CHAPTER 1

The Cauchy problem for linear nonlocal diffusion

The aim of this chapter is to begin the study of the nonlocal evolution problems by the analysis of the asymptotic behaviour of solutions of nonlocal linear diffusion problems in the whole $\mathbb{R}^N$. First, we deal with the simplest model,

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t),$$

and after that we also treat a nonlocal analog of higher order problems,

$$u_t(x,t) = (-1)^{n-1}\,(J * \mathrm{Id} - 1)^n\,(u(x,t)).$$

We focus our attention on existence and uniqueness of solutions, their asymptotic behaviour as $t \to \infty$ and the convergence of solutions of these nonlocal evolution equations to solutions of classical models, such as the heat equation, when the nonlocal equation is rescaled in an appropriate way. As it happens in the study of the Cauchy problem for the heat equation, the Fourier transform will play a fundamental role, allowing us to obtain an explicit formula for the solution to the nonlocal equation in Fourier variables.

1.1. The Cauchy problem

We consider the linear nonlocal diffusion problem presented in the Preface,

$$(1.1)\qquad \begin{cases} u_t(x,t) = J * u(x,t) - u(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \\ u(x,0) = u_0(x), \end{cases}$$

for $x \in \mathbb{R}^N$ and $t > 0$. Here J satisfies the following hypothesis, which will be assumed throughout this chapter:

(H) $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative radial function with $J(0) > 0$ and

$$\int_{\mathbb{R}^N} J(x)\,dx = 1.$$

This means that J is a radial probability density.

As we have mentioned in the Preface, this equation has been used to model diffusion processes. More precisely (see [**106**]), if $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x

to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1).

A solution of (1.1) is understood as a function $u \in C^0([0,+\infty); L^1(\mathbb{R}^N))$ that satisfies (1.1) in the integral sense; that is, u satisfies

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)u(y,s)\,dy - u(x,s)\,ds.$$

This definition of solution is quite natural since every term in the equation is well defined. In Theorem 1.4, it is shown that existence and uniqueness hold for this kind of solutions. As we are dealing with nonlocal diffusion problems, searching for a solution in some Lebesgue space seems the appropriate thing to do.

We shall make an extensive use of the Fourier transform in order to obtain explicit solutions in frequency formulation. Moreover, the main result in this section states that the decay rate as t goes to infinity of solutions of this nonlocal problem is determined by the behaviour of the Fourier transform of J near the origin, and the asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the Laplacian. Therefore, we begin with some preliminaries concerning the Fourier transform. We assume that the reader is familiar with them and hence we refer to [**113**] or [**148**] for details.

In the sequel, $\hat{f}$ denotes the Fourier transform of f, which is given by the following definition.

Definition 1.1. For $f \in L^1(\mathbb{R}^N)$, the *Fourier transform* of f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-i\langle x,\xi\rangle} f(x)\,dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$. And the *inverse Fourier transform* of f is given by

$$\check{f}(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle} f(\xi)\,d\xi.$$

We set $\mathcal{F} : L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ to be the Fourier transformation, i.e., the linear bounded operator defined by $\mathcal{F}(f) := \hat{f}$.

By $\mathcal{S}(\mathbb{R}^N)$ we denote the space of rapidly decreasing functions, that is, the set of all $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\sup |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}^N$ and $\beta = (\beta_1,\dots,\beta_N) \in \mathbb{N}^N$, with the usual multi-index notation $|\alpha| = \sum_{i=1}^N \alpha_i$, $\alpha! = \alpha_1!\cdots\alpha_N!$, $x^\beta = x_1^{\beta_1}\cdots x_N^{\beta_N}$ and $\partial^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_N}x_N}$.

In the following proposition we list some of the main properties of the Fourier transform.

Proposition 1.2.

(1) *For $f \in L^1(\mathbb{R}^N)$, $\hat{f}$ and $\check{f}$ are bounded and continuous. Moreover,*

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \quad \textit{and} \quad \lim_{|x|\to\infty} \check{f}(x) = 0.$$

(2) $\|\hat{f}\|_{L^\infty(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)}$.

(3) *If* $f \in C^k(\mathbb{R}^N)$ *with* $D^\beta f \in L^1(\mathbb{R}^N)$ *for all* β, $|\beta| \le k$, *then* $\xi^\beta \hat{f} \in L^\infty(\mathbb{R}^N)$ *and* $\widehat{D^\beta f} = i^{|\beta|}\xi^\beta \hat{f}$.

