An Overview

Let Φ be a C^1-functional defined on a real Banach space W and satisfying the (PS) condition. In Morse theory the local behavior of Φ near an isolated critical point u is described by the sequence of critical groups

$$C^q(\Phi, u) = H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad q \geqslant 0 \tag{1}$$

where $c = \Phi(u)$ is the corresponding critical value, Φ^c is the sublevel set $\{u \in W : \Phi(u) \leqslant c\}$, U is a neighborhood of u containing no other critical points, and H denotes cohomology. They are independent of U by the excision property. When the critical values are bounded from below, the global behavior of Φ can be described by the critical groups at infinity

$$C^q(\Phi, \infty) = H^q(W, \Phi^a), \quad q \geqslant 0$$

where a is less than all critical values. They are independent of a by the second deformation lemma and the homotopy invariance of cohomology groups.

When Φ has only a finite number of critical points $u_1, \dots, u_k$, their critical groups are related to those at infinity by

$$\sum_{i=1}^{k} \operatorname{rank} C^q(\Phi, u_i) \geqslant \operatorname{rank} C^q(\Phi, \infty) \quad \forall q$$

(see Proposition 3.16). Thus, if $C^q(\Phi, \infty) \neq 0$, then Φ has a critical point u with $C^q(\Phi, u) \neq 0$. If zero is the only critical point of Φ and $\Phi(0) = 0$, then taking $U = W$ in (1), and noting that Φ^0 is a deformation retract of W and $\Phi^0 \setminus \{0\}$ deformation retracts to Φ^a by the second deformation lemma, gives

$$\begin{aligned} C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}) &\approx H^q(W, \Phi^0 \setminus \{0\}) \\ &\approx H^q(W, \Phi^a) = C^q(\Phi, \infty) \quad \forall q. \end{aligned}$$

Thus, if $C^q(\Phi, 0) \not\approx C^q(\Phi, \infty)$ for some q, then Φ has a critical point $u \neq 0$. Such ideas have been used extensively in the literature to obtain multiple nontrivial solutions of semilinear elliptic boundary value problems (see, e.g., Mawhin and Willem [**81**], Chang [**28**], Bartsch and Li [**14**], and their references).

Now consider the eigenvalue problem

$$\begin{cases} -\Delta_p \, u = \lambda \, |u|^{p-2} \, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$,

$$\Delta_p\, u = \operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right)$$

is the p-Laplacian of u, and $p \in (1, \infty)$. The eigenfunctions coincide with the critical points of the C^1-functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda\, |u|^p$$

defined on the Sobolev space $W^{1,\,p}_0(\Omega)$ with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}.$$

When λ is not an eigenvalue, zero is the only critical point of Φ_λ and we may take

$$U = \left\{u \in W^{1,\,p}_0(\Omega) : \|u\| \leqslant 1\right\}$$

in the definition (1). Since Φ_λ is positive homogeneous, $\Phi_\lambda^0 \cap U$ radially contracts to the origin and $\Phi_\lambda^0 \cap U \setminus \{0\}$ radially deformation retracts onto

$$\Phi_\lambda^0 \cap S = \Psi^\lambda$$

where S is the unit sphere in $W^{1,\,p}_0(\Omega)$ and

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^p}, \quad u \in S.$$

It follows that

$$C^q(\Phi_\lambda, 0) \approx \begin{cases} \delta_{q0}\, \mathcal{G}, & \Psi^\lambda = \varnothing \\[2ex] \widetilde{H}^{q-1}(\Psi^\lambda), & \Psi^\lambda \neq \varnothing \end{cases} \tag{2}$$

where δ is the Kronecker delta, $\mathcal{G}$ is the coefficient group, and $\widetilde{H}$ denotes reduced cohomology. Note also that the eigenvalues coincide with the critical values of Ψ by the Lagrange multiplier rule.

