CHAPTER 3

Mutual intersection: large deviations

3.1. High moment asymptotics

One of the major goals of this chapter is to provide a precise estimate to the tail probability

$$\mathbb{P}\Big\{\alpha\big([0,1]^p\big) \ge t\Big\}$$

as $t \to \infty$. By Theorem 1.2.8, the problem is in connection to the study of the *high moment asymptotics* posted as

$$\mathbb{E}\Big[\alpha([0,1]^p)^m\Big] \qquad (m \to \infty).$$

By comparing (2.2.8) and (2.2.19), it seems that the high moment asymptotics for

$$\mathbb{E}\Big[\alpha\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)^m\Big]$$

more tractable. The question is: What can we say about the quantity

$$\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \bigg[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\bigg]^p$$

when $m \to \infty$?

We generalize this problem to the study of

$$\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\bigg[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\bigg]^p$$

where $(E, \mathcal{E}, \pi)$ is a measure space, $K(x,y)$ is a non-negative measurable function on $E \times E$, and $x_{\sigma(0)} = x_0$ is an arbitrary but fixed point in E.

THEOREM 3.1.1. *Let $p > 1$ be fixed. Assume that the kernel satisfies:*

(1) *Irreducibility: $K(x,y) > 0$ for every $(x,y) \in E \times E$.*
(2) *Symmetry: $K(x,y) = K(y,x)$ for every $(x,y) \in E \times E$.*
(3) *Integrability:*

$$\varrho \equiv \sup_f \iint_{E\times E} K(x,y) f(x) f(y) \pi(dx)\pi(dy) < \infty$$

where the supremum is taken over all f satisfying

$$\int_E |f(x)|^{\frac{2p}{2p-1}} \pi(dx) = 1.$$

Then

$$\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p$$

$$\geq p\log\varrho. \tag{3.1.1}$$

Proof. Let $f\in\mathcal{L}^{\frac{2p}{2p-1}}(E,\mathcal{E},\pi)$ be a bounded, non-negative function such that

$$\delta\equiv\inf_{x;\, f(x)>0}K(x_0,x)>0 \tag{3.1.2}$$

and that

$$\int_E f^{\frac{2p}{2p-1}}(x)\pi(dx)=1.$$

Let $g(x)=f^{\frac{2(p-1)}{2p-1}}(x)$ and $h(x)=f^{\frac{p}{2p-1}}(x)$. Then $f(x)=\sqrt{g(x)}h(x)$ and

$$\int_E g^{\frac{p}{p-1}}(x)\pi(dx)=\int_E h^2(x)\pi(dx)=1.$$

By boundedness of $f(x)$, there is $\epsilon>0$ such that $g(x)\geq\epsilon h^2(x)$. By Hölder inequality, we get

$$\begin{aligned}
&\Big\{\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\Big\}^{1/p} \qquad (3.1.3)\\
&\geq\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big(\prod_{k=1}^m g(x_k)\Big)\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\\
&=m!\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big(\prod_{k=1}^m g(x_k)\Big)\prod_{k=1}^m K(x_{k-1},x_k)\\
&=m!\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)K(x_0,x_1)\sqrt{g(x_1)}\\
&\quad\times\Big[\prod_{k=2}^m\sqrt{g(x_{k-1})}K(x_{k-1},x_k)\sqrt{g(x_k)}\Big]\sqrt{g(x_m)}\\
&\geq\delta\epsilon m!\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)h(x_1)\\
&\quad\times\Big[\prod_{k=2}^m\sqrt{g(x_{k-1})}K(x_{k-1},x_k)\sqrt{g(x_k)}\Big]h(x_m)\\
&=\delta\epsilon m!\langle h,T^{m-1}h\rangle
\end{aligned}$$

where the linear operator T: $\mathcal{L}^2(E,\mathcal{E},\pi)\longrightarrow\mathcal{L}^2(E,\mathcal{E},\pi)$ is defined as

$$(T\tilde{h})(x)=\sqrt{g(x)}\int_E K(x,y)\sqrt{g(y)}\tilde{h}(y)\pi(dy)\qquad\tilde{h}\in\mathcal{L}^2(E,\mathcal{E},\pi).$$

For any $h_1,h_2\in\mathcal{L}^2(E,\mathcal{E},\pi)$ with $||h_1||=||h_2||=1$, we get

$$\begin{aligned}
\langle h_1,Th_2\rangle&=\iint_{E\times E}K(x,y)\sqrt{g(x)}h_1(x)\sqrt{g(y)}h_2(y)\pi(dx)\pi(dy)\\
&=\iint_{E\times E}K(y,x)\sqrt{g(x)}h_1(x)\sqrt{g(y)}h_2(y)\pi(dx)\pi(dy)=\langle Th_1,h_2\rangle.
\end{aligned}$$

That is, T is symmetric. In addition,

$$\langle h_1, Th_2\rangle = \frac{1}{4}\Big\{\langle h_1+h_2, T(h_1+h_2)\rangle - \langle h_1-h_2, T(h_1-h_2)\rangle\Big\}.$$

Notice that

$$\begin{aligned}
&\Big|\langle h_1 \pm h_2, T(h_1 \pm h_2)\rangle\Big| \\
&\le \iint_{E\times E} K(x,y)\sqrt{g(x)}\big|h_1(x)\pm h_2(x)\big|\sqrt{g(y)}\big|h_1(y)\pm h_2(y)\big|\pi(dx)\pi(dy).
\end{aligned}$$

By Hölder inequality, we get

$$\begin{aligned}
&\int_E \big|\sqrt{g(x)}\big(h_1(x)\pm h_2(x)\big)\big|^{\frac{2p}{2p-1}}\pi(dx) \\
&\le \Big\{\int_E g^{\frac{p}{p-1}}(x)\pi(dx)\Big\}^{\frac{p-1}{2p-1}}\Big\{\int_E |h_1(x)\pm h_2(x)|^2\pi(dx)\Big\}^{\frac{p}{2p-1}} \\
&= \Big\{\int_E f^{\frac{2p}{2p-1}}(x)\pi(dx)\Big\}^{\frac{p-1}{2p-1}}\Big\{\int_E |h_1(x)\pm h_2(x)|^2\pi(dx)\Big\}^{\frac{p}{2p-1}} \\
&= \Big\{\int_E |h_1(x)\pm h_2(x)|^2\pi(dx)\Big\}^{\frac{p}{2p-1}}.
\end{aligned}$$

Consequently,

$$\Big|\langle h_1 \pm h_2, T(h_1 \pm h_2)\rangle\Big| \le \varrho\int_E |h_1(x)\pm h_2(x)|^2\pi(dx).$$

Hence,

$$\langle h_1, Th_2\rangle \le \frac{1}{4}\varrho\big\{||h_1+h_2||^2 + ||h_1-h_2||^2\big\} = \frac{\varrho}{2}\big(||h_1||^2+||h_2||^2\big) = \varrho.$$

Therefore, T is a bounded linear operator. This, together with symmetry, implies that T is self-adjoint.