(4) *If* $x^\alpha f \in L^1(\mathbb{R}^N)$ *for all* α, $|\alpha| \le k$, *then* $\hat{f} \in C^k(\mathbb{R}^N)$ *and* $D^\alpha \hat{f} = (-i)^{|\alpha|}\widehat{x^\alpha f}$.

(5) $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

(6) *The Fourier transform operator* $\mathcal{F}$ *is an isomorphism from* $\mathcal{S}(\mathbb{R}^N)$ *onto* $\mathcal{S}(\mathbb{R}^N)$, *and its inverse is given by the inversion formula*

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle}\hat{f}(\xi)\,d\xi. \tag{1.2}$$

(7) *For* $f \in L^1(\mathbb{R}^N)$ *such that* $\hat{f} \in L^1(\mathbb{R}^N)$, *the inversion formula* (1.2) *holds for almost every* $x \in \mathbb{R}^N$.

(8) *(Plancherel) There is an isomorphism* $\mathcal{P} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *such that* $\mathcal{P}(f) = \hat{f}$ *for all* $f \in \mathcal{S}(\mathbb{R}^N)$. *Moreover, the following identity holds:*

$$\|\mathcal{P}(f)\|_{L^2(\mathbb{R}^N)} = (2\pi)^{\frac{N}{2}}\|f\|_{L^2(\mathbb{R}^N)}, \qquad \forall f \in L^2(\mathbb{R}^N).$$

(9) *(Hausdorff-Young) If* $1 \le p \le 2$, *there is a linear bounded operator* $\mathcal{F}_p : L^p(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$, $\frac{1}{p} + \frac{1}{p'} = 1$, *such that* $\mathcal{F}_p(f) = \hat{f}$ *for all* $f \in \mathcal{S}(\mathbb{R}^N)$. *Moreover, the following inequality holds:*

$$\|\mathcal{F}_p(f)\|_{L^{p'}(\mathbb{R}^N)} \le (2\pi)^{\frac{N}{p'}}\|f\|_{L^p(\mathbb{R}^N)}, \qquad \forall f \in L^p(\mathbb{R}^N).$$

Throughout this chapter we denote by G_A^s, $A > 0$, the inverse Fourier transform of $e^{-A|\xi|^s}$, that is,

$$\widehat{G_A^s}(\xi) = e^{-A|\xi|^s}. \tag{1.3}$$

The fractional Laplacian of a function $f : \mathbb{R}^N \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^{\frac{s}{2}} f(x) := C_{N,s}\int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+s}}\,dy,$$

where the parameter s is a real number $0 < s \le 2$, and $C_{N,s}$ is a normalization constant given by

$$C_{N,s} = \frac{\Gamma(\frac{N+s}{2})}{2^{-s}\pi^{\frac{N}{2}}|\Gamma(-\frac{s}{2})|}.$$

It can also be defined as a pseudo-differential operator,

$$\mathcal{F}\left((-\Delta)^{\frac{s}{2}} f\right)(\xi) = |\xi|^s \hat{f}(\xi).$$

The fractional Laplacian can be defined in a distributional sense for functions that are not differentiable, as long as f is not too singular at the origin or, in terms of the x variable, as long as

$$\int_{\mathbb{R}^N} \frac{f(x)}{(1+|x|)^{N+s}}\,dx < \infty.$$

For references concerning the fractional Laplacian see for instance [**130**] or [**147**].

We will use the following notation throughout this monograph:

$$g(\xi) = h(\xi) + o(|\xi|^s) \qquad \text{as } \xi \to 0$$

means

$$\lim_{\xi \to 0} \frac{g(\xi) - h(\xi)}{|\xi|^s} = 0.$$

The hypothesis (H) for J immediately implies that

$$|\hat{J}(\xi)| \le 1 \qquad \text{and} \qquad \hat{J}(0) = 1.$$

The main result of this section reads as follows.

THEOREM 1.3. *Assume there exist $A > 0$ and $0 < s \le 2$ such that*

$$\hat{J}(\xi) = 1 - A|\xi|^s + o(|\xi|^s) \quad \text{as } \xi \to 0. \tag{1.4}$$

For any nonnegative u_0 such that $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$, there exists a unique solution $u(x,t)$ of (1.1).