In the semilinear case $p = 2$, the spectrum $\sigma(-\Delta)$ consists of isolated eigenvalues λ_k, repeated according to their multiplicities, satisfying

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \to \infty.$$

If $\lambda < \lambda_1 = \inf \Psi$, then $\Psi^\lambda = \varnothing$ and hence

$$C^q(\Phi_\lambda, 0) \approx \delta_{q0}\, \mathcal{G} \tag{3}$$

by (2). If $\lambda_k < \lambda < \lambda_{k+1}$, then we have the orthogonal decomposition

$$H^1_0(\Omega) = H^- \oplus H^+, \quad u = v + w \tag{4}$$

where H^- is the direct sum of the eigenspaces corresponding to $\lambda_1, \ldots, \lambda_k$ and H^+ is its orthogonal complement, and

$$\dim H^- = k$$

is called the Morse index of zero. It is easy to check that

$$\eta(u,t) = \frac{v + (1-t)\,w}{\|v + (1-t)\,w\|}, \quad (u,t) \in \Psi^\lambda \times [0,1]$$

is a deformation retraction of Ψ^λ onto $H^- \cap S$, so

$$C^q(\Phi_\lambda, 0) \approx \widetilde{H}^{q-1}(H^- \cap S) \approx \delta_{qk}\,\mathcal{G}$$

by (2).

The quasilinear case $p \neq 2$ is far more complicated. Very little is known about the spectrum $\sigma(-\Delta_p)$ itself. The first eigenvalue λ_1 is positive, simple, and has an associated eigenfunction φ_1 that is positive in Ω (see Anane [**9**] and Lindqvist [**68, 69**]). Moreover, λ_1 is isolated in the spectrum, so the second eigenvalue $\lambda_2 = \inf\, \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is also well-defined. In the ODE case $n = 1$, where Ω is an interval, the spectrum consists of a sequence of simple eigenvalues $\lambda_k \nearrow \infty$, and the eigenfunction φ_k associated with λ_k has exactly $k-1$ interior zeroes (see, e.g., Drábek [**46**]). In the PDE case $n \geqslant 2$, an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselskii's genus, but it is not known whether this gives a complete list of the eigenvalues.

If $\lambda < \lambda_1$, then (3) holds as before. It was shown in Dancer and Perera [**40**] that

$$C^q(\Phi_\lambda, 0) \approx \delta_{q1}\,\mathcal{G}$$

if $\lambda_1 < \lambda < \lambda_2$ and that

$$C^q(\Phi_\lambda, 0) = 0, \quad q = 0, 1$$

if $\lambda > \lambda_2$. Thus, the question arises as to whether there is a nontrivial critical group when $\lambda > \lambda_2$. An affirmative answer was given in Perera [**98**] where a new sequence of eigenvalues was constructed using a minimax scheme involving the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [**49**] as follows.

Let $\mathcal{F}$ denote the class of symmetric subsets of S, let $i(M)$ denote the cohomological index of $M \in \mathcal{F}$, and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geqslant k}} \; \sup_{u \in M} \; \Psi(u).$$

Then $\lambda_k \nearrow \infty$ is a sequence of eigenvalues, and if $\lambda_k < \lambda_{k+1}$, then

$$i(\Psi^{\lambda_k}) = i(S \backslash \Psi_{\lambda_{k+1}}) = k \tag{5}$$

where

$$\Psi^{\lambda_k} = \big\{u \in S : \Psi(u) \leqslant \lambda_k\big\}, \qquad \Psi_{\lambda_{k+1}} = \big\{u \in S : \Psi(u) \geqslant \lambda_{k+1}\big\}$$

(see Theorem 4.6). Thus, if $\lambda_k < \lambda < \lambda_{k+1}$, then

$$i(\Psi^\lambda) = k$$

by the monotonicity of the index, which implies that

$$\widetilde{H}^{k-1}(\Psi^\lambda) \neq 0$$

(see Proposition 2.14) and hence

$$C^k(\Phi_\lambda, 0) \neq 0 \tag{6}$$

by (2).

The structure provided by this new sequence of eigenvalues is sufficient to adapt many of the standard variational methods for solving semilinear problems to the quasilinear case. In particular, we will construct new linking sets and local linkings that are readily applicable to quasilinear problems. Of course, such constructions cannot be based on linear subspaces since we no longer have eigenspaces to work with. They will instead use nonlinear splittings generated by the sub- and superlevel sets of Ψ that appear in (5), and the indices given there will play a key role in these new topological constructions as we will see next.

