Contents of Part III of Volume Two

Well, you know, we're doing what we can.

– From "Revolution" by The Beatles

Chapter 17. Perelman's entropy $\mathcal{W}$ leads to the μ-invariant. We discuss qualitative properties of the μ-invariant such as lower and upper bounds and we give a proof of the fact that $\lim_{\tau\to 0^+}\mu(g,\tau)=0$. We also discuss applications of the μ-invariant monotonicity formula. This includes the recent classification by Z.-L. Zhang of compact finite time singularity models as shrinking gradient Ricci solitons. We revisit the proof of the existence of a smooth minimizer for $\mathcal{W}$, providing more details than in Part I, and we also show that when the isometry group acts transitively, the minimizer is not unique for sufficiently small τ. Related to renormalization group considerations, some low-loop calculations are presented.

Chapter 18. We discuss some tools used in the study of the Ricci flow including the changing distances estimate for solutions of Ricci flow, point picking methods, rough monotonicity of the size of necks in complete noncompact manifolds with positive sectional curvature, and a local form of the weakened no local collapsing theorem.

Chapter 19. With the goal of understanding compactness in higher dimensions, we introduce the notion of 'κ-solution with Harnack', which is a variant of Perelman's notion of κ-solution. In dimensions 2 and 3 we show that κ-solutions with Harnack must have bounded curvature. We also discuss the construction of Perelman's rotationally symmetric ancient solution on $\mathcal{S}^n$, the result that κ-solutions with Harnack must have bounded curvature, the existence of an asymptotic shrinker in a κ-solution (correcting a gap (no pun intended) in Part I), and the κ-gap theorem.

Chapter 20. We show that noncompact κ-solutions have asymptotic scalar curvature ratio ASCR $=\infty$ and asymptotic volume ratio AVR $=0$; the latter result does not require the κ-noncollapsed at all scales assumption. We show that solutions which are almost ancient and have bounded nonnegative curvature operator are collapsed at large scales and we obtain a curvature estimate in noncollapsed balls. We prove that the collection of κ-solutions with Harnack is compact modulo scaling. In dimension 3 this is equivalent to Perelman's compactness theorem and implies scaled derivative of curvature estimates.

Chapter 21. We discuss Perelman's pseudolocality theorem. Assuming an initial ball with scalar curvature bounded from below and which is almost Euclidean isoperimetrically, we obtain a curvature estimate in a smaller ball; this estimate gets worse as time approaches the initial time. One may consider this as sort of a pseudolocalization of the curvature doubling time estimate. One of the ideas in the proof is that one can localize the entropy monotonicity formula by multiplying the integrand by a suitable time-dependent cutoff function. In the setting of a proof by contradiction, a main idea is to use point picking methods to locate an infinite sequence of 'good' high curvature points and to study a local entropy in their neighborhoods via Perelman's Harnack-type estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow.

Chapter 22. We discuss tools used in the proof of the pseudolocality theorem such as the point picking 'Claims 1 and 2', convergence of heat kernels under Cheeger–Gromov convergence, a uniform negative upper bound for the local entropies centered at the well-chosen bad points at time zero, and a sharp form of the logarithmic Sobolev inequality related to the isoperimetric inequality.

Chapter 23. We discuss existence and asymptotics for heat kernels with respect to *static* metrics. We follow the parametrix method of Levi and its Riemannian adaptation by Minakshisundaram and Pleijel. Starting with a good approximation to the heat kernel, we prove the existence of the heat kernel by establishing the convergence of the 'convolution series'. With this construction we compute some low-order asymptotics for the heat kernel.

Chapter 24. We adapt the methods of the previous chapter to study the existence and asymptotics for heat kernels with respect to *evolving* metrics. We consider aspects of the adjoint heat kernel for evolving metrics related to §9.6 of Perelman's paper [**152**]. We also discuss the existence of Dirichlet heat kernels on compact manifolds with boundary and heat kernels on noncompact manifolds with respect to evolving metrics.

Chapter 25. We discuss estimates for solutions to the heat equation with respect to evolving metrics including the parabolic mean value property for solutions to heat equations and the Li–Yau differential Harnack estimate for positive solutions to heat equations.

Chapter 26. Applying the estimates of the previous chapter, we discuss estimates for heat kernels with respect to evolving metrics including upper and lower bounds and the space-time mean value property. We also discuss the existence of distance-type functions on complete noncompact Riemannian manifolds with bounded gradient and Laplacian.

Appendix G. With Perelman's work, the space-time of a solution of the Ricci flow is given a quasi-length space structure. This geometric structure is foundational in the understanding of singularity formation under the

Ricci flow. We discuss notions of (quasi-)metric and (quasi-)length spaces, Gromov–Hausdorff convergence, and Aleksandrov spaces.

Appendix H. We discuss convex analysis on Euclidean spaces and on locally convex subsets in Riemannian manifolds.

Appendix I. We discuss the points at infinity for nonnegatively curved manifolds, the Sharafutdinov retraction theorem, and some consequences.

Appendix J. We provide solutions to some of the exercises in the book.

CHAPTER 17

Entropy, μ-invariant, and Finite Time Singularities

I'll tip my hat to the new constitution.

– From "Won't Get Fooled Again" by The Who

Monotonicity formulas may be used to understand the qualitative behavior of solutions of the Ricci flow. As an example, in this chapter we consider the μ-invariant monotonicity formula and its applications to singularity analysis.

In §1 we discuss lower and upper bounds for the μ-invariant. As an application, there is a lower bound for the volume of solutions $g(t)$ of the Ricci flow with nonpositive λ-invariant. This implies that the corresponding finite time singularity models are noncompact in this case. As a further application, we discuss the classification of *compact* finite time singularity models as shrinking gradient Ricci solitons with no assumption on the sign of the λ-invariant.

In §2 we prove the fact that $\lim_{\tau \to 0^+} \mu(g, \tau) = 0$. This result was stated as Lemma 6.33(ii) in Part I but was not proved there. As an application we show that, for a closed Riemannian manifold on which the isometry group acts transitively, the minimizer for $\mathcal{W}$ is *not* unique for sufficiently small τ.

In §3 we revisit the proof of the existence of a smooth minimizer for $\mathcal{W}$ while completing some additional details not discussed earlier in this book series.

One may hope to extend Perelman's energy and entropy monotonicity formulas. In §4 we discuss formulas relating Perelman's energy $\mathcal{F}$, the linear trace Harnack quadratic, Hamilton's matrix quadratic, the 2-tensor $R_i^{k\ell m} R_{jk\ell m}$, and the functional $\int_{\mathcal{M}} |\mathrm{Rm}|^2 e^{-f} d\mu$.

Throughout this chapter we assume that $\mathcal{M}^n$ is a closed manifold unless otherwise indicated.

1. Compact finite time singularity models are shrinkers

In this section and the next we discuss properties of the μ-invariant of a metric g at a scale $\tau > 0$. This invariant is the infimum of Perelman's entropy functional $\mathcal{W}(g, f, \tau)$, under a constraint, considered in Chapter 6 of Part I. In this section we present the results and proofs of Z.-L. Zhang on bounds for the μ-invariant and their geometric application to the classification of compact finite time singularity models.

1.1. Perelman's entropy and its associated invariants.

In this subsection we recall some basic facts regarding Perelman's energy and entropy functionals, including the logarithmic Sobolev inequality, which shall be used in this chapter.

1.1.1. *Energy, entropy, and their minimizers.*

Let g be a C^∞ Riemannian metric on a closed manifold $\mathcal{M}^n$, let $f : \mathcal{M} \to \mathbb{R}$ be a C^∞ function, and let $\tau \in (0, \infty)$. Perelman's energy functional is defined by (see (5.1) in Part I)

$$\mathcal{F}(g, f) \doteqdot \int_{\mathcal{M}} \left(R + |\nabla f|^2 \right) e^{-f} d\mu = \int_{\mathcal{M}} (R + \Delta f) e^{-f} d\mu. \tag{17.1}$$

We may rewrite $\mathcal{F}$ as

$$\mathcal{F}(g, f) = \int_{\mathcal{M}} \left(Rv^2 + 4 |\nabla v|^2 \right) d\mu \doteqdot \mathcal{G}(g, v), \tag{17.2}$$

where $v \doteqdot e^{-f/2}$. The associated λ-invariant is (see (5.45) in Part I)

$$\begin{aligned} \lambda(g) &\doteqdot \inf \left\{ \mathcal{F}(g, f) : f \in C^\infty(\mathcal{M}), \int_{\mathcal{M}} e^{-f} d\mu = 1 \right\} \\ &= \inf \left\{ \mathcal{G}(g, v) : \int_{\mathcal{M}} v^2 d\mu = 1 \right\}. \end{aligned} \tag{17.3}$$

There exists a unique C^∞ minimizer f_0 of $\mathcal{F}(g, f)$ subject to the constraint $\int_{\mathcal{M}} e^{-f} d\mu = 1$. Moreover, f_0 satisfies the Euler–Lagrange equation (see Lemma 5.23 in Part I)

$$2\Delta f_0 - |\nabla f_0|^2 + R = \lambda(g). \tag{17.4}$$

Perelman's entropy functional $\mathcal{W}$ is given by (see (6.1) in Part I)

$$\mathcal{W}(g, f, \tau) \doteqdot \int_{\mathcal{M}} \left(\tau \left(R + |\nabla f|^2 \right) + f - n \right) u \, d\mu, \tag{17.5}$$

where

$$u \doteqdot (4\pi\tau)^{-n/2} e^{-f} \doteqdot w^2. \tag{17.6}$$

We may rewrite $\mathcal{W}$ as

$$\begin{aligned} \mathcal{W}(g, f, \tau) &= \int_{\mathcal{M}} \left(\begin{array}{c} \tau \left(Rw^2 + 4 |\nabla w|^2 \right) \\ - \left(\log \left(w^2 \right) + \frac{n}{2} \log(4\pi\tau) + n \right) w^2 \end{array} \right) d\mu \\ &\doteqdot \mathcal{K}(g, w, \tau). \end{aligned} \tag{17.7}$$

Recall that the associated μ-invariant is defined by (see (6.49) in Part I)

$$\mu(g, \tau) \doteqdot \inf \left\{ \mathcal{W}(g, f, \tau) : f \in C^\infty(\mathcal{M}), \int_{\mathcal{M}} u \, d\mu = 1 \right\} \tag{17.8}$$

$$= \inf \left\{ \mathcal{K}(g, w, \tau) : \int_{\mathcal{M}} w^2 d\mu = 1 \right\}. \tag{17.9}$$

The ν-invariant is

$$\nu(g) \doteqdot \inf\left\{\mu(g,\tau) : \tau \in \mathbb{R}^+\right\}. \tag{17.10}$$

1.1.2. *Monotonicity of Perelman's entropy.*

Under the coupled Ricci flow system

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{17.11a}$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \tag{17.11b}$$

$$\frac{d\tau}{dt} = -1, \tag{17.11c}$$

we have

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = \int_{\mathcal{M}} 2\tau \left|\operatorname{Rc} + \nabla\nabla f - \frac{1}{2\tau} g\right|^2 u\, d\mu \geq 0. \tag{17.12}$$

Indeed, let

$$v \doteqdot \left[\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right] u,$$

which satisfies $\mathcal{W}(g, f, \tau) = \int_{\mathcal{M}} v\, d\mu$. We have (see Lemma 6.8 in Part I)

$$\Box^* v = -2\tau \left|\operatorname{Rc} + \nabla\nabla f - \frac{1}{2\tau} g\right|^2 u, \tag{17.13}$$

where $\Box^* = -\frac{\partial}{\partial t} - \Delta + R$ is the adjoint heat operator. This implies (17.12) since

$$\frac{d\mathcal{W}}{dt} = \int_{\mathcal{M}} \left(\frac{\partial}{\partial t} - R\right) v\, d\mu = -\int_{\mathcal{M}} \Box^* v\, d\mu.$$

We remark that formula (17.13) is central to the proof of Perelman's differential Harnack estimate (see Chapter 16 in Part II).

The functional $\mathcal{W}(g, \cdot, \tau)$ is bounded from below under the constraint $\int_{\mathcal{M}} u\, d\mu = 1$ and there exists a smooth minimizer f_τ, which satisfies the equation

$$\tau\left(2\Delta f_\tau - |\nabla f_\tau|^2 + R\right) + f_\tau - n = \mu(g, \tau) \tag{17.14}$$

(see Proposition 17.24 below). In terms of $w_\tau = (4\pi\tau)^{-n/4} e^{-f_\tau/2}$, this is

$$\tau\left(-4\Delta w_\tau + R w_\tau\right) - w_\tau \log\left(w_\tau^2\right) - \left(\frac{n}{2}\log(4\pi\tau) + n\right) w_\tau = \mu(g, \tau)\, w_\tau. \tag{17.15}$$

1.1.3. *Logarithmic Sobolev inequality.*

The logarithmic Sobolev inequality is intimately tied to Perelman's entropy functional due to their related forms. In our discussion of this, we shall assume $n \geq 3$; we leave it to the reader to verify that these results carry over to the case $n = 2$ with only minor adjustments. The following is given as Lemma 6.36 in Part I.

LEMMA 17.1 (Logarithmic Sobolev inequality). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. For any $a > 0$, there exists a constant $C(a, g)$ such that if $\varphi > 0$ satisfies $\int_{\mathcal{M}} \varphi^2 d\mu_g = 1$, then*

$$\int_{\mathcal{M}} \varphi^2 \log \varphi d\mu_g \leq a \int_{\mathcal{M}} |\nabla \varphi|_g^2 \, d\mu_g + C(a, g), \tag{17.16}$$

where[1]

$$C(a, g) = a \operatorname{Vol}(g)^{-2/n} + \frac{n^2}{4ae^2 C_s(\mathcal{M}, g)}. \tag{17.17}$$

Here $C_s(\mathcal{M}, g)$ denotes the L^2 **Sobolev constant**, which we define to be the best (largest) positive constant such that (see Lemma 2 in [**114**])

$$\int_{\mathcal{M}} |\nabla \varphi|_g^2 \, d\mu_g \geq C_s(\mathcal{M}, g) \left(\int_{\mathcal{M}} \varphi^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{n}} - \operatorname{Vol}(g)^{-\frac{2}{n}} \int_{\mathcal{M}} \varphi^2 d\mu_g \tag{17.18}$$

for any C^∞ function φ on $\mathcal{M}$.

We have the following elementary properties.

LEMMA 17.2 (Sobolev constants under scaling the metric). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold.*

(i) *The L^2 Sobolev constant has the property that for any $\lambda > 0$,*

$$C_s(\mathcal{M}, \lambda^2 g) = C_s(\mathcal{M}, g).$$

(ii) *The logarithmic Sobolev constant has the property that for any $a > 0$ and $\lambda \geq 1$,*

$$C(a, \lambda^2 g) \leq C(a, g).$$

PROOF. **(i)** Let $\tilde{g} = \lambda^2 g$ and $\tilde{\varphi} = \lambda^{-n/2} \varphi$. Then $d\mu_{\tilde{g}} = \lambda^n d\mu_g$, $\tilde{\varphi}^2 d\mu_{\tilde{g}} = \varphi^2 d\mu_g$, and

$$\begin{aligned}
\int_{\mathcal{M}} \left| \tilde{\nabla} \tilde{\varphi} \right|_{\tilde{g}}^2 d\mu_{\tilde{g}} &= \lambda^{-2} \int_{\mathcal{M}} |\nabla \varphi|_g^2 \, d\mu_g \\
&\geq \lambda^{-2} C_s(\mathcal{M}, g) \left(\int_{\mathcal{M}} \varphi^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{n}} - \lambda^{-2} \operatorname{Vol}(g)^{-\frac{2}{n}} \int_{\mathcal{M}} \varphi^2 d\mu_g \\
&= C_s(\mathcal{M}, g) \left(\int_{\mathcal{M}} \tilde{\varphi}^{\frac{2n}{n-2}} d\mu_{\tilde{g}} \right)^{\frac{n-2}{n}} - \operatorname{Vol}(\tilde{g})^{-\frac{2}{n}} \int_{\mathcal{M}} \tilde{\varphi}^2 d\mu_{\tilde{g}}.
\end{aligned}$$

From this we can easily deduce that

$$C_s(\mathcal{M}, \lambda^2 g) = C_s(\mathcal{M}, g), \tag{17.19}$$

i.e., the L^2 Sobolev constant is invariant under scaling the metric.

