# Introduction

**1.** Scattering theory has its origin in quantum mechanics and is intimately related to the theory of partial differential equations. From the mathematical point of view it can be considered as perturbation theory of self-adjoint operators on the (absolutely) continuous spectrum. In general, perturbation theory draws conclusions about a self-adjoint operator $H$ (acting in a Hilbert space $\mathcal{H}$) given information regarding a simpler operator $H_0$. Thereby it is required that the operators $H_0$ and $H$ be close in a sense depending on a particular problem. In physical terms the operator $H_0$ (the free Hamiltonian) describes a system of non-interacting particles (or clusters of particles), while the "full" Hamiltonian $H$ describes the real system including interactions.

Scattering theory is concerned with a study of the behavior for large times of solutions of the time-dependent equation

$$i\partial u/\partial t = Hu, \quad u(0) = f,$$

in terms of the free equation $i\partial u_0/\partial t = H_0 u_0$. It turns out that under appropriate assumptions on the perturbation $V = H - H_0$, for every vector $f$ orthogonal to eigenvectors of $H$, there exists a vector $f_0^{(\pm)}$ orthogonal to eigenvectors of $H_0$ such that

$$\lim_{t\to\pm\infty} \|u(t) - u_0(t)\| = 0,$$

if $u_0(0) = f_0^{(\pm)}$. Since $u(t) = \exp(-iHt)f$ and $u_0(t) = \exp(-iHt)f_0^{(\pm)}$, the initial data $f$ and $f_0^{(\pm)}$ are related by the equality

$$f = \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)f_0^{(\pm)}.$$

This motivates the following definition of the wave operator (WO)

$$W_\pm = W_\pm(H, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)P_0^{(a)} \tag{1}$$

provided of course that this strong limit exists. Here $P_0^{(a)}$ is the orthogonal projection on the absolutely continuous subspace $\mathcal{H}_0^{(a)}$ of the operator $H_0$. WO are automatically isometric on $\mathcal{H}_0^{(a)}$. Moreover, under the assumption of their existence, WO enjoy the intertwining property $HW_\pm = W_\pm H_0$. Therefore the range $\operatorname{Ran} W_\pm$ of $W_\pm$ belongs to the absolutely continuous subspace $\mathcal{H}^{(a)}$ of $H$. The operator $W_\pm$ is called complete if the equality $\operatorname{Ran} W_\pm = \mathcal{H}^{(a)}$ holds. Then the absolutely continuous parts of the Hamiltonians $H_0$ and $H$ are unitarily equivalent. In physical applications Hamiltonians do not usually have singular continuous spectrum.

Another important object is the scattering operator $\mathbf{S} = W_+^* W_-$ which connects directly the asymptotic behavior of a quantum system as $t \to -\infty$ and $t \to \infty$ in terms of the free problem, that is $\mathbf{S}\colon f_0^{(-)} \mapsto f_0^{(+)}$. The scattering operator is of great interest in mathematical physics problems, because it relates the "initial" and the "final" characteristics of the process directly, bypassing its consideration for finite times. This also explains the term "scattering theory" which is borrowed from physics.

Since the operator $\mathbf{S}$ commutes with $H_0$, it reduces to multiplication by an operator-valued function $S(\lambda)$, known as the scattering matrix (SM), in a diagonal representation of the free operator $H_0$. The operators $S(\lambda)$ are unitary for almost all $\lambda$ if both WO $W_\pm$ exist, are isometric and complete.

If $H_0$ and $H$ are not close enough and even act in different spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, then it is sometimes still possible to prove the existence of more general WO

$$W_\pm = W_\pm(H, H_0; J) = \operatorname*{s-lim}_{t\to\pm\infty} \exp(iHt) J \exp(-iH_0 t) P_0^{(a)}, \tag{2}$$

where the "identification" $J : \mathcal{H}_0 \to \mathcal{H}$ is a bounded operator. These limits exist if the "effective perturbation" $HJ - JH_0$ is in some sense small. The intertwining property remains true for WO $W_\pm(H, H_0; J)$, but their isometricity may be lost. Sometimes it is convenient to choose different operators $J_\pm$ for $t \to \pm\infty$.

**2.** The Schrödinger operator $H = -\Delta + v(x)$ in the space $\mathcal{H} = L_2(\mathbb{R}^d)$ with a potential $v$ decaying at infinity is a typical object of scattering theory. More general differential operators whose coefficients have limits as $|x| \to \infty$ can be treated in a similar way. The operator $H$ describes two interacting particles which may be either in a bound state or asymptotically (as the time $t \to \infty$ or $t \to -\infty$) free. This statement is called asymptotic completeness. If $v(x)$ decays with power $\rho$, that is,

$$|v(x)| \le C(1 + |x|)^{-\rho}, \tag{3}$$

and $\rho > 1$ (the short-range case), then the kinetic energy operator $H_0 = -\Delta$ plays the role of the unperturbed operator. In this case the WO $W_\pm$ exist and are complete. For the Schrödinger operator, as well as for more general differential operators which can be considered as perturbations of differential operators with constant coefficients, the existence of WO is usually an easy fact while their completeness is a substantial mathematical problem.