According to Theorem E.2 in the Appendix, the self-adjoint operator T admits the spectral integral representation

$$T = \int_{-\infty}^{\infty} \lambda E(d\lambda). \tag{3.1.4}$$

Using Corollary E.5 in the Appendix, we have

$$T^{m-1} = \int_{-\infty}^{\infty} \lambda^{m-1} E(d\lambda). \tag{3.1.5}$$

By (E.15) in the Appendix, the above representations lead to

$$\langle h, Th\rangle = \int_{-\infty}^{\infty} \lambda\mu_h(d\lambda), \tag{3.1.6}$$

$$\langle h, T^{m-1}h\rangle = \int_{-\infty}^{\infty} \lambda^{m-1}\mu_h(d\lambda) \tag{3.1.7}$$

where μ_h is a measure on $\mathbb{R}$ with

$$\mu_h(\mathbb{R}) = \int_E h^2(x)\pi(dx) = 1$$

(i.e., μ_h is a probability measure).

When m is odd (so $m-1$ is even), by Jensen inequality, we have

$$\begin{aligned}\langle h, T^{m-1}h\rangle &\geq \Big(\int_{-\infty}^{\infty}\lambda\mu_h(d\lambda)\Big)^{m-1} = \Big(\langle h, Th\rangle\Big)^{m-1}\\ &= \Big(\iint_{E\times E}K(x,y)\sqrt{g(x)}h(x)\sqrt{g(y)}h(y)\pi(dx)\pi(dy)\Big)^{m-1}\\ &= \Big(\iint_{E\times E}K(x,y)f(x)f(y)\pi(dx)\pi(dy)\Big)^{m-1}.\end{aligned}$$

Summarizing our argument, when m is odd, we have

$$\begin{aligned}&\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\ &\geq (\delta\epsilon)^p(m!)^p\Big(\iint_{E\times E}K(x,y)f(x)f(y)\pi(dx)\pi(dy)\Big)^{p(m-1)}.\end{aligned}$$

When m is even, a slight modification of (3.1.3) gives us

$$\begin{aligned}&\Big\{\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\Big\}^{1/p}\\ &\geq c(\delta,\epsilon)m!\langle h, T^{m-2}h\rangle\end{aligned}$$

for some $c(\delta,\epsilon)>0$. So the argument based on spectral representation leads to the lower bound

$$\begin{aligned}&\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\ &\geq c^p(\delta,\epsilon)(m!)^p\Big(\iint_{E\times E}K(x,y)f(x)f(y)\pi(dx)\pi(dy)\Big)^{p(m-2)}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}&\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\ &\geq p\log\iint_{E\times E}K(x,y)f(x)f(y)\pi(dx)\pi(dy).\end{aligned}$$

Finally, the desired conclusion follows from the following two facts:

First, the supremum in the definition of ϱ can be taken over the non-negative f.

Second, by irreducibility assumption the set of functions f satisfying (3.1.2) is dense in non-negative functions in $\mathcal{L}^{\frac{2p}{2p-1}}(E,\mathcal{E},\pi)$. $\square$

The upper bound is much more difficult. Indeed, we are able to establish the upper bound only in the case when E is a finite set.

THEOREM 3.1.2. *Fix $p > 1$, let E be a finite set and let $K\colon E \times E \longrightarrow \mathbb{R}^+$ be non-negative function such that $K(x,y) = K(y,x)$ for any $x, y \in E$. Let π be a non-negative function on E. Then*

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^p} \sum_{x_1,\cdots,x_m\in E} \pi(x_1)\cdots\pi(x_m) \\
&\quad\times \Big[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p \qquad (3.1.8)\\
&\le p \log \sup_f \sum_{x,y\in E} K(x,y) f(x) f(y) \pi(x)\pi(y)
\end{aligned}$$

where the supremum is taken over all functions f on Ω satisfying

$$\sum_{x\in E} |f(x)|^{\frac{2p}{2p-1}} \pi(x) = 1.$$

Proof. We may assume that $\pi(x) > 0$ for every $x \in E$, for otherwise we can remove all zero points of π from E. Let

$$\mu = L_m^{\mathbf{x}} = \frac{1}{m}\sum_{k=1}^m \delta_{x_k}$$

be the empirical measure generated by $\mathbf{x} = (x_1, \cdots, x_m)$. Notice that for each $\sigma \in \Sigma_m$, we have

$$\sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} = 1.$$

We have

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)}) \\
&= \sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} \sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)}) \\
&= \sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} \prod_{k=1}^m K(y_{k-1}, y_k) \sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}}.
\end{aligned}$$

We claim that

$$\sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} = \prod_{x\in E} \big(m\mu(x)\big)!.$$

Indeed, for each $x \in E$ there are, respectively, exactly $m\mu(x)$ of $x_1, \cdots, x_m$ and exactly $m\mu(x)$ of $y_1, \cdots, y_m$ which are equal to x. Therefore, there are $\big(m\mu(x)\big)!$ ways to match each x-valued component of $\mathbf{y}$ with each x-valued component of $\mathbf{x}$. Thus, the claim follows from multiplication principle.