[1] In (17.17) we correct formula (6.67) in Part I, where the $n^2/4$ factor originally appeared in the denominator; after (6.66) it should read $c_n = \frac{e}{2n}$.

(ii) Hence

$$C\left(a,\lambda^2 g\right) = a\,\mathrm{Vol}\left(\lambda^2 g\right)^{-2/n} + \frac{n^2}{4ae^2 C_s\left(\mathcal{M},\lambda^2 g\right)}$$
$$\leq C\left(a,g\right)$$

if $\lambda \geq 1$. □

1.2. Lower and upper bounds for the μ-invariant.

In this subsection we prove lower and upper bounds for the μ-invariant in terms of τ and certain geometric invariants of $(\mathcal{M}^n, g)$.

1.2.1. *Upper bounds for μ.*

Taking $f = c$ to be constant in (17.5), we obtain the following elementary upper bound for μ:

$$\mu(g,\tau) \leq \tau R_{\mathrm{avg}} + \log \mathrm{Vol}\left(g\right) - \frac{n}{2}\log(4\pi\tau) - n, \tag{17.20}$$

where R_{avg} is the average scalar curvature of g.

On the other hand, we may choose f in terms of the minimizer of $\mathcal{F}$. The following is inequality (2) in Lemma 2.1 of [**197**].

Lemma 17.3 (Upper bound for μ in terms of λ, Vol, τ, and n). *For any closed Riemannian manifold $(\mathcal{M}^n, g)$ and $\tau > 0$*

$$\mu(g,\tau) \leq \tau\lambda\left(g\right) + \frac{1}{e}\mathrm{Vol}\left(g\right) - \frac{n}{2}\log\left(4\pi\tau\right) - n. \tag{17.21}$$

Proof. For any w with $\int_{\mathcal{M}} w^2 d\mu = 1$, by (17.7) and $\log\left(w^2\right)w^2 \geq -\frac{1}{e}$, we have

$$\mathcal{W}\left(g,f,\tau\right) = \tau\int_{\mathcal{M}}\left(Rw^2 + 4\left|\nabla w\right|^2\right)d\mu - \int_{\mathcal{M}}\log\left(w^2\right)w^2 d\mu - \frac{n}{2}\log(4\pi\tau) - n$$
$$\leq \tau\mathcal{G}\left(g,w\right) + \frac{1}{e}\mathrm{Vol}\left(g\right) - \frac{n}{2}\log(4\pi\tau) - n,$$

where $\mathcal{G}$ is defined by (17.2). Choosing $w_0 \doteqdot (4\pi\tau)^{-n/4}e^{-f_0/2}$ to be the minimizer of the functional $\mathcal{G}\left(g,w\right)$, we conclude that

$$\mu(g,\tau) \leq \mathcal{W}\left(g,f_0,\tau\right) \leq \tau\lambda\left(g\right) + \frac{1}{e}\mathrm{Vol}\left(g\right) - \frac{n}{2}\log(4\pi\tau) - n.$$

□

When $\lambda\left(g\right) \leq 0$, one may apply the scaling property of μ to obtain the following, which is Corollary 2.3 of [**197**].

Corollary 17.4 (Upper bound for μ when $\lambda \leq 0$). *If $\lambda\left(g\right) \leq 0$, then*

$$\mu\left(g,\tau\right) \leq \log\mathrm{Vol}\left(g\right) - \frac{n}{2}\log\left(4\pi\tau\right) - n + \frac{1}{e}. \tag{17.22}$$

PROOF. By the scaling invariance of μ (see property (iii) on p. 236 of Part I) and (17.21), we have for any $c \in (0, \infty)$,

$$\mu(g, \tau) = \mu(cg, c\tau) \leq c\tau\lambda(cg) + \frac{1}{e}\operatorname{Vol}(cg) - \frac{n}{2}\log(4\pi c\tau) - n.$$

Taking $c = \operatorname{Vol}(g)^{-2/n}$, we obtain

$$\mu(g, \tau) \leq c\tau\lambda(cg) + \frac{1}{e} - \frac{n}{2}\log(4\pi\tau) + \log\operatorname{Vol}(g) - n$$

and (17.22) follows from $\lambda(cg) = c^{-1}\lambda(g) \leq 0$. □

As a consequence of Corollary 17.4 we have that if $\lambda(g) \leq 0$, then $\lim_{\tau\to\infty}\mu(g, \tau) = -\infty$. This improves Exercise 6.32 in Part I.

1.2.2. *Lower bounds for μ.*

Now we consider lower bounds for μ using the logarithmic Sobolev inequality; we have the following.

LEMMA 17.5 (Lower bound for μ). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold and let $\tau > 0$. We have*

$$\mu(g, \tau) \geq \tau R_{\min}(g) - 2C(2\tau, g) - \frac{n}{2}\log(4\pi\tau) - n, \tag{17.23}$$

where $R_{\min}(g) \doteqdot \min_{x\in\mathcal{M}} R(x)$ and the constant $C(2\tau, g)$ is given by (17.17).

PROOF. By Lemma 17.1, for any $w \geq 0$ with $\int_{\mathcal{M}} w^2 d\mu_g = 1$ we have

$$\int_{\mathcal{M}} w^2 \log w\, d\mu \leq 2\tau\int_{\mathcal{M}} |\nabla w|^2 d\mu + C(2\tau, g).$$

Substituting this into (17.7), we obtain

$$\begin{aligned}\mathcal{K}(g, w, \tau) &= \int_{\mathcal{M}}\left(\tau R w^2 + 4\tau|\nabla w|^2\right)d\mu - \frac{n}{2}\log(4\pi\tau) - n \\ &\quad - 2\int_{\mathcal{M}} w^2\log w\, d\mu \\ &\geq \tau R_{\min}(g) - 2C(2\tau, g) - \frac{n}{2}\log(4\pi\tau) - n\end{aligned}$$

since $\int_{\mathcal{M}}\tau R w^2 d\mu \geq \tau R_{\min}(g)$. Taking the infimum over w, we obtain the desired inequality (17.23). □

Let $C_s(g)$ denote the L^2 Sobolev constant: the smallest number such that

$$\|\varphi\|_{L^{\frac{2n}{n-2}}(g)} \leq C_s(g)\|\varphi\|_{W^{1,2}(g)} \doteqdot C_s(g)\left(\int_{\mathcal{M}}\left(|\nabla\varphi|^2 + \varphi^2\right)d\mu\right)^{1/2} \tag{17.24}$$

for all $\varphi \in C^\infty(\mathcal{M})$. By (17.18), we have

$$C_s(g)^2 \leq \frac{1}{C_s(\mathcal{M}, g)}\max\left\{\operatorname{Vol}(g)^{-\frac{2}{n}}, 1\right\}.$$

The following is inequality (3) in Lemma 2.1 of [**197**] (compare with (6.63) in Part I); again we assume $n \geq 3$ for simplicity.

LEMMA 17.6 (Lower bound for the μ-invariant). *If* $\tau \geq \frac{n}{8}$, *then*

$$\mu(g,\tau) \geq \left(\tau - \frac{n}{8}\right) \lambda(g) - \frac{n}{2}\log(4\pi\tau) - n - n\log C_s(g) + \frac{n}{8} R_{\min}(g). \tag{17.25}$$

REMARK 17.7.

(1) Note that estimate (17.25) is not true for sufficiently small τ. One reason is because the limit as $\tau \to 0_+$ of the RHS of (17.25) is equal to $+\infty$, contradicting (17.47) below.
(2) We see from (17.25) that if $\lambda(g) > 0$, then $\lim_{\tau\to\infty} \mu(g,\tau) = \infty$; see also Lemma 6.30 in Part I.

PROOF. We consider the term $-\int_{\mathcal{M}} \log(w^2) w^2 d\mu$ on the RHS of (17.7). For any w such that $\displaystyle\int_{\mathcal{M}} w^2 d\mu = 1$, Jensen's inequality says that if $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $f \in L^1\left(w^{4/n} g\right)$, then

$$\int_{\mathcal{M}} (\phi \circ f)\, w^2 d\mu \geq \phi\left(\int_{\mathcal{M}} f\, w^2 d\mu\right).$$

In particular, taking $f = w^{\frac{4}{n-2}}$ and $\phi(u) = -\log u$, we have

$$\begin{aligned}
-\int_{\mathcal{M}} w^2 \log\left(w^2\right) d\mu &= -\frac{n-2}{2}\int_{\mathcal{M}} \log\left(w^{\frac{4}{n-2}}\right) w^2 d\mu \\
&\geq -\frac{n-2}{2}\log\left(\int_{\mathcal{M}} w^{\frac{2n}{n-2}} d\mu\right) \\
&= -n\log\|w\|_{L^{\frac{2n}{n-2}}(g)} \\
&\geq -n\log\left(C_s(g)\,\|w\|_{W^{1,2}(g)}\right) \\
&= -n\log C_s(g) - \frac{n}{2}\log\left(1 + \int_{\mathcal{M}} |\nabla w|^2 d\mu\right),
\end{aligned} \tag{17.26}$$

where we used the L^2 Sobolev inequality (17.24).

Since $\log(1+x) \leq x$ for $x \geq 0$, we have

$$-\int_{\mathcal{M}} w^2 \log\left(w^2\right) d\mu \geq -n\log C_s(g) - \frac{n}{2}\int_{\mathcal{M}} |\nabla w|^2 d\mu. \tag{17.27}$$

Combining this with (17.7), we have

$$\begin{aligned}
\mathcal{W}(g,f,\tau) \geq \tau\mathcal{G}(g,w) - \frac{n}{2}\log(4\pi\tau) - n - n\log C_s(g) \\
- \frac{n}{2}\int_{\mathcal{M}} |\nabla w|^2 d\mu.
\end{aligned}$$

On the other hand,

$$\int_{\mathcal{M}} 4 |\nabla w|^2 \, d\mu \leq \int_{\mathcal{M}} \left(4 |\nabla w|^2 + (R - R_{\min}(g)) w^2 \right) d\mu$$
$$= \mathcal{G}(g, w) - R_{\min}(g),$$

so that

$$\mathcal{W}(g, f, \tau) \geq \left(\tau - \frac{n}{8} \right) \mathcal{G}(g, w) - \frac{n}{2} \log(4\pi\tau) - n - n \log C_s(g) + \frac{n}{8} R_{\min}(g).$$

Choosing f_τ to be the constrained minimizer of $\mathcal{W}(g, \cdot, \tau)$, we have

$$\mu(g, \tau) \geq \left(\tau - \frac{n}{8} \right) \mathcal{G}(g, w_\tau) - \frac{n}{2} \log(4\pi\tau) - n - n \log C_s(g) + \frac{n}{8} R_{\min}(g),$$

where $w_\tau \doteqdot (4\pi\tau)^{-n/4} e^{-f_\tau/2}$. If $\tau \geq \frac{n}{8}$, then we obtain (17.25). This completes the proof of the lemma. □

1.3. Volume lower bound for Ricci flow solutions with $\lambda \leq 0$.

As a geometric application of the upper and lower bounds for μ, we have the following, which is Lemma 3.1 in [**197**].

Lemma 17.8 (Lower bound for the volume of a solution when $\lambda \leq 0$). *If $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, is a solution to the Ricci flow on a closed manifold with $\lambda(g(t)) \leq 0$ for all $t \in [0, T)$, then there exists $c_1, c_2 \in (0, \infty)$ depending only on $g(0)$ such that*

$$\operatorname{Vol}(g(t)) \geq c_1 e^{-c_2 t} \tag{17.28}$$

for all $t \in [0, T)$.

Proof. By taking $g = g(0)$ and $\tau = \frac{n}{8} + t$ in (17.25) and by Perelman's μ-invariant monotonicity formula (see Lemma 6.26 in Part I), we have

$$\mu\left(g(t), \frac{n}{8}\right) \geq \mu\left(g(0), \frac{n}{8} + t\right)$$
$$\geq t\lambda(g(0)) - \frac{n}{2} \log\left(4\pi\left(\frac{n}{8} + t\right)\right)$$
$$- n - n \log C_s(g(0)) + \frac{n}{8} R_{\min}(g(0)).$$

On the other hand, since $\lambda(g(t)) \leq 0$, by (17.22) we have

$$\mu\left(g(t), \frac{n}{8}\right) \leq \log \operatorname{Vol}(g(t)) - \frac{n}{2} \log\left(\frac{\pi n}{2}\right) - n + 1.$$

Hence

$$\log \operatorname{Vol}(g(t)) \geq t\lambda(g(0)) - \frac{n}{2} \log\left(4\pi\left(\frac{n}{8} + t\right)\right) + c,$$

where

$$c \doteqdot -n \log C_s(g(0)) + \frac{n}{8} R_{\min}(g(0)) + \frac{n}{2} \log\left(\frac{\pi n}{2}\right) - 1.$$

We conclude that

$$\operatorname{Vol}(g(t)) \geq e^c \left(4\pi\left(\frac{n}{8} + t\right)\right)^{-n/2} e^{t\lambda(g(0))}.$$

The lemma follows easily. □

REMARK 17.9. For an elementary upper bound of the volume of a solution when $\lambda > 0$, see the notes and commentary at the end of this chapter.

Recall that a complete (**finite time**) **singularity model** $(\mathcal{M}^n_\infty, g_\infty(t))$, $t \in (-\infty, 0]$, is obtained from taking the limit of rescalings of a finite time singular solution of the Ricci flow $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, on a closed manifold (see §1 in Chapter 19 in this volume and Chapter 8 of [**45**]).

The volume lower bound has the following consequence for singularity models associated to solutions with $\lambda \leq 0$; this is Corollary 3.2 in [**197**].

LEMMA 17.10 (Finite time singularity models are noncompact when $\lambda \leq 0$). *If $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, where $T < \infty$, is a singular solution to the Ricci flow on a closed manifold with $\lambda(g(t)) \leq 0$ for all $t \in [0, T)$, then any corresponding singularity model is noncompact.*

PROOF. This follows since by (17.28) there exists $c > 0$ such that $\mathrm{Vol}(g(t)) \geq c$ for all $t \in [0, T)$, whereas for any blow-up sequence the dilation factors tend to infinity. More explicitly, suppose that $t_i \nearrow T$ and $p_i \in \mathcal{M}$ are such that $K_i \doteqdot \left|\mathrm{Rm}_{g(t_i)}(p_i)\right| \to \infty$ and suppose that the sequence $(\mathcal{M}^n, g_i(t), p_i)$, where

$$g_i(t) \doteqdot K_i \cdot g\left(t_i + \frac{t}{K_i}\right),$$

converges to a complete ancient solution $(\mathcal{M}^n_\infty, g_\infty(t), p_\infty)$ to the Ricci flow in the sense of C^∞ Cheeger–Gromov convergence. Then

$$\begin{aligned}\lim_{i\to\infty} \mathrm{Vol}(g_i(0)) &= \lim_{i\to\infty} K_i^{n/2}\,\mathrm{Vol}(g(t_i)) \\ &= \infty\end{aligned}$$

since $\mathrm{Vol}(g(t_i)) \geq c > 0$, independent of i.