The operator $H_0$ reduces by the Fourier transform to multiplication by the independent variable $\lambda$ (which plays the role of the energy) in the space $L_2(\mathbb{R}_+; L_2(\mathbb{S}^{d-1}))$. Therefore the SM $S(\lambda)$ for the pair $H_0$, $H$ acts in the space $L_2(\mathbb{S}^{d-1})$ and is a unitary operator for all $\lambda > 0$. The SM can also be defined in terms of solutions of the stationary Schrödinger equation

$$-\Delta\psi + v(x)\psi = \lambda\psi. \tag{4}$$

If condition (3) is satisfied for $\rho > (d+1)/2$, then, for any $\lambda > 0$ and any unit vector $\omega \in \mathbb{S}^{d-1}$, equation (4) has a (unique) solution $\psi(x, \omega, \lambda)$ with the asymptotics

$$\exp(i\lambda^{1/2}\langle\omega, x\rangle) + a|x|^{-(d-1)/2} \exp(i\lambda^{1/2}|x|) e^{-\pi i(d-3)/4} + o(|x|^{-(d-1)/2}) \tag{5}$$

as $|x| \to \infty$. The coefficient $a = a(\hat{x}, \omega, \lambda)$ depends on the incident direction $\omega$ of the incoming plane wave $\psi_0(x, \omega, \lambda) = \exp(i\lambda^{1/2}\langle\omega, x\rangle)$, its energy $\lambda$ and the direction $\hat{x} = x|x|^{-1}$ of observation of the outgoing spherical wave

$$|x|^{-(d-1)/2} \exp(i\lambda^{1/2}|x|) e^{-\pi i(d-3)/4}.$$

The function $a(\phi,\omega,\lambda)$ is called the scattering amplitude, and $S(\lambda)-I$ is the integral operator with kernel

$$i\lambda^{(d-1)/4}(2\pi)^{-(d-1)/2}a(\phi,\omega,\lambda).$$

It can be recovered by the formula

$$a(\phi,\omega,\lambda)=-2^{-1}(2\pi)^{-(d-1)/2}\lambda^{(d-3)/4}\int_{\mathbb{R}^d}\exp(-i\lambda^{1/2}\langle\phi,x\rangle)v(x)\psi(x,\omega,\lambda)dx. \quad (6)$$

From the point of view of quantum mechanics the plane wave describes a beam of particles incident on a scattering center, and the outgoing spherical wave corresponds to scattered particles. The ratio of the flux density of the scattered particles to that of the incident beam is $|a(\phi,\omega,\lambda)|^2$. This quantity is the main observable in scattering experiments. The probability

$$d\boldsymbol{\sigma}(\phi,\omega,\lambda)=|a(\phi,\omega,\lambda)|^2d\phi$$

of scattered particles to pass through the solid angle $d\phi$ is called the differential scattering cross section. Integrating it over $\phi$, we obtain the total cross section

$$\boldsymbol{\sigma}(\omega,\lambda)=\int_{\mathbb{S}^{d-1}}|a(\phi,\omega,\lambda)|^2d\phi$$

for the energy $\lambda$ and the incident direction $\omega$. Further details on the quantum mechanical picture of scattering may be found in the textbooks [**20**] by L. D. Faddeev and O. A. Yakubovski, [**31**] by L. D. Landau and E. M. Lifshitz, and [**37**] by R. Newton.

**3.** The approach in scattering theory relying on definition (1) is called time-dependent. An alternative possibility is to change the definition of WO replacing the unitary groups by the corresponding resolvents $R_0(z)=(H_0-z)^{-1}$ and $R(z)=(H-z)^{-1}$. They are related by a simple identity

$$R(z)=R_0(z)-R_0(z)VR(z)=R_0(z)-R(z)VR_0(z), \quad (7)$$

where $V=H-H_0$ and $\operatorname{Im} z\neq 0$. In the stationary approach in place of limits as $t\to\pm\infty$ one has to study the boundary values (in a suitable topology) of the resolvents as the spectral parameter $z$ tends to the real axis. An important advantage of the stationary approach is that it gives convenient formulas for the WO and the SM. Iterating (7), we obtain an expansion of $R(z)$ in powers of $R_0(z)$. This (Born) series plays a consequential role in practical calculations.

Let us discuss here the stationary formulation of the scattering problem for the Schrödinger operator in terms of solutions $\psi(x,\omega,\lambda)$ of differential equation (4) with asymptotics (5) as $|x|\to\infty$. Actually, we consider two sets of scattering solutions, or eigenfunctions of the continuous spectrum, $\psi_-(x,\omega,\lambda)=\psi(x,\omega,\lambda)$ and $\psi_+(x,\omega,\lambda)=\overline{\psi(x,-\omega,\lambda)}$. In terms of boundary values of the resolvent, the functions $\psi_\pm(\omega,\lambda)$ can be defined by the formula

$$\psi_\pm(\omega,\lambda)=\psi_0(\omega,\lambda)-R(\lambda\mp i0)V\psi_0(\omega,\lambda). \quad (8)$$

Using the resolvent identity, it is easy to deduce the Lippmann-Schwinger equation

$$\psi_\pm(\omega,\lambda)=\psi_0(\omega,\lambda)-R_0(\lambda\mp i0)V\psi_\pm(\omega,\lambda) \quad (9)$$

for the scattering solutions $\psi_\pm(\omega,\lambda)$.

