Preface

Among the most notable events in nonlinear functional analysis for the last three decades one should mention the development of the theory of differentiable measures and the Malliavin calculus. These two closely related theories can be regarded at the same time as infinite dimensional analogues of such classical fields as geometric measure theory, the theory of Sobolev spaces, and the theory of generalized functions.

The theory of differentiable measures was suggested by S.V. Fomin in his address at the International Congress of Mathematicians in Moscow in 1966 as a candidate for an infinite dimensional substitute of the Sobolev–Schwartz theory of distributions (see [**440**]–[**443**], [**82**], [**83**]). It was Fomin who realized that in the infinite dimensional case instead of a pair of spaces (test functions – generalized functions) it is natural to consider four spaces: a certain space of functions S, its dual S', a certain space of measures M, and its dual M'. In the finite dimensional case, M is identified with S by means of Lebesgue measure which enables us to represent measures via their densities. The absence of translationally invariant nonzero measures destroys this identification in the infinite dimensional case. Fomin found a convenient definition of a smooth measure on an infinite dimensional space corresponding to a measure with a smooth density on $\mathbb{R}^n$. By means of this definition one introduces the space M of smooth measures. The Fourier transform then acts between S and M and between S' and M'. In this way a theory of pseudo-differential operators can be developed: the initial objective of Fomin was a theory of infinite dimensional partial differential equations. However, as it often happens with fruitful ideas, the theory of differentiable measures overgrew the initial framework. This theory has become an efficient tool in a wide variety of the most diverse applications such as stochastic analysis, quantum field theory, and nonlinear analysis. At present it is a rapidly developing field abundant with many challenging problems of great importance for better understanding of the nature of infinite dimensional phenomena.

It is worth mentioning that before the pioneering works of Fomin related ideas appeared in several papers by T. Pitcher on the distributions of random processes in functional spaces. T. Pitcher investigated an even more general situation, namely he studied the differentiability of a family of measures $\mu_t = \mu \circ T_t^{-1}$ generated by a family of transformations T_t of a fixed measure μ. Fomin's differentiability corresponds to the case where the measures μ_t are the shifts μ_{th} of a measure μ on a linear space. However, the theory constructed for this more special case is much richer. Let us also mention similar ideas of L. Gross [**322**] developed in the analysis on Wiener space.

In the mid 1970s P. Malliavin [**751**] suggested an elegant method for proving the smoothness of the transition probabilities of finite dimensional diffusions. The essence of the method is to consider the transition probabilities P_t as images of the Wiener measure under some nonlinear transformations (generated by stochastic differential equations) and then apply an integration by parts formula on the Wiener space leading to certain estimates of the generalized derivatives of P_t which ensure the membership in C^∞. As an application, a probabilistic proof of Hörmander's theorem on hypoelliptic second order operators was given. Malliavin's method (now known as *the Malliavin calculus* or *the stochastic variational calculus*) attracted considerable attention and was refined and developed by many authors from different countries, including P. Malliavin himself. It was soon realized that the Malliavin calculus is a powerful tool for proving the differentiability of the densities of measures induced by finite dimensional nonlinear mappings of measures on infinite dimensional spaces (under suitable assumptions of regularity of the mappings and measures). Moreover, the ideas of the Malliavin calculus have proved to be very efficient in many applications such as stochastic differential equations, filtering of stochastic processes, analysis on infinite dimensional manifolds, asymptotic behavior of heat kernels on finite dimensional Riemannian manifolds (including applications to the index theorems), and financial mathematics.

Although the theory of differentiable measures and the Malliavin calculus have a lot in common and complement each other in many respects, their deep connection was not immediately realized and explored. The results of this exploration have turned out to be exciting and promising.

Let us consider several examples illuminating, on the one hand, the basic ideas of the theory of differentiable measures and the Malliavin calculus, and, on the other hand, explaining why the analysis on the Wiener space (or the Gaussian analysis) is not sufficient for dealing with smooth measures in infinite dimensions. First of all, which measures, say, on a separable Hilbert space X, should be regarded as infinite dimensional analogues of absolutely continuous measures on $\mathbb{R}^n$? It is well-known that there is no canonical infinite dimensional substitute for Lebesgue measure. For example, no nonzero locally finite Borel measure μ on l^2 is invariant or quasi-invariant under translations, that is, if $\mu \sim \mu(\cdot + h)$ for every h, then μ is identically zero. Similarly, if μ is not the Dirac measure at the origin, it cannot be invariant under all orthogonal transformations of l^2. In the finite dimensional case, instead of referring to Lebesgue measure one can fix a nondegenerate Gaussian measure γ. Then such properties as the existence of densities or regularity of densities can be described via densities with respect to γ. For example, it is possible to introduce Sobolev classes over γ as completions of the class of smooth compactly supported functions with respect to certain natural Sobolev norms. The Malliavin calculus pursues this possibility and enables us to study measures whose Radon–Nikodym densities with respect to γ are elements of such Sobolev classes. However, in the infinite dimensional case, there is a continuum of mutually singular nondegenerate Gaussian measures (e.g., all measures $\gamma_t(A) = \gamma(tA)$, $t > 0$, are mutually singular). Therefore, by fixing one of them, one restricts significantly the class of measures which can be investigated. Moreover, even if we do not fix a particular Gaussian measure, but consider measures for which there exist dominating Gaussian measures, we exclude from our considerations wide classes of smooth measures arising in applications. Let us consider an example of such a measure with an especially simple structure.