Now assume $\mathcal{M}_\infty$ is compact. Then

$$\lim_{i\to\infty} \mathrm{Vol}(g_i(0)) = \mathrm{Vol}(g_\infty(0)) < \infty,$$

which is a contradiction. □

REMARK 17.11 (Noncompact singularity models have infinite volume). For any finite time noncompact singularity model $(\mathcal{M}^n_\infty, g_\infty(t))$ with bounded curvature we have $\mathrm{Vol}(g_\infty(t)) = \infty$. This is because for each t there exists $\kappa > 0$ such that

$$\mathrm{Vol}_{g_\infty(t)} B_{g_\infty(t)}(x, 1) \geq \kappa$$

for all $x \in \mathcal{M}_\infty$ (by Perelman's no local collapsing theorem).

Lemma 17.10 implies

COROLLARY 17.12 (Compact singularity model implies $\lambda > 0$). *Given a finite time singular solution $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, of the Ricci flow, if some associated singularity model is compact, then $\lambda(g(t_0)) > 0$ for some $t_0 \in [0, T)$; by the λ-monotonicity formula (see Lemma 5.25 in Part I) we then have $\lambda(g(t)) > 0$ for all $t \in [t_0, T)$.*

On the other hand, it is possible for a finite time singular solution on a closed manifold that $\lambda(g(t)) > 0$ for all $t \in [0, T)$ and that all singularity models are noncompact. Such an example on $\mathcal{S}^n$ is given by Angenent and one of the authors [**7**], where a finite time neckpinch (singularity model is $\mathcal{S}^{n-1} \times \mathbb{R}$) is exhibited for a class of rotationally symmetric solutions with $R(g(t)) > 0$ (which implies $\lambda(g(t)) > 0$).

1.4. Classification of compact finite time singularity models.

In this subsection we discuss the following application of bounds for the μ-invariant to the classification of *compact* finite time singularity models as shrinking gradient Ricci solitons by Z.-L. Zhang (see Theorem 1.1 in [**197**]). In the a priori special case of singularity models of *Type I* singular solutions, this result was proved earlier by Šešum [**169**].

THEOREM 17.13 (Compact finite time singularity models are shrinkers). *If* $(\mathcal{M}^n_\infty, g_\infty(t))$, $t \in (-\infty, 0]$, *is a finite time singularity model, where* $\mathcal{M}_\infty$ *is a closed manifold, then* $g_\infty(t)$ *is a shrinking gradient Ricci soliton.*

PROOF. STEP 1. $\lim_{t\to T} \nu(g(t))$ *exists.* By assumption, there exists a singular solution to the Ricci flow on a closed manifold $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, where $T < \infty$, and there exists a sequence (x_i, t_i) with $t_i \to T$ such that

$$\left(\mathcal{M}^n, Q_i g\left(t_i + Q_i^{-1} t\right)\right) \to (\mathcal{M}^n_\infty, g_\infty(t)) \tag{17.29}$$

in the sense of C^∞ Cheeger–Gromov convergence for $t \in (-\infty, 0]$ and for

$$Q_i \doteqdot |\mathrm{Rm}|(x_i, t_i) \to \infty.$$

By Corollary 17.12, we may assume by translating time that

$$\lambda(g(t)) > 0$$

for all $t \geq 0$. By

$$-\infty < \nu(g(t)) < 0 \tag{17.30}$$

(see Remark 17.7(2) above and (17.47) and (17.42) below) and the monotonicity of $\nu(g(t))$ (see Lemma 6.35(1) in Part I), we have that

$$\nu_T \doteqdot \lim_{t\to T} \nu(g(t)) \in (-\infty, 0] \tag{17.31}$$

exists.

STEP 2. $\lambda(g_\infty(t)) > 0$. Since $\mathcal{M}_\infty$ is compact, $\mathcal{M}_\infty$ is diffeomorphic to $\mathcal{M}$ and there exist diffeomorphisms $\varphi_i : \mathcal{M}_\infty \to \mathcal{M}$ such that

$$\varphi_i^*\left(Q_i g\left(t_i + Q_i^{-1} t\right)\right) \to g_\infty(t)$$

converges pointwise in C^k on $\mathcal{M}_\infty \times [-k, 0]$ for each $k \in \mathbb{N}$. Hence, by Lemma 5.24 in Part I,

$$\begin{aligned}\lambda(g_\infty(t)) &= \lim_{i\to\infty} \lambda\left(\varphi_i^*\left(Q_i g\left(t_i + Q_i^{-1} t\right)\right)\right) \\ &= \lim_{i\to\infty} Q_i^{-1} \lambda\left(g\left(t_i + Q_i^{-1} t\right)\right) \\ &\geq 0.\end{aligned}$$

We claim that

$$\lambda\left(g_\infty\left(t\right)\right) > 0 \tag{17.32}$$

for all $t \in (-\infty, 0]$.

To prove this, suppose by contradiction that $\lambda\left(g_\infty\left(t'\right)\right) = 0$ for some $t' \in (-\infty, 0]$. Then by the λ-monotonicity formula,

$$\lambda\left(g_\infty\left(t\right)\right) \equiv 0$$

for all $t \in (-\infty, t']$. This implies that $g_\infty\left(t\right)$ is a steady gradient Ricci soliton for $t \in (-\infty, t']$ (see Lemma 5.28 in Part I). Since $\mathcal{M}_\infty$ is compact, $g_\infty\left(t\right)$ is Ricci flat for $t \in (-\infty, 0]$ (see Proposition 1.13 in Part I; we also use uniqueness to extend to the whole time interval). This contradicts the claim that *any compact finite time singularity model cannot be Ricci flat.*

To see this claim, recall that (see Theorem 6.74 in Part I) any finite time singularity model is κ-noncollapsed at all scales for some $\kappa > 0$ in the sense that if $B_{g_\infty(t)}(x, r)$, $r \in (0, \infty)$, is a metric ball such that

$$R_{g_\infty(t)} \leq r^{-2} \quad \text{for all } y \in B_{g_\infty(t)}(x, r),$$

then

$$\operatorname{Vol}_{g_\infty(t)} B(x, r) \geq \kappa r^n.$$

Since $\operatorname{Rc}_{g_\infty(t)} \equiv 0$ on $\mathcal{M}_\infty$, this implies

$$\operatorname{Vol}_{g_\infty(t)}\left(\mathcal{M}_\infty\right) \geq \kappa r^n$$

for all $r \in (0, \infty)$ and $t \in (-\infty, 0]$, which is a contradiction since the LHS is finite. This completes the proof of (17.32).

STEP 3. *$g_\infty\left(t\right)$ is a shrinker.* Now by (17.29) we have (we justify the first equality in Lemma 17.14 below)

$$\begin{aligned}
\nu\left(g_\infty\left(t\right)\right) &= \lim_{i\to\infty} \nu\left(\varphi_i^*\left(Q_i g\left(t_i + Q_i^{-1} t\right)\right)\right) \\
&= \lim_{i\to\infty} \nu\left(g\left(t_i + Q_i^{-1} t\right)\right) \\
&= \lim_{t\to T} \nu\left(g\left(t\right)\right) \\
&= \nu_T
\end{aligned}$$

for all $t \in (-\infty, 0]$, where we used the fact that $\nu(cg) = \nu(g)$ for any $c \in (0, \infty)$. Since $\nu\left(g_\infty\left(t\right)\right)$ is identically a constant and $\lambda\left(g_\infty\left(t\right)\right) > 0$, by the equality case of the ν-monotonicity formula (see Lemma 6.35(2) in Part I), we conclude that $g_\infty\left(t\right)$ is a shrinking gradient Ricci soliton. □

The following fact is used in the proof above.

LEMMA 17.14. *If $\mathcal{M}_\infty^n$ is a closed manifold and $g_i \to g_\infty$ pointwise in C^∞ on $\mathcal{M}_\infty$, where $\lambda\left(g_\infty\right) > 0$, then*

$$\nu\left(g_\infty\right) = \lim_{i\to\infty} \nu\left(g_i\right). \tag{17.33}$$

PROOF. STEP 1. *$\nu(g_i)$ is bounded above by a negative constant.* By (17.42) below there exists $\bar{\tau} > 0$ such that

$$\mu(g_\infty, \bar{\tau}) < 0.$$

On the other hand, by Lemma 17.15 below,

$$\lim_{i\to\infty} \mu(g_i, \bar{\tau}) = \mu(g_\infty, \bar{\tau}),$$

so that

$$\nu(g_i) \leq \mu(g_i, \bar{\tau}) \leq \frac{1}{2}\mu(g_\infty, \bar{\tau}) < 0$$

for i sufficiently large. Thus

$$\nu(g_i) \leq -\varepsilon_0 \tag{17.34}$$

for all $i \in \mathbb{N} \cup \{\infty\}$ and some $\varepsilon_0 > 0$.

STEP 2. *Properties of $\mu(g_i, \tau)$.*

(i) By Lemma 17.15 again, we have for any $C > 1$,

$$\mu(g_i, \tau) \to \mu(g_\infty, \tau) \tag{17.35}$$

uniformly with respect to $\tau \in [C^{-1}, C]$.

(ii) Since $\lambda(g_i) \to \lambda(g_\infty) > 0$ and $C_s(g_i)$ and $R_{\min}(g_i)$ are uniformly bounded, Lemma 17.6 implies that there exists $C_1 \in \left(\frac{n}{8}, \infty\right)$ such that

$$\mu(g_i, \tau) \geq 0 \tag{17.36}$$

for all $\tau \geq C_1$ and $i \in \mathbb{N} \cup \{\infty\}$.

(iii) By the proof of Proposition 17.20 below, we have that for any sequence $\tau_i \to 0$ there exists a subsequence such that

$$\lim_{i\to\infty} \mu(g_i, \tau_i) = 0.$$

This implies that for any $\varepsilon > 0$ there exists $\tau(\varepsilon) > 0$ such that

$$\mu(g_i, \tau) \geq -\varepsilon \tag{17.37}$$

for all $i \in \mathbb{N}$ and $\tau \in (0, \tau(\varepsilon)]$.

STEP 3. *Completion of the proof.* Equation (17.33) now follows from combining (17.34), (17.35), (17.36), and (17.37). □

Finally, we give the proof of

LEMMA 17.15 (Continuous dependence of $\mu(g, \tau)$ on g). *For any $n \geq 2$, $C < \infty$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{M}^n$ is a closed manifold and if g and $\tilde{g}$ are Riemannian metrics such that*

(1) $R_{\text{avg}}(\tilde{g}) \leq C$,
(2) $\text{Vol}(\tilde{g}) \leq C$,
(3) $|R_g - R_{\tilde{g}}| \leq \delta$,
(4) $|g - \tilde{g}|_{\tilde{g}} \leq \delta$,

then

$$\mu(g, \tau) - \mu(\tilde{g}, \tau) \leq \varepsilon$$

for $\tau \in [C^{-1}, C]$.

PROOF. Given $\tau \in \left[C^{-1}, C\right]$, let $\tilde{w}$ with $\int_{\mathcal{M}} \tilde{w}^2 d\mu_{\tilde{g}} = 1$ be a minimizer of the entropy $\mathcal{K}(\tilde{g}, \cdot, \tau)$ in (17.7). Note that by (17.20) and assumptions (1) and (2),

$$\begin{aligned}\mu(\tilde{g}, \tau) &= \mathcal{K}(\tilde{g}, \tilde{w}, \tau) \\ &\leq \tau R_{\mathrm{avg}}(\tilde{g}) + \log \mathrm{Vol}(\tilde{g}) + \frac{n}{2}\log C \\ &\leq \mathrm{const}(n, C)\end{aligned}$$

for $\tau \in \left[C^{-1}, C\right]$. Define $c \in \mathbb{R}_+$ so that

$$\int_{\mathcal{M}} (c\tilde{w})^2 \, d\mu_g = 1\,;$$

we may make c arbitrarily close to 1 by choosing δ sufficiently small. We have

$$\begin{aligned}\mu(g, \tau) &\leq \mathcal{K}(g, c\tilde{w}, \tau) \\ &= \int_{\mathcal{M}} \left(\tau\left(R_g (c\tilde{w})^2 + 4\left|\nabla (c\tilde{w})\right|_g^2\right) - \left(\log\left((c\tilde{w})^2\right)\right)(c\tilde{w})^2\right) d\mu_g \\ &\quad - \frac{n}{2}\log(4\pi\tau) - n \\ &\leq \mu(\tilde{g}, \tau) + \tau\int_{\mathcal{M}} \tilde{w}^2\left(c^2 R_g d\mu_g - R_{\tilde{g}} d\mu_{\tilde{g}}\right) \\ &\quad + 4\tau\int_{\mathcal{M}}\left(c^2\left|\nabla\tilde{w}\right|_g^2 d\mu_g - \left|\nabla\tilde{w}\right|_{\tilde{g}}^2 d\mu_{\tilde{g}}\right) \\ &\quad + \int_{\mathcal{M}} \tilde{w}^2\log\left(\tilde{w}^2\right)\left(d\mu_{\tilde{g}} - c^2 d\mu_g\right) - c^2\log\left(c^2\right)\int_{\mathcal{M}}\tilde{w}^2 d\mu_g\end{aligned}$$

since $\mathcal{K}(\tilde{g}, \tilde{w}, \tau) = \mu(\tilde{g}, \tau)$. Thus for any $\varepsilon > 0$, by taking δ sufficiently small in assumptions (3) and (4) and by making c sufficiently close enough to 1, we obtain

$$\mu(g, \tau) - \mu(\tilde{g}, \tau) \leq \varepsilon.$$

Here we used the fact that the logarithmic Sobolev inequality implies that

$$\int_{\mathcal{M}} \left|\nabla\tilde{w}\right|_{\tilde{g}}^2 d\mu_{\tilde{g}} \quad \text{and} \quad \int_{\mathcal{M}} \tilde{w}^2\log\left(\tilde{w}^2\right) d\mu_{\tilde{g}}$$

are bounded by $\mu(\tilde{g}, \tau) + \mathrm{const}(n, C)$, which in turn is uniformly bounded (see the proof of Lemma 6.24 in Part I or (17.58) below). □

In dimension 3 any shrinking gradient Ricci soliton on a closed 3-manifold is a constant positive sectional curvature solution (see Theorem 9.79 in [**45**] for example; note that compact quotients of $\mathcal{S}^2 \times \mathbb{R}$ cannot be κ-noncollapsed at scales), so that we have the following.