The WO $W_\pm(H, H_0)$ can be constructed in terms of the solutions $\psi_\pm$. Set $\xi = \lambda^{1/2}\omega$ ($\xi$ is the momentum variable), write $\psi_\pm(x,\xi)$ instead of $\psi_\pm(x,\omega,\lambda)$ and consider two transformations

$$(\Phi_\pm f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \overline{\psi_\pm(x,\xi)} f(x) dx \tag{10}$$

(defined initially, for example, on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$) of the space $L_2(\mathbb{R}^d)$ into itself. The operators $\Phi_\pm$ can be regarded as generalized Fourier transforms, and both of them coincide with the usual Fourier transform $\Phi_0$ if $v = 0$. Under the action of $\Phi_\pm$ the operator $H$ goes over into multiplication by $|\xi|^2$, i.e.,

$$(\Phi_\pm H f)(\xi) = |\xi|^2 (\Phi_\pm f)(\xi).$$

Moreover, it can be shown that $\Phi_\pm$ is an isometry on $\mathcal{H}^{(a)}$, it is zero on $\mathcal{H} \ominus \mathcal{H}^{(a)}$ and its range $\operatorname{Ran}\Phi_\pm = L_2(\mathbb{R}^d)$. This is equivalent to the equations

$$\Phi_\pm^* \Phi_\pm = P^{(a)}, \quad \Phi_\pm \Phi_\pm^* = I,$$

where $P^{(a)}$ is the orthogonal projection on the subspace $\mathcal{H}^{(a)}$. Hence any function $f \in \mathcal{H}^{(a)}$ admits the expansion in the generalized Fourier integral

$$f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi_\pm(x,\xi)(\Phi_\pm f)(\xi) d\xi.$$

It can also be deduced from asymptotics (5) as $|x| \to \infty$ of the functions $\psi(x,\omega,\lambda)$ that the vectors

$$\left(\Phi_\pm^* - \Phi_0^*\right) \exp(-i|\xi|^2 t)\hat{f}, \quad \hat{f} = \Phi_0 f,$$

tend to zero as $t \to \pm\infty$ for all $\hat{f} \in L_2(\mathbb{R}^d)$. This implies the existence of the WO $W_\pm = W_\pm(H, H_0)$ for the pair $H_0 = -\Delta$, $H = -\Delta + v(x)$ and gives the representation

$$W_\pm = \Phi_\pm^* \Phi_0. \tag{11}$$

This formula is an example of a stationary representation for the WO. It formally implies that

$$W_\pm : \psi_0(\omega,\lambda) \mapsto \psi_\pm(\omega,\lambda),$$

which means that each WO establishes a one-to-one correspondence between eigenfunctions of the continuous spectrum of the operators $H_0$ and $H$. Completeness of $W_\pm$ follows from the equation $\Phi_\pm^*\Phi_\pm = P^{(a)}$. The equality $\Phi_\pm\Phi_\pm^* = I$ is equivalent to the isometricity of $W_\pm$.

For any dimension $d$, equation (9) is the Fredholm integral equation which is relatively difficult to study. On the other hand, for $d = 1$ the specific methods of ordinary differential equations can be used. In particular, in this case (9) can be replaced by a Volterra integral equation. The one-dimensional problem is treated in Chapters 4 and 5 where various analytic problems are also considered. The same problems in the multidimensional case are studied in subsequent chapters.

**4.** The methods used in scattering theory are naturally subdivided into two groupes: smooth and trace class. The smooth method (its scheme was briefly described in the previous subsection) makes essential use of an explicit spectral analysis of the unperturbed operator $H_0$; for example, $H_0 = -\Delta$ is diagonalized by the Fourier transform. This approach requires that the perturbation $V = H - H_0$ be sufficiently "regular" in the spectral decomposition of the operator $H_0$. There are different ways to understand regularity. For example, in the Friedrichs-Faddeev model $H_0$ acts as multiplication by the independent variable in the space $\mathcal{H} = L_2(\Lambda; \mathfrak{h})$ where $\Lambda \subset \mathbb{R}$ is an interval and $\mathfrak{h}$ is an auxiliary Hilbert space. The perturbation $V$ is an integral operator with a sufficiently smooth matrix-valued kernel.

Another possibility is to use the concept of $H$-smoothness introduced by T. Kato. An $H$-bounded operator $G$ is called $H$-smooth if, for all $f \in \mathcal{D}(H)$,

$$\int_{-\infty}^{\infty} \|G \exp(-iHt) f\|^2 dt \leq C \|f\|^2. \tag{12}$$

It is important that this definition admits equivalent reformulations in terms of the resolvent and of the spectral family. Thus, $G$ is $H$-smooth if and only if

$$\sup_{\lambda \in \mathbb{R}, \varepsilon > 0} \|G\big(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)\big)G^*\| < \infty.$$

In applications, the assumption of $H$-smoothness of the operator $G$ imposes very stringent conditions on the operator $H$. In particular, the operator $H$ is necessarily absolutely continuous if the kernel of $G$ is trivial. This assumption excludes eigenvalues and other singular points in the spectrum of $H$, for example, the bottom of the continuous spectrum for the Schrödinger operator with decaying potential or edges of bands if the spectrum has the band structure. The notion of local $H$-smoothness is considerably more flexible. Let $E(\cdot)$ be the spectral projection of the operator $H$. By definition, $G$ is called $H$-smooth on a Borel set $X \subset \mathbb{R}$ if the operator $GE(X)$ is $H$-smooth.

The following theorem of T. Kato and R. Lavine is simple but very useful. Suppose that

$$HJ - JH_0 = G^* G_0,$$

where the operators $G_0$ and $G$ are $H_0$-smooth and $H$-smooth, respectively, on an arbitrary compact subinterval of some interval $\Lambda$. Then the WO

$$W_\pm(H, H_0; JE_0(\Lambda)) \quad \text{and} \quad W_\pm(H_0, H; J^* E(\Lambda))$$

exist (and are adjoint to each other).