Consequently,

$$\begin{aligned}&\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\\&=\prod_{x\in E}\big(m\mu(x)\big)!\sum_{y_1,\cdots,y_m\in E}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k).\end{aligned}$$

Let $q>1$ be the conjugate number of p and define $\varphi_\mu(x)=\mu(x)^{1/q}\pi(x)^{1/p}$. Then

$$\begin{aligned}&\sum_{y_1,\cdots,y_m\in E}\varphi_\mu(y_1)\cdots\varphi_\mu(y_m)\prod_{k=1}^m K(y_{k-1},y_k)\\&\ge\sum_{y_1,\cdots,y_m\in E}\varphi_\mu(y_1)\cdots\varphi_\mu(y_m)1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\\&=\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)\sum_{y_1,\cdots,y_m\in E}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\end{aligned}$$

where the last step follows from the fact that when $L_m^{\mathbf{y}}=\mu$, there are $m\mu(x)$ factors in the product $\varphi_\mu(x_1)\cdots\varphi_\mu(x_m)$ which are equal to $\varphi_\mu(x)$ for any $x\in E$.

Summarizing the above steps, we have

$$\begin{aligned}&\frac{1}{(m!)^p}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\&\le\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\Big]^p\\&\times\Big[\sum_{y_1,\cdots,y_m\in E}\varphi_\mu(y_1)\cdots\varphi_\mu(y_m)\prod_{k=1}^m K(y_{k-1},y_k)\Big]^p.\end{aligned}$$

Define $g_\mu(x)=\mu(x)^{1/q}\pi(x)^{-1/q}$. Then $\varphi_\mu(x)=g_\mu(x)\pi(x)$ for every $x\in E$ and

$$\sum_{x\in E}g_\mu^q(x)\pi(x)=\sum_{x\in E}\mu(x)=1,\qquad\sup_{x\in E}g_\mu(x)\le\sup_{x\in E}\pi(x)^{-1/q}.$$

Consequently,

$$\begin{aligned}&\frac{1}{(m!)^p}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\&\le\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\Big]^p\\&\times\Big[\sup_g\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k)\Big]^p\end{aligned}\tag{3.1.9}$$

where the supremum on the right hand side is taken for all non-negative functions g on E satisfying

$$\sum_{x\in E}g^q(x)\pi(x)=1,\qquad\sup_{x\in E}g(x)\le c$$

and where $c = \sup_{x\in E} \pi(x)^{-1/q}$.

For each g,

$$\begin{aligned}
&\sum_{y_1,\cdots,y_m\in E} \pi(y_1)\cdots\pi(y_m)\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k)\\
&= \sum_{y_1,\cdots,y_m\in E} \pi(y_1)\cdots\pi(y_m)K(x_0,y_1)\sqrt{g(y_1)}\\
&\times\Big[\sqrt{g(y_{k-1})}\prod_{k=2}^m K(y_{k-1},y_k)\sqrt{g(x_k)}\Big]\sqrt{g(x_m)}\\
&\le c\big(\sup_{x,y\in E} K(x,y)\big)\Big(\sum_{x\in E}\pi(x)\Big)\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)h_0(y_1)\\
&\times\Big[\sqrt{g(y_{k-1})}\prod_{k=2}^m K(y_{k-1},y_k)\sqrt{g(x_k)}\Big]h_0(x_m)\\
&= c\big(\sup_{x,y\in E} K(x,y)\big)\Big(\sum_{x\in E}\pi(x)\Big)\langle h_0, T^{m-1}h_0\rangle
\end{aligned}$$

where $h_0(y)\equiv\Big(\sum_{x\in E}\pi(x)\Big)^{-1/2}$ on E and the bounded self-adjoint linear operator T: $\mathcal{L}^2(E,\mathcal{E},\pi)\longrightarrow\mathcal{L}^2(E,\mathcal{E},\pi)$ is defined by

$$(Th)(x) = \sqrt{g(x)}\sum_{y\in E} K(x,y)\sqrt{g(y)}h(y)\pi(y) \qquad h\in\mathcal{L}^2(E,\mathcal{E},\pi).$$

Similarly to (3.1.7), we have

$$\langle h_0, T^{m-1}h_0\rangle = \int_{-\infty}^{\infty}\lambda^{m-1}\mu_{h_0}(d\lambda).$$

By the fact that $||h_0||=1$, μ_{h_0} is a probability measure on $\mathbb{R}$. By Theorem E.3 in the Appendix, μ_{h_0} is supported on the interval

$$\Big[\inf_{||h||=1}\langle h, Th\rangle, \quad \sup_{||h||=1}\langle h, Th\rangle\Big].$$

Thus,

$$\begin{aligned}
\langle h_0, T^{m-1}h_0\rangle &\le \left(\max\left\{\Big|\inf_{||h||=1}\langle h,Th\rangle\Big|,\ \Big|\sup_{||h||=1}\langle h,Th\rangle\Big|\right\}\right)^{m-1}\\
&= \sup_{||h||=1}\left|\sum_{x,y\in E} K(x,y)\sqrt{g(x)}h(x)\sqrt{g(y)}h(y)\pi(x)\pi(y)\right|^{m-1}.
\end{aligned}$$

Write $f(x)=\sqrt{g(x)}h(x)$. Then $f(x)\ge 0$ and

$$\sum_{x\in E} f^{\frac{2p}{2p-1}}(x)\pi(x) \le \Big\{\sum_{x\in E} g^q(x)\pi(x)\Big\}^{\frac{p-1}{2p-1}}\Big\{\sum_{x\in E} h^2(x)\pi(x)\Big\}^{\frac{p}{2p-1}} = 1.$$

Consequently,

$$\langle h_0, T^{m-1}h_0\rangle \le \varrho^{m-1}$$

where

$$\varrho = \sup\Big\{\sum_{x,y\in E} K(x,y)f(x)f(y)\pi(x)\pi(y);\quad \sum_{x\in E}|f(x)|^{\frac{2p}{2p-1}}\pi(x)=1\Big\}.$$

Summarizing our argument, we have

$$\limsup_{m\to\infty}\frac{1}{m}\log\sup_g \sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m) \times\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k)\le \log\varrho. \tag{3.1.10}$$

In view of (3.1.9), it remains to show that

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m) \times\bigg[\frac{1}{m!}\Big(\prod_{x\in E}(m\mu(x))!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\bigg]^p\le 0. \tag{3.1.11}$$