COROLLARY 17.16 (Singularity models on closed 3-manifolds are round). *If $\left(\mathcal{M}^3_\infty, g_\infty(t)\right)$, $t \in (-\infty, 0]$, is a finite time singularity model on a closed 3-manifold, then $g_\infty(t)$ is a shrinking spherical space form.*

Šešum [**169**] considers immortal solutions $g(t)$ to the 'Ricci flow with cosmological constant 1'

$$\frac{\partial}{\partial t}g = -2\operatorname{Rc} + g \tag{17.38}$$

with

$$|\operatorname{Rm}| \leq C \quad \text{and} \quad \operatorname{diam} \leq C \tag{17.39}$$

on $\mathcal{M} \times [0,\infty)$, where $C < \infty$. Making the change of time variable $t\left(\hat{t}\right) \doteqdot -\ln\left(1-\hat{t}\right)$, i.e., $\hat{t}(t) \doteqdot 1 - e^{-t}$, and rescaling the solution by defining $\widehat{g}\left(\hat{t}\right) \doteqdot \left(1-\hat{t}\right) g\left(t\left(\hat{t}\right)\right)$, we have

$$\frac{\partial}{\partial \hat{t}}\widehat{g} = -2\widehat{\operatorname{Rc}}$$

on $\mathcal{M} \times [0,1)$. The conditions (17.39) correspond to the Type I condition

$$\left|\widehat{\operatorname{Rm}}\right|\left(\hat{t}\right) \leq \frac{C}{1-\hat{t}}$$

and the diameter estimate

$$\widehat{\operatorname{diam}} \leq C\sqrt{1-\hat{t}}. \tag{17.40}$$

This implies

$$\max_{\mathcal{M}}\left|\widehat{\operatorname{Rm}}\right|\left(\hat{t}\right) \to \infty \quad \text{as } \hat{t} \to 1. \tag{17.41}$$

For if the (Ricci) curvature were uniformly bounded, then the diameter could not tend to zero as $\hat{t} \to 1$.[2] Moreover, we then also have

$$\max_{\mathcal{M}}\left|\widehat{\operatorname{Rm}}\right|\left(\hat{t}\right) \geq \frac{1}{8\left(1-\hat{t}\right)}$$

(see Lemma 8.19 in Volume One and Lemma 8.7 in [**45**]).

In turn, if one does a Type I rescaling by defining

$$\tilde{g}_i\left(\tilde{t}\right) = \frac{1}{1-\hat{t}_i}\widehat{g}\left(\hat{t}_i + \left(1-\hat{t}_i\right)\tilde{t}\right)$$

for some $\hat{t}_i \to 1$, then one obtains uniform bounds for $\left|\widetilde{\operatorname{Rm}}_i\right|\left(\tilde{t}\right)$ and $\widetilde{\operatorname{diam}}_i\left(\tilde{t}\right)$ for $\tilde{t} \leq 0$.

Problem 17.17. Show that if $(\mathcal{M}^n, g(t))$, $t \in [0,T)$, is a finite time singular solution forming a singularity model on a closed manifold (which then must be diffeomorphic to $\mathcal{M}$), then $g(t)$ is Type I.

[2] This implies

$$\max_{\mathcal{M}}\left|\widehat{\operatorname{Rm}}\right|\left(\hat{t}_i\right) \to \infty \quad \text{for some } \hat{t}_i \to 1,$$

from which (17.41) follows.

One could conceivably have the strange situation of a Type IIa singular solution which forms a Type I singularity model which is a *compact* shrinking gradient Ricci soliton. Certainly, one can have a Type IIa singular solution which forms a Type I singularity model which is a *noncompact* shrinking gradient Ricci soliton as evidenced by a degenerate neckpinch which includes the shrinking round cylinder as one of its singularity models.

MINI-PROBLEM 17.18 (Compact factors of singularity models are shrinkers). *Show that if the universal cover of a (finite time) singularity model splits as* $(\mathcal{N}^m, h(t)) \times \mathbb{R}^{n-m}$*, where* $\mathcal{N}$ *is compact, then* $(\mathcal{N}^m, h(t))$ *is a shrinking gradient Ricci soliton.*

2. Behavior of $\mu(g,\tau)$ for τ small

In this section we present a detailed discussion of the limiting behavior of the μ-invariant as τ tends to 0. As a consequence, we shall show that for a closed Riemannian manifold on which the isometry group acts transitively, the minimizer f_τ of $\mathcal{W}(g,\cdot,\tau)$ is not unique for τ sufficiently small.

2.1. Behavior of $\mu(g,\tau)$ for τ small.

In the next lemma and proposition we give a belated proof of Lemma 6.33(i), (ii) in Part I regarding the behavior of $\mu(g,\tau)$ for τ sufficiently small (this is a result of Perelman; see §3.1 of [**152**]).

LEMMA 17.19 ($\mu(g,\tau)$ is negative for τ small). *If* $(\mathcal{M}^n, g)$ *is a closed Riemannian manifold, then there exists* $\bar{\tau} > 0$ *such that*

$$\mu(g,\tau) < 0 \quad \textit{for all } \tau \in (0,\bar{\tau}). \tag{17.42}$$

PROOF. Since $\mathcal{M}$ is closed, by the short time existence theorem, there is a $\bar{\tau} > 0$ such that a (unique) solution $g(t)$ to the Ricci flow with $g(0) = g$ exists for $t \in [0,\bar{\tau}]$. Let $\tau(t) \doteqdot \bar{\tau} - t$ and $x_0 \in \mathcal{M}$ and consider the corresponding fundamental solution

$$u(x,t) \doteqdot (4\pi\tau(t))^{-n/2} e^{-f(x,t)}, \quad x \in \mathcal{M},\ t \in [0,\bar{\tau}),$$

to the adjoint heat equation

$$\frac{\partial u}{\partial t} = -\Delta_{g(t)} u + R_{g(t)} u$$

centered at $(x_0,\bar{\tau})$ (i.e., $\lim_{t \nearrow \bar{\tau}} u(\cdot,t) = \delta_{x_0}$; note that $\tau(\bar{\tau}) = 0$).

As Perelman says in §3.1 of [**152**] and as we have seen in Chapter 16 of Part II (where we discussed Perelman's differential Harnack estimate $v \leq 0$), we have

$$\lim_{t \nearrow \bar{\tau}} \mathcal{W}(g(t), f(t), \tau(t)) = 0. \tag{17.43}$$

Hence, by the monotonicity of the entropy functional,

$$\mu(g,\bar{\tau}) = \mu(g,\tau(0)) \leq \mathcal{W}(g(0), f(0), \tau(0)) \leq \lim_{t \nearrow \bar{\tau}} \mathcal{W}(g(t), f(t), \tau(t)) = 0. \tag{17.44}$$

We now rule out $\mu(g,\bar{\tau}) = 0$, from which the lemma follows. Suppose $\mu(g,\bar{\tau}) = 0$. Since $\mathcal{W}$ is monotone, we then have

$$\mathcal{W}(g(t), f(t), \tau(t)) = \mu(g(t), \tau(t)) \equiv 0$$

for all $t \in [0, \bar{\tau})$. Hence, by (17.12) we have

$$\left(\mathrm{Rc} + \nabla\nabla f - \frac{g}{2\tau}\right)(t) \equiv 0$$

for $t \in [0, \bar{\tau}]$, so that $g(t)$ is a shrinking gradient Ricci soliton with singular time $t = \bar{\tau}$. In particular,

$$\tau(t) \max_{\mathcal{M}} |\mathrm{Rm}(g(t))| \equiv \mathrm{const}$$

for $t \in [0, \bar{\tau}]$. On the other hand, since $g(\bar{\tau})$ is a smooth metric (and recall that $\tau(\bar{\tau}) = 0$), we conclude that $|\mathrm{Rm}(g(t))| \equiv 0$ for $t \in [0, \bar{\tau}]$. We obtain a contradiction because there are no flat shrinking Ricci solitons on closed manifolds. □

Recall that Gross's Euclidean logarithmic Sobolev inequality (see Corollary 6.40 in Part I or Theorem 22.15 below) says that if $f_0 : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with

$$\int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-f_0} d\mu_{\mathbb{R}^n} = 1,$$

then

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f_0|^2 + f_0 - n\right)(2\pi)^{-n/2} e^{-f_0} d\mu_{\mathbb{R}^n} \geq 0, \tag{17.45}$$

with equality if $f_0(x) = \frac{|x-x_0|^2}{2}$ for some $x_0 \in \mathbb{R}^n$. That is, for Euclidean space, the entropy is nonnegative and the μ-invariant is zero. Note that if we let $w_0 \doteqdot (2\pi)^{-n/4} e^{-f_0/2}$, then $\int_{\mathbb{R}^n} w_0^2 d\mu_{\mathbb{R}^n} = 1$ and we may rewrite (17.45) as

$$\int_{\mathbb{R}^n} \left(|\nabla w_0|^2 - w_0^2 \log(w_0) - \left(\frac{n}{2} + \frac{n}{4}\log(2\pi)\right) w_0^2\right) d\mu_{\mathbb{R}^n} \geq 0. \tag{17.46}$$

Roughly speaking, since Riemannian manifolds are almost geometrically Euclidean on small scales, the Euclidean logarithmic Sobolev inequality implies that the entropy on small scales (τ small) is almost nonnegative and the corresponding μ-invariant is almost zero. The following is in §3.1 of [**152**] (see also Proposition 3.2 in Šešum, Tian, and Wang [**170**]).

PROPOSITION 17.20 ($\mu(g,\tau) \to 0$ as $\tau \to 0$). *If $(\mathcal{M}^n, g)$ is a closed Riemannian manifold, then*

$$\lim_{\tau \to 0^+} \mu(g,\tau) = 0. \tag{17.47}$$

PROOF. Suppose that (17.47) is not true. Then there exist $\varepsilon > 0$ and a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ with $\tau_i \searrow 0$ such that

$$\mu(g,\tau_i) \leq -\varepsilon.$$

We assume that $\tau_i \leq \frac{1}{2}$. We shall derive a contradiction to (17.46).

Consider the rescaled metrics

$$g_i \doteqdot \frac{1}{2\tau_i} g \tag{17.48}$$

(note that $\mu\left(g_i, \frac{1}{2}\right) = \mu(g,\tau_i) \leq -\varepsilon$). By Proposition 17.24 below, there exists a corresponding sequence $\{f_i\}_{i\in\mathbb{N}}$ of minimizers of

$$\mathcal{W}(g,\,\cdot\,,\tau_i) = \mathcal{W}\left(g_i,\,\cdot\,,\frac{1}{2}\right)$$

subject to the constraints

$$\int_{\mathcal{M}} (4\pi\tau_i)^{-n/2} e^{-f_i} d\mu_g = \int_{\mathcal{M}} (2\pi)^{-n/2} e^{-f_i} d\mu_{g_i} = 1. \tag{17.49}$$

Then

$$\mathcal{W}(g,f_i,\tau_i) = \mathcal{W}\left(g_i,f_i,\frac{1}{2}\right) = \mu(g,\tau_i). \tag{17.50}$$

Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of points in $\mathcal{M}$ such that

$$f_i(x_i) = \min_{x\in\mathcal{M}} f_i(x). \tag{17.51}$$

The pointed sequence of Riemannian manifolds $\{(\mathcal{M}^n, g_i, x_i)\}_{i\in\mathbb{N}}$ converges in the C^∞ Cheeger–Gromov sense to Euclidean n-space $(\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$. That is, there exists an exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ by relatively compact open sets ($\overline{U_i} \subset U_{i+1}$) and embeddings $\Phi_i : U_i \to \mathcal{M}$ such that $\Phi_i(0) = x_i$ and

$$\tilde{g}_i \doteqdot \Phi_i^* g_i \to g_{\mathbb{R}^n} \tag{17.52}$$

uniformly in C^∞ on compact subsets of $\mathbb{R}^n$.[3]

Consider the positive functions

$$w_i \doteqdot (2\pi)^{-n/4} e^{-f_i/2},$$

which by (17.15) satisfy

$$-2\Delta_{g_i} w_i + \frac{1}{2} R_{g_i} w_i - 2w_i \log w_i - \left(\frac{n}{2}\log(2\pi) + n\right) w_i = \mu\left(g_i, \frac{1}{2}\right) w_i \tag{17.53}$$

with the constraint (17.49), i.e.,

$$\int_{\mathcal{M}} w_i^2 d\mu_{g_i} = 1. \tag{17.54}$$

The contradiction to (17.45) is obtained via the following steps.

[3] See the notes and commentary at the end of this chapter for explicit choices of U_i and Φ_i.

STEP 1. For a subsequence, the functions

$$\tilde{w}_i \doteqdot w_i \circ \Phi_i : U_i \to \mathbb{R}$$

converge in $C^{1,\alpha}$ on compact subsets to a $C^{1,\alpha}$ function $\tilde{w}_\infty$ on $\mathbb{R}^n$, for some $\alpha \in (0,1)$. Moreover, $\tilde{w}_\infty \in W^{1,2}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \tilde{w}_\infty^2 d\mu_{\mathbb{R}^n} \leq 1. \tag{17.55}$$

STEP 2. The limit function $\tilde{w}_\infty$ is a weak solution to the elliptic equation

$$2\Delta_{\mathbb{R}^n} \tilde{w}_\infty = -\left(\mu_\infty + \frac{n}{2}\log(2\pi) + n + 2\log \tilde{w}_\infty\right)\tilde{w}_\infty. \tag{17.56}$$

STEP 3. The function $\tilde{w}_\infty$ is positive (by Step 1, this implies the $C^{1,\alpha}$ convergence of $\tilde{w}_i^2 \log \tilde{w}_i$ to $\tilde{w}_\infty^2 \log \tilde{w}_\infty$).

STEP 4. The function $\tilde{w}_\infty$ is C^∞ and from the elliptic equation (17.56) that $\tilde{w}_\infty$ satisfies, we obtain a contradiction to (17.45) since (17.55) holds.

Now we prove Steps 1–4. From (17.50) and (17.7) we have for the minimizers w_i of $\mathcal{K}\left(g_i, \cdot, \frac{1}{2}\right)$ that

$$\begin{aligned} \mu\left(g_i, \frac{1}{2}\right) &= \mathcal{K}\left(g_i, w_i, \frac{1}{2}\right) \\ &= \int_{\mathcal{M}} \begin{pmatrix} \frac{1}{2}\left(R_{g_i} w_i^2 + 4\left|\nabla w_i\right|_{g_i}^2\right) \\ -\left(\log\left(w_i^2\right) + \frac{n}{2}\log(2\pi) + n\right) w_i^2 \end{pmatrix} d\mu_{g_i}. \end{aligned} \tag{17.57}$$

Proof of Step 1. By (17.57) and (17.54) we have

$$\begin{aligned} 2\int_{\mathcal{M}} \left|\nabla w_i\right|_{g_i}^2 d\mu_{g_i} &= 2\int_{\mathcal{M}} w_i^2 \log w_i d\mu_{g_i} - \int_{\mathcal{M}} \frac{1}{2} R_{g_i} w_i^2 d\mu_{g_i} \\ &\quad + \mu\left(g_i, \frac{1}{2}\right) + \frac{n}{2}\log(2\pi) + n \\ &\leq C_1, \end{aligned} \tag{17.58}$$

where C_1 is independent of i; here we used the logarithmic Sobolev inequality, Lemma 17.2, and $R_{g_i} = 2\tau_i R_g$. Hence there exists $C_2 < \infty$ such that

$$\|w_i\|_{W^{1,2}(\mathcal{M}, g_i)} \leq C_2 \tag{17.59}$$

for all i. By the L^2 Sobolev inequality, i.e., (17.18), we then have[4]

$$\|w_i\|_{L^{\frac{2n}{n-2}}(\mathcal{M}, g_i)} \leq C_3 \tag{17.60}$$

when $n \geq 3$ and we have $\|w_i\|_{L^p(\mathcal{M}, g_i)} \leq C_4(p)$ for all $p \in [1, \infty)$ when $n = 2$.

Now consider

$$\tilde{w}_i = w_i \circ \Phi_i : U_i \to \mathbb{R}.$$

[4]From (17.19), the L^2 Sobolev constant is independent of scaling and $\mathrm{Vol}(g_i)^{-2/n} \leq \mathrm{Vol}(g)^{-2/n}$.