Example 1. Let μ be the countable product of measures m_n on the real line given by densities $p_n(t) = 2^n p(2^n t)$, where p is a smooth probability density with bounded support and finite Fisher's information, i.e., the integral of $(p')^2 p^{-1}$ is finite. It is clear that μ is a probability measure on l^2. The measure μ is mutually singular with *every* Gaussian measure (which will be proven in §5.3).

Certainly, this measure μ can be regarded and has the same property on $\mathbb{R}^\infty$ (the countable power of the real line). Note that μ is an invariant probability for the diffusion process ξ_t governed by the following stochastic differential equation (on the space $\mathbb{R}^\infty$):

$$d\xi_t = dW_t + \frac{1}{2}\beta(\xi_t)dt, \tag{1}$$

where the Wiener process W_t in $\mathbb{R}^\infty$ is a sequence (w_t^n) of independent real Wiener processes, and the vector field β is given by

$$\beta(x) = \bigl(p_n'(x_n)/p_n(x_n)\bigr)_{n=1}^\infty.$$

Therefore, in order to study differential properties of the transition probabilities and invariant measures of infinite dimensional diffusions we have to admit measures which cannot be characterized via Gaussian measures. Recent investigations of infinite dimensional diffusions have attracted considerable attention to the theory of differentiable measures and stimulated further progress in this field. One of the key achievements in this direction is the characterization of the drifts of symmetrizable diffusions as logarithmic gradients of measures. In the finite dimensional case, the logarithmic gradient of a measure μ with density p in the Sobolev class $W^{1,1}(\mathbb{R}^n)$ is given by $\beta = \nabla p/p$. It is known that in this case there is a diffusion ξ_t determined by equation (1) for which μ is a symmetric invariant measure. Conversely, if μ is a symmetric invariant measure for a diffusion ξ_t generated by equation (1), where β is μ-square integrable, then μ has a density $p \in W^{1,1}(\mathbb{R}^n)$ and $\beta = \nabla p/p$ μ-a.e. As we already know, in infinite dimensions, neither p nor ∇p are meaningful. Surprisingly enough, it is possible to give a rigorous meaning to the expression $\beta = \nabla p/p$ without defining its entries separately in such a way that in the finite dimensional case one obtains the usual interpretation. We shall discuss logarithmic gradients in Chapters 7 and 12 and see there that in the situation of Example 1 one gets exactly the mapping $\beta(x) = \bigl(p_n'(x_n)/p_n(x_n)\bigr)$. Moreover, we shall also see that in the infinite dimensional case logarithmic gradients of measures coincide with mappings β in equation (1) giving rise to symmetrizable diffusions.

Another related important problem, which both the theory of differentiable measures and the Malliavin calculus deal with, is the smoothness of the transition probabilities $P(t,x,\,\cdot\,)$ of the diffusion process ξ_t as set functions. In the finite dimensional case this corresponds to the differentiability of the transition density $p(t,x,y)$ with respect to y. In infinite dimensions such a problem in principal cannot be reduced to the study of densities because typically all measures $P(t,x,\,\cdot\,)$ are mutually singular for different t. For example, the transition probabilities $P(t,0,\,\cdot\,)$ of an infinite dimensional Wiener process W_t are given by $P(t,0,A) = \gamma(A/t)$, where γ is a Gaussian measure with infinite dimensional support. One can show that there is no σ-finite measure with respect to which all of these transition probabilities would be absolutely continuous. However, the theory of differentiable measures provides adequate tools for dealing with such examples.

Now we address the following problem. Let μ be a measure on a measurable space X and let F be a real function on X or an $\mathbb{R}^n$-valued mapping. Does the induced measure $\mu\circ F^{-1}$ have a density? Is this density smooth? Questions of this kind arise both in probability theory and analysis. The following reasoning in the spirit of the theory of differentiable measures and the Malliavin calculus illuminates one of the links between the two subjects.

Example 2. Let μ be a measure on $\mathbb{R}^n$ given by a density p from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and let F be a polynomial on $\mathbb{R}^n$. Suppose that p and all of its derivatives vanish on the set $Z = \{x\colon\ \nabla F(x) = 0\}$. For example, let μ be the standard Gaussian measure and $Z = \varnothing$. Then the measure $\nu = \mu\circ F^{-1}$ admits an infinitely differentiable density q.