Consider the boundary value problem

$$\begin{cases} -\Delta_p\, u = f(x, u) & \text{in } \Omega \\[2ex] \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

where the nonlinearity f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|f(x, t)| \leqslant C\left(|t|^{r-1} + 1\right) \quad \forall (x, t) \in \Omega \times \mathbb{R}$$

for some $r \in (1, p^*)$. Here

$$p^* = \begin{cases} \dfrac{np}{n - p}, & p < n \\[2ex] \infty, & p \geqslant n \end{cases}$$

is the critical exponent for the Sobolev imbedding $W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega)$. Weak solutions of this problem coincide with the critical points of the C^1-functional

$$\Phi(u) = \int_\Omega |\nabla u|^p - p\, F(x, u), \quad u \in W^{1,p}_0(\Omega)$$

where

$$F(x, t) = \int_0^t f(x, s)\, ds$$

is the primitive of f.

It is customary to roughly classify problem (7) according to the growth of f as

(i) p-sublinear if

$$\lim_{t \to \pm\infty} \frac{f(x, t)}{|t|^{p-2}\, t} = 0 \quad \forall x \in \Omega,$$

(ii) asymptotically p-linear if

$$0 < \liminf_{t \to \pm\infty} \frac{f(x, t)}{|t|^{p-2}\, t} \leqslant \limsup_{t \to \pm\infty} \frac{f(x, t)}{|t|^{p-2}\, t} < \infty \quad \forall x \in \Omega,$$

(*iii*) p-superlinear if

$$\lim_{t \to \pm\infty} \frac{f(x,t)}{|t|^{p-2}\, t} = \infty \quad \forall x \in \Omega.$$

Consider the asymptotically p-linear case where

$$\lim_{t \to \pm\infty} \frac{f(x,t)}{|t|^{p-2}\, t} = \lambda, \text{ uniformly in } x \in \Omega$$

with $\lambda_k < \lambda < \lambda_{k+1}$, and assume $\lambda \notin \sigma(-\Delta_p)$ to ensure that Φ satisfies the (PS) condition.

In the semilinear case $p = 2$, let

$$A = \big\{v \in H^- : \|v\| = R\big\}, \qquad B = H^+$$

with $H^\pm$ as in (4) and $R > 0$. Then

$$\max \Phi(A) < \inf \Phi(B) \tag{8}$$

if R is sufficiently large, and A cohomologically links B in dimension $k-1$ in the sense that the homomorphism

$$\widetilde{H}^{k-1}(H^1_0(\Omega)\backslash B) \to \widetilde{H}^{k-1}(A)$$

induced by the inclusion is nontrivial. So it follows that problem (7) has a solution u with $C^k(\Phi, u) \neq 0$ (see Proposition 3.25).

We may ask whether this well-known argument can be modified to obtain the same result in the quasilinear case $p \neq 2$ where we no longer have the splitting given in (4). We will give an affirmative answer as follows. Let

$$A = \big\{Ru : u \in \Psi^{\lambda_k}\big\}, \qquad B = \big\{tu : u \in \Psi_{\lambda_{k+1}},\ t \geqslant 0\big\}$$

with $R > 0$. Then (8) still holds if R is sufficiently large, and A cohomologically links B in dimension $k-1$ by (5) and the following theorem proved in Section 3.7, so problem (7) again has a solution u with $C^k(\Phi, u) \neq 0$.

THEOREM 1. *Let A_0 and B_0 be disjoint nonempty closed symmetric subsets of the unit sphere S in a Banach space such that*

$$i(A_0) = i(S\backslash B_0) = k$$

where i denotes the cohomological index, and let

$$A = \big\{Ru : u \in A_0\big\}, \qquad B = \big\{tu : u \in B_0,\ t \geqslant 0\big\}$$

with $R > 0$. Then A cohomologically links B in dimension $k-1$.