This result cannot usually be applied directly since the verification of $H_0$- and especially of $H$-smoothness may be a difficult problem. However, the operator $\langle x \rangle^{-\alpha}$ for $\alpha > 1/2$ is smooth with respect to $H_0 = -\Delta$ on any compact subinterval of $\mathbb{R}_+$. This result is equivalent to the existence of traces in the space $L_2(\mathbb{S}^{d-1})$ of functions from the Sobolev space $\mathsf{H}^\alpha(\mathbb{R}^d)$. The proof of $H$-smoothness of $\langle x \rangle^{-\alpha}$, $\alpha > 1/2$, for the operator $H = -\Delta + v(x)$, relies on the resolvent identity (7) considered as an equation for the resolvent $R(z)$. Then the analytic Fredholm alternative is applied to this equation. This method shows that for the Schrödinger operator, the WO $W_\pm(H, H_0)$ exist and are complete if condition (3) is satisfied for $\rho > 1$. The last result is optimal because for the Coulomb potential $v(x) = v_0 |x|^{-1}$ the limits (1) do not exist.

**5.** The main advantage of the trace class method compared to the smooth one is that it does not require constructing an explicit spectral representation for the free operator $H_0$. To put it differently, the trace class theory relies on the fact that in a weak sense an arbitrary Hilbert-Schmidt operator is smooth with respect to an arbitrary self-adjoint operator (see §6.1 of [I] and compare estimates (12) and (13) below). This allows one to extend considerably the class of free operators $H_0$. On the other hand, compared to the smooth approach the trace class one demands more restrictive conditions on the falloff of the perturbation at infinity. Its another drawback is that it does not give any information on the singular part of the spectrum.

The fundamental result of the trace class method is the following theorem of T. Kato and M. Rosenblum. If $V = H - H_0$ belongs to the trace class $\mathfrak{S}_1$, then the WO $W_\pm(H, H_0)$ exist and are complete. In particular, the operators $H_0^{(a)}$ and $H^{(a)}$ are unitarily equivalent. This can be considered as a far advancement of the H. Weyl theorem, which states the stability of the essential spectrum under compact perturbations.

Although sharp in the abstract framework, the Kato-Rosenblum theorem cannot directly be applied to the theory of differential operators where a perturbation is usually an operator of multiplication and hence is not even compact. We mention two generalizations of this theorem applicable under such circumstances. The first, the Birman-Kato-Kreĭn theorem, guarantees that the WO $W_\pm(H, H_0)$ exist and are complete, provided that

$$R^m(z) - R_0^m(z) \in \mathfrak{S}_1$$

for some $m = 1, 2, \dots$ and all $z$ with $\operatorname{Im} z \neq 0$. The second, the Birman theorem, asserts that the same is true if $\mathcal{D}(H) = \mathcal{D}(H_0)$ or $\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2})$ and

$$E(X)(H - H_0)E_0(X) \in \mathfrak{S}_1$$

for all bounded intervals $X$.

A direct generalization of the Kato-Rosenblum theorem to the operators acting in different spaces is due to D. Pearson. Suppose that $H_0$ and $H$ are self-adjoint operators in spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, $J : \mathcal{H}_0 \to \mathcal{H}$ is a bounded operator and $V = HJ - JH_0 \in \mathfrak{S}_1$. Then the WO $W_\pm(H, H_0; J)$ and $W_\pm(H_0, H; J^*)$ exist.

Although rather sophisticated, Pearson's proof relies only on the following elementary lemma of Rosenblum. For a self-adjoint operator $H$, consider the set $\mathfrak{R} \subset \mathcal{H}^{(a)}$ of elements $f$ (dense in $\mathcal{H}^{(a)}$) such that

$$r_H^2(f) := \operatorname{ess\,sup} d(E(\lambda)f, f)/d\lambda < \infty.$$

If $G$ is a Hilbert-Schmidt operator, then for all $f \in \mathfrak{R}$,

$$\int_{-\infty}^{\infty} \|G \exp(-iHt)f\|^2 dt \le 2\pi r_H^2(f)\|G\|_2^2. \tag{13}$$

Estimates (12) and (13) look rather similar although the first of them is uniform with respect to all $f \in \mathcal{D}(H)$ while the second is uniform with respect to all $G \in \mathfrak{S}_2$.

A typical application of the trace class theory is the following result. Suppose that

$$H_0 = -\Delta + v_0(x), \quad H = -\Delta + v(x), \tag{14}$$

where the functions $v_0$ and $v$ are real, $v_0, v \in L_\infty(\mathbb{R}^d)$ and the difference $\tilde{v} = v - v_0$ satisfies estimate (3) for some $\rho > d$. Then the WO $W_\pm(H, H_0)$ exist and are complete. It is an open problem whether WO for the pair (14) exist for an arbitrary bounded function $v_0$ if $v$ satisfies (3) for $\rho > 1$ only. Presumably the answer to this question is negative.

**6.** Let us return to the Schrödinger operator$H = -\Delta = v(x)$. As a by-product of the stationary version of the smooth approach, we obtain that the operator-valued function

$$\mathcal{R}(z) = \langle x \rangle^{-\alpha} R(z) \langle x \rangle^{-\alpha}, \quad \langle x \rangle = (1 + |x|^2)^{1/2}, \tag{15}$$

for all $\alpha > 1/2$ is norm-continuous in the complex plane up to the cut along $\mathbb{R}_+$ except, possibly, the point $z = 0$. This result is known as the limiting absorption principle (LAP). Actually, using the analytic Fredholm alternative, one first deduces from equation (7) that the function $\mathcal{R}(z)$ is continuous away from some closed set $\mathcal{N} \subset \mathbb{R}_+$ of Lebesgue measure zero. Then one proves that $\mathcal{N}$ consists of eigenvalues of the operator $H$. This implies, in particular, that the operator $H$ does not have singular continuous spectrum. Finally, T. Kato's theorem shows that the operator $H$ does not have positive eigenvalues.