Let $\mathcal{P}_m(E)$ be the set of the probability measures ν on E such that for each $x\in E$, $\nu(x)=k/n$ for some integer $0\le k\le m$.

$$\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\Big(\prod_{x\in E}(m\mu(x))!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\bigg]^p\\
&=\sum_{\nu\in\mathcal{P}_m(E)}1_{\{\mu=\nu\}}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\\
&\times\bigg[\frac{1}{m!}\Big(\prod_{x\in E}(m\nu(x))!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\bigg]^p\\
&\le\sum_{\nu\in\mathcal{P}_m(E)}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\\
&\times\bigg[\frac{1}{m!}\Big(\prod_{x\in E}(m\nu(x))!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\bigg]^p.
\end{aligned}$$

By Stirling formula, we get

$$m!\sim\sqrt{2\pi m}m^m e^{-m}\quad\text{and}\quad \big(m\nu(x)\big)!\le C\sqrt{m\nu(x)}\big(m\nu(x)\big)^{m\nu(x)}e^{-m\nu(x)}$$

where $x\in E$. Hence,

$$\frac{1}{m!}\Big(\prod_{x\in E}(m\nu(x))!\Big)\le Cm^{\#(E)/2}\prod_{x\in E}\nu(x)^{m\nu(x)}.$$

Recall that $\varphi_\nu(x)=\nu(x)^{1/q}\pi(x)^{1/p}$. Therefore,

$$\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}(m\nu(x))!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\Big]^p\\
&\le C^p m^{p\#(E)/2}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\prod_{x\in E}\Big(\frac{\nu(x)}{\pi(x)}\Big)^{m\nu(x)}\\
&=C^p m^{p\#(E)/2}\sum_{x_1,\cdots,x_m\in E}\nu(x_1)\cdots\nu(x_m)=C^p m^{p\#(E)/2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}(m\mu(x))!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\Big]^p\\
&\le C^p m^{p\#(E)/2}\#\{\mathcal{P}_m(E)\}.
\end{aligned}$$

Finally, (3.1.11) follows from the fact that $\#\{\mathcal{P}_m(E)\}$ is equal to the number of the non-negative lattice solutions $(k(x);\quad x\in E)$ of the equation

$$\sum_{x\in E}k(x)=m$$

and that the latter is equal to

$$\binom{m+\#(E)-1}{\#(E)-1}.$$

□

3.2. High moment of $\alpha([0,\tau_1]\times\cdots\times[0,\tau_p])$

We now return to the intersection local time $\alpha(A)$ of independent d-dimensional Brownian motions $W_1(t),\cdots,W_p(t)$. Throughout we assume that $p(d-2)<d$. Define

$$\rho=\sup\Big\{\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy;\quad\int_{\mathbb{R}^d}|f(x)|^{\frac{2p}{2p-1}}dx=1\Big\}. \tag{3.2.1}$$

LEMMA 3.2.1. *Under $p(d-2)<d$, we get*

$$0<\rho\le\Big(\int_{\mathbb{R}^d}G^p(x)dx\Big)^{1/p}<\infty.$$

Proof. The lower bound $\rho>0$ is obvious. We now prove the upper bound. Let the non-negative f on $\mathbb{R}^d$ satisfy

$$\int_{\mathbb{R}^d}f^{\frac{2p}{2p-1}}(x)dx=1.$$

We finally reach the bound

$$\Big(\int_0^M |\bar{g}(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2 dx$$
$$\le \epsilon_M + (M^{-1/2}+1)^2\theta_M^{\frac{2p}{p+1}}\sup_{f\in\mathcal{F}}\bigg\{\Big(\int_{-\infty}^{\infty}|f(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|f'(x)|^2dx\bigg\}$$

uniformly over $g\in\mathcal{F}$, which leads to the desired conclusion. □

In the case $d\ge 2$, the local time $L(t,x)$ no longer exists. We may introduce

$$L(t,x,\epsilon) = \int_0^t p_\epsilon\big(W(s)-x\big)ds$$

instead. With some slight modification on the proof of (4.2.3), we can prove the following.

THEOREM 4.2.3. *For each $d\ge 1$, $p>1$ and $\epsilon,\theta>0$,*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\theta\Big(\int_{\mathbb{R}^d}L^p(t,x,\epsilon)dx\Big)^{1/p}\bigg\}$$
$$= \sup_{g\in\mathcal{F}_d}\bigg\{\theta\Big(\int_{\mathbb{R}^d}\big[g(x,\epsilon)\big]^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}$$

where $\mathcal{F}_d$ is defined in (4.1.22) and

$$g(x,\epsilon) = \bigg\{\int_{\mathbb{R}^d}p_\epsilon(y)g^2(x-y)dx\bigg\}^{1/2} \qquad x\in\mathbb{R}^d.$$

4.3. Two-dimensional case

Recall that in Section 2.4 we constructed the renormalized self-intersection local time $\gamma\big([0,t]^2_<\big)$ (formally written in (2.4.1)) run by a 2-dimensional Brownian motion $W(t)$. The discussion naturally leads to the "renormalized" polymer models

$$\widehat{\mathbb{P}}_\lambda(A) = \widehat{C}_\lambda^{-1}\mathbb{E}\bigg(\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}1_{\{W(\cdot)\in A\}}\bigg) \qquad A\subset C\big\{[0,1],\mathbb{R}^2\big\}, \tag{4.3.1}$$

$$\widetilde{\mathbb{P}}_\lambda(A) = \widetilde{C}_\lambda^{-1}\mathbb{E}\bigg(\exp\Big\{-\lambda\gamma\big([0,1]^2_<\big)\Big\}1_{\{W(\cdot)\in A\}}\bigg) \quad A\subset C\big\{[0,1],\mathbb{R}^2\big\} \tag{4.3.2}$$

where

$$\widehat{C}_\lambda = \mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\} \quad\text{and}\quad \widetilde{C}_\lambda = \mathbb{E}\exp\Big\{-\lambda\gamma\big([0,1]^2_<\big)\Big\}$$

are normalizers.

