Introduction

Motivation, subject and method. Stationary phenomena in structural mechanics and physics can be frequently modelled by solutions of elliptic equations in domains with edges and vertices on the boundary. It is known that standard facts of classical elliptic theory (normal solvability, smoothness of solutions etc.) fail for these domains, because the solutions acquire singularities even if the coefficients and other function data are regular. This stimulated the development of a theory of elliptic boundary value problems in polyhedral domains during the last 40 years. A variant of such a theory is presented in this book together with applications to particular boundary value problems of mathematical physics. The results obtained belong to the following four areas:

- Pointwise estimates for the Green and Poisson kernels
- Solvability in weighted and nonweighted Sobolev spaces
- Solvability in weighted and nonweighted Hölder spaces
- Miranda-Agmon type maximum principles

The inclusion of the L_p and Hölder scales is useful for applications to nonlinear problems and one of them, a mixed boundary value problem for the Navier-Stokes system in a polyhedral domain, is treated in detail in the third part of the book.

Dealing with polyhedral domains, one is able to study derivatives of solutions of arbitrarily high order. The weights in the norms of the above mentioned function spaces are products of powers of distances from a point to the vertices and edges. The use of such weights is natural because it enables one to control the singularities of solutions and their derivatives in a very efficient way.

In short, the approach to boundary value problems employed here is as follows. We pass subsequently from simpler geometrical configurations to more complicated ones, beginning with dihedral angles, then turning to polyhedral cones, and ending up with domains diffeomorphic to polyhedra. We systematically start with a variational statement of the problem and investigate the dependence of regularity of its weak solutions on the regularity of the data using only L_2 Sobolev type norms. This is attained by more or less standard techniques based on localization and the use of the Mellin and Fourier transforms. With sharp L_2 estimates at hand, we evaluate the kernels of the inverse integral operators of boundary value problems, so that the majorants depend explicitly on the position of the arguments of these kernels with respect to edges and vertices. This information about Green's and Poisson's kernels is a powerful tool leading to the aforementioned L_p and C^α estimates for derivatives of the solutions, both local and global.

Although the approach just described is applicable to quite general elliptic boundary value problems, we implement it exclusively for the Dirichlet, Neumann and certain mixed boundary operators. This restriction is made in order to state solvability conditions explicitly, without referring to requirements of triviality of kernels and cokernels of auxiliary operators (operator Fourier-Mellin symbols) arising inevitably in the treatment of arbitrary boundary value problems.

Illustrative examples. As a rule, we demonstrate applications of general results to special geometric configurations. To give an idea of the level of understanding which the analytic machinery developed in the book allows to achieve, we quote Section 8.3, where the Neumann problem for the equation

$$-\Delta u = f$$

with zero Neumann data on the boundary of a convex bounded three-dimensional polyhedron $\mathcal{G}$ is considered. We show that, for every $p \geq 2$, this problem has a unique (up to a constant term) solution in the Sobolev space $W^{1,p}(\mathcal{G})$ provided f is a distribution in the dual space $(W^{1,p'}(\mathcal{G}))^*$ orthogonal to 1.

Another typical result is the following assertion on the regularity of solutions to the above Neumann problem in a convex polyhedron. Let $\{\mathcal{O}\}$ be the collection of all vertices and let $\{U_{\mathcal{O}}\}$ be an open finite covering of $\overline{\mathcal{G}}$ such that $\mathcal{O}$ is the only vertex in $U_{\mathcal{O}}$. Let also $\{E\}$ be the collection of all edges and let α_E denote the opening of the dihedral angle with edge E, $0 < \alpha_E < \pi$. The notation $r_E(x)$ stands for the distance between $x \in U_{\mathcal{O}}$ and the edge E such that $\mathcal{O} \in \overline{E}$.

