Preface

The p-Laplacian operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right), \quad p \in (1, \infty)$$

arises in non-Newtonian fluid flows, turbulent filtration in porous media, plasticity theory, rheology, glaciology, and in many other application areas; see, e.g., Esteban and Vázquez [**48**] and Padial, Takáč, and Tello [**90**]. Problems involving the p-Laplacian have been studied extensively in the literature during the last fifty years. However, only a few papers have used Morse theoretic methods to study such problems; see, e.g., Vannella [**130**], Cingolani and Vannella [**29, 31**], Dancer and Perera [**40**], Liu and Su [**74**], Jiu and Su [**58**], Perera [**98, 99, 100**], Bartsch and Liu [**15**], Jiang [**57**], Liu and Li [**75**], Ayoujil and El Amrouss [**10, 11, 12**], Cingolani and Degiovanni [**30**], Guo and Liu [**55**], Liu and Liu [**73**], Degiovanni and Lancelotti [**43, 44**], Liu and Geng [**70**], Tanaka [**129**], and Fang and Liu [**50**]. The purpose of this monograph is to fill this gap in the literature by presenting a Morse theoretic study of a very general class of homogeneous operators that includes the p-Laplacian as a special case.

Infinite dimensional Morse theory has been used extensively in the literature to study semilinear problems (see, e.g., Chang [**28**] or Mawhin and Willem [**81**]). In this theory the behavior of a C^1-functional defined on a Banach space near one of its isolated critical points is described by its critical groups, and there are standard tools for computing these groups for the variational functional associated with a semilinear problem. They include the Morse and splitting lemmas, the shifting theorem, and various linking and local linking theorems based on eigenspaces that give critical points with nontrivial critical groups. Unfortunately, none of them apply to quasilinear problems where the Euler functional is no longer defined on a Hilbert space or is C^2 and there are no eigenspaces to work with. We will systematically develop alternative tools, such as nonlinear linking and local linking theories, in order to effectively apply Morse theory to such problems.

A complete description of the spectrum of a quasilinear operator such as the p-Laplacian is in general not available. Unbounded sequences of eigenvalues can be constructed using various minimax schemes, but it is generally not known whether they give a full list, and it is often unclear whether different schemes give the same eigenvalues. The standard eigenvalue sequence based on the Krasnoselskii genus is not useful for obtaining nontrivial critical groups or for constructing linking sets or local linkings. We will work with a new sequence of eigenvalues introduced by the first author in [**98**] that uses the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz. The necessary background material on algebraic topology and the cohomological index will be given in order to make the text as self-contained as possible.

One of the main points that we would like to make here is that, contrary to the prevailing sentiment in the literature, the lack of a complete list of eigenvalues is not a serious obstacle to effectively applying critical point theory. Indeed, our sequence of eigenvalues is sufficient to adapt many of the standard variational methods for solving semilinear problems to the quasilinear case. In particular, we will obtain nontrivial critical groups and use the stability and piercing properties of the cohomological index to construct new linking sets that are readily applicable to quasilinear problems. Of course, such constructions cannot be based on linear subspaces since we no longer have eigenspaces. We will instead use nonlinear splittings based on certain sub- and superlevel sets whose cohomological indices can be precisely calculated. We will also introduce a new notion of local linking based on these splittings.

We will describe the general setting and give some examples in Chapter 1, but first we give an overview of the theory developed here and a preliminary survey chapter on Morse theoretic methods used in variational problems in order to set up the history and context.