The asymptotic behaviour of $u(x,t)$ is given by

$$\lim_{t\to+\infty} t^{\frac{N}{s}} \max_x |u(x,t) - v(x,t)| = 0, \tag{1.5}$$

where v is the solution of

$$v_t(x,t) = -A(-\Delta)^{\frac{s}{2}} v(x,t)$$

with initial condition $v(x,0) = u_0(x)$. Moreover,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-\frac{N}{s}},$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{s}} u(yt^{\frac{1}{s}}, t) - \|u_0\|_{L^1} G_A^s(y) \right| = 0.$$

The assumption $u_0 \ge 0$ can be easily removed, but it is adopted here for simplicity.

Condition (1.4) can be reduced to the existence of $A, s > 0$ such that

$$\hat{J}(\xi) = 1 - A|\xi|^s + o(|\xi|^s), \qquad \xi \to 0,$$

since (H) implies $s \le 2$ (see Lemma 1.8 below).

In the special case $s = 2$, the decay rate is $t^{-\frac{N}{2}}$ and the asymptotic profile is a Gaussian

$$G_A^2(y) = (4\pi A)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4A}},$$

where $A \cdot \mathrm{Id} = -\frac{1}{2} D^2 \hat{J}\,(0)$. Note that in this case (which occurs, for example, when J is compactly supported) the asymptotic behaviour is the same as the one that holds for the solutions of the heat equation and, as it happens for the heat equation, the asymptotic profile is a Gaussian.

The decay of the solutions in L^∞ together with the conservation of mass (that holds trivially for solutions to the nonlocal evolution problem (1.1)) give the decay in the L^p-norms by interpolation. As a consequence of the previous theorem, we find that this decay is analogous to the decay of the evolution given by the fractional Laplacian, that is,

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le C\,t^{-\frac{N}{s}\left(1-\frac{1}{p}\right)};$$

see Corollary 1.11. We refer to [**74**] for the decay in the L^p-norms for the fractional Laplacian; see also [**66**], [**96**] and [**98**] for finer decay estimates in the L^p-norms for solutions of the heat equation.

1.1.1. Existence and uniqueness. Let us begin by proving existence and uniqueness of solutions using Fourier variables.

THEOREM 1.4. *Let $u_0 \in L^1(\mathbb{R}^N)$ such that $\widehat{u_0} \in L^1(\mathbb{R}^N)$. There exists a unique solution $u \in C^0([0,\infty); L^1(\mathbb{R}^N))$ of* (1.1)*, and it is given by*

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\widehat{u_0}(\xi).$$

PROOF. We have, formally,

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t) = J * u - u(x,t).$$

Applying the Fourier transform to this equation we obtain

$$\hat{u}_t(\xi,t) = \hat{u}(\xi,t)(\hat{J}(\xi)-1).$$

Consequently,

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\widehat{u_0}(\xi).$$

Since $\widehat{u_0} \in L^1(\mathbb{R}^N)$ and $e^{(\hat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse Fourier transform. □

REMARK 1.5. One can also understand solutions of (1.1) directly in Fourier variables. This concept of solution is equivalent to the integral solution in the original variables under our hypotheses on the initial condition.

Now we prove a lemma concerning the fundamental solution of (1.1), that is, the solution of (1.1) with initial condition $u_0 = \delta_0$, the Dirac measure at zero.

LEMMA 1.6. *Let $J \in \mathcal{S}(\mathbb{R}^N)$. The fundamental solution of* (1.1) *can be decomposed as*

$$w(x,t) = e^{-t}\delta_0(x) + K_t(x), \tag{1.6}$$

where $K_t(x) = K(x,t)$ is a smooth function defined in Fourier variables by

$$\widehat{K_t}(\xi) = e^{-t}(e^{\hat{J}(\xi)t} - 1).$$

Moreover, if u is a solution of (1.1) *with $u_0 \in L^1(\mathbb{R}^N)$, it can be written as*

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^N} w(x-z,t)u_0(z)\,dz.$$

PROOF. As in the previous result we have

$$\hat{w}_t(\xi,t) = \hat{w}(\xi,t)(\hat{J}(\xi)-1).$$

Hence, as the initial datum satisfies $\hat{\delta}_0 = 1$,

$$\hat{w}(\xi,t) = e^{(\hat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\hat{J}(\xi)t} - 1).$$

The first part of the lemma is proved by applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^N)$.