Since $\tilde{g}_i = \Phi_i^* g_i \to g_{\mathbb{R}^n}$, by (17.116) below and (17.60), we have for any compact domain $\Omega \subset \mathbb{R}^n$ and any $2 < p < \frac{2n}{n-2}$ (when $n = 2$, define $\frac{2n}{n-2} \doteqdot \infty$)

$$\|\tilde{w}_i \log \tilde{w}_i\|_{L^p(\Omega, g_{\mathbb{R}^n})} \leq C(\Omega, p) < \infty, \tag{17.61}$$

where $C(\Omega,p) < \infty$ is independent of i. Since (17.53) says

$$-2\Delta_{g_i} w_i + \frac{1}{2} R_{g_i} w_i - \left(\frac{n}{2}\log(2\pi) + n\right) w_i - \mu\left(g_i, \frac{1}{2}\right) w_i = 2 w_i \log w_i, \tag{17.62}$$

by the L^p estimate for solutions to second-order elliptic equations (see Theorem 9.11 in Gilbarg and Trudinger [**71**]) applied to $\tilde{w}_i$ on Ω, we have

$$\|\tilde{w}_i\|_{W^{2,p}(\Omega, g_{\mathbb{R}^n})} \leq C(\Omega, p) \tag{17.63}$$

for any $2 < p < \frac{2n}{n-2}$ and all i.

By the Sobolev inequality, we have

$$\frac{1}{C}\|\tilde{w}_i\|_{L^{p\left(\frac{n}{n-2}\right)^2}(\Omega, g_{\mathbb{R}^n})} \leq \|\tilde{w}_i\|_{W^{1,\frac{np}{n-p}}(\Omega, g_{\mathbb{R}^n})} \leq C\|\tilde{w}_i\|_{W^{2,p}(\Omega, g_{\mathbb{R}^n})},$$

where $C = C(\Omega,p) < \infty$ is independent of i (note that $\frac{n}{n-p} \geq \frac{n}{n-2}$ and $\int_{\mathcal{M}} w_i^2 d\mu_{g_i} = 1$). Thus, by applying (17.115) with $\delta > 0$ arbitrarily small, we have

$$\int_\Omega |\tilde{w}_i \log \tilde{w}_i|^q \, d\mu_{\mathbb{R}^n} \leq C(\Omega, q)$$

for $2 \leq q < 2\left(\frac{n}{n-2}\right)^3$, independent of i. From this and the standard elliptic L^p estimate for (17.62), we obtain the stronger (as compared to (17.63)) estimate

$$\|\tilde{w}_i\|_{W^{2,q}(\Omega, g_{\mathbb{R}^n})} \leq C(\Omega, q)$$

for $2 \leq q < 2\left(\frac{n}{n-2}\right)^3$. By iterating the above argument (easy exercise), we see that for any $q \in (1,\infty)$

$$\|\tilde{w}_i\|_{W^{2,q}(\Omega, g_{\mathbb{R}^n})} \leq C(\Omega, q), \tag{17.64}$$

where $C(\Omega,q)$ is independent of i.[5]

Because we have (17.64) with $q > n$, by the Sobolev inequality it follows that

$$\|\tilde{w}_i\|_{C^{1,\alpha}(\Omega, g_{\mathbb{R}^n})} \leq C(\Omega)$$

for some $\alpha \in (0,1)$ and where $C(\Omega) < \infty$ is independent of i. By the Arzela–Ascoli theorem and a diagonalization argument, passing to a subsequence, we have that for some $\alpha \in (0,1)$ there is a nonnegative function $\tilde{w}_\infty$ in $C^{1,\alpha}(\mathbb{R}^n, g_{\mathbb{R}^n})$ such that

$$\tilde{w}_i \to \tilde{w}_\infty$$

[5] Note that if $\|\tilde{w}_i \log \tilde{w}_i\|_{L^q(\Omega, g_{\mathbb{R}^n})} \leq C$ for some $q \in (1,\infty)$, then $\|\tilde{w}_i\|_{W^{2,q}(\Omega, g_{\mathbb{R}^n})} \leq C'$.

in $C^{1,\alpha}(\Omega, g_{\mathbb{R}^n})$ for all compact domains $\Omega \subset \mathbb{R}^n$. Note that now we have $\tilde{w}_i \log \tilde{w}_i \to \tilde{w}_\infty \log \tilde{w}_\infty$ in $C^0(\Omega, g_{\mathbb{R}^n})$.

Moreover, (17.54) implies

$$\int_\Omega \tilde{w}_i^2 d\mu_{\tilde{g}_i} \leq 1$$

for i large enough, so that

$$\int_{\mathbb{R}^n} \tilde{w}_\infty^2 d\mu_{\mathbb{R}^n} \leq 1.$$

Now by (17.58), we have

$$\int_{\mathbb{R}^n} |\nabla \tilde{w}_\infty|^2 d\mu_{\mathbb{R}^n} \leq C$$

for some $C < \infty$. In particular, $\tilde{w}_\infty \in W^{1,2}(\mathbb{R}^n)$.

Proof of Step 2. First we show that, after passing to a subsequence, the limit

$$\mu_\infty \doteqdot \lim_{i\to\infty} \mu\left(g_i, \frac{1}{2}\right) \leq -\varepsilon \tag{17.65}$$

exists. By (17.23),

$$\begin{aligned} \mu\left(g_i, \frac{1}{2}\right) &\geq \frac{1}{2} R_{\min}(g_i) - 2C(1, g_i) - \frac{n}{2}\log(2\pi) - n \\ &\geq \tau_i R_{\min}(g) - 2C(1, g) - \frac{n}{2}\log(2\pi) - n \end{aligned}$$

is uniformly bounded from below since by Lemma 17.2 we have $C\left(1, \frac{1}{2\tau_i} g\right) \leq C(1, g)$ because $\tau_i \leq \frac{1}{2}$.

Now we integrate the equations (which follow from (17.53))

$$2\Delta_{\tilde{g}_i} \tilde{w}_i = -\mu\left(g_i, \frac{1}{2}\right)\tilde{w}_i + \frac{1}{2} R_{\tilde{g}_i}\tilde{w}_i - \left(\frac{n}{2}\log(2\pi) + n\right)\tilde{w}_i - 2\tilde{w}_i \log \tilde{w}_i$$

in U_i against a compactly supported test function. Taking the limit of the integrations, we obtain

$$2\int_{\mathbb{R}^n} \langle \nabla \tilde{w}_\infty, \nabla \varphi \rangle d\mu_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \left(\mu_\infty + \frac{n}{2}\log(2\pi) + n + 2\log \tilde{w}_\infty\right) \tilde{w}_\infty \varphi \, d\mu_{\mathbb{R}^n} \tag{17.66}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ since $R_{\tilde{g}_i} \to 0$. That is, $\tilde{w}_\infty$ is a weak solution of (17.56).

Proof of Step 3. By (17.88) below, we have

$$\max_{\mathcal{M}} w_i \geq \exp\left(\frac{1}{4} R_{\min}(g_i) - \frac{n}{4}\log(2\pi) - \frac{n}{2} - \frac{1}{2}\mu\left(g_i, \frac{1}{2}\right)\right). \tag{17.67}$$

Since $\tau_i \leq \frac{1}{2}$, we have

$$R_{\min}(g_i) = 2\tau_i R_{\min}(g) \geq -|R_{\min}(g)| \tag{17.68}$$

for all i. Hence, by (17.67), (17.51), and $\Phi_i(0) = x_i$,
(17.69)

$$\tilde{w}_i(0) = w_i(x_i) = \max_{\mathcal{M}} w_i \geq \exp\left(-\frac{1}{4}|R_{\min}(g)| - \frac{n}{4}\log(2\pi) - \frac{n}{2}\right).$$

Hence

$$\tilde{w}_\infty(0) > 0. \tag{17.70}$$

By (17.70) and by the strong maximum principle for weak solutions (Lemma 17.26 below) applied to (17.56), we have

$$\tilde{w}_\infty > 0 \quad \text{on } \mathbb{R}^n$$

and $\tilde{w}_\infty \log \tilde{w}_\infty$ is contained in the local Hölder space $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$.

Proof of Step 4. Since $\tilde{w}_\infty$ is a weak solution of (17.56), where the RHS is contained in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$, by the regularity theorem for weak solutions we have that $\tilde{w}_\infty$ is a classical solution of (17.56). Now by Schauder theory, we have for any $k \in \mathbb{N}$

$$\|\tilde{w}_\infty\|_{C^{k,\alpha}(\Omega, g_{\mathbb{R}^n})} \leq C_k(\Omega)$$

for some $C_k(\Omega) < \infty$. In particular, $\tilde{w}_\infty \in W^{1,2}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

Now we complete the proof of the proposition. For any $R > 0$ let $\eta_R : \mathbb{R}^n \to [0,1]$ be a radial cutoff function with

$$\eta_R(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq R, \\ 0 & \text{if } |x| \geq R+1, \end{cases}$$

and with $-2 \leq \frac{\partial}{\partial r}\eta_R \leq 0$. Then by (17.66) with $\varphi = \eta_R^2 \tilde{w}_\infty$, we have

$$\int_{\mathbb{R}^n} \left\langle \nabla \tilde{w}_\infty, \nabla\left(\eta_R^2 \tilde{w}_\infty\right)\right\rangle d\mu_{\mathbb{R}^n}$$
$$= \int_{\mathbb{R}^n} \left(\frac{\mu_\infty}{2} + \frac{n}{4}\log(2\pi) + \frac{n}{2} + \log \tilde{w}_\infty\right)(\eta_R \tilde{w}_\infty)^2 \, d\mu_{\mathbb{R}^n},$$

so that

$$0 \leq \int_{\mathbb{R}^n} |\nabla(\eta_R \tilde{w}_\infty)|^2 \, d\mu_{\mathbb{R}^n}$$
$$- \int_{\mathbb{R}^n} \left(\frac{n}{4}\log(2\pi) + \frac{n}{2} + \log(\eta_R \tilde{w}_\infty)\right)(\eta_R \tilde{w}_\infty)^2 \, d\mu_{\mathbb{R}^n}$$
$$= \frac{\mu_\infty}{2}\int_{\mathbb{R}^n} (\eta_R \tilde{w}_\infty)^2 \, d\mu_{\mathbb{R}^n} + \int_{\mathbb{R}^n} \left(|\nabla \eta_R|^2 - \eta_R^2 \log \eta_R\right)\tilde{w}_\infty^2 d\mu_{\mathbb{R}^n}, \tag{17.71}$$

where the inequality is true by (17.46), which holds since $\int_{\mathbb{R}^n} \tilde{w}_\infty^2 d\mu_{\mathbb{R}^n} \leq 1$. Taking R sufficiently large, the RHS of (17.71) is arbitrarily close to

$$\frac{\mu_\infty}{2}\int_{\mathbb{R}^n} \tilde{w}_\infty^2 d\mu_{\mathbb{R}^n} < 0,$$

where this inequality is true because $\mu_\infty < 0$ and $\int_{\mathbb{R}^n} \tilde{w}_\infty^2 d\mu_{\mathbb{R}^n} > 0$. This is a contradiction. □

PROBLEM 17.21. In view of Proposition 17.20, can one determine the more precise asymptotic behavior of $\mu(g,\tau)$ for τ near 0?

2.2. Possible nonuniqueness of minimizers of $\mathcal{W}$ for τ small.

Proposition 17.20 has the following consequence for the nonuniqueness of certain minimizers for τ sufficiently small. This contrasts with the case of the energy functional $\mathcal{F}$, for which the minimizer *is* unique.

LEMMA 17.22 (For small τ the minimizer is nonconstant and may not be unique). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold.*

(1) *Any minimizer f_τ of $\mathcal{W}(g,\cdot,\tau)$ cannot be a constant function for τ sufficiently small.*
(2) *If the isometry group of $(\mathcal{M}, g)$ acts transitively, then for τ sufficiently small any minimizer f_τ of $\mathcal{W}(g,\cdot,\tau)$ is not unique.*

PROOF. We prove both statements by contradiction.

(1) Suppose that there exists a sequence $\tau_i \to 0$ such that for each i, there is a minimizer f_{τ_i} of $\mathcal{W}(g,\cdot,\tau_i)$ which is constant. Then by (17.14), i.e.,

$$\tau_i\left(2\Delta f_{\tau_i} - |\nabla f_{\tau_i}|^2 + R_g\right) + f_{\tau_i} - n = \mu(g,\tau_i),$$

we have that R_g is constant. Hence we have for all i,

$$\begin{aligned} \mu(g,\tau_i) &= \tau_i R_g + f_{\tau_i} - n \\ &= \tau_i R_g - \frac{n}{2}\log(4\pi\tau_i) + \log \operatorname{Vol}(g) - n, \end{aligned} \tag{17.72}$$

where the second equality follows from the constraint in (17.8).[6] We obtain

$$\lim_{i\to\infty} \mu(g,\tau_i) = \lim_{i\to\infty}\left(\tau_i R_g - \frac{n}{2}\log(4\pi\tau_i) + \log\operatorname{Vol}(g) - n\right) = \infty$$

since $\tau_i \to 0$. This contradicts (17.47).

(2) Let τ be sufficiently small so that part (1) holds for $(\mathcal{M}, g)$. Then suppose that the minimizer f_τ of $\mathcal{W}(g,\cdot,\tau)$ is unique. Since the isometry group of g acts transitively on $\mathcal{M}$, for every $x, y \in \mathcal{M}$ there exists an isometry

$$\phi : \mathcal{M} \to \mathcal{M}$$

of the metric g with $\phi(x) = y$. By the diffeomorphism invariance of the $\mathcal{W}$-functional, we have that $f_\tau \circ \phi$ is also a minimizer of $\mathcal{W}(g,\cdot,\tau)$. Thus, by our uniqueness assumption, $f_\tau \circ \phi = f_\tau$, which implies $f_\tau(x) = f_\tau(y)$. Since x and y are arbitrary, we conclude that f_τ is constant, a contradiction to our assumption on τ. □

[6] When f_τ is constant, the constraint $\int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-f_\tau} d\mu = 1$ implies $f_\tau = -\frac{n}{2}\log(4\pi\tau) + \log\operatorname{Vol}(g)$.

Now suppose that $(\mathcal{M}^n, g(\tau), f(\tau))$, $\tau \in (0, \infty)$, is an expanding gradient soliton solution to the backward Ricci flow (i.e., $\tilde{g}(t) \doteqdot g(-t)$ is a shrinking gradient Ricci soliton) satisfying

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0.$$

By the proof of Theorem 6.29 in Part I, $f(\tau)$ is a minimizer for $\mathcal{W}(g(\tau), \cdot, \tau)$. Note that $g(\tau)$ is isometric to $\tau g(1)$, so that

$$\mu(g(\tau), \tau) = \mu(g(1), 1)$$

for all $\tau \in (0, \infty)$.

In the special case of an Einstein solution, where

$$R_{ij} - \frac{1}{2\tau} g_{ij} = 0,$$

we have that a minimizer for $\mathcal{W}(g(\tau), \cdot, \tau)$ is a constant function.[7] On the other hand, by Lemma 17.22(1), given $g(\tau)$, for $\tilde{\tau}$ sufficiently small, any minimizer $f_{\tilde{\tau}}$ of $\mathcal{W}(g(\tau), \cdot, \tilde{\tau})$ is not constant.

PROBLEM 17.23. It would be interesting to understand the behavior of minimizers in some special cases.

(1) For $\tau > 0$ what are the minimizers of $\mathcal{W}(g_{\mathcal{S}^n}, \cdot, \tau)$ on the unit n-sphere? Are they always radial functions about some point in $\mathcal{S}^n$? How do the cases $\tau < \frac{1}{2(n-1)}$ and $\tau > \frac{1}{2(n-1)}$ compare? (Note that $R_{g_{\mathcal{S}^n}} = n(n-1)$, so that in the case $\tau = \frac{1}{2(n-1)}$ we have a constant minimizer.)
(2) For which manifolds can one find minimizers with nice properties such as having a certain amount of symmetry? For example, one may consider the minimizers on complex projective space $\mathbb{C}P^n$.

