable, locally bounded, even function on $\mathbb{R}$, satisfying the condition

$$A_1|k|^{m_1} \leq \alpha(k) \leq A_2(1+|k|)^{m_2}$$

for $m_1 \leq m_2$, $|k| \geq k_0$, $\alpha(k) > b$ for all $k \in \mathbb{Z}$, and $A_i > 0$. The travelling wave solutions

$$u(x,t) = \varphi_c(x - ct)$$

have a profile φ_c being a real-valued smooth periodic function with an *a priori* fundamental period $2L$, $L > 0$. Hence substituting the form of u given above and integrating once (assuming the constant of integration to be zero), one obtains that $\varphi = \varphi_c$ is a solution of the equation

$$(M + c)\varphi_c - \frac{1}{p+1}\varphi_c^{p+1} = 0. \tag{1.16}$$

So we find the associated linear, closed, unbounded, self-adjoint operator $\mathcal{L} : D(\mathcal{L}) \to L^2_{\text{per}}([-L, L])$ defined on a dense subspace of $L^2_{\text{per}}([-L, L])$ by

$$\mathcal{L}_M u = (M + c)u - \varphi_c^p u. \tag{1.17}$$

The problem about the existence of a nontrivial smooth curve of periodic solutions for (1.16) above presents new and delicate issues that need to be handled. The possibility of explicitly finding solutions will naturally depend on the form of M. If it is a differential operator of the form $M = -\partial_x^2$, the use of the quadrature method and the theory of elliptic functions has been shown to be a main tool, and so the solutions will depend on a general form of the Jacobian elliptic functions of the *snoidal*, *cnoidal*, and *dnoidal* types (see [**25**], [**26**], [**27**], [**45**]). Since the period of these functions depends on the complete elliptic integral of the first kind $K(k)$, we have that the modulus k will depend on the velocity c and therefore we have that *a priori* the period of φ_c will depend on c. So, the required smooth branch of periodic travelling wave solutions with a fixed minimal period has been obtained in many cases by using the Implicit Function Theorem. We note that this procedure based on the quadrature method obviously in general does not work if M is a pseudo-differential operator (see Albert [**7**] in which it is shown that the travelling wave equations associated to the Intermediate Long Wave (ILW) and the Benjamin-Ono (BO) equations can be reduced to a quadrature form).

In Angulo and Natali [**37**] a different approach was established to obtain explicit solutions of (1.16) for a specific form of M and values of p. This approach is based on the classical *Poisson Summation Theorem* [**258**]. There are at least two important advantages to this new approach. The first one is that it can be used for obtaining solutions when M is a pseudo-differential operator, for example in the case of the Benjamin-Ono equation. The other one is related to computing the expression

$$\frac{d}{dc}\int \varphi_c^2(x)dx. \tag{1.18}$$

In general, determining the sign of the expression in (1.18) is very difficult in the periodic case. As has been shown in the literature, the use of nontrivial identities for the complete elliptic integrals of the first and the second kinds sometimes comes on the scene as a fundamental piece in the analysis, and so verifying this property can become a challenge. As we will see, the verification of (1.18) can be very easily obtained by using a combination of the Poisson Summation Theorem and the *Parseval Theorem*.

With regard to the spectral conditions associated with (1.17), the problem of obtaining (1.12) is very delicate. One of the most remarkable results in the theory of stability of solitary wave solutions was given by Albert and by Albert and Bona in [**6**] and [**10**], in which sufficient conditions were given to obtain the properties in (1.12) associated with $\mathcal{L}_M$. The advantage of Albert's approach is that it does not require an explicit computation of the spectrum of the linear operator (1.17), since it is obtained exclusively from positivity properties of the Fourier transform of the solitary wave in question. The Angulo-Natali theory establishes an extension of the theory in [**6**] and [**10**] for the case of positive even periodic travelling wave solutions. The periodic problem has new points not encountered when considering issues related to the solitary waves. The analysis in [**37**] also relies upon the theory of totally positive operators and so the class $PF(2)$ defined by Karlin in [**149**] is used.