To finish the proof we observe that $w * u_0$ is a solution of (1.1) (just use Fubini's theorem) with $(w * u_0)(x, 0) = u_0(x)$. □

REMARK 1.7. The above proof together with the fact that $\hat{J}(\xi) \to 0$ as $|\xi| \to +\infty$ (since $J \in L^1(\mathbb{R}^N)$) shows that if $\hat{J} \in L^1(\mathbb{R}^N)$, then the same decomposition (1.6) holds and the result also applies.

1.1.2. Asymptotic behaviour. We begin by collecting some properties of the function J.

LEMMA 1.8. *Under the hypothesis* (H) *for* J, *we have*

i) $|\hat{J}(\xi)| \le 1$, $\hat{J}(0) = 1$.
ii) *If* $\int_{\mathbb{R}^N} J(x)|x|\,dx < +\infty$, *then*

$$\frac{\partial \hat{J}}{\partial \xi_i}(0) = \left(\nabla_\xi \hat{J}\right)_i(0) = -i\int_{\mathbb{R}^N} x_i J(x)\,dx = 0,$$

and if $\int_{\mathbb{R}^N} J(x)|x|^2\,dx < +\infty$,

$$\left(D^2\hat{J}\right)_{ij}(0) = -\int_{\mathbb{R}^N} x_i x_j J(x)\,dx;$$

therefore $\left(D^2\hat{J}\right)_{ij}(0) = 0$ *when* $i \ne j$ *and* $\left(D^2\hat{J}\right)_{ii}(0) \ne 0$. *Hence the Hessian matrix of* $\hat{J}$ *at the origin is given by*

$$D^2\hat{J}\,(0) = -\left(\frac{1}{N}\int_{\mathbb{R}^N} |x|^2 J(x)\,dx\right) \cdot \mathrm{Id}.$$

iii) *If* $\hat{J}(\xi) = 1 - A|\xi|^s + o(|\xi|^s)$ *as* $\xi \to 0$, *then necessarily* $s \in (0, 2]$, *and if* J *has a first momentum, then* $s \ne 1$. *Finally, if* $s = 2$,

$$A \cdot \mathrm{Id} = -\frac{1}{2}\,D^2\hat{J}\,(0).$$

iv) *If* $\hat{J}(\xi) = 1 - A|\xi|^s + o(|\xi|^s)$ *as* $\xi \to 0$, *then*

$$|\hat{J}(\xi) - 1 + A|\xi|^s| \le |\xi|^s h(\xi), \tag{1.7}$$

where $h \ge 0$ *is bounded and* $h(\xi) \to 0$ *as* $\xi \to 0$. *Moreover, given* $D > 0$ *there exist* $a > 0$ *and* $0 < \delta < 1$ *such that*

$$|\hat{J}(\xi) - 1 + A|\xi|^s| \le D|\xi|^s \qquad \text{for } |\xi| \le a, \tag{1.8}$$

and

$$|\hat{J}(\xi)| \le 1 - \delta \qquad \text{for } |\xi| \ge a. \tag{1.9}$$

PROOF. Points i) and ii) are rather straightforward (recall that J is a radial probability density). Now we turn to iii). Observe that if $\hat{J}$ has an expansion of the form

$$\hat{J}(\xi) = 1 + i\langle \mathbf{a}, \xi\rangle - \frac{1}{2}\langle \xi, B\xi\rangle + o(|\xi|^2),$$

where $\mathbf{a} = (a_1, \ldots, a_N)$ and $B = (B_{ij})_{i,j=1,\ldots,N}$, then J has a second momentum and

$$a_i = \int x_i J(x)dx, \qquad B_{ij} = \int x_i x_j J(x)dx < \infty.$$

Thus if iii) held for some $s > 2$, it would turn out that the second moment of J is null, which would imply that $J \equiv 0$, a contradiction. When $s = 2$, since by (ii) $B_{ij} = -\big(D^2 \hat{J}\big)_{ij}(0)$, the Hessian is diagonal. Finally, (1.7) is evident and implies the existence of $a > 0$ satisfying (1.8). Once a is fixed, on account that J is radial and $|\hat{J}(\xi)| \le 1$, there exists $\delta > 0$ such that (1.9) holds. □

Observe that $\hat{J}(\xi)$ is real for $\xi \in \mathbb{R}^N$ due to the symmetry of J.