3. Existence of a minimizer for the entropy

In this section we discuss the proof of the existence of a minimizer for $\mathcal{W}$ which supplements the proof of Lemma 6.24 in Part I; we adopt here the notation used there. Included in our discussion is a proof of a strong maximum principle for *weak* solutions. We also consider a lower bound for the maximum value of the minimizer.

3.1. Proof of the existence of a minimizer for $\mathcal{W}$.

The following result is due to Rothaus and we follow his paper (see §1 of [**161**]).

PROPOSITION 17.24 (Existence of a smooth minimizer for $\mathcal{W}$). *For any metric g on a closed manifold $\mathcal{M}^n$ and for any $\tau > 0$, there exists a smooth minimizer f_τ of $\mathcal{W}(g, \cdot, \tau)$ which satisfies* (17.14).

[7]Note that the scale τ is related to the scalar curvature by $\tau = \frac{n}{2R_{g(\tau)}}$.

Without loss of generality, we may assume $\tau = 1$. Recall from (17.8) and (17.7) that

$$\mu(g,1) = \inf\left\{\mathcal{H}(g,w) : w \in W^{1,2}(\mathcal{M},g),\ \int_{\mathcal{M}} w^2 d\mu = 1\right\},$$

where

$$\mathcal{H}(g,w) \doteqdot \int_{\mathcal{M}} \left(4|\nabla w|^2 + \left(R - \log\left(w^2\right) - \frac{n}{2}\log(4\pi) - n\right) w^2\right) d\mu$$

for $w \in W^{1,2}(\mathcal{M},g)$. By Lemma 17.5, the functional $\mathcal{H}(g,\cdot)$ is bounded from below.

STEP 1. *There exists a minimizer* $0 \leq w_\infty \in W^{1,2}$ *of* $\mathcal{H}$. Let $\{w_i\}_{i\in\mathbb{N}}$ be a minimizing sequence of $W^{1,2}$ functions for the functional $\mathcal{H}(g,\cdot)$ with $\int_{\mathcal{M}} w_i^2 d\mu = 1$ for all $i \in \mathbb{N}$. We may assume that $w_i \geq 0$ for the following reason. If $w \in W^{1,2}$, then $|w| \in W^{1,2}$ and

$$|\nabla |w|| \leq |\nabla w|$$

(see Corollary 2.1.8 of Ziemer [**198**]), so that $\mathcal{H}(g,|w|) \leq \mathcal{H}(g,w)$. Thus, if $\{w_i\}$ is a minimizing sequence in $W^{1,2}$, then so is $\{|w_i|\}$.

Recall that we proved that there exists $C < \infty$ (independent of i) such that

$$\|w_i\|_{W^{1,2}(\mathcal{M},g)} \leq C \tag{17.73}$$

for all $i \in \mathbb{N}$ (we leave this as an exercise; see p. 238 in Part I or (17.58) above). By the Banach–Alaoglu theorem,[8] there exists

$$w_\infty \in W^{1,2}(\mathcal{M},g)$$

and a subsequence such that w_i converges to w_∞ weakly in $W^{1,2}(\mathcal{M},g)$, i.e., for every $v \in W^{1,2}(\mathcal{M},g)$

$$\lim_{i\to\infty} \langle w_i, v\rangle_{W^{1,2}(\mathcal{M},g)} = \langle w_\infty, v\rangle_{W^{1,2}(\mathcal{M},g)}.$$

As a standard consequence, we have (see also p. 205 in Part I)

$$\|w_\infty\|_{W^{1,2}(\mathcal{M},g)} \leq \liminf_{i\to\infty} \|w_i\|_{W^{1,2}(\mathcal{M},g)}. \tag{17.74}$$

By (17.73) and the Rellich–Kondrachov compactness theorem, for every $\varepsilon \in (0, \frac{n+2}{n-2}]$ we have that w_i converges to w_∞ in $L^{\frac{2n}{n-2}-\varepsilon}(\mathcal{M},g)$. In particular, $\int_{\mathcal{M}} w_\infty^2 d\mu = 1$ and $w_\infty \geq 0$. By this convergence and by Lemma 17.25 below, we have

$$\mu(g,1) \leq \mathcal{H}(g,w_\infty) \leq \lim_{i\to\infty} \mathcal{H}(g,w_i) = \mu(g,1).$$

We conclude that

$$\mathcal{H}(g,w_\infty) = \mu(g,1). \tag{17.75}$$

[8] See Theorems 3.15 and 3.17 in Rudin [**164**].

STEP 2. *w_∞ is a weak solution of* (17.77). Since w_∞ is a minimizer in $W^{1,2}$ of $\mathcal{H}(g,w)$ subject to the constraint $\int_{\mathcal{M}} w^2 d\mu = 1$, for any $W^{1,2}$ function $\phi : \mathcal{M} \to \mathbb{R}$ such that $\int_{\mathcal{M}} w_\infty \phi \, d\mu = 0$, we have

(17.76)
$$\begin{aligned}
0 &= \left.\frac{d}{ds}\right|_{s=0} \mathcal{H}(g, w_\infty + s\phi) \\
&= \left.\frac{d}{ds}\right|_{s=0} \int_{\mathcal{M}} \left(4\left|\nabla (w_\infty + s\phi)\right|^2\right) d\mu \\
&\quad + \left.\frac{d}{ds}\right|_{s=0} \int_{\mathcal{M}} \left(R - \log (w_\infty + s\phi)^2 - \frac{n}{2}\log(4\pi) - n\right)(w_\infty + s\phi)^2 \, d\mu \\
&= 2\int_{\mathcal{M}} \left(4\langle \nabla w_\infty, \nabla \phi\rangle + \left(R - \log\left(w_\infty^2\right) - \frac{n}{2}\log(4\pi) - n\right) w_\infty\phi - w_\infty\phi\right) d\mu.
\end{aligned}$$

That is, by definition w_∞ is a weak solution to the following second-order elliptic equation

$$-4\Delta w_\infty + R w_\infty - w_\infty \log\left(w_\infty^2\right) - \left(\frac{n}{2}\log(4\pi) + n\right) w_\infty = \mu(g,1) w_\infty. \tag{17.77}$$

The constant $\mu(g,1)$ is determined by (17.75) and by substituting $\phi = w_\infty$ in (17.76).

STEP 3. *w_∞ is a positive C^∞ minimizer.* Define

$$P(w_\infty) \doteqdot R w_\infty - w_\infty \log\left(w_\infty^2\right) - \left(\frac{n}{2}\log(4\pi) + n + \mu(g,1)\right) w_\infty. \tag{17.78}$$

Since $w_\infty \in L^{\frac{2n}{n-2}}$, by (17.116) below we have $P(w_\infty) \in L^p$ for any $p \in [1, \frac{2n}{n-2})$ (here $\frac{2n}{n-2} \doteqdot \infty$ if $n = 2$), so that $w_\infty \in W^{2,p}$ by the standard (interior) L^p estimate for weak solutions to second-order elliptic equations (see Theorem 9.11 in [**71**]). Bootstrapping, we obtain $w_\infty \in W^{2,q}$ for all $q \in [1,\infty)$. By the Sobolev embedding theorem, this implies that $w_\infty \in C^{1,\alpha}$ for some $\alpha \in (0,1)$.

Since $w_\infty \geq 0$ everywhere and $w_\infty > 0$ somewhere (since $\int_{\mathcal{M}} w_\infty^2 d\mu = 1$), by the strong maximum principle for weak solutions of (17.77) (see Lemma 17.26 below), we have $w_\infty > 0$ everywhere on $\mathcal{M}$. Therefore $w_\infty \log w_\infty \in C^{1,\alpha}$, so that

$$4\Delta w_\infty = P(w_\infty) \in C^{1,\alpha}.$$

By the regularity theory for weak solutions of the Poisson equation, w_∞ is a classical solution and we may apply Schauder theory to conclude that $w_\infty \in C^{k,\alpha}$ for all $k \in \mathbb{N}$. Hence w_∞ is C^∞, so that $f_1 \doteqdot -\frac{n}{2}\log(4\pi) - 2\log w_\infty$ is a C^∞ minimizer of $\mathcal{K}(g, \cdot, 1)$. This completes the proof of Proposition 17.24. □

To conclude this subsection, we prove the following result, which was used in the proof of Proposition 17.24 (see p. 112 of [**161**]).

LEMMA 17.25. *Let* $(\mathcal{M}^n, g)$ *be a closed Riemannian manifold. The quantity* $\int_{\mathcal{M}} w^2 \log(w^2)\, d\mu$ *depends continuously on* w *with respect to the* $L^{2(1+\delta)}$*-norm for any* $\delta > 0$. *In particular, the dependence is continuous with respect to the* $W^{1,2}$*-norm.*

PROOF. To see this, suppose $w_1, w_2 \in L^{2(1+\delta)}(\mathcal{M})$, where $\delta > 0$. At each $x \in \mathcal{M}$ we have

$$\begin{aligned} & w_2^2 \log(w_2^2) - w_1^2 \log(w_1^2) \\ &= \int_{|w_1|}^{|w_2|} \frac{d}{du}\left(u^2 \log(u^2)\right) du \\ &= \int_{|w_1|}^{|w_2|} 2u\left(1 + \log(u^2)\right) du. \end{aligned}$$

Applying the mean value theorem for integrals to this, we have

$$w_2^2 \log(w_2^2) - w_1^2 \log(w_1^2) = (|w_2| - |w_1|) \cdot 2a\left(1 + \log(a^2)\right),$$

where $a : \mathcal{M} \to [|w_1|, |w_2|]$ and $a \in L^{2(1+\delta)}(\mathcal{M})$. Hence

$$\begin{aligned} & \int_{\mathcal{M}} \left(w_2^2 \log(w_2^2) - w_1^2 \log(w_1^2)\right) d\mu \\ & \leq \left(\int_{\mathcal{M}} |w_2 - w_1|^2 d\mu\right)^{1/2} \left(\int_{\mathcal{M}} 4a^2\left(1 + \log(a^2)\right)^2 d\mu\right)^{1/2} \end{aligned} \tag{17.79}$$

since $(|w_2| - |w_1|)^2 \leq |w_2 - w_1|^2$. Note that

$$|a \log a| \leq \max\left\{\frac{1}{e}, \frac{1}{\delta e} a^{1+\delta}\right\} \tag{17.80}$$

since for $a, \delta > 0$ we have $a \log a \geq -\frac{1}{e}$ and $\log a \leq \frac{1}{\delta e} a^{\delta}$. Therefore we can bound $\int_{\mathcal{M}} 4a^2\left(1 + \log(a^2)\right)^2 d\mu$ in terms of $\|w_1\|_{L^{2(1+\delta)}(\mathcal{M},g)}$ and $\|w_2\|_{L^{2(1+\delta)}(\mathcal{M},g)}$. The lemma now follows from (17.79). □

3.2. Strong maximum principle for weak solutions.

We now give the proof of the strong maximum principle for weak solutions,[9] which is used in the proofs of Proposition 17.20 and Proposition 17.24. *The following proof is on pp. 114–116 of Rothaus* [**161**] (*we also follows his notation for the most part*).

LEMMA 17.26 (Strong maximum principle for weak solutions). *Suppose* $(\mathcal{M}^n, g)$ *is a complete Riemannian manifold. Let* $w_\infty \geq 0$ *be a* $C^{1,\alpha}$ *function which is a weak solution to* (17.77). *If* $w_\infty(p) = 0$ *for some* $p \in \mathcal{M}$, *then* $w_\infty = 0$ *in a neighborhood of* p.

[9] On the other hand, Calabi [**21**] proved strong maximum principles for sub- and super-solutions *in the support sense*; see also Trudinger [**181**] and Theorem 2.4 of Andersson, Galloway, and Howard [**5**].

REMARK 17.27. Note that (17.77) is the same as (17.15) with $\tau = 1$. By scaling, one sees that the lemma holds for solutions of (17.15) with arbitrary $\tau > 0$.

The proof is via a monotonicity formula in the radial direction emanating from p. Denote by $S(p, r)$ the geodesic sphere of radius r centered at p. Let $\mathrm{J}(\theta, r)$, where $\theta \in \mathcal{S}^{n-1}(1)$ and $r \in (0, \mathrm{inj}(p))$, be the Jacobian of the exponential map in spherical coordinates, so that

$$d\mu\left(\exp_p(r\theta)\right) = \mathrm{J}(\theta, r)\left(\exp_p\right)_*\left(d\sigma_{\mathcal{S}^{n-1}(1)}\right) \wedge dr,$$

where $\mathcal{S}^{n-1}(1) \subset T_p\mathcal{M}$ is the unit sphere and $d\sigma_{\mathcal{S}^{n-1}(1)}$ is its volume $(n-1)$-form. Note that $\lim_{r\to 0} \frac{\mathrm{J}}{r^{n-1}} = 1$.

Now define

$$F : (0, \mathrm{inj}(p)) \to \mathbb{R}$$

by

$$F(r) = \frac{\int_{S(p,r)} w_\infty \, d\sigma}{\int_{S(p,r)} d\sigma} = \frac{\int_{\mathcal{S}^{n-1}(1)} w_\infty(\theta, r)\, \mathrm{J}(\theta, r)\, d\sigma_{\mathcal{S}^{n-1}(1)}}{\int_{\mathcal{S}^{n-1}(1)} \mathrm{J}(\theta, r)\, d\sigma_{\mathcal{S}^{n-1}(1)}},$$

where $d\sigma$ denotes the induced volume $(n-1)$-form on $S(p, r)$.[10] Since $w_\infty(p) = 0$ and w_∞ is continuous, we have

$$\lim_{r\to 0^+} F(r) = 0.$$

We shall derive a differential inequality for $F(r)$ to show that $F(r) = 0$ for r sufficiently small. Since $w_\infty \geq 0$, this implies $w_\infty = 0$ in a neighborhood of p.