The theory in [**37**] leads to a significant simplification of some recent proofs of stability of periodic travelling wave solutions of KdV-type equations, such as in the case of the Korteweg-de Vries and the modified Korteweg-de Vries equations, since in those cases the verification of the spectral conditions requires the determination of the instability intervals associated with the Lamé equation above and of an explicit formula of at least the first three eigenvalues ρ. Our analysis does not require this information.

The theory in [**37**] has established two very interesting and novel results. The first one is about the stability of the periodic travelling wave solutions found by Benjamin in [**41**] for the Benjamin-Ono equation

$$u_t + uu_x - \mathcal{H}u_{xx} = 0, \tag{1.19}$$

where $\mathcal{H}$ denotes the periodic Hilbert transform defined by

$$\mathcal{H}f(x) = \frac{1}{2L}\ \text{p.v.}\int_{-L}^{L} \text{cotg}\Big[\frac{\pi(x-y)}{2L}\Big] f(y)\, dy.$$

The associated periodic waves for (1.19) with a minimal period $2L$ are given for $c > \frac{\pi}{L}$ as

$$\varphi_c(x) = \frac{2\pi}{L}\frac{\sinh(\gamma)}{\cosh(\gamma) - \cos\left(\frac{\pi x}{L}\right)},$$

such that $\gamma > 0$ satisfies $\tanh(\gamma) = \dfrac{\pi}{cL}$. The second result is the existence and stability/instability of positive periodic travelling waves for the critical KdV equation

$$u_t + 5u^4u_x + u_{xxx} = 0 \tag{1.20}$$

and the critical nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^4 u = 0.$$

More exactly, in [**38**] there is shown the existence of a smooth curve $c \in \left(\frac{\pi^2}{L^2}, +\infty\right) \to \varphi_c$ of periodic travelling wave solutions for (1.20) where the profile φ_c is given by

$$\varphi_c(z) = \frac{\sqrt{\eta_3}\, dn(\frac{4}{3g}z; k)}{\sqrt{1 + \beta^2 sn^2(\frac{4}{3g}z; k)}}$$

with η_3, g, and k smooth functions depending on c. Moreover, there is a unique (threshold) value c^* of the speed-wave c of φ_c, which separates two different global scenarios of the evolution of a localized initial perturbation of φ_c: for $c < c^*$ we have nonlinear stability and for $c > c^*$ we have nonlinear instability.

We note that several instability analyses for evolution partial differential equations found in the literature have been carried out in a general setting, in which the periodic waves have a distinguished asymptotic limit (see Gardner in [**115**] and Sandstede and Scheel in [**247**]). The results presented in [**37**] do not fall into this class. More precisely, the instability result for the critical KdV and the critical Schrödinger equations is not obtained from an instability linearized analysis or from an analysis of the spectra of large wavelength periodic waves which follow the unstable solitary wave. We believe that detecting instabilities via the last approach is a difficult task. Moreover, from the instability results obtained in Gardner [**115**] it is possible to say that the approaches presented for the critical KdV equation complete the picture about the existence of a branch of periodic travelling wave solutions, which are unstable for the generalized Korteweg-de Vries equation provided $p \geqq 4$.

It is important to note that the stability results about periodic waves are obtained in general by periodic initial disturbance having the same minimal period of the periodic travelling wave solutions being studied. The results of stability in $H^1_{\mathrm{per}}([0, L])$ for the orbit generated by the cnoidal waves φ in (1.15),

$$\Omega_\varphi \equiv \{\varphi(\cdot + s)\}_{s \in \mathbb{R}},$$

for the flow of the periodic KdV equation, are obtained for initial disturbances of φ having the same period L. It is a conjecture on the part of Benjamin [**45**] that cnoidal waves of minimal period L are unstable by perturbations, for example, of period $2L$. Some evidence in favor of this scenario was recently shown by Angulo [**25**] in the case of the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0.$$

Now we describe how this book has been divided. The following chapters will present many ideas which may determine or at least suggest whether solitary wave solutions or periodic travelling wave solutions for a given nonlinear dispersive equation are or are not stable.