The LAP is violated for function (15) if the parameter $\alpha = 1/2$. However, it can be somewhat improved in terms of the Agmon-Hörmander classes $\mathbf{B}$ (which is slightly smaller than the space $L_2^{(1/2)}$) and $\mathbf{B}^*$ (which is slightly bigger than the space $L_2^{(-1/2)}$). The exact form of the LAP (see §6.3) is the statement that the operator-valued function $R(z) : \mathbf{B} \to \mathbf{B}^*$ is continuous weakly (but not even strongly) up to the cut along $\mathbb{R}_+$.

Other approaches to the proof of the LAP rely on different versions of the commutator method. Roughly speaking, the underlying idea is that the commutator of the operator $H$ with a specially chosen first-order differential operator $A$ is essentially positive. In the original approach by T. Kato, R. Lavine and C. R. Putnam the operator $A$ had bounded coefficients, and the results were basically limited to Schrödinger operators with repulsive potentials.

The most powerful version of the commutator method is due to E. Mourre who took the generator $\mathbb{A}$ of the group of dilations for the operator $A$. The important advantage of the Mourre method is that it applies automatically to long-range potentials and can even be extended to multiparticle systems. Although it is already thoroughly described in books [**5**] by W. O. Amrein, A. Boutet de Monvel, V. Georgescu, [**14**] by H. Cycon, R. Froese, W. Kirsch, B. Simon and [**16**] by J. Dereziński, C. Gérard, we need to discuss it again in §6.9.

Moreover, we present in Chapter 11 an ingenious version of the commutator method due to A. F. Vakulenko. In this version the Schrödinger operator is commuted with a first-order differential (not symmetric) operator with bounded coefficients specially adapted to the potential $v(x)$. This method allows us to efficiently control the norm of the operator $\mathcal{R}(z) - \mathcal{R}(\bar{z})$ which seems to be impossible using other methods. Moreover, it gives a new proof of the Kato theorem on the absence of positive eigenvalues of the Schrödinger operator.

It turns out that one can improve the behavior of the resolvent $R(z)$ as $z$ approaches the continuous spectrum of $H$ if $R(z)$ is sandwiched additionally by some specially chosen differential or pseudodifferential (PDO) operators $B_\pm$. If the supports of their symbols $b_\pm(x, \xi)$ are contained in the cones $\mp\langle x, \xi\rangle \geq \epsilon > 0$, then the operator-valued function $\langle x\rangle^\alpha B_+^* R(z) B_- \langle x\rangle^\alpha$ is continuous as $z$ approaches the cut from the upper half-plane for all *positive* $\alpha$. A naive explanation of this astonishing fact is that the operators $B_\pm$ remove a part of the phase space where a classical particle propagates. The proof of such resolvent estimates relies again on the Mourre method.

Another important example of a differential operator with such "improving" property is the angular part $\nabla^\perp$ of the gradient. In this case the symbol

$$b(x, \xi) = \xi - |x|^{-2}\langle \xi, x\rangle x$$

of the PDO $B = \nabla^{\perp}$ equals zero if $x = c\xi$ for some $c \in \mathbb{R}$, that is, in the region of the phase space where a free classical particle "lives". The $H$-smoothness of the operator $\langle x\rangle^{-1/2}\nabla^{\perp}$ on any compact interval $X \subset \mathbb{R}_+$ is known as the radiation estimate. Its proof hinges again on the commutation of the operator $H$ with a suitable first-order differential operator. Together with the LAP, the radiation estimate implies the asymptotic completeness for long-range potentials (see subs. 9).

**7.** Scattering theory has many faces. In particular, as was discussed already, its main results can be formulated either in operator terms (WO (1), etc.) or in terms of solutions of the homogeneous stationary Schrödinger equation (4) distinguished by their asymptotic behavior as $|x| \to \infty$.

Actually, there are several possibilities to distinguish solutions of equation (4). One of them (see §6.7) is to impose asymptotics (5) which works if $\rho > (d+1)/2$ in assumption (3). Solutions $\psi(x, \omega, \lambda)$ allow us to define the generalized Fourier transforms $\Phi_\pm$ by formula (10) which gives a diagonalization of the operator $H$. The operators $\Phi_\pm$ are related to the WO by formula (11). Another possibility is to require that asymptotically a solution is a sum of incoming and outgoing spherical waves. Such solutions are constructed in §6.5 under the weaker assumption $\rho > 1$. An expansion theorem with respect to these solutions is proven in §6.6. The solutions of this type determine the action of the SM, whereas the solutions with asymptotics (5) determine its integral kernel (see subs. 2).

As far as the nonhomogeneous equation

$$-\Delta u + v(x)\psi = \lambda u + f, \quad \lambda > 0, \tag{16}$$

is concerned, its solution is uniquely determined by one of the radiation conditions

$$\partial_r u(x) \mp i\lambda^{1/2} u(x) = o(|x|^{-(d-1)/2}), \quad |x| \to \infty,$$

playing the role of boundary conditions at infinity. We prove in §6.4 that solutions of equation (16) with such asymptotics exist and can be obtained by the formula $u = R(\lambda \pm i0)f$. This assertion is also traditionally called the LAP. As shown in §11.3, this result remains true for long-range potentials.