If we set

$$G(r) = \frac{\int_{S(p,r)} \frac{\partial w_\infty}{\partial r} \, d\sigma}{\int_{S(p,r)} d\sigma} = \frac{\int_{\mathcal{S}^{n-1}(1)} \frac{\partial w_\infty(\theta,r)}{\partial r}\, \mathrm{J}(\theta, r)\, d\sigma_{\mathcal{S}^{n-1}(1)}}{\int_{\mathcal{S}^{n-1}(1)} \mathrm{J}(\theta, r)\, d\sigma_{\mathcal{S}^{n-1}(1)}},$$

then

$$\begin{aligned} F'(r) &= \frac{\int_{S(p,r)} \frac{\partial w_\infty}{\partial r}\, d\sigma}{\int_{S(p,r)} d\sigma} + \frac{\int_{S(p,r)} w_\infty \frac{\partial}{\partial r} \log \mathrm{J}\; d\sigma}{\int_{S(p,r)} d\sigma} \\ &\quad - \frac{\int_{S(p,r)} w_\infty\, d\sigma \int_{S(p,r)} \frac{\partial}{\partial r} \log \mathrm{J}\; d\sigma}{\left(\int_{S(p,r)} d\sigma\right)^2} \\ &= G(r) + \frac{\int_{S(p,r)} w_\infty \left(\frac{\partial}{\partial r} \log \mathrm{J} - \frac{\int_{S(p,r)} \frac{\partial}{\partial r} \log \mathrm{J}\; d\sigma}{\int_{S(p,r)} d\sigma}\right) d\sigma}{\int_{S(p,r)} d\sigma}. \end{aligned}$$

Since

$$\frac{\partial}{\partial r} \log \mathrm{J} - \frac{\int_{S(p,r)} \frac{\partial}{\partial r} \log \mathrm{J}\; d\sigma}{\int_{S(p,r)} d\sigma} = O(r),$$

[10] That is, $F(r)$ is the average value of w_∞ on $S(p, r)$.

there exists a constant $C < \infty$ such that

$$F'(r) \leq G(r) + CrF(r) . \tag{17.81}$$

Now we proceed to estimate $G(r)$ from above. Let $\varphi \in C_0^\infty(B)$ be a radial function (i.e., a function of $d(\cdot, p)$), where $B = B(p, \operatorname{inj}(p))$. We have

$$\begin{aligned}\int_B \nabla w_\infty \cdot \nabla \varphi \, d\mu &= \int_0^{\operatorname{inj}(p)} dr \int_{S(p,r)} \frac{\partial w_\infty}{\partial r} \varphi'(r) \, d\sigma \\ &= \int_0^{\operatorname{inj}(p)} \varphi'(r) A(r) G(r) \, dr,\end{aligned}$$

where

$$A(r) = \int_{S(p,r)} d\sigma$$

is the $(n-1)$-dimensional volume of $S(p,r)$. Since w_∞ is a weak solution to (17.77),

$$\begin{aligned}&\int_0^{\operatorname{inj}(p)} \varphi'(r) A(r) G(r) \, dr \\ &\quad = \frac{1}{2} \int_B w_\infty \log(w_\infty) \, \varphi \, d\mu \\ &\qquad + \frac{1}{4} \int_B \left(-R + \frac{n}{2} \log(4\pi) + n + \mu(g,1) \right) w_\infty \varphi \, d\mu.\end{aligned} \tag{17.82}$$

Let

$$L(r) = \frac{1}{2A(r)} \int_{S(p,r)} w_\infty \log w_\infty \, d\sigma \tag{17.83}$$

and

$$K(r) = \frac{1}{4A(r)} \int_{S(p,r)} \left(R - \frac{n}{2} \log(4\pi) - n - \mu(g,1) \right) w_\infty \, d\sigma, \tag{17.84}$$

so that (17.82) implies that $G(r)$ satisfies

$$\int_0^{\operatorname{inj}(p)} \left(\varphi'(r) G(r) - \varphi(r) L(r) + \varphi(r) K(r) \right) A(r) \, dr = 0$$

for all radial functions $\varphi \in C_0^\infty(B)$. This implies (Rothaus says, 'By the usual one-dimensional regularity result, ...')

$$\frac{d}{dr}(G(r) A(r)) = (K(r) - L(r)) A(r) . \tag{17.85}$$

Now from definition (17.84) we have

$$\begin{aligned}K(r) &\leq \frac{C}{A(r)} \int_{S(p,r)} w_\infty \, d\sigma \\ &= CF(r) ,\end{aligned}$$

where $C = \frac{1}{4}\left(\max_{\bar{B}(p,\mathrm{inj}(p))} R - \frac{n}{2}\log(4\pi) - n - \mu(g,1)\right)$. Moreover, Jensen's inequality says that if $\varphi : [0,\infty) \to \mathbb{R}$ is convex, then

$$\frac{1}{A(r)} \int_{S(p,r)} \varphi \circ w_\infty \, d\sigma \geq \varphi\left(\frac{1}{A(r)} \int_{S(p,r)} w_\infty \, d\sigma\right) = \varphi(F(r)).$$

Applying Jensen's inequality with $\varphi(x) = x \log x$ to (17.83), we have

$$2L(r) \geq F(r) \log F(r).$$

Applying these two inequalities to (17.85), we obtain

$$\frac{d}{dr}(G(r) A(r)) \leq \left(CF(r) - \frac{1}{2} F(r) \log F(r)\right) A(r). \tag{17.86}$$

Since $w_\infty \geq 0$, $w_\infty(p) = 0$, and $w_\infty \in C^{1,\alpha}$, there exists a sequence $r_i \to 0^+$ such that

$$\lim_{i\to\infty} G(r_i) A(r_i) = 0.$$

Thus integrating (17.86) on the interval $[r_i, r]$ and taking $i \to \infty$, we have

$$G(r) \leq \frac{1}{A(r)} \int_0^r \left(CF(s) - \frac{1}{2} F(s) \log F(s)\right) A(s) \, ds.$$

Substituting this into (17.81) yields

$$F'(r) \leq CrF(r) + \frac{1}{A(r)} \int_0^r \left(CF(s) - \frac{1}{2} F(s) \log F(s)\right) A(s) \, ds.$$

Since $\lim_{r\to 0^+} F(r) = 0$, we obtain

$$F(t) \leq C \int_0^t rF(r) \, dr + \int_0^t \frac{dr}{A(r)} \int_0^r \left(CF(s) - \frac{1}{2} F(s) \log F(s)\right) A(s) \, ds$$

for $t \in (0, \mathrm{inj}(p))$.

Now assume that $t_0 \leq \mathrm{inj}(p)$ is small enough so that for $t \in (0, t_0]$,

(1) $C_1 t^{n-1} \leq A(t) \leq C_2 t^{n-1}$, where $C_1 > 0$ and $C_2 < \infty$,

(2) $0 \leq F(t) \leq 1$.

Then for $t \in (0, t_0]$,

$$F(t) \leq C\left(\int_0^t rF(r) \, dr - \int_0^t \frac{dr}{r^{n-1}} \int_0^r s^{n-1} F(s) \log F(s) \, ds\right) + C \int_0^t \frac{dr}{r^{n-1}} \int_0^r s^{n-1} F(s) \, ds$$

for some $C < \infty$. Now for $a \in (0, t_0]$ there exists $b = b(a) < \infty$ (where $\lim_{a\to 0^+} b(a) = 0$) such that

$$F(s) \leq b$$

for $s \in (0, a]$. If $t \in (0, a]$, then

$$F(t) \leq C\left(\int_0^t br\,dr + \left(\frac{1}{e} + b\right)\int_0^t \frac{dr}{r^{n-1}}\int_0^r s^{n-1}ds\right) = C\left(b + \frac{\frac{1}{e}+b}{n}\right)\frac{t^2}{2}$$

since $-x\log x \leq \frac{1}{e}$. Choosing a small enough, we have

$$F(t) \leq t$$

for $t \in (0, a]$.

In general, if we have $F(t) \leq t^k$ on $(0, a]$ for some $k \geq 1$, then

$$F(t) \leq C\left(\int_0^t r^{k+1}dr - \int_0^t \frac{dr}{r^{n-1}}\int_0^r s^{n-1+k}k\log s\,ds\right) + C\int_0^t \frac{dr}{r^{n-1}}\int_0^r s^{n-1+k}ds.$$

Applying $-x\log x \leq \frac{1}{e}$ again and integrating, we obtain

$$F(t) \leq C\left(\frac{t^{k+2}}{k+2} + \frac{k}{e}\int_0^t \frac{dr}{r^{n-1}}\int_0^r s^{n-2+k}ds + \int_0^t \frac{r^{k+1}}{n+k}dr\right)$$

$$= C\left(\frac{t^{k+2}}{k+2} + \frac{k}{e}\frac{t^{k+1}}{(k+1)(n+k-1)} + \frac{t^{k+2}}{(k+2)(n+k)}\right) \tag{17.87}$$

provided $t \in (0, \min\{a, 1/e\}]$, where C is independent of k. Now (17.87) implies that there exists $a_0 \in (0, \min\{a, 1/e\})$ independent of $k \geq 1$ such that[11]

$$F(t) \leq t^{k+1/2}$$

for $t \in (0, a_0]$. By induction, we have

$$F(t) \leq t^{\ell}$$

for $t \in (0, a_0]$ and all $\ell \geq 1$. This implies $F(t) = 0$ for $t \in (0, a_0]$ and Lemma 17.26 is proved. □

3.3. The maximum value of a minimizer.

Under the constraint $\int_{\mathcal{M}} w^2 d\mu_g = 1$, let w_τ be a minimizer of the functional $\mathcal{K}(g, w, \tau)$ defined by (17.7). The maximum value of w_τ is related to an upper bound for the μ-invariant as follows (this is used in the proof of Proposition 17.20).

[11]Indeed, we just need a small enough so that

$$C\left(\frac{t^2}{k+2} + \frac{k}{e}\frac{t}{(k+1)(n+k-1)} + \frac{t^2}{(k+2)(n+k)}\right) \leq \sqrt{t}$$

for $t \in (0, a]$ and $k \geq 1$.

LEMMA 17.28 (Lower bound for the maximum value of a minimizer). *On a closed manifold we have*

$$\max_{\mathcal{M}} w_\tau \geq \exp\left(\frac{\tau}{2} R_{\min}(g) - \frac{n}{4}\log(4\pi\tau) - \frac{n}{2} - \frac{1}{2}\mu(g,\tau)\right). \tag{17.88}$$

PROOF. By (17.15), a minimizer w_τ satisfies

$$\tau\left(-4\Delta w_\tau + R w_\tau\right) - w_\tau \log\left(w_\tau^2\right) - \left(\frac{n}{2}\log(4\pi\tau) + n\right) w_\tau = \mu(g,\tau)\, w_\tau.$$

At a point $x_\tau \in \mathcal{M}$ where w_τ attains its maximum, we have $(\Delta w_\tau)(x_\tau) \leq 0$, so that

$$\tau R w_\tau - 2 w_\tau \log w_\tau - \left(\frac{n}{2}\log(4\pi\tau) + n\right) w_\tau \leq \mu(g,\tau)\, w_\tau.$$

Hence

$$\max_{\mathcal{M}} w_\tau = w_\tau(x_\tau) \geq \exp\left(\frac{\tau}{2} R - \frac{n}{4}\log(4\pi\tau) - \frac{n}{2} - \frac{1}{2}\mu(g,\tau)\right)$$

and the lemma follows. □

4. 1- and 2-loop variation formulas related to RG flow

In this section we discuss formulas related to Perelman's energy functional and its variation.[12] One may wish that some of these formulas are related to 'renormalization group flow' (RG flow) in physics; the 'loop' terminology is from there. However the point of view we take is simply to calculate first variation formulas for certain Riemannian geometric invariants and to look for structure in these formulas. We leave the calculations as exercises for the reader.

4.1. Some 1-loop formulas.

Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold and let f be a function on $\mathcal{M}$. Recall from (17.1) that Perelman's energy functional is

$$\mathcal{F}_1(g,f) = \int_{\mathcal{M}} \left(R + 2\Delta f - |\nabla f|^2\right) e^{-f} d\mu \tag{17.89}$$

(we add the subscript 1 to $\mathcal{F}$ with the hope that this is the first in an infinite sequence of functionals). The integrand in (17.89) appears in a contracted second Bianchi-type identity (see §1.3 of [**152**]):

$$\operatorname{div}\left(\left(\mathrm{Rc} + \nabla\nabla f\right) e^{-f}\right) = \frac{1}{2} e^{-f} \nabla\left(R + 2\Delta f - |\nabla f|^2\right). \tag{17.90}$$

Let v be a symmetric 2-tensor on $\mathcal{M}$ and let X be a vector field on $\mathcal{M}$. The **linear trace Harnack quadratic** is given by

$$L(v,X) \doteqdot \operatorname{div}(\operatorname{div} v) + \langle v, \mathrm{Rc}\rangle - 2\langle \operatorname{div} v, X\rangle + v(X,X) \tag{17.91}$$

[12] We would like to thank Shengli Kong for helpful discussions.

(see Theorem A.57 in Part I for the corresponding linear trace Harnack estimate). When $X = \nabla f$, we may rewrite this as

$$L(v, \nabla f) = (\operatorname{div} - \iota_{\nabla f}) \circ (\operatorname{div} - \iota_{\nabla f}) v + \langle \operatorname{Rc} + \nabla\nabla f, v \rangle \tag{17.92}$$

$$= e^f (\operatorname{div} \circ \operatorname{div} + \operatorname{Rc} + \nabla\nabla f) \left(e^{-f} v\right). \tag{17.93}$$

Observe also that

$$L(2\operatorname{Rc}, X) = \frac{\partial R}{\partial t} - 2\langle \nabla R, X\rangle + 2\operatorname{Rc}(X, X)$$

is Hamilton's trace Harnack quadratic (see (15.17) in Part II).

Let $V = g^{ij} v_{ij}$.Two of the above quantities are related by the following (see also Lemma 6.82 in Part I)

LEMMA 17.29 (Variation of Perelman's modified scalar curvature). *If $\frac{\partial}{\partial s} g = v$ and $\frac{\partial}{\partial s} f = V$ (so that $\frac{\partial}{\partial s}\left(e^{-f} d\mu\right) = 0$), then*

$$\frac{\partial}{\partial s}\left(R + 2\Delta f - |\nabla f|^2\right) = L(v, \nabla f) - 2\langle v, \operatorname{Rc} + \nabla\nabla f\rangle. \tag{17.94}$$

Integrating the above formula by parts, we obtain (see §1.1 of [**152**]; compare with Exercise 6.16 in Part I)

LEMMA 17.30 (Perelman's first variation formula for $\mathcal{F}$). *If $\frac{\partial}{\partial s} g = v$ and $\frac{\partial}{\partial s} f = V$, then*

$$\frac{\partial}{\partial s}\mathcal{F}_1(g, f) = -\int_{\mathcal{M}} \langle v, \operatorname{Rc} + \nabla\nabla f\rangle e^{-f} d\mu \tag{17.95}$$

$$= -\int_{\mathcal{M}} L(v, \nabla f) e^{-f} d\mu. \tag{17.96}$$

4.2. Some 2-loop formulas.

Let

$$(\Delta_L h)_{ij} \doteqdot \Delta h_{ij} + 2R_{kij\ell} h_{k\ell} - R_{ik} h_{kj} - R_{jk} h_{ki}$$

be the Lichnerowicz Laplacian, which acts on symmetric 2-tensors. Given a function f on $\mathcal{M}$, define the symmetric 2-tensor $H_{\nabla f}$, which is a form of **Hamilton's matrix Harnack quadratic**, by (compare with the expression in (15.11) of Part II)

(17.97)

$$H_{\nabla f}(X, Y) \doteqdot \left(\Delta_L \operatorname{Rc} - \frac{1}{2}\nabla\nabla R + \operatorname{Rc}^2\right)(X, Y) + P(X, \nabla f, Y) + P(Y, \nabla f, X) + \operatorname{Rm}(\nabla f, X, Y, \nabla f),$$

where

$$P(X, Y, Z) \doteqdot (\nabla_X \operatorname{Rc})(Y, Z) - (\nabla_Y \operatorname{Rc})(X, Z)$$

for tangent vectors X, Y, Z (see Chapter 15 in Part II for the proof of Hamilton's matrix Harnack estimate). This may be rewritten as

$$H_{\nabla f} = e^f (\operatorname{div} \circ \operatorname{div} + \operatorname{Rc} + \nabla\nabla f)_{1,4} \left(e^{-f} \operatorname{Rm}\right), \tag{17.98}$$

where the subscripts $1, 4$ denote the components on which the operator is acting. Note that if $\mathrm{Rc} + \nabla\nabla f \equiv 0$, then $H_{\nabla f} \equiv 0$.