Next, we prove Theorem 1.3. Throughout this chapter we denote by C any constant independent of the relevant quantities that may vary from line to line.

Proof of Theorem 1.3. By Theorem 1.4 we have

$$\hat{u}(\xi, t) = e^{(\hat{J}(\xi)-1)t} \widehat{u_0}(\xi)$$

and

$$\hat{u}_t(\xi, t) = \hat{u}(\xi, t)(\hat{J}(\xi) - 1).$$

On the other hand, let $v(x, t)$ be a solution of

$$v_t(x, t) = -A(-\Delta)^{s/2} v(x, t),$$

with the same initial datum $v(x, 0) = u_0(x)$. Solutions of this equation are understood in the sense that

$$\hat{v}(\xi, t) = e^{-A|\xi|^s t} \widehat{u_0}(\xi); \tag{1.10}$$

see [**21**]. Hence in Fourier variables,

$$\int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t)\, d\xi = \int_{\mathbb{R}^N} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^s t} \right) \widehat{u_0}(\xi) \right| d\xi$$

$$\le \int_{|\xi| \ge r(t)} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^s t} \right) \widehat{u_0}(\xi) \right| d\xi$$

$$+ \int_{|\xi| < r(t)} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^s t} \right) \widehat{u_0}(\xi) \right| d\xi = [I] + [II],$$

where $[I]$ and $[II]$ denote the first and second integral, respectively, in the left hand side of the last identity, and $r(t)$, with $r(t) \to 0$, will be determined later.

To get a bound for $[I]$ we proceed as follows. We decompose it into two parts,

$$t^{\frac{N}{s}} [I] \le t^{\frac{N}{s}} \int_{|\xi| \ge r(t)} \left| e^{-A|\xi|^s t} \widehat{u_0}(\xi) \right| d\xi + t^{\frac{N}{s}} \int_{|\xi| \ge r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi.$$

First, we deal with the first term on the right hand side in the above expression. Changing variables, $\eta = \xi t^{1/s}$, we have

$$t^{\frac{N}{s}} \int_{|\xi| > r(t)} e^{-A|\xi|^s t} |\widehat{u_0}(\xi)| d\xi \le \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| > r(t) t^{1/s}} e^{-A|\eta|^s} \to 0$$

as $t \to \infty$ if we impose that

$$r(t)t^{\frac{1}{s}} \to \infty \qquad \text{as } t \to \infty. \tag{1.11}$$

Let us now deal with the second term. Let a and δ be as in Lemma 1.8 iv) for $D = \frac{A}{2}$. By (1.9),

$$\begin{aligned}
& t^{\frac{N}{s}} \int_{|\xi| \geq r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi \\
& \leq t^{\frac{N}{s}} \int_{a \geq |\xi| \geq r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi + t^{\frac{N}{s}} \int_{|\xi| \geq a} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi \\
& = t^{\frac{N}{s}} \int_{a \geq |\xi| \geq r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi + t^{\frac{N}{s}} e^{-t} \int_{|\xi| \geq a} e^{t|\hat{J}(\xi)|} \left| \widehat{u_0}(\xi) \right| d\xi \\
& \leq t^{\frac{N}{s}} \int_{a \geq |\xi| \geq r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi + C t^{\frac{N}{s}} e^{-\delta t}.
\end{aligned}$$

Using the above inequality, (1.8) and changing variables, $\eta = \xi t^{1/s}$,

$$\begin{aligned}
& t^{\frac{N}{s}} \int_{|\xi| \geq r(t)} \left| e^{t(\hat{J}(\xi)-1)} \widehat{u_0}(\xi) \right| d\xi \\
& \leq t^{\frac{N}{s}} \int_{a \geq |\xi| \geq r(t)} e^{-tA|\xi|^s} e^{t\left|\hat{J}(\xi)-1+A|\xi|^s\right|} \left| \widehat{u_0}(\xi) \right| d\xi + C t^{\frac{N}{s}} e^{-\delta t} \\
& \leq t^{\frac{N}{s}} \int_{a \geq |\xi| \geq r(t)} e^{-t\frac{A}{2}|\xi|^s} \left| \widehat{u_0}(\xi) \right| d\xi + C t^{\frac{N}{s}} e^{-\delta t} \\
& \leq \int_{a t^{1/s} \geq |\eta| \geq t^{1/s} r(t)} e^{-\frac{A}{2}|\eta|^s} \left| \widehat{u_0}(\eta t^{-1/s}) \right| d\eta + C t^{\frac{N}{s}} e^{-\delta t} \\
& \leq C \int_{|\eta| \geq t^{1/s} r(t)} e^{-\frac{A}{2}|\eta|^s} d\eta + C t^{\frac{N}{s}} e^{-\delta t} \quad \to 0
\end{aligned}$$