In view of Theorem 2.4.2, $\widetilde{C}_\lambda<\infty$ for all $\lambda>0$. In the term of physics, it shows that there is no phase transition in the self-repelling polymer model given in (4.3.2). On the other hand, we shall show that there is a $\lambda_0>0$ such that $\widehat{C}_\lambda=\infty$ for sufficiently large $\lambda>0$. An important question is to find the $\lambda_0>0$ such that

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}\begin{cases} <\infty & \lambda<\lambda_0,\\[2ex] =\infty & \lambda>\lambda_0.\end{cases} \tag{4.3.3}$$

In physics, the value λ_0 is critical to the "melt-down" of a self-attracting polymer. We shall identify λ_0 in terms of the Gagliardo-Nirenberg constant.

THEOREM 4.3.1. *Let $\kappa(2,2)>0$ be the best constant of the Gagliardo-Nirenberg inequality*

$$||f||_4 \le C\sqrt{||\nabla f||_2}\sqrt{||f||_2} \qquad f\in W^{1,2}(\mathbb{R}^2) \tag{4.3.4}$$

where

$$W^{1,2}(\mathbb{R}^2)=\big\{f\in\mathcal{L}^2(\mathbb{R}^2);\quad \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\big\}.$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1]^2_<\big)\ge t\Big\}=-\kappa(2,2)^{-4}.$$

Proof. Let $0\le\lambda_0\le\infty$ be defined by (4.3.3). Then

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1]^2_<\big)\ge t\Big\}=-\lambda_0.$$

Theorem 2.4.2 implies that $\lambda_0>0$. To establish the upper bound, we need to show that $\lambda_0\ge\kappa(2,2)^{-4}$. For this purpose we may assume that $\lambda_0<\infty$.

Consider the decomposition

$$\gamma\big([0,1]^2_<\big)=\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)+\gamma\big([0,1/2]\times[1/2,1]\big).$$

It gives that for each $\epsilon>0$,

$$\begin{aligned}-\lambda_0\le\max\bigg\{&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\ge\frac{1+\epsilon}{2}t\Big\},\\ &\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1/2]\times[1/2,1]\big)\ge\frac{1-\epsilon}{2}t\Big\}\bigg\}.\end{aligned}$$

By Proposition 2.3.4,

$$\gamma\big([0,1/2]\times[1/2,1]\big)\overset{d}{=}\frac{1}{2}\Big\{\alpha\big([0,1]^2\big)-\mathbb{E}\alpha\big([0,1]^2\big)\Big\}.$$

Taking $d=p=2$ in (3.3.4),

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\alpha\big([0,1]^2\big)\ge t\Big\}=-\kappa(2,2)^{-4}.$$

Consequently,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1/2]\times[1/2,1]\big)\ge\frac{1-\epsilon}{2}t\Big\}=-(1-\epsilon)\kappa(2,2)^{-4}.$$

Observe that $\gamma\big([0,1/2]^2_<\big)$ and $\gamma\big([1/2,1]^2_<\big)$ are independent and have the same law as $(1/2)\gamma\big([0,1]^2_<\big)$. According to the definition of λ_0,

$$\mathbb{E}\exp\Big\{2\lambda\Big[\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\Big]\Big\}=\Big(\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}\Big)^2<\infty$$

for every $\lambda<\lambda_0$. A standard application of Chebyshev inequality leads to

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\ge\frac{1+\epsilon}{2}t\Big\}\le-(1+\epsilon)\lambda_0.$$

Summarizing our argument, we have

$$-\lambda_0 \le \max\Big\{ -(1+\epsilon)\lambda_0, \quad -(1-\epsilon)\kappa(2,2)^{-4}\Big\}.$$

By the fact that $\lambda_0 > 0$, we must have $\lambda_0 \ge (1-\epsilon)\kappa(2,2)^{-4}$. Letting $\epsilon \to 0^+$ we have proved the upper bound

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big\{\gamma\big([0,1]^2_<\big) \ge t\Big\} \le -\kappa(2,2)^{-4}. \tag{4.3.5}$$

To establish the lower bound, recall that

$$L(t,x,\epsilon) = \int_0^t p_\epsilon\big(W(s)-x\big)ds$$

and notice that

$$\int_{\mathbb{R}^2} L^2(t,x,\epsilon)dx = 2\iint_{\{0\le r<s\le t\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds.$$

By Theorem 4.2.7 (with $p=2$) for any $\theta > 0$,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\bigg(\iint_{\{0\le r<s\le t\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds\bigg)^{1/2}\right\}\\
&= \sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\bigg(\int_{\mathbb{R}^2} g^4(x,\epsilon)dx\bigg)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.
\end{aligned}$$

Let $t=n$ be a positive integer and write

$$\begin{aligned}
&\iint_{\{0\le r<s\le n\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds\\
&= \sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big) + \sum_{k=1}^{n}\beta_\epsilon\big([k-1,k]^2_<\big).
\end{aligned}\tag{4.3.6}$$

Here we recall that

$$\beta_\epsilon(A) = \int_A p_{2\epsilon}\big(W(r)-W(s)\big)drds,$$

$$\beta(A) = \int_A \delta_0\big(W(r)-W(s)\big)drds$$

are the random measures discussed in Section 2.3 (with $p=d=2$). By Theorem 2.3.2 and by a treatment similar to the one used in Proposition 2.3.4 we conclude that for any fixed n,

$$\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big) \overset{\epsilon\to 0^+}{\longrightarrow} \sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)$$

in $\mathcal{L}^m(\Omega,\mathcal{A},\mathbb{P})$ for all $m>0$.