With every vertex $\mathcal{O}$ and edge E we associate real numbers $\beta_{\mathcal{O}}$ and δ_E, and we introduce the weighted L_p-norm

$$\|v\|_{L_p(\mathcal{G};\{\beta_{\mathcal{O}}\},\{\delta_E\})} := \Big(\sum_{\{\mathcal{O}\}} \int_{U_{\mathcal{O}}} |x-\mathcal{O}|^{p\beta_{\mathcal{O}}} \prod_{\{E:\mathcal{O}\in\overline{E}\}} \Big(\frac{r_E(x)}{|x-\mathcal{O}|}\Big)^{p\delta_E} |v(x)|^p dx\Big)^{1/p},$$

where $1 < p < \infty$. Under the conditions

$$\begin{aligned} 3/p' &> \beta_{\mathcal{O}} > -2 + 3/p', \\ 2/p' &> \delta_E > -\min\{2,\, \pi/\alpha_E\} + 2/p', \end{aligned}$$

the inclusion of the function f in $L_p(\mathcal{G};\{\beta_{\mathcal{O}}\},\{\delta_E\})$ implies the unique solvability of the Neumann problem in the class of functions with all derivatives of the second order belonging to $L_p(\Omega;\{\beta_{\mathcal{O}}\},\{\delta_E\})$. An important particular case when all $\beta_{\mathcal{O}}$ and δ_E vanish, i.e. when we deal with a standard Sobolev space $W^{2,p}(\Omega)$, is also included here. To be more precise, if

$$1 < p < \min\Big\{3, \frac{2\alpha_E}{(2\alpha_E - \pi)_+}\Big\} \tag{1}$$

for all edges E, then the inverse operator of the Neumann problem:

$$L_p(\mathcal{G}) \ni f \to u \in W^{2,p}(\mathcal{G})$$

is continuous whatever the convex polyhedron $\mathcal{G} \subset \mathbb{R}^3$ is. The bounds for p in (1) are sharp for the class of all convex polyhedra.

A deeper illustrative example can be found in Section 11.3, where the Navier-Stokes system

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \qquad \nabla\cdot u = 0$$

with the zero Dirichlet condition is considered in the complements of five regular Plato's polyhedra. We prove in particular that the variational solution (u,p) belongs to the Cartesian product $W^{2,s}\times W^{1,s}$ of Sobolev spaces in a neighborhood of all vertices and edges provided $f\in L_s$, $s>1$, where the best possible values for s are as follows (with all digits shown correct):

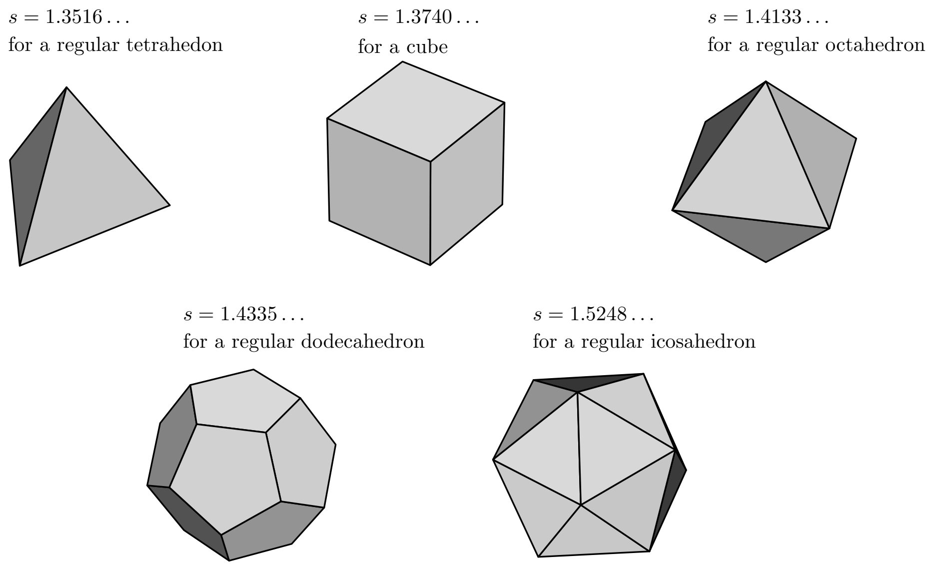

FIGURE 1. The $W^{2,s}$-regularity for solutions of the Navier-Stokes system outside of Plato's polyhedra

Monographic literature. The classical boundary value problems for harmonic functions in nonsmooth domains attracted attention of some of the best mathematicians such as POINCARÉ, PLEMELJ, RADON, CARLEMAN, LEBESGUE, WIENER at the end of the 19th and beginning of the 20th century. Afterwards, a gap followed, until the development of functional analytic methods led since the 1960's to a considerable progress in the study of special and general elliptic differential equations in domains with nonsmooth boundaries. Long lists of references related to domains with isolated point singularities can be found, for example, in [**84**] and [**85**]. Bibliographical information concerning piecewise smooth domains with boundary singularities of positive dimension and other classes of nonsmooth domains is collected at the end of the present volume.