**8.** In Chapter 7 we study the behavior of the resolvent $R(z)$ of the Schrödinger operator at low (as $|z| \to 0$) and high (as $|z| \to \infty$) energies.

Although the proof of the existence and completeness of WO under assumption (3) where $\rho > 1$ does not require these results, singularities of the resolvent at the point $z = 0$ (at the bottom of the continuous spectrum) are responsible for some new phenomena in more complicated scattering problems. Actually, they produce (see [**315, 319**]) new channels of scattering for anisotropic as well as for time-dependent potentials. Moreover, the behavior of the resolvent kernel $R(x, x'; z)$ as $|z| \to 0$ is closely connected with a decay of the integral kernel of the operator $\exp(-iHt)$ (and of the operator $\exp(-Ht)$, $t > 0$) as $|t| \to \infty$ for *bounded* values of $x$ and $x'$. We emphasize that the last problem is marginal for scattering theory since the function $(\exp(-iHt)f)(x)$ "lives" in the region of the space $L_2(\mathbb{R}^d)$ where $|x|$ and $|t|$ are of the same orders.

The behavior of the resolvent kernel $R(x, x'; z)$ as $|z| \to \infty$ *away* from the spectrum is related by the Laplace transform with the behavior of the parabolic Green function $G(x, x'; t)$ (integral kernel of the operator $\exp(-Ht)$) as $t \to 0$. Usually it is more convenient to treat local problems in terms of the semigroup

$\exp(-Ht)$, whereas the asymptotic expansion as $t \to 0$ of the trace

$$\mathrm{Tr}\left(e^{-Ht} - e^{-H_0 t}\right)$$

is natural to deduce from the asymptotic expansion as $|z| \to \infty$ of the trace

$$\mathrm{Tr}\left(R^m(z) - R_0^m(z)\right), \quad 2m > d-2, \tag{17}$$

($R^m - R_0^m \in \mathfrak{S}_1$ for such $m$).

**9.** Potentials $v(x)$ decaying slower than (or as) the Coulomb potential $v_0|x|^{-1}$ at infinity are called long-range. More precisely, it is required that

$$|\partial^\kappa v(x)| \le C(1+|x|)^{-\rho-|\kappa|}, \quad \rho \in (0,1],$$

for all derivatives of $v$ up to some order. In the long-range case, the WO $W_\pm(H, H_0)$ do not exist, and the asymptotic dynamics should be properly modified. It can be done in a time-dependent way either in the coordinate or momentum representations. For example, in the coordinate representation the free evolution $\exp(-iH_0 t)$ in the definition (1) of WO should be replaced (see §1.5) by unitary operators $\mathcal{U}_0(t)$ defined by the formula

$$(\mathcal{U}_0(t)f)(x) = \exp\left(i\Xi(x,t)\right)(2it)^{-d/2}\hat{f}(x/(2t)). \tag{18}$$

For short-range potentials we can set $\Xi(x,t) = (4t)^{-1}|x|^2$. In the long-range case the phase function $\Xi(x,t)$ should be chosen as a (perhaps, approximate) solution of the eikonal equation

$$\partial\Xi/\partial t + |\nabla\Xi|^2 + v = 0.$$

For example, for $\rho > 1/2$ we can set

$$\Xi(x,t) = (4t)^{-1}|x|^2 - t\int_0^1 v(sx)ds.$$

Formula (18) shows that both in short- and long-range cases solutions of the time-dependent Schrödinger equation "live" in a region of the configuration space where $|x|$ and $|t|$ are of the same order. Long-range potentials change only phases of these solutions.

Another possibility is a time-independent modification in the phase space. Actually, we consider in §10.2 WO (2) where $J$ is a PDO with oscillating symbol $\exp(i\Phi(x,\xi))$. Put

$$\Theta(x,\xi) = \langle x, \xi\rangle + \Phi(x,\xi).$$

The perturbation $HJ - JH_0$ is also a PDO with symbol $t(x,\xi)$ which is a short-range function of $x$ if $\exp(i\Theta(x,\xi))$ is an approximate eigenfunction of the operator $H$ corresponding to the "eigenvalue" $|\xi|^2$. This leads to the eikonal equation

$$|\nabla_x\Theta(x,\xi)|^2 + v(x) = |\xi|^2.$$

The notorious difficulty (for $d \ge 2$) of this approach is that the eikonal equation does not have (approximate) solutions satisfying it up to short-range terms. However, it is easy to construct functions $\Theta = \Theta_\pm$ satisfying this condition if a conical neighborhood of the direction $\mp\xi$ is removed from $\mathbb{R}^d$. For example, we can set

$$\Phi_\pm(x,\xi) = \pm 2^{-1}\int_0^\infty \left(v(x \pm \tau\xi) - v(\pm\tau\xi)\right)d\tau$$

if $\rho > 1/2$. Thus, we are induced to multiply the symbol of $J = J_\pm$ by a cut-off function $\zeta(x,\xi) = \zeta_\pm(x,\xi)$ which is a homogeneous function of order zero in the variable $x$ and "kills" bad regions of the phase space. We emphasize that now we consider the WO $W_\pm(H, H_0; J_\pm)$ with the identifications $J_\pm$ different for $t \to \infty$ and $t \to -\infty$.