Notice that the second term on the right hand side of (4.3.6) is bounded by $C_\epsilon n$ for some constant $C_\epsilon > 0$. So we have

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\exp\left\{\theta\Big(\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big)^{1/2}\right\}$$
$$=\sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\Big(\int_{\mathbb{R}^2}g^4(x,\epsilon)dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

By Lemma 1.2.6, this leads to

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\left\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m\right\}^{1/2}\tag{4.3.7}$$
$$=\frac{1}{2}\sup_{g\in\mathcal{F}_2}\left\{\sqrt{2}\theta\Big(\int_{\mathbb{R}^2}g^4(x,\epsilon)dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

We now claim that for any $\epsilon>0$, integers $m,n\ge 1$,

$$\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m\le\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m.\tag{4.3.8}$$

Indeed, write

$$D=\bigcup_{k=1}^{n-1}[0,k]\times[k,k+1].$$

By Fourier inversion (Theorem B.1, Appendix), we get

$$p_{2\epsilon}(x)=(2\pi)^{-2}\int_{\mathbb{R}^2}e^{-i\lambda\cdot x}\exp\big\{-\epsilon|\lambda|^2\big\}d\lambda.$$

Thus,

$$\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)=\int_D drdsp_{2\epsilon}\big(W(r)-W(s)\big)drds$$
$$=\int_{\mathbb{R}^2}d\lambda\exp\big\{-\epsilon|\lambda|^2\big\}\int_D drds\exp\big\{-i\lambda\cdot\big(W(r)-W(s)\big)\big\}.$$

Consequently,

$$\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m$$
$$=\int_{(\mathbb{R}^2)^m}d\lambda_1\cdots d\lambda_m\exp\Big\{-\epsilon\sum_{k=1}^m|\lambda_k|^2\Big\}$$
$$\times\int_{D^m}dr_1ds_1\cdots dr_mds_m\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot\big(W(r_k)-W(s_k)\big)\Big)\Big\}.$$

Therefore, for any $0<\epsilon'<\epsilon$,

$$\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m\le\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_{\epsilon'}\big([0,k]\times[k,k+1]\big)\Big]^m,$$

which leads to (4.3.8) by letting $\epsilon'\to 0$.

By (4.3.7) and (4.3.8),

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\bigg\}^{1/2}$$
$$\geq\frac{1}{2}\sup_{g\in\mathcal{F}_2}\bigg\{\sqrt{2}\theta\Big(\int_{\mathbb{R}^2}g^4(x,\epsilon)dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}.$$

When $\epsilon\to 0^+$, the right hand side converges to

$$\frac{1}{2}\sup_{g\in\mathcal{F}_2}\bigg\{\sqrt{2}\theta\Big(\int_{\mathbb{R}^2}|g(x)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}=\frac{1}{2}\theta^2\kappa(2,2)^4$$

where the equality follows from Theorem C.1, Appendix (with $d=p=2$). Thus,

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\bigg\}^{1/2}\geq\frac{1}{2}\theta^2\kappa(2,2)^4\qquad(\theta>0).\tag{4.3.9}$$

Notice that (4.3.9) alone is not enough for the lower bound of the large deviation. In the following effort, we strengthen (4.3.9) into an equality.

By the scaling property, (4.3.5) can be rewritten as

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\gamma\big([0,n]^2_<\big)\geq\lambda n^2\Big\}\leq-\lambda\kappa(2,2)^{-4}\qquad(\lambda>0).$$

Similarly to the decomposition (4.3.6), we have

$$\gamma\big([0,n]^2_<\big)=\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)-\sum_{k=1}^{n-1}\mathbb{E}\beta\big([0,k]\times[k,k+1]\big)+\sum_{k=1}^{n}\gamma\big([k-1,k]^2_<\big).\tag{4.3.10}$$

Notice that

$$\sum_{k=1}^{n-1}\mathbb{E}\beta\big([0,k]\times[k,k+1]\big)=\frac{1}{2\pi}\sum_{k=1}^{n-1}\int_0^k\int_k^{k+1}\frac{1}{s-r}dsdr=\frac{1}{2\pi}\sum_{k=1}^{n-1}\big\{\log(k+1)-\log k\big\}=\frac{1}{2\pi}n\log n.\tag{4.3.11}$$

The random variables $\gamma\big([k-1,k]^2_<\big)$ $(k=1,2\cdots)$ form an i.i.d. exponentially integrable sequence. By a standard application of Chebyshev inequality, we get

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\Big|\sum_{k=1}^n\gamma\big([k-1,k]^2_<\big)\Big|\geq\epsilon n^2\Big\}=-\infty.\tag{4.3.12}$$

Summarizing our argument, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\geq\lambda n^2\Big\}\leq-\lambda\kappa(2,2)^{-4}.$$

By Lemma 1.2.9 (part (2)) and Lemma 1.2.10, this leads to

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\bigg\}^{1/2}$$
$$\le\sup_{\lambda>0}\Big\{\theta\lambda^{1/2}-\frac{1}{2}\lambda\kappa(2,2)^{-4}\Big\}=\frac{1}{2}\theta^2\kappa(2,2)^4.$$

Combining this with (4.3.9) gives us

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\bigg\}^{1/2}=\frac{1}{2}\theta^2\kappa(2,2)^4.$$

Applying Theorem 1.2.7, we get

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\ge\lambda n^2\Big\}=-\lambda\kappa(2,2)^{-4}.$$

We bring this back to the decomposition (4.3.10). By (4.3.11) and (4.3.12),

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\gamma\big([0,n]^2_<\big)\ge\lambda n^2\Big\}=-\lambda\kappa(2,2)^{-4},$$

which ends the proof. □

Comparing Theorem 4.3.1 with (3.3.4) (with $d=p=2$), one can see that the intersection local time $\alpha\big([0,1/2]^2\big)$ obeys the same large deviation principle as $\gamma\big([0,1]^2_<\big)$. This observation is crucial in our proof of the upper bound part of Theorem 4.3.1, since $\alpha\big([0,1/2]^2\big)\overset{d}{=}2^{-1}\alpha\big([0,1]^2\big)$; so the tail of $\alpha\big([0,1/2]^2\big)$ is given by Theorem 3.3.2. We try to turn our mathematical argument into the following intuitive explanation:

Cut the Brownian path $\big\{W(t);\ 0\le t\le 1\big\}$ into two paths at the middle point $W(1/2)$ and shift the whole system so the cutting point $W(1/2)$ becomes the origin after shifting. If we reverse the direction of the path before the time $t=1/2$, then the resulted two paths are trajectories of two independent Brownian motions running up to time $1/2$.