Now suppose $f(x, 0) \equiv 0$, so that problem (7) has the trivial solution $u(x) \equiv 0$. Assume that

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{p-2}\, t} = \lambda, \text{ uniformly in } x \in \Omega, \tag{9}$$

$\lambda_k < \lambda < \lambda_{k+1}$, and the sign condition

$$p\, F(x,t) \geqslant \lambda_{k+1}\, |t|^p \quad \forall (x,t) \in \Omega \times \mathbb{R} \tag{10}$$

holds. In the p-superlinear case it is customary to also assume the following Ambrosetti-Rabinowitz type condition to ensure that Φ satisfies the (PS) condition:

$$0 < \mu F(x,t) \leqslant t f(x,t) \quad \forall x \in \Omega,\ |t| \text{ large} \tag{11}$$

for some $\mu > p$.

In the semilinear case $p = 2$, we can then obtain a nontrivial solution of problem (7) using the well-known saddle point theorem of Rabinowitz as follows. Fix a $w_0 \in H^+ \setminus \{0\}$ and let

$$X = \big\{u = v + s\,w_0 : v \in H^-,\ s \geqslant 0,\ \|u\| \leqslant R\big\},$$

$$A = \big\{v \in H^- : \|v\| \leqslant R\big\} \cup \big\{u \in X : \|u\| = R\big\},$$

$$B = \big\{w \in H^+ : \|w\| = r\big\}$$

with $H^{\pm}$ as in (4) and $R > r > 0$. Then

$$\max \Phi(A) \leqslant 0 < \inf \Phi(B) \tag{12}$$

if R is sufficiently large and r is sufficiently small, and A homotopically links B with respect to X in the sense that

$$\gamma(X) \cap B \neq \varnothing \quad \forall \gamma \in \Gamma$$

where

$$\Gamma = \big\{\gamma \in C(X, H^1_0(\Omega)) : \gamma|_A = id_A\big\}.$$

So it follows that

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u)$$

is a positive critical level of Φ (see Proposition 3.21).

Again we may ask whether linking sets that would enable us to use this argument in the quasilinear case $p \neq 2$ can be constructed. In Perera and Szulkin [**105**] the following such construction based on the piercing property of the index (see Proposition 2.12) was given. Recall that the cone CA_0 on a topological space A_0 is the quotient space of $A_0 \times [0,1]$ obtained by collapsing $A_0 \times \{1\}$ to a point. We identify $A_0 \times \{0\}$ with A_0 itself. Fix an $h \in C(C\Psi^{\lambda_k}, S)$ such that $h(C\Psi^{\lambda_k})$ is closed and $h|_{\Psi^{\lambda_k}} = id_{\Psi^{\lambda_k}}$, and let

$$X = \big\{tu : u \in h(C\Psi^{\lambda_k}),\ 0 \leqslant t \leqslant R\big\},$$

$$A = \big\{tu : u \in \Psi^{\lambda_k},\ 0 \leqslant t \leqslant R\big\} \cup \big\{u \in X : \|u\| = R\big\},$$

$$B = \big\{ru : u \in \Psi_{\lambda_{k+1}}\big\}$$

with $R > r > 0$. Then (12) still holds if R is sufficiently large and r is sufficiently small, and A homotopically links B with respect to X by (5) and the following theorem proved in Section 3.6, so Φ again has a positive critical level.

THEOREM 2. *Let A_0 and B_0 be disjoint nonempty closed symmetric subsets of the unit sphere S in a Banach space such that*

$$i(A_0) = i(S \backslash B_0) < \infty,$$

$h \in C(CA_0, S)$ be such that $h(CA_0)$ is closed and $h|_{A_0} = id_{A_0}$, and let

$$X = \{tu : u \in h(CA_0),\ 0 \leqslant t \leqslant R\},$$

$$A = \{tu : u \in A_0,\ 0 \leqslant t \leqslant R\} \cup \{u \in X : \|u\| = R\},$$

$$B = \{ru : u \in B_0\}$$

with $R > r > 0$. Then A homotopically links B with respect to X.