For the proof we denote by D_v the differentiation along the vector field $v = \nabla F$. For any $\varphi \in C_0^\infty(\mathbb{R}^1)$ the integration by parts formula yields

$$\int \varphi'(y)\,\nu(dy) = \int \varphi'\bigl(F(x)\bigr)\,\mu(dx)$$
$$= \int D_v(\varphi\circ F)\frac{1}{D_vF}\,p\,dx = -\int \varphi\circ F D_v\frac{p}{D_vF}\,dx.$$

The right-hand side is estimated by $\sup_t |\varphi(t)| \, \|D_v(p/D_vF)\|_{L^1(\mathbb{R}^1)}$, whence it follows that the derivative of ν in the sense of generalized functions is a measure and so ν has a density of bounded variation. For justification of integration by parts we observe that $D_vF = |\nabla F|^2$ and the function

$$D_v(p/D_vF) = (\nabla p, \nabla F)|\nabla F|^{-2} - p|\nabla F|^{-4}\big(\nabla F, \nabla(|\nabla F|^2)\big)$$

has the form $|\nabla F|^{-2}(\nabla p, \nabla F) + |\nabla F|^{-4}gp$, where g is a polynomial. The principal problem is to show that for every partial derivative $\partial^r p$ of the function p and every $m \geqslant 1$ the function $\partial^r p/|\nabla F|^m$ belongs to $L^1(\mathbb{R}^n)$. This can be done by means of the well-known Seidenberg–Tarski theorem (see Trèves [**1122**, Lemma 5.7, p. 315]). For example, if the set Z is empty, then this theorem gives two positive numbers a and b for which one has the inequality

$$|\nabla F(x)| \geqslant a(1+|x|)^{-b}.$$

In this case,

$$f/|\nabla F|^m \in L^1(\mathbb{R}^n) \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad D_v(p/D_vF) \in L^1(\mathbb{R}^n).$$

In the general case,

$$|\nabla F(x)| \geqslant a \,\mathrm{dist}\,(x, Z)^{b_1}(1+|x|)^{-b_2},$$

which also ensures the required integrability due to our condition on p. Repeating this procedure we obtain that the integral of $\varphi^{(r)}$ against the measure ν is estimated by $C_r \sup_t |\varphi(t)|$, where C_r is independent of φ. Hence ν is an element of the Sobolev class $W^\infty(\mathbb{R}^n)$, which yields the existence of a smooth density q of the measure ν.

Trying to follow the same plan in the infinite dimensional case, we at once encounter the obvious difficulty that the last equality in the integration by parts formula makes no sense due to the lack of infinite dimensional analogues of Lebesgue measure. This difficulty can be overcome if we define the action of differential operators directly on measures. This is, in fact, the essence of Fomin's theory of differentiable measures and the Malliavin calculus. Then a more delicate problem arises of finding vector fields for which the integration by parts holds. Most of the other problems discussed below are connected in one way or another to this central one.

The goal of this book is to present systematically the basic facts of the theory of differentiable measures, to give an introduction to the Malliavin calculus with emphasizing Malliavin's method of studying nonlinear images of measures and its connections with the theory of differentiable measures, and to discuss applications of both theories in stochastic analysis. The applications we consider include the study of some problems in the theory of diffusion processes (such as the existence and symmetrizability of diffusions, regularity of invariant measures and transition probabilities), and some standard applications of the Malliavin calculus. As compared to the finite dimensional case, one can say that we deal with geometric measure theory and Sobolev classes. Naturally, not all aspects of infinite dimensional analysis are touched upon, in particular, we do not discuss at all or just briefly mention such directions as harmonic analysis, white noise analysis, generalized functions, and differential equations in infinite dimensional spaces.

Chapter 1 provides some background material which may be useful for reading the subsequent chapters. Chapter 2 gives a concise introduction into the theory of classical Sobolev spaces. Almost all results in this chapter are given with proofs in view of the fact that very similar problems are studied further for measures on infinite dimensional spaces and hence it is useful to have at hand not only the formulations, but also the details of proofs from the finite dimensional case. In Chapters 3–7 we discuss the properties of measures on infinite dimensional spaces that are analogous to such properties as the existence and smoothness of densities of measures with respect to Lebesgue measure on $\mathbb{R}^n$. Chapter 8 is concerned with Sobolev classes over infinite dimensional spaces. Chapters 9–11 are devoted to the Malliavin calculus and nonlinear transformations of smooth measures, in particular, to the regularity of measures induced by smooth functionals (which is the central problem in the Malliavin calculus). In addition, we discuss measurable manifolds (measurable spaces with some differential structure). Finally, in Chapter 12 diverse applications are considered.

Dependence between the chapters. Chapters 1 and 2 are auxiliary, one can consult their sections when needed. Chapter 3 is one of the key chapters; the concepts introduced there are used throughout. Adjacent are Chapter 4 and Chapter 5, which are not essentially used in the subsequent exposition. Chapters 6 and 7 are used in Chapters 8, 9, and 12. Chapter 8 is important for Chapters 9, 10, and 12. Finally, Chapter 11 is relatively independent and self-contained; it employs only some general ideas from the previous chapters along with a few concrete results from them.

For all statements and formulas we use the triple enumeration: the chapter number, section number, and assertion number (all statements are numbered independently of their type within each section); numbers of formulas are given in parenthesis. For every work in the references all page numbers are indicated in square brackets where the corresponding work is cited.

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