Calculating the perturbation $HJ_\pm - J_\pm H_0$, we see that it is a sum of two PDO. The first of them is short-range, and thus the LAP can be applied to it. The symbol of the second one contains the first derivatives (in the variable $x$) of the cut-off function $\zeta_\pm(x, \xi)$ and hence decays at infinity as $|x|^{-1}$ only. Nevertheless, this operator can be factored into a product of $H_0$- and $H$-smooth operators if the radiation estimate (see subs. 6) is taken into account. Thus, all WO $W_\pm(H, H_0; J_\pm)$ and $W_\pm(H_0, H; J_\pm^*)$ exist. These operators are isometric since the operators $J_\pm$ are in some sense close to unitary operators. The isometricity of $W_\pm(H_0, H; J_\pm^*)$ is equivalent to the completeness of $W_\pm(H, H_0; J_\pm)$.

Although the modified WO enjoy basically the same properties as in the short-range case, properties of the SM in the short- and long-range cases are drastically different.

**10.** The SM $S(\lambda)$ studied in Chapters 8 and 10 is one of the main objects of this book. Considered as an integral operator, it typically has a smooth kernel $s(\omega, \omega', \lambda)$ away from the diagonal $\omega = \omega'$ but may be very singular on the diagonal. In the short-range case the leading diagonal singularity is the Dirac delta function, and the operator $S(\lambda) - I$ is compact (see §8.1). Hence the spectrum of $S(\lambda)$ consists of eigenvalues of finite multiplicity (except, possibly, the eigenvalue 1) lying on the unit circle and accumulating at the point 1 only. The asymptotics of these eigenvalues is determined by the behavior of the potential at infinity. The SM can be expressed (see formulas (6) and (8)) via the resolvent $R(z)$ of the operator $H$ which in view of the resolvent identity (7) yields the Born expansion of $S(\lambda)$ valid for small potentials as well as for high energies.

Furthermore, the operator $S(\lambda) - I$ belongs to a Schatten-von Neumann class $\mathfrak{S}_p$ where $p$ depends on the falloff of the potential at infinity and gets smaller as the power $\rho$ in estimate (3) increases. In particular, $S(\lambda) - I \in \mathfrak{S}_2$ if $\rho > (d+1)/2$. In this case one introduces the total scattering cross section

$$\boldsymbol{\sigma}(\lambda) = |\mathbb{S}^{d-1}|^{-1}(2\pi)^{d-1}\lambda^{-(d-1)/2}\|S(\lambda) - I\|_2^2$$

averaged over all incident directions. Roughly speaking, $\sigma(\lambda)$ shows how strongly a free particle of energy $\lambda$ is perturbed by a potential $v$. It turns out that this quantity can be efficiently estimated in terms of the potential $v$ only, although the norm of operator (15) is not controllable in these terms. For example, if $d = 3$, then

$$\boldsymbol{\sigma}(\lambda) \leq (2^{1/3} + 1)^3 (4\pi\lambda)^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v(x)\, v(x')||x - x'|^{-2} dx dx'.$$

This and other bounds on the total scattering cross section can be found in §§8.3 and 8.6.

The high-energy behavior of the SM and, in particular, of the forward scattering amplitude $a(\omega, \omega, \lambda)$ and of the total scattering cross section is studied in §§8.4 and 8.5. This requires some analytical methods which rely on a construction of special approximate but explicit solutions of the Schrödinger equation (4). In a standard way, such construction reduces to a study of the eikonal and transport equations. Actually, in these sections we consider two-parameter asymptotics where the coupling constant $\gamma$ (when $v$ is replaced by $\gamma v$) might tend to infinity as well as the energy $\lambda$.

In the long-range case, the diagonal singularity of the scattering amplitude $s(\omega, \omega', \lambda)$ becomes very wild, but can be explicitly described. In fact, as shown in §10.4, for potentials which asymptotically are homogeneous functions of order $-\rho$ where $\rho < 1$, we have

$$s(\omega, \omega', \lambda) \sim c(\lambda)|\omega - \omega'|^{-(d-1)(1+\beta)/2} \exp\left(i\Psi(\omega, \omega', \lambda)|\omega - \omega'|^{1-\beta}\right), \quad \beta = \rho^{-1};$$

here $\Psi(\omega, \omega', \lambda)$ is a sufficiently regular function as $\omega' \to \omega$. Since $\beta > 1$, the singularity of $|s(\omega, \omega', \lambda)|$ as $\omega' \to \omega$ is stronger than that of the kernel of the singular integral operator. Nevertheless the operator $S(\lambda)$ remains bounded due to the oscillating factor in its kernel. Note that the Dirac delta function disappears from the kernel of $S(\lambda)$. In the long-range case the operator $S(\lambda) - I$ is not compact, and the spectrum of the SM $S(\lambda)$ covers the whole unit circle.

**11.** The Lifshitz-Kreĭn spectral shift function (SSF) $\xi(\lambda)$ is introduced by the relation (trace formula)

$$\operatorname{Tr}\left(f(H) - f(H_0)\right) = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) d\lambda \tag{19}$$

for the trace of the difference of rather arbitrary functions $f$ of the operators $H_0$ and $H$. Of course, $f(H) - f(H_0) \in \mathfrak{S}_1$ if, for example, $H - H_0 \in \mathfrak{S}_1$ and $f \in C_0^\infty(\mathbb{R})$, but both of these conditions can be significantly relaxed. If $H - H_0 \in \mathfrak{S}_1$, the SSF can be expressed via the perturbation determinant (PD)

$$D(z) = D_{H/H_0}(z) = \operatorname{Det}(I + VR_0(z)), \quad z \in \rho(H_0),$$

for the pair $H_0$, $H$ by the relation

$$\xi(\lambda) = \pi^{-1} \lim_{\varepsilon \to +0} \arg D(\lambda + i\varepsilon).$$

These limits exist for almost all $\lambda \in \mathbb{R}$ and $\xi \in L_1(\mathbb{R})$. On the continuous spectrum, $\xi(\lambda)$ is connected with the SM by the Birman-Kreĭn formula

$$\operatorname{Det} S(\lambda) = e^{-2\pi i \xi(\lambda)}. \tag{20}$$

On the discrete spectrum, $\xi(\lambda)$ depends on the shift of the eigenvalues of the operator $H$ relative to the eigenvalues of the operator $H_0$. This explains the name of the SSF.