Different aspects of the theory of elliptic boundary value problems in domains with vertices and edges were discussed in several monographs (GRISVARD [**59, 60**], REMPEL, SCHULZE [**175**], DAUGE [**30**], MAZ'YA, NAZAROV, PLAMENEVSKIĬ [**110**], SCHULZE [**186**], NICAISE [**162**], NAZAROV, PLAMENEVSKIĬ [**160**], KOZLOV, Maz'ya, Rossmann [**84, 85**], Borsuk, Kondrat'ev [**14**], Nazaikinski, Savin, Schulze, Sternin [**154**] *et al.*)

The present volume which is mostly based on our papers [**133**]–[**140**], has no essential intersections with the just mentioned books. None the less, there are close relations of this book with [**84**] and [**85**]. In fact, the monographs [**84**] and [**85**] can be considered as the first two volumes of a trilogy, of which the present book is the third volume.

In the first part [**84**] of the trilogy, a systematic exposition of the theory of general elliptic boundary value problems for domains with isolated singularities is given. This theory becomes satisfactory only being completed by information on the spectrum of model operator pencils (Mellin operator symbols) corresponding to the original problem. Obtaining this information is the goal of the second part [**85**] of the trilogy.

The theme of the present volume - analysis of boundary value problems for domains with boundary singularities of positive dimension - depends crucially on results in [**84**] concerning domains with isolated singularities. These results form the induction base when considering multi-dimensional geometric singularities. Furthermore, the systematic use in the present book of explicit estimates of eigenvalues of Mellin symbols obtained in [**85**] leads to the best known and sometimes optimal regularity results for solutions of boundary value problems in polyhedral domains. In the sequel, we only formulate auxiliary results proved in [**84**] and [**85**] but this does not prevent the text from being self-contained.

It is necessary to say that the large theme of asymptotic representations for solutions near edges and vertices is not in the scope of the present book. Its comprehensive exposition would require another monograph.

Structure of the book. The book consists of three parts. Part 1 is dedicated to the Dirichlet problem for strongly elliptic systems of arbitrary order. In Part 2 we are concerned with mixed boundary value problems for a class of second order elliptic systems which contains, for example, the system of linear elasticity. Mixed problems for the Stokes and Navier-Stokes systems are treated in Part 3.

There are five chapters in Part 1. Chapter 1 is auxiliary. It contains formulations of well-known properties of elliptic boundary value problems in domains with smooth boundaries and in infinite angles and cones. Chapters 2–4 are dedicated respectively to the Dirichlet problem in a dihedron, a cone with edges and a bounded polyhedral domain. Here we justify the solvability of the problem in weighted Sobolev and Hölder spaces and obtain regularity results for the variational solution. In Chapter 5 we obtain maximum modulus estimates for the solutions and their derivatives in a three-dimensional polyhedral domain.

In the second part of the book consisting of Chapters 6–8 we treat mixed problems for second order systems with Dirichlet and Neumann conditions prescribed on different faces of the boundary. The so-called homogeneous weighted norms employed in the case of the Dirichlet problem prove to be insufficient for the Neumann problem. This difficulty is overcome by using nonhomogeneous weighted norms.

The third part includes Chapters 9–11. Here we investigate mixed boundary value problems for the Stokes and Navier-Stokes systems with four boundary conditions given in arbitrary combinations on different faces of the polyhedron. It is again one of our main goals to study the regularity of the variational solutions. We conclude this part with maximum modulus estimates, first for solutions of the linear system and later of the nonlinear system of hydrodynamics in bounded polyhedral domains.

Readership. This volume is addressed to mathematicians who work in partial differential equations, spectral analysis, asymptotic methods and their applications. It will be also of interest for those who work in numerical analysis, mathematical elasticity and hydrodynamics. A prerequisite for this book is advanced calculus. The reader should be familiar with basic fasts of functional analysis and the theory of partial differential equations.

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