Recall that under

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}, \tag{17.99a}$$

$$\frac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2, \tag{17.99b}$$

we have (see Proposition 6.95 in Part I)

(17.100)

$$\left(\frac{\partial}{\partial t} + \Delta_L - 2\nabla f \cdot \nabla\right)(\mathrm{Rc} + \nabla\nabla f) = 2H_{\nabla f} - (\mathrm{Rc} + \nabla\nabla f) \circledcirc (\mathrm{Rc} - \nabla\nabla f),$$

where, for $(1,1)$-tensors A_i^j and B_i^j, we define $(A \circledcirc B)_i^j \doteqdot A_i^k B_k^j + B_i^k A_k^j$.

Define

$$P^*(X, Y, Z) \doteqdot P(Z, Y, X),$$

so that (see (15.50) in Part II)

$$\operatorname{div}(\mathrm{Rm}) = P^*,$$

where $\operatorname{div}(\mathrm{Rm}) = \operatorname{trace}_g^{1,2}(\nabla\,\mathrm{Rm})$ and where the superscripts $1, 2$ indicate that the first two components are traced.

Under the Ricci flow, the Riemann curvature $(3,1)$-tensor Rm evolves by (see (6.2) in Volume One)

$$\frac{\partial}{\partial t}\mathrm{Rm} = -[\nabla, \nabla]\,\mathrm{Rc} + d_\nabla P^*, \tag{17.101}$$

where $(d_\nabla P^*)(X, Y, Z, W) = (\nabla_X P^*)(Y, Z, W) - (\nabla_Y P^*)(X, Z, W)$.

Define the symmetric 2-tensor

$$\alpha \doteqdot \sum_{i,j,k} \mathrm{Rm}(\cdot, e_i, e_j, e_k)\,\mathrm{Rm}(\cdot, e_i, e_j, e_k), \tag{17.102}$$

where $\{e_\ell\}$ is an orthonormal frame, i.e., $\alpha_{ij} = R_{ik\ell p} R_{jk\ell p}$. Then α satisfies the contracted second Bianchi-type identity

$$\operatorname{div}\left(\alpha - \frac{1}{4}|\mathrm{Rm}|^2 g\right) = \mathrm{Rm}(\cdot, e_i, e_j, e_k)\,P^*(e_i, e_j, e_k). \tag{17.103}$$

Now we discuss some calculations due to two of the authors [**44**]. By (17.101), under the Ricci flow

$$\begin{aligned}\frac{1}{4}\frac{\partial}{\partial t}|\mathrm{Rm}|^2 &= \langle \mathrm{Rc}, \alpha\rangle + \langle \mathrm{Rm}, \nabla P^*\rangle \\ &= \langle \mathrm{Rc}, \alpha\rangle + \operatorname{div}(\mathrm{Rm}(\cdot, e_i, e_j, e_k)\,P^*(e_i, e_j, e_k)) - |P|^2.\end{aligned} \tag{17.104}$$

We then compute that under (17.99)

$$\frac{1}{4}\Box^*\left(|\mathrm{Rm}|^2 e^{-f}\right) = e^{-f}\left(-L(\alpha, \nabla f) + |P^* - \iota_{\nabla f}\,\mathrm{Rm}|^2\right), \tag{17.105}$$

where

$$\square^* \doteqdot -\frac{\partial}{\partial t} - \Delta + R$$

and where $(\iota_{\nabla f} \operatorname{Rm})(X, Y, Z) = \operatorname{Rm}(\nabla f, X, Y, Z)$.

If $\frac{\partial}{\partial s} g = v$ and $\frac{\partial}{\partial s} f = \frac{V}{2}$, then

$$\begin{aligned} \frac{1}{4}\frac{d}{ds} \int |\operatorname{Rm}|^2 e^{-f} d\mu & \\ = \int v \cdot \left(-H_{\nabla f} - \frac{1}{2}\alpha + \left\langle \operatorname{Rm}_{(1,4)}, \operatorname{Rc} + \nabla\nabla f \right\rangle \right) e^{-f} d\mu, \end{aligned} \tag{17.106}$$

where the subscript $(1, 4)$ denotes the components on which the inner product is acting. As a special case, if $\operatorname{Rc} + \nabla\nabla f \equiv 0$, then $\frac{1}{4}\frac{d}{ds} \int |\operatorname{Rm}|^2 e^{-f} d\mu = -\frac{1}{2} \int \langle v, \alpha \rangle e^{-f} d\mu$.

We summarize the above formulas. For $a = 1, 2$, define the 2-tensors $\beta^{(a)}$, the functions $\gamma^{(a)}$, and the functionals $\mathcal{F}_a(g, f)$ by

$$\beta^{(1)} = -2(\operatorname{Rc} + \nabla\nabla f), \tag{17.107a}$$

$$\beta^{(2)} = -\alpha, \tag{17.107b}$$

$$\gamma^{(1)} = -R - \Delta f, \tag{17.107c}$$

$$\gamma^{(2)} = -\frac{1}{2} |\operatorname{Rm}|^2, \tag{17.107d}$$

$$\mathcal{F}_1(g, f) = \int_{\mathcal{M}} \left(R + |\nabla f|^2 \right) e^{-f} d\mu, \tag{17.107e}$$

$$\mathcal{F}_2(g, f) = \frac{1}{4} \int_{\mathcal{M}} |\operatorname{Rm}|^2 e^{-f} d\mu. \tag{17.107f}$$

Given a functional $\mathcal{G}(g, f)$, let $\delta_{(\beta^{(a)}, \gamma^{(a)})} \mathcal{G}(g, f)$ denote $\frac{d}{ds}\mathcal{G}(g(s), f(s))$ under $\frac{\partial}{\partial s} g(s) = \beta^{(a)}$ and $\frac{\partial}{\partial s} f(s) = \gamma^{(a)}$. We have

$$\begin{aligned} \delta_{(\beta^{(1)}, \gamma^{(1)})} \left(\frac{1}{4} |\operatorname{Rm}|^2 e^{-f} d\mu \right) & \\ = -\frac{1}{4} \operatorname{div}\left(e^{-f} \nabla |\operatorname{Rm}|^2 \right) d\mu + L(\alpha, \nabla f) e^{-f} d\mu & \\ - |P^* - \iota_{\nabla f} \operatorname{Rm}|^2 e^{-f} d\mu & \end{aligned} \tag{17.108}$$

and

$$\begin{aligned} \delta_{(\beta^{(2)}, \gamma^{(2)})} \left(\left(R + 2\Delta f - |\nabla f|^2 \right) e^{-f} d\mu \right) = -L(\alpha, \nabla f) e^{-f} d\mu & \\ + 2 \langle \alpha, \operatorname{Rc} + \nabla\nabla f \rangle e^{-f} d\mu, & \end{aligned} \tag{17.109}$$

so that

(17.110)
$$\begin{aligned}&\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\left(-\frac{1}{4}\left|\mathrm{Rm}\right|^2 e^{-f}d\mu\right)+\delta_{\left(\beta^{(2)},\gamma^{(2)}\right)}\left(\left(R+2\Delta f-\left|\nabla f\right|^2\right)e^{-f}d\mu\right)\\&=-\operatorname{div}\left(\frac{1}{4}e^{-f}\nabla\left|\mathrm{Rm}\right|^2\right)d\mu+2\left\langle\alpha,\mathrm{Rc}+\nabla\nabla f\right\rangle e^{-f}d\mu\\&\quad-\left|P^*-\iota_{\nabla f}\mathrm{Rm}\right|^2 e^{-f}d\mu.\end{aligned}$$

We obtain

$$\begin{aligned}&\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_2(g,f)+\delta_{\left(\beta^{(2)},\gamma^{(2)}\right)}\mathcal{F}_1(g,f)\\&=-\int_{\mathcal{M}}\left|P^*-\iota_{\nabla f}\mathrm{Rm}\right|^2 e^{-f}d\mu+2\int_{\mathcal{M}}\left\langle\alpha,\mathrm{Rc}+\nabla\nabla f\right\rangle e^{-f}d\mu.\end{aligned}\tag{17.111}$$

Note that

(17.112)
$$-\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_2(g,f)+\delta_{\left(\beta^{(2)},\gamma^{(2)}\right)}\mathcal{F}_1(g,f)=\int_{\mathcal{M}}\left|P^*-\iota_{\nabla f}\mathrm{Rm}\right|^2 e^{-f}d\mu\geq 0.$$

REMARK 17.31. If one defines $\beta_{ij}^{(0)}=g_{ij}$, $\gamma^{(0)}=\frac{n}{2}$, and $\mathcal{F}_0(g,f)=-\int_{\mathcal{M}}fe^{-f}d\mu$, then

$$\delta_{\left(\beta^{(0)},\gamma^{(0)}\right)}\mathcal{F}_1(g,f)+\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_0(g,f)=0.\tag{17.113}$$

PROBLEM 17.32. Do any of the above formulas fit into an infinite sequence of formulas? One would like to obtain a monotonicity formula extending Perelman's entropy monotonicity formula.

In particular, one may wish to consider an expression of the form

(17.114)
$$\begin{aligned}&\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_1(g,f)+\lambda\left(\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_2(g,f)+\delta_{\left(\beta^{(2)},\gamma^{(2)}\right)}\mathcal{F}_1(g,f)\right)\\&+\lambda^2\left(\delta_{\left(\beta^{(1)},\gamma^{(1)}\right)}\mathcal{F}_3(g,f)+\delta_{\left(\beta^{(2)},\gamma^{(2)}\right)}\mathcal{F}_2(g,f)+\delta_{\left(\beta^{(3)},\gamma^{(3)}\right)}\mathcal{F}_1(g,f)\right)\\&+\cdots,\end{aligned}$$

where $\beta^{(k)}$, $\gamma^{(k)}$, $\mathcal{F}_k$ are suitably defined for $k\in\mathbb{N}$ and $\lambda\in\mathbb{R}_+$. However it is not clear whether or not one should introduce new fields in addition to g and f.

Some related calculations are in Oliynyk, Suneeta, and Woolgar [**145**]; see also Zamolodchikov [**194**] and Tseytlin [**182**].[13] In the physics literature there are various 3- and 4-loop calculations.

[13]We would like to thank C. Vafa for discussions at the California Institute of Technology during January and February of 2003. The first author would also like to thank A. Tseytlin for discussions at Ohio State University during May of 2003.

5. Notes and commentary

The monotonicity formulas which are applied in this chapter were discovered partly based on the notion of self-similarity.

On a more philosophical note, we venture to ask the following general question.

PROBLEM 17.33. In Ricci flow some of the guiding ideas and principles, among others, used to study the geometry of solutions are

(1) analogies with the heat equation and its smoothing aspects,
(2) applications of the maximum principle and monotonicity,
(3) self-similar solutions, i.e., Ricci solitons, used to find quantities to estimate,
(4) natural space-time quantities such as the reduced distance,
(5) point picking and determining location and scale.

Can one discover new guiding principles and ideas?

(A) One goal (espoused by Hamilton) is to localize various formulas. (Perelman's pseudolocality in effect does this for the curvature evolution under certain hypotheses.) Can one find new quantities on space-time (or some larger system) to help accomplish this?
(B) An important problem (espoused by Perelman) is to formulate a notion of weak solution which enables the flow to continue past singularities.
(C) Largely uncharted territory is the role of Riemannian Ricci flow in higher dimensions. Note that Lie algebra aspects of the evolution of the curvature operator, which began with Hamilton's work on 4-manifolds, have been studied by Böhm and Wilking with applications to the classification of closed manifolds with 2-positive curvature operator. In addition, Hamilton and Perelman proved a number of results in arbitrary dimensions in their works.
(D) Is it fruitful to study Ricci flow on spin manifolds or some other large class of manifolds?

§1. In contrast to the lower bound given by Lemma 17.8 for the volume of a solution $g(t)$ to Ricci flow with $\lambda(g(t)) \leq 0$, we have the following.

Upper bound for the volume of a solution with $\lambda > 0$. Suppose that a solution $(\mathcal{M}^n, g(t))$ of the Ricci flow on a closed manifold and maximal time interval $[0, T)$ has $\lambda(g(0)) > 0$. Since $\frac{d}{dt}\lambda(g(t)) \geq \frac{2}{n}\lambda(g(t))^2$ (see Lemma 5.25 in Part I), we have

$$\lambda(g(t)) \geq \frac{1}{\lambda(g(0))^{-1} - \frac{2}{n}t},$$

so that $T \leq \frac{n}{2}\lambda(g(0))^{-1} < \infty$. Since

$$\frac{d}{dt}\log \mathrm{Vol}(g(t)) = -R_{\mathrm{avg}}(g(t)) \leq -\lambda(g(t)),$$

we also obtain

$$\operatorname{Vol}(g(t)) \leq \left(1 - \frac{2}{n}\lambda(g(0))\,t\right)^{n/2} \operatorname{Vol}(g(0)).$$

§2. (1) The following is used in the proof of Proposition 17.20. From the formula

$$\frac{d}{dx}\left(x^{-\delta}\log x\right) = x^{-\delta-1}(1 - \delta\log x)$$

for $\delta > 0$, we see that the maximum of $x^{-\delta}\log x$ on $(0,\infty)$ is $\frac{1}{\delta}e^{-1}$ which occurs at $x = e^{1/\delta}$. Hence

$$-\frac{1}{e} \leq w\log w \leq \frac{1}{\delta e}w^{1+\delta}.$$

Thus for any $q > 0$ and $\delta > 0$,

$$\int_{\mathcal{M}} w^q (\log w)^q \, d\mu \leq \frac{1}{(\delta e)^q}\int_{\mathcal{M}} w^{q(1+\delta)} d\mu + \frac{1}{e^q}\operatorname{Vol}(g). \tag{17.115}$$

Note that we have for any $p < \frac{2n}{n-2}$

$$\int_{\mathcal{M}} \varphi^p (\log\varphi)^p \, d\mu \leq \left(\frac{2ne}{(n-2)\,p} - e\right)^{-p}\int_{\mathcal{M}} \varphi^{\frac{2n}{(n-2)}} d\mu + \frac{1}{e^p}\operatorname{Vol}(g), \tag{17.116}$$

where we have used $\log x \leq \frac{1}{\delta e}x^\delta$ with $\delta = \frac{2n}{(n-2)p} - 1 > 0$.

Following the proof of Lemma 6.36 in Part I, we conclude that for $a > 0$

$$\int_{\mathcal{M}} \varphi^2 (\log\varphi)^2 \, d\mu_g \leq a\int_{\mathcal{M}} |\nabla\varphi|_g^2 \, d\mu_g + \frac{n^4}{ae^4 C_s(\mathcal{M}, g)} + a\operatorname{Vol}(g)^{-2/n} + \frac{1}{e^2}\operatorname{Vol}(g). \tag{17.117}$$

(2) We can define U_i and Φ_i in (17.52) explicitly as follows. Let $\operatorname{inj}(g)$ denote the injectivity radius of g (note that $\operatorname{inj}(g_i) = \frac{\operatorname{inj}(g)}{\sqrt{2\tau_i}} \to \infty$ as $i \to \infty$). Choose orthonormal frames $\left\{e_\alpha^i\right\}_{\alpha=1}^n$ at x_i with respect to g_i and let

$$\psi_i : (\mathbb{R}^n, g_{\mathbb{R}^n}) \to (T_{x_i}\mathcal{M}, g_i(x_i))$$

be the linear isometry taking the standard basis in $\mathbb{R}^n$ to $\left\{e_\alpha^i\right\}_{\alpha=1}^n$. Let

$$U_i \doteqdot B\left(\vec{0}, \frac{\operatorname{inj}(g)}{\sqrt{2\tau_i}}\right) \subset \mathbb{R}^n$$

and define $\Phi_i : U_i \to \mathcal{M}$ by

$$\Phi_i = \exp_{x_i}^g \circ \psi_i,$$

where $\exp_{x_i}^g$ denotes the exponential map of g at x_i. Then (17.52) holds.