In §3.8 we illustrate the abstract results of [I] on the SSF on the example of pair (14) where $v_0 \in L_\infty(\mathbb{R}^d)$ and the difference $\tilde{v} = v - v_0$ satisfies condition (3) for $\rho > d$. However, our main goal is to study properties of the SSF specific for the Schrödinger operator $H = -\Delta + v(x)$; in this case $H_0 = -\Delta$. In §9.1 we introduce regularized PD $D_p(z)$ and SSF $\xi_p(\lambda)$ (the index $p = 2, 3, \ldots$ stands for the class $\mathfrak{S}_p$). These functions possess basically the same properties as $D(z)$ and $\xi(\lambda)$, but are well defined, for a sufficiently large $p = p(\rho, d)$, for potentials satisfying condition (3) with an arbitrary $\rho > 1$; moreover, they are sometimes easier to handle (especially for $p = 2$). In the case $\rho \leq d$, formulas (19) and (20) require some nontrivial regularization because neither $f(H) - f(H_0) \notin \mathfrak{S}_1$ nor $S(\lambda) - I \notin \mathfrak{S}_1$. The regularized SSF $\xi_p(\lambda)$ turns out to be useful for the study of the usual SSF. For example, it is used for the proof that, for all dimensions $d$, the SSF $\xi(\lambda)$ is a continuous function of $\lambda > 0$ if $\rho > d$. We emphasize that this fact is not a consequence of the general theory where $\xi \in L_1^{(loc)}$ only.

Our next aim in Chapter 9 is to find the asymptotic expansion of the SSF $\xi(\lambda)$ as $\lambda \to \infty$ which yields also the asymptotic expansion of the trace (17) as

$|z| \to \infty$ in the whole complex plane cut along $\mathbb{R}_+$. Here we proceed from the high-energy asymptotic expansion of the SM and use formula (20). Unfortunately, this procedure gives rather complicated expressions for asymptotic coefficients of function (17). One obtains much simpler expressions using the expansion of this function obtained in Chapter 7 away from the spectrum of $H$.

Roughly speaking, the trace identities give explicit representations in terms of the potential $v$ for the expressions "Tr"$(H^n - H_0^n)$ where the traces are properly regularized and $n$ is an integer or a half-integer number. To be more precise, if $n$ is an integer, we have an identity relating $\xi(\lambda)$, eigenvalues of the operator $H$ and some functionals of $v$. In the case of a half-integer $n$, the role of $\xi(\lambda)$ in such identities is played by $\ln|D(\lambda + i0)|$ for $d = 1$ or by $\ln|D_2(\lambda + i0)|$ for $d = 2, 3$. The proof of trace identities essentially uses the high-energy asymptotic expansion of function (17).

In the one-dimensional case all of these problems are considered in Chapters 4 and 5.

**12.** Another important example of this book is first order symmetric hyperbolic systems

$$M(x)\partial u(x,t)/\partial t = \sum_{j=1}^{d} B_j \partial u(x,t)/\partial x_j, \tag{22}$$

where $M(x)$ and $B_j$ are symmetric $n \times n$ matrices. It is supposed that the matrices $B_j$ do not depend on $x$ and $M(x)$ is uniformly bounded and positive definite on $\mathbb{R}^d$. Systems (22) describe the wave propagation phenomena of classical physics. Maxwell's equations are a famous example of such a system.

Set

$$H_{00} = i \sum_{j=1}^{d} B_j \partial/\partial x_j.$$

The operator $H = M(x)^{-1} H_{00}$ is self-adjoint in the Hilbert space $\mathcal{H} = \mathcal{H}_M$ with scalar product

$$(f, g)_M = \int_{\mathbb{R}^d} \langle M(x) f(x), g(x) \rangle dx.$$

If $M(x) = M_0$ does not depend on $x \in \mathbb{R}^d$, system (22) is said to be homogeneous. The operator $H_0 = M_0^{-1} H_{00}$ is self-adjoint in the Hilbert space $\mathcal{H}_0 = \mathcal{H}_{M_0}$, which can obviously be identified with $\mathcal{H}$. This allows one to discuss the WO $W_\pm(H, H_0; I_0)$ where $I_0 : \mathcal{H}_0 \to \mathcal{H}$ is the identity operator.

An important condition which we require is that the rank of the symbol $A(\xi) = -\sum_{j=1}^{d} B_j \xi_j$ of the operator $H_{00}$ does not depend on $\xi \neq 0$. Then under the assumption

$$M(x) - M_0 = O(|x|^{-\rho}), \quad |x| \to \infty, \tag{23}$$

where $\rho > 1$, the smooth approach yields the existence and completeness of the WO $W_\pm(H, H_0; I_0)$.

If $M_0$ is also a function of $x \in \mathbb{R}^d$, then only the trace class method works. It allows us to prove that the WO $W_\pm(H, H_0; I_0)$ exist and are complete if $A(\xi)$ is nondegenerate, that is $\operatorname{Det} A(\xi) \neq 0$ for $\xi \neq 0$, and estimate (23) is satisfied for $\rho > d$.