Preface

The goal in this monograph is to present recent results concerning nonlocal evolution equations with different boundary conditions. We deal with existence and uniqueness of solutions and their asymptotic behaviour. We also give some results concerning limits of solutions to nonlocal equations when a rescaling parameter goes to zero. We recover in these limits some of the most frequently used diffusion models such as the heat equation, the p-Laplacian evolution equation, the porous medium equation, the total variation flow and a convection-diffusion equation. This book is based mainly on results from the papers [**14**], [**15**], [**16**], [**17**], [**68**], [**78**], [**79**], [**80**], [**120**], [**121**] and [**140**].

First, let us briefly introduce the prototype of nonlocal problems that will be considered in this monograph. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function with

$$\int_{\mathbb{R}^N} J(z)\, dz = 1.$$

Nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\, dy - u(x,t), \tag{0.1}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [**106**], if $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\, dy = (J * u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\, dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (0.1).

Equation (0.1) is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t depends not only on $u(x,t)$ and its derivatives, but also on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed, [**106**]. However, there is no regularizing effect in general.

Let us fix a bounded domain Ω in $\mathbb{R}^N$. For local problems the two most common boundary conditions are Neumann's and Dirichlet's. When looking at boundary conditions for nonlocal problems, one has to modify the usual formulations for local problems. As an analog for nonlocal problems of Neumann boundary conditions we propose

$$(0.2)\qquad \begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))\,dy, & x\in\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega. \end{cases}$$

In this model, the integral term takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving at or leaving position x from other places. Since we are integrating over Ω, we are assuming that diffusion takes place only in Ω. The individuals may not enter or leave the domain. This is analogous to what is called homogeneous Neumann boundary conditions in the literature.

As the homogeneous Dirichlet boundary conditions for nonlocal problems we consider

$$(0.3)\qquad \begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x\in\Omega,\ t>0,\\ u(x,t) = 0, & x\notin\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega. \end{cases}$$

In this model, diffusion takes place in the whole $\mathbb{R}^N$, but we assume that u vanishes outside Ω. In the biological interpretation, we have a hostile environment outside Ω, and any individual that jumps outside dies instantaneously. This is the analog of what is called homogeneous Dirichlet boundary conditions for the heat equation. However, the boundary datum is not understood in the usual sense, since we are not imposing that $u|_{\partial\Omega} = 0$.

The nonlocal problems of the type of (0.1), (0.2) and (0.3) have been used to model very different applied situations, for example in biology ([**65**], [**133**]), image processing ([**110**], [**129**]), particle systems ([**51**]), coagulation models ([**108**]), nonlocal anisotropic models for phase transition ([**1**], [**2**]), mathematical finances using optimal control theory ([**50**], [**126**]), etc.

We have to mention the close relation between this kind of evolution problems and probability theory. In fact, when one looks at a Levy process ([**48**]), the nonlocal operator that appears naturally is a fractional power of the Laplacian. This approach is out of the scope of this monograph, and we refer to [**21**] for a reference concerning the interplay between nonlocal partial differential equations and probability. Nevertheless, let us explain briefly why the concrete problem (0.1) has a clear probabilistic interpretation.

Let $(E,\mathcal{E})$ be a measurable space and $P : E\times\mathcal{E}\to[0,1]$ a transition probability on E. Then we define a Markovian transition function as follows: for any $x\in E$, $\mathcal{A}\in\mathcal{E}$, let

$$P_t(x,\mathcal{A}) = e^{-t}\sum_{n=0}^{+\infty}\frac{t^n}{n!}P^{(n)}(x,\mathcal{A}), \qquad t\in\mathbb{R}_+,$$

where $P^{(n)}$ denotes the n-th iterate of P. The associated family of Markovian operators, $P_tf(x) = \int f(y)P_t(x,dy)$, satisfy

$$\frac{\partial}{\partial t}P_tf(x) = \int P_tf(y)P(x,dy) - P_tf(x).$$

If we consider a Markov process $(Z_t)_{t\geq 0}$ associated to the transition function $(P_t)_{t\geq 0}$, and if we denote by μ_t the distribution of Z_t, then the family $(\mu_t)_{t\geq 0}$ satisfies also a linear equation of the form

$$\frac{\partial}{\partial t}\mu_t = \int P(y,\cdot)\mu_t(dy) - \mu_t.$$

In particular, for $E = \mathbb{R}^N$, if the transition probability $P(x,dy)$ has a density $y \mapsto J(x,y)$, and μ_t has a density $y \mapsto u(y,t)$, then the following equation is satisfied:

$$\frac{\partial}{\partial t}u(x,t) = \int J(x,y)u(y,t)\,d\lambda(y) - u(x,t). \tag{0.4}$$

With different particular choices of P we recover the equation studied in the Cauchy, Dirichlet and Neumann cases. For example, if $P(x,dy) = J(y-x)dy$ is the transition probability of a random walk, equation (0.4) is just equation (0.1). In this particular case, the asymptotic behaviour, described in the first chapter, can be obtained as a consequence of the so-called Local Limit Theorem for Random Walks, which is a classical result in probability theory ([**104**, Theorems 1 & 2]).