The sign condition (10) can be removed by using a comparison of the critical groups of Φ at zero and infinity instead of the above linking argument. First consider the nonresonant case where (9) holds with $\lambda \in (\lambda_k, \lambda_{k+1}) \backslash \sigma(-\Delta_p)$. Then

$$C^q(\Phi, 0) \approx C^q(\Phi_\lambda, 0) \quad \forall q$$

by the homotopy invariance of the critical groups and hence

$$C^k(\Phi, 0) \neq 0 \tag{13}$$

by (6). On the other hand, a simple modification of an argument due to Wang [**132**] shows that

$$C^q(\Phi, \infty) = 0 \quad \forall q$$

when (11) holds (see Example 5.14). So Φ has a nontrivial critical point by the remarks at the beginning of the chapter.

In the p-sublinear case, where Φ is bounded from below, we can use (13) to obtain multiple nontrivial solutions of problem (7). Indeed, the three critical points theorem (see Corollary 3.32) gives two nontrivial critical points of Φ when $k \geqslant 2$.

Note that we do not assume that there are no other eigenvalues in the interval $(\lambda_k, \lambda_{k+1})$, in particular, λ may be an eigenvalue. Our results hold as long as $\lambda_k < \lambda < \lambda_{k+1}$, even if the entire interval $[\lambda_k, \lambda_{k+1}]$ is contained in the spectrum. Thus, eigenvalues that do not belong to the sequence (λ_k) are not that important in this context. In fact, we will see that the cohomological index of sublevel sets changes only when crossing an eigenvalue from this particular sequence.

Now we consider the resonant case where (9) holds with $\lambda \in [\lambda_k, \lambda_{k+1}] \cap \sigma(-\Delta_p)$, and ask whether we still have (13). We will show that this is indeed the case when a suitable sign condition holds near $t = 0$. Write f as

$$f(x, t) = \lambda\, |t|^{p-2}\, t + g(x, t),$$

so that

$$\lim_{t \to 0} \frac{g(x, t)}{|t|^{p-2}\, t} = 0, \quad \text{uniformly in } x \in \Omega,$$

set

$$G(x,t) = \int_0^t g(x,s)\, ds,$$

and assume that either

$$\lambda = \lambda_k, \qquad G(x,t) \geqslant 0 \quad \forall x \in \Omega,\ |t| \text{ small},$$

or

$$\lambda = \lambda_{k+1}, \qquad G(x,t) \leqslant 0 \quad \forall x \in \Omega,\ |t| \text{ small}.$$

In the semilinear case $p = 2$, let

$$A = \big\{v \in H^- : \|v\| \leqslant r\big\}, \qquad B = \big\{w \in H^+ : \|w\| \leqslant r\big\}$$

with $H^\pm$ as in (4) and $r > 0$. Then

$$\Phi|_A \leqslant 0 < \Phi|_{B\setminus\{0\}} \tag{14}$$

if r is sufficiently small (see Li and Willem [**67**]), so Φ has a local linking near zero in dimension k and hence (13) holds (see Liu [**71**]).

So we may ask whether the notion of a local linking can be generalized to apply in the quasilinear case $p \neq 2$ as well. We will again give an affirmative answer. Let

$$A = \big\{tu : u \in \Psi^{\lambda_k},\ 0 \leqslant t \leqslant r\big\}, \qquad B = \big\{tu : u \in \Psi_{\lambda_{k+1}},\ 0 \leqslant t \leqslant r\big\}$$

with $r > 0$. Then (14) still holds if r is sufficiently small (see Degiovanni, Lancelotti, and Perera [**42**]), so Φ has a cohomological local splitting near zero in dimension k in the sense of the following definition given in Section 3.11. Hence (13) holds again (see Proposition 3.34).

Definition 3. We say that a C^1-functional Φ defined on a Banach space W has a cohomological local splitting near zero in dimension k if there is an $r > 0$ such that zero is the only critical point of Φ in

$$U = \big\{u \in W : \|u\| \leqslant r\big\}$$

and there are disjoint nonempty closed symmetric subsets A_0 and B_0 of ∂U such that

$$i(A_0) = i(S\setminus B_0) = k$$

and

$$\Phi|_A \leqslant 0 < \Phi|_{B\setminus\{0\}}$$

where

$$A = \big\{tu : u \in A_0,\ 0 \leqslant t \leqslant 1\big\}, \qquad B = \big\{tu : u \in B_0,\ 0 \leqslant t \leqslant 1\big\}.$$