The second chapter provides a set of equations which give the reader a clear understanding of the applicability and physical relevance of nonlinear dispersive evolution equations. In the next chapters, many of these models have been chosen to exemplify the theories of stability or instability. Chapter 3, we consider the principal issue of our study, namely, travelling wave solutions. Here we give a good set of explicit formulas for solitary waves and periodic travelling waves for nonlinear equations of interest.

Chapter 4 contains results on the well-posedness of the Cauchy problem and initial boundary value problem belonging to some equations put forward in Chapter 2. In particular we present an idea of the proof of the theory of local well-posedness in H^s_{per}, $s \geq 0$, for the KdV equaton in the Bourgain spaces. Chapter 5 establishes the definition of stability, which is understood in the Liapunov sense.

Chapter 6 develops the theory of stability of solitary wave solutions for the Generalized Korteweg-de Vries equations. In this case we apply in detail the germinal ideas of Benjamin, Bona, and Weinstein. We have called this approach the "classical method". Also, we include an application of this local method by obtaining a theory of "*stability of the blow-up*" for a class of Kortweg-de Vries equations. The chapter finishes with a small review of improvements in the theory of solitary wave solutions and solitons for KdV-type equations.

Chapter 7 is devoted to exemplifying the Grillakis, Shatah, and Strauss stability theory. Special attention is given to model equations that arise in the study of long wave propagation. The chapter finishes with some comments and other applications of their theory. It includes the basic theorems of the theory of Albert and Bona, with regard to sufficient conditions for the required spectral information in the stability theory of solitary wave solutions.

In Chapter 8, we consider the Concentration-Compactness Principle and its applicability to the study of the existence and stability of solitary wave solutions of nonlinear evolution equations. Special attention is given to the existence and stability of solitary wave solutions for the Generalized Benjamin-Ono equations. Using the theory of symmetric decreasing rearrangements of functions and Levandosky's approach, we present a *new proof of the stability of Benjamin-Ono's solitary waves*. In Chapter 9, we consider other applications of the Concentration-Compactness principle. In particular, we review the stability theory for Kadomtsev-Petviashvili equations.

Chapter 10 deals with the instability of solitary wave solutions. Special attention is devoted to the solitary wave solutions for Generalized Benjamin equations. Here the solitary wave solutions in consideration are those obtained via a variational approach, and the method to be established does not depend on spectral or convexity conditions. The chapter contains other applications of the theory. In particular, we review the instability theory for fifth-order Korteweg-de Vries equations. We also show a linearized instability theory for weakly coupled Korteweg-de Vries systems.

Chapter 11 is dedicated exclusively to the study of the existence and stability of periodic travelling wave solutions. Here we present an extension of the Grillakis, Shatah, and Strauss theory to the case when the *travelling wave solutions do not have the property of being critical points*. We apply it to the stability of cnoidal wave solutions with mean zero with regard to the flow of the KdV and Benney-Luke equations. We also establish the basic theory of Angulo and Natali and apply it to obtain the existence and stability of positive cnoidal wave solutions for the Korteweg-de Vries and Benjamin-Ono equation. We also apply it to show the stability/instability of a smooth curve of periodic travelling wave solutions for the critical Korteweg-de Vries and the critical nonlinear Schrödinger equations.

We finish these notes with two appendices. The first one establishes the basic tools of the Fourier transform, the space of tempered distributions, the theory of Sobolev spaces $H^s(\mathbb{R}^n)$ of type $L^2(\mathbb{R})$, as well as the periodic Sobolev spaces on $\mathbb{R}$.

Also, we pull together some facts about the symmetric decreasing rearrangement of a function in $\mathbb{R}$ and the theory of Jacobian elliptic functions. The second appendix contains a detailed and clear explanation of the theory of closed linear operators in Hilbert spaces that is used in this work. Special attention is given to the spectral theory required in the stability approach. A careful treatment is given to the Sturm-Liouville theory on $\mathbb{R}$ of second-order differential operators and to the Floquet theory associated with the Lamé equation.

The literature concerned with the subject of this book is immense. Therefore here we cite those papers most closely connected with the text of the book without claiming completeness. An updated and vast bibliography is included to stimulate new interest in future study and research.