In the Dirichlet and Neumann cases, the results described here also give interesting information on the asymptotic behaviour of some natural Markov process in the space.

Let us now summarize the contents of this book.

The book contains two main parts. The first, which consists of Chapters 1 to 4, deals mainly with linear problems, and in this case the main tool to get existence and uniqueness of solutions is the Fourier transform for the Cauchy problem and a fixed point argument for the Dirichlet and Neumann problems. The second part, Chapters 5, 6, 7 and 8, is concerned with nonlinear problems, and here the main tool for proving existence and uniqueness is Nonlinear Semigroup Theory.

For several classical partial differential equations the solutions belong to appropriate Sobolev spaces. Hence, Poincaré type inequalities play a key role in the analysis. When considering nonlocal problems, it is natural to look for solutions in L^p spaces; however, we prove nonlocal analogs of Poincaré type inequalities that also play a fundamental role in this monograph.

Chapter 1 is devoted to the study of the Cauchy problem for a linear nonlocal operator. In this chapter we make an extensive use of the Fourier transform. We show existence and uniqueness of solutions and study their asymptotic behaviour. In addition, we prove convergence to solutions of local equations when the kernel of the nonlocal operator is rescaled in a suitable way. We also deal with nonlocal analogs of linear higher order evolution problems.

In Chapters 2 and 3 we study the analogs for linear nonlocal diffusion of the Dirichlet and Neumann problems for both the homogeneous and the nonhomogeneous case. For these nonlocal problems we find, besides existence and uniqueness, the asymptotic behaviour as well as convergence, under rescaling, to the usual boundary value problems for the heat equation.

The next chapter contains the study of a nonlocal analog of a convection-diffusion problem taking into account a nonsymmetric kernel to model the convective part of the equation.

Chapter 5 deals with the nonlocal Neumann problem for a nonlinear diffusion equation. The local counterpart serves as a model for many applications, for instance, diffusion in porous media and changes of phases (the multiphase Stefan problem and the Hele-Shaw problem). Here we use the Crandall-Liggett Theorem to prove existence and uniqueness of solutions.

In Chapter 6 we study a nonlocal analog of the p-Laplacian evolution equation for $1 < p < \infty$. We deal here with the Cauchy problem as well as Dirichlet or Neumann boundary conditions. As in the previous chapter, one of the main tools is Nonlinear Semigroup Theory. The main ingredient for the proof of convergence to the local problem is a precompactness lemma inspired by a result due to Bourgain, Brezis and Mironescu, [**52**].

Motivated by problems in image processing, in recent years there has been an increasing interest in the study of the Total Variation Flow, [**7**]. Chapter 7 is devoted to the Dirichlet and Neumann problems for the nonlocal version of this evolution. After proving existence and uniqueness of solutions, we analyze their asymptotic behaviour as well as the convergence to local problems when the kernel is rescaled.

In the last chapter we present two nonlocal versions of models for the evolution of a sandpile. The first model corresponds to a nonlocal version of the Aronsson-Evans-Wu model obtained as limit as $p \to \infty$ in the local Cauchy problem for the p-Laplacian evolution equation, and the second corresponds to the Prigozhin model. The local sandpile models are based on the requirement that the slope of sandpile is at most one. However, a more realistic model would require the slope constraint only on a larger scale, with no slope requirements on a smaller scale. This is exactly the case for the nonlocal model presented in this chapter. The main tools for the analysis here are convex analysis and accretive operators. In this chapter we also present some explicit formulae for solutions of the nonlocal sandpile models that illustrate the results.

The book ends with an appendix, in which we outline some of the main tools from Nonlinear Semigroup Theory used in the above chapters. This theory has shown to be a very useful technique to deal with nonlinear evolution equations, and it is well suited to treat nonlocal evolution problems.

The Bibliography of this monograph does not escape the usual rule of being incomplete. In general, we have listed those papers which are closer to the topics discussed here. But, even for those papers, the list is far from being exhaustive and we apologize for omissions. At the end of each chapter we have included some bibliographical notes concerning the references used in that chapter and related ones.

It is a pleasure to acknowledge here the gratitude to our coauthors, namely C. Cortazar, M. Elgueta, M. Chaves, E. Chasseigne, L. Ignat, C. Schonlieb and

N. Wolanski. This monograph could not have been written without their contribution. We would also like to thank M. Pérez-Llanos for her continuous encouragement. Finally, we want to thank B. L. G. for the spirit imbued in us at the beginning of this story.

F. Andreu, J. M. Mazón and J. Toledo have been partially supported by the Spanish MEC and FEDER, project MTM2008-03176. J. D. Rossi has been partially supported by MEC project MTM2004-02223, UBA X066 and CONICET (Argentina).

F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo
Valencia, November 2008

Fuensanta Andreu died 26 December of 2008, before this book was published.
All of us, Julio, Julián and, especially, Mazón are greatly indebted to her.
Thanks for such good friendship and mathematics.