These constructions, which were based on the existence of a sequence of eigenvalues satisfying (5), can be extended to situations involving indefinite eigenvalue problems such as

$$\begin{cases} -\Delta_p\, u = \lambda\, V(x)\, |u|^{p-2}\, u & \text{in } \Omega \\[2ex] \qquad u = 0 & \text{on } \partial\Omega \end{cases}$$

where the weight function $V \in L^\infty(\Omega)$ changes sign. Here the eigenfunctions are the critical points of the functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - \lambda\, V(x)\, |u|^p, \quad u \in W_0^{1,\,p}(\Omega)$$

and the positive and negative eigenvalues are the critical values of

$$\Psi^\pm(u) = \frac{1}{J(u)}, \quad u \in S^\pm,$$

respectively, where

$$J(u) = \int_\Omega V(x)\, |u|^p$$

and

$$S^\pm = \big\{u \in S : J(u) \gtrless 0\big\}.$$

Let $\mathcal{F}^\pm$ denote the class of symmetric subsets of $S^\pm$, respectively, and set

$$\lambda_k^+ := \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geqslant k}} \sup_{u \in M} \Psi^+(u),$$

$$\lambda_k^- := \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geqslant k}} \inf_{u \in M} \Psi^-(u).$$

We will show that $\lambda_k^+ \nearrow +\infty$ and $\lambda_k^- \searrow -\infty$ are sequences of positive and negative eigenvalues, respectively, and if $\lambda_k^+ < \lambda_{k+1}^+$ (resp. $\lambda_{k+1}^- < \lambda_k^-$), then

$$i((\Psi^+)^{\lambda_k^+}) = i(S^+ \backslash (\Psi^+)_{\lambda_{k+1}^+}) = k$$

$$(\text{resp. } i((\Psi^-)_{\lambda_k^-}) = i(S^- \backslash (\Psi^-)^{\lambda_{k+1}^-}) = k).$$

In particular, if $\lambda_k^+ < \lambda < \lambda_{k+1}^+$ or $\lambda_{k+1}^- < \lambda < \lambda_k^-$, then

$$C^k(\Phi_\lambda, 0) \neq 0.$$

Finally we will present an extension of our theory to anisotropic p-Laplacian systems of the form

$$\text{(15)} \qquad \left\{\begin{aligned} -\Delta_{p_i} u_i &= \frac{\partial F}{\partial u_i}(x,u) && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega, \end{aligned}\right. \qquad i = 1, \dots, m$$

where each $p_i \in (1, \infty)$, $u = (u_1, \dots, u_m)$, and $F \in C^1(\Omega \times \mathbb{R}^m)$ satisfies the subcritical growth conditions

$$\left|\frac{\partial F}{\partial u_i}(x,u)\right| \leqslant C \left(\sum_{j=1}^m |u_j|^{r_{ij}-1} + 1\right) \quad \forall (x,u) \in \Omega \times \mathbb{R}^m,\ i = 1, \dots, m$$

for some $r_{ij} \in (1, 1 + (1 - 1/p_i^*)\, p_j^*)$. Weak solutions of this system are the critical points of the functional

$$\Phi(u) = I(u) - \int_\Omega F(x,u), \quad u \in W = W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_m}(\Omega)$$

where

$$I(u) = \sum_{i=1}^m \frac{1}{p_i} \int_\Omega |\nabla u_i|^{p_i}.$$

Unlike in the scalar case, here I is not homogeneous except when $p_1 = \cdots = p_m$. However, it still has the following weaker property. Define a continuous flow on W by

$$\mathbb{R} \times W \to W, \quad (\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1}\, \alpha\, u_1, \dots, |\alpha|^{1/p_m - 1}\, \alpha\, u_m).$$