The total self-intersection of the original path $\big\{W(t);\ 0\le t\le 1\big\}$ is the sum of the intersection within each sub-path and the intersection between two sub-paths. When $d=2$, the first type intersection out-numbers the second type intersection (that is the main reason why $\beta\big([0,1]^2_<\big)=\infty$ (Proposition 2.3.6)). The renormalization subdues the short range intersection so that these two intersections are comparable. By chance, the ratio is 1 to 1 here in the sense of large deviations. That is to say, about half of $\gamma\big([0,1]^2_<\big)$ is made of a random quantity distributed as $\alpha\big([0,1/2]^2\big)$. As for another half, it is equal to the sum of the (renormalized) self-intersection local times of two independent paths and each of them can be analyzed in the way proposed above.

The proportion between these two types of intersection varies in different settings. From later development, we shall see that finding the ratio is an important part of establishing the large deviations related to self intersections.

The following theorem shows a completely different tail asymptotic behavior.

THEOREM 4.3.2. *There is a constant $0 < L < \infty$ such that for any $\theta > 0$,*

$$\lim_{t\to\infty} t^{-2\pi\theta}\log \mathbb{P}\Big\{ -\gamma\big([0,1]^2_<\big) \ge \theta\log t\Big\} = -L.$$

Proof. One needs only to prove Theorem 4.3.2 in the case $\theta = (2\pi)^{-1}$. That is,

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big\{ -\gamma\big([0,1]^2_<\big) \ge (2\pi)^{-1}\log t\Big\} = -L \tag{4.3.13}$$

for some $0 < L < \infty$. Indeed, the general statement will follows from (4.3.13) if we substitute t by $t^{2\pi\theta}$.

The argument for (4.3.13) is based on sub-additivity. Define

$$Z_t = -\frac{1}{2\pi} t\log t - \gamma\big([0,t]^2_<\big) \qquad t \ge 0. \tag{4.3.14}$$

For any $s, t > 0$,

$$\begin{aligned} Z_{s+t} &= -\frac{1}{2\pi}(s+t)\log(s+t) - \gamma\big([0,s]^2_<\big) - \gamma\big([s,s+t]^2_<\big)\\ &\quad - \beta\big([0,s]\times[s,s+t]\big) + \mathbb{E}\beta\big([0,s]\times[s,s+t]\big)\\ &\le -\frac{1}{2\pi}(s+t)\log(s+t) - \gamma\big([0,s]^2_<\big)\\ &\quad - \gamma\big([s,s+t]^2_<\big) + \mathbb{E}\beta\big([0,s]\times[s,s+t]\big). \end{aligned}$$

By Proposition 2.3.4, we get

$$\begin{aligned} &\mathbb{E}\beta\big([0,s]\times[s,s+t]\big) = \mathbb{E}\alpha\big([0,s]\times[0,t]\big)\\ &= \int_{\mathbb{R}^2} dx\Big[\int_0^s p_u(x)du\Big]\Big[\int_0^t p_u(x)du\Big] = \int_0^s\int_0^t p_{u+v}(0)dudv \end{aligned}$$

where the second equality follows from Le Gall's moment formula in the special case $d = p = 2$ and $m = 1$. By a straightforward calculation, we have

$$\begin{aligned} &\int_0^s\int_0^t p_{u+v}(0)dudv = \frac{1}{2\pi}\int_0^s\int_0^t \frac{1}{u+v}dudv\\ &= \frac{1}{2\pi}\Big[(s+t)\log(s+t) - s\log s - t\log t\Big]. \end{aligned}$$

Summarizing our argument gives us

$$Z_{s+t} \le Z_s + Z'_t \tag{4.3.15}$$

where

$$Z'_t = -\frac{1}{2\pi}t\log t - \gamma\big([s,s+t]^2_<\big)$$

is independent of Z_s and $Z'_t \overset{d}{=} Z_t$. This means that Z_t is sub-additive.

By Theorem 2.4.7, and the fact $\gamma\big([0,\delta]^2_<\big) \overset{d}{=} \delta\gamma\big([0,1]^2_<\big)$, for any $\lambda > 0$, one can take $\delta > 0$ sufficiently small so that

$$\mathbb{E}\exp\Big\{\lambda Z_\delta\Big\} < \infty.$$

By sub-additivity given in (4.3.15), therefore, we conclude that[1]

$$\mathbb{E}\exp\Big\{\lambda Z_t\Big\} < \infty \qquad \forall \lambda, t > 0. \tag{4.3.16}$$

By (4.3.15) again, for any $s, t > 0$,

$$\mathbb{E}\exp\Big\{2\pi Z_{s+t}\Big\} \le \mathbb{E}\exp\Big\{2\pi Z_s\Big\}\mathbb{E}\exp\Big\{2\pi Z_t\Big\}$$

or,

$$\begin{aligned}&(s+t)^{-(s+t)}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,s+t]_<^2\big)\Big\}\\ &\le \bigg(s^{-s}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,s]_<^2\big)\Big\}\bigg)\bigg(t^{-t}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,t]_<^2\big)\Big\}\bigg).\end{aligned}$$

By Lemma 1.3.1, the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\bigg(t^{-t}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,t]_<^2\big)\Big\}\bigg) = A \tag{4.3.17}$$

exists with $A < \infty$. By Lemma 4.3.3 below, $A > -\infty$. Let $t = n$ be integer. By Stirling formula and by scaling, we get

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\mathbb{E}\exp\Big\{-2\pi n\gamma\big([0,1]_<^2\big)\Big\} = A+1.$$

Finally, applying Theorem 1.2.8 to the non-negative random variable

$$Y = \exp\Big\{-2\pi\gamma\big([0,1]_<^2\big)\Big\}$$

leads to (4.3.13) with

$$L = \exp\{-1-A\}. \tag{4.3.18}$$

□

We end this section by establishing the following lemma.

Lemma 4.3.3.

$$\liminf_{n\to\infty}\frac{1}{n}\log\bigg(n^{-n}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,n]_<^2\big)\Big\}\bigg) > -\infty.$$

Proof. By (4.3.10) and (4.3.11)

$$\begin{aligned}&n^{-n}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,n]_<^2\big)\Big\}\\ &= \exp\bigg\{-2\pi\sum_{k=1}^n\gamma\big([k-1,k]_<^2\big) - 2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\bigg\}\\ &\ge e^{-2\pi Mn}\mathbb{E}\exp\bigg\{-2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\bigg\}\\ &- e^{-2\pi Mn}\mathbb{P}\bigg\{\sum_{k=1}^n\gamma\big([k-1,k]_<^2\big)\ge Mn\bigg\}.\end{aligned}$$

[1]Here we are not allowed to apply Theorem 1.3.5 directly for lack of monotonicity.

By Chebyshev inequality, for any $0 < \lambda < \kappa(2,2)^{-4}$,

$$\mathbb{P}\Big\{ \sum_{k=1}^{n} \gamma\big([k-1,k]_<^2\big) \ge Mn \Big\} \le e^{-n\lambda M} \bigg(\mathbb{E}\exp\Big\{ \lambda\gamma\big([0,1]_<^2\big)\Big\}\bigg)^n$$

and the right hand side can be exponentially small to a requested level by choosing sufficiently large M. To complete the proof, therefore, we need only to establish the lower bound

$$\mathbb{E}\exp\bigg\{ -2\pi \sum_{k=1}^{n-1} \beta\big([0,k]\times[k,k+1]\big)\bigg\} \ge c_1^n \tag{4.3.19}$$

for some constant $c_1 > 0$ and sufficiently large n.

Let the 1-dimensional Brownian motion $W_0(t)$ be the first component of $W(t)$ and write

$$D_n = \Big\{ \sup_{0\le s\le n} |W_0(s) - s| \le \delta \Big\}$$

where $0 < \delta < 1/2$ is fixed. Notice that on D_n,

$$\sum_{k=1}^{n-1} \beta\big([0,k]\times[k,k+1]\big) = \sum_{k=1}^{n-1} \beta\big([k-1,k]\times[k,k+1]\big).$$

Consequently, for any $N > 0$,

$$\begin{aligned}
&\mathbb{E}\exp\bigg\{ -2\pi \sum_{k=1}^{n-1} \beta\big([0,k]\times[k,k+1]\big)\bigg\} \\
&\ge \mathbb{E}\bigg[\exp\bigg\{ -2\pi \sum_{k=1}^{n-1} \beta\big([k-1,k]\times[k,k+1]\big)\bigg\} 1_{D_n}\bigg] \\
&\ge e^{-2\pi Nn}\bigg\{ \mathbb{P}(D_n) - \mathbb{P}\Big\{ \sum_{k=1}^{n-1} \beta\big([k-1,k]\times[k,k+1]\big) \ge Nn\Big\}\bigg\}.
\end{aligned} \tag{4.3.20}$$

Write

$$\begin{aligned}
&\sum_{k=1}^{n-1} \beta\big([k-1,k]\times[k,k+1]\big) \\
&= \sum_k \beta\big([2(k-1),2k-1]\times[2k-1,2k]\big) \\
&+ \sum_k \beta\big([2k-1,2k]\times[2k,2k+1]\big).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\mathbb{P}\Big\{ \sum_{k=1}^{n-1} \beta\big([k-1,k]\times[k,k+1]\big) \ge Nn\Big\} \\
&\le \mathbb{P}\Big\{ \sum_k \beta\big([2(k-1),2k-1]\times[2k-1,2k]\big) \ge 2^{-1}Nn\Big\} \\
&+ \mathbb{P}\Big\{ \sum_k \beta\big([2k-1,2k]\times[2k,2k+1]\big)) \ge 2^{-1}Nn\Big\}.
\end{aligned}$$

By the fact that the sequence

$$\beta\big([2(k-1), 2k-1] \times [2k-1, 2k]\big) \qquad k = 1, 2, \cdots$$

is an i.i.d. with common distribution the same as $\alpha([0,1]^2)$,

$$\mathbb{P}\Big\{\sum_k \beta\big([2(k-1), 2k-1] \times [2k-1, 2k]\big) \ge 2^{-1} N n\Big\}$$
$$\le e^{-2^{-1} n\lambda N}\bigg(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\bigg)^n$$

where $\lambda > 0$ is chosen in order that

$$\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\} < \infty.$$

Similarly,

$$\mathbb{P}\Big\{\sum_k \beta\big([2k-1, 2k] \times [2k, 2k+1]\big) \ge 2^{-1} N n\Big\}$$
$$\le e^{-2^{-1} n\lambda N}\bigg(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\bigg)^n.$$

Thus, by triangular inequality, we get

$$\mathbb{P}\Big\{\sum_{k=1}^{n-1} \beta\big([k-1, k] \times [k, k+1]\big) \ge N n\Big\}$$
$$\le 2e^{-2^{-1} n\lambda N}\bigg(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\bigg)^n.$$

On the other hand,

$$D_n = \Big\{\sup_{0\le s\le n-1} |W_0(s) - s| \le \delta\Big\}$$
$$\cap \Big\{\sup_{0\le s\le 1} \big|\big(W_0(n-1+s) - W_0(n-1)\big) - s + \big(W_0(n-1) - (n-1)\big)\big| \le \delta\Big\}.$$

By independence of increments,

$$\mathbb{P}(D_n) \ge \mathbb{P}(D_{n-1}) \inf_{|x|\le\delta} \mathbb{P}_x\Big\{\sup_{0\le s\le 1} \big|W_0(s) - s\big| \le \delta\Big\}.$$

Repeating our argument,

$$\mathbb{P}(D_n) \ge \bigg(\inf_{|x|\le\delta} \mathbb{P}_x\Big\{\sup_{0\le s\le 1} \big|W_0(s) - s\big| \le \delta\Big\}\bigg)^n = c_2^n,$$

where

$$c_2 \equiv \inf_{|x|\le\delta} \mathbb{P}_x\Big\{\sup_{0\le s\le 1} \big|W_0(s) - s\big| \le \delta\Big\} > 0.$$

Thus, one can take $N > 0$ sufficiently large so that

$$\mathbb{P}\Big\{\sum_{k=1}^{n-1} \beta\big([k-1, k] \times [k, k+1]\big) \ge N n\Big\} \le \frac{1}{2}\mathbb{P}(D_n).$$

Hence, (4.3.19) follows from (4.3.20). $\square$