Then

$$I(u_\alpha) = |\alpha|\, I(u) \quad \forall \alpha \in \mathbb{R},\ u \in W.$$

This suggests that the appropriate class of eigenvalue problems to study here are of the form

$$(16) \qquad \begin{cases} -\Delta_{p_i} u_i = \lambda\, \dfrac{\partial J}{\partial u_i}(x,u) & \text{in } \Omega \\ \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \dots, m$$

where $J \in C^1(\Omega \times \mathbb{R}^m)$ satisfies

$$(17) \qquad J(x, u_\alpha) = |\alpha|\, J(x,u) \quad \forall \alpha \in \mathbb{R},\ (x,u) \in \Omega \times \mathbb{R}^m.$$

For example,

$$J(x,u) = V(x)\, |u_1|^{r_1} \cdots |u_m|^{r_m}$$

where $r_i \in (1, p_i)$ with $r_1/p_1 + \cdots + r_m/p_m = 1$ and $V \in L^\infty(\Omega)$. Note that (17) implies that if u is an eigenvector associated with λ, then so is u_α for any $\alpha \neq 0$.

The eigenfunctions of problem (16) are the critical points of the functional

$$\Phi_\lambda(u) = I(u) - \lambda\, J(u), \quad u \in W$$

where

$$J(u) = \int_\Omega J(x,u).$$

Let

$$\mathcal{M} = \{u \in W : I(u) = 1\}$$

and suppose that

$$\mathcal{M}^\pm = \{u \in \mathcal{M} : J(u) \gtrless 0\} \neq \varnothing.$$

Then $\mathcal{M} \subset W \backslash \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere in W, $\mathcal{M}^\pm$ are symmetric open submanifolds of $\mathcal{M}$, and the positive and negative eigenvalues are given by the critical values of

$$\Psi^{\pm}(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M}^{\pm},$$

respectively.

Let $\mathcal{F}^{\pm}$ denote the class of symmetric subsets of $\mathcal{M}^{\pm}$, respectively, and set

$$\lambda_k^+ := \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geqslant k}} \sup_{u \in M} \Psi^+(u),$$

$$\lambda_k^- := \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geqslant k}} \inf_{u \in M} \Psi^-(u).$$

We will again show that $\lambda_k^+ \nearrow +\infty$ and $\lambda_k^- \searrow -\infty$ are sequences of positive and negative eigenvalues, respectively, and if $\lambda_k^+ < \lambda_{k+1}^+$ (resp. $\lambda_{k+1}^- < \lambda_k^-$), then

$$i((\Psi^+)^{\lambda_k^+}) = i(\mathcal{M}^+ \backslash (\Psi^+)_{\lambda_{k+1}^+}) = k$$

$$(\text{resp. } i((\Psi^-)_{\lambda_k^-}) = i(\mathcal{M}^- \backslash (\Psi^-)^{\lambda_{k+1}^-}) = k),$$

in particular, if $\lambda_k^+ < \lambda < \lambda_{k+1}^+$ or $\lambda_{k+1}^- < \lambda < \lambda_k^-$, then

$$C^k(\Phi_\lambda, 0) \neq 0.$$

This will allow us to extend our existence and multiplicity theory for a single equation to systems.

For example, suppose $F(x, 0) \equiv 0$, so that the system (15) has the trivial solution $u(x) \equiv 0$. Assume that

$$F(x, u) = \lambda\, J(x, u) + G(x, u)$$

where λ is not an eigenvalue of (16) and

$$|G(x, u)| \leqslant C \sum_{i=1}^{m} |u_i|^{s_i} \quad \forall (x, u) \in \Omega \times \mathbb{R}^m$$

for some $s_i \in (p_i, p_i^*)$. Further assume the following superlinearity condition: there are $\mu_i > p_i$, $i = 1, \ldots, m$ such that $\displaystyle\sum_{i=1}^{m} \left(\frac{1}{p_i} - \frac{1}{\mu_i}\right) u_i \, \frac{\partial F}{\partial u_i}$ is bounded from below and

$$0 < F(x, u) \leqslant \sum_{i=1}^{m} \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x, u) \quad \forall x \in \Omega, \ |u| \text{ large}.$$

We will obtain a nontrivial solution of (15) under these assumptions in Sections 10.2 and 10.3.

All this machinery can be adapted to many other p-Laplacian like operators as well. Therefore we will develop our theory in an abstract operator setting that includes many of them as special cases.