as $t \to \infty$ if (1.11) holds.

Now we estimate $[II]$ as follows: by (1.7) and the elementary inequality

$$|e^y - 1| \leq C|y|$$

for $|y|$ bounded,

$$\begin{aligned}
t^{\frac{N}{s}} [II] &= t^{\frac{N}{s}} \int_{|\xi| < r(t)} \left| e^{(\hat{J}(\xi)-1+A|\xi|^s)t} - 1 \right| e^{-A|\xi|^s t} \left| \widehat{u_0}(\xi) \right| d\xi \\
& \leq C t^{\frac{N}{s}} \int_{|\xi| < r(t)} t|\xi|^s h(\xi) e^{-A|\xi|^s t} \, d\xi,
\end{aligned}$$

provided we impose

$$t\,(r(t))^s h(r(t)) \to 0 \qquad \text{as } t \to \infty. \tag{1.12}$$

In this case, changing variables $\eta = \xi t^{1/s}$ again, we have

$$t^{\frac{N}{s}} [II] \leq C \int_{|\eta| < r(t) t^{1/s}} |\eta|^s h(\eta / t^{1/s}) e^{-A|\eta|^s} d\eta.$$

Since the integrand is dominated by $\|h\|_\infty |\eta|^s \exp(-c|\eta|^s)$, a function which belongs to $L^1(\mathbb{R}^N)$, and $h(\eta/t^{1/s}) \to 0$ as $t \to \infty$, using the Dominated Convergence Theorem, we obtain

$$\lim_{t\to+\infty} t^{\frac{N}{s}}[II] = 0.$$

This shows that

$$t^{\frac{N}{s}}([I]+[II]) \to 0 \quad \text{as } t \to \infty, \tag{1.13}$$

provided we can find a $r(t) \to 0$ as $t \to \infty$ which fulfils both conditions (1.11) and (1.12). This is proved in Lemma 1.9, which is postponed to just after the end of the present proof. To conclude (1.5), we only have to observe that from (1.13) we get

$$t^{\frac{N}{s}} \max_x |u(x,t) - v(x,t)| \le (2\pi)^{-N} t^{\frac{N}{s}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi,t)\, d\xi \to 0 \quad \text{as } t \to \infty.$$

From the above, we obtain that the asymptotic behaviour is the same as the one for solutions of the evolution given by the fractional Laplacian. Now, it is easy to check that this asymptotic behaviour is exactly the one described in the statement. Indeed, in Fourier variables (see (1.10)) we have for $t \to \infty$

$$\hat{v}(t^{-\frac{1}{s}}\eta, t) = e^{-A|\eta|^s}\widehat{u_0}(\eta t^{-\frac{1}{s}}) \longrightarrow e^{-A|\eta|^s}\widehat{u_0}(0) = \widehat{G^s_A}(\eta)\|u_0\|_{L^1(\mathbb{R}^N)}.$$

Therefore

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{s}} v(yt^{\frac{1}{s}}, t) - \|u_0\|_{L^1} G^s_A(y) \right| = 0.$$

□

The following lemma shows that there exists a function $r(t)$ satisfying (1.11) and (1.12), as required in the proof of the previous theorem.

LEMMA 1.9. *Given a function $h \in C(\mathbb{R},\mathbb{R})$ such that $h(\rho) \to 0$ as $\rho \to 0$ with $h(\rho) > 0$ for small ρ, there exists a function r with $r(t) \to 0$ as $t \to \infty$ which satisfies*

$$\lim_{t\to\infty} r(t) t^{\frac{1}{s}} = \infty$$

and

$$\lim_{t\to\infty} t(r(t))^s h(r(t)) = 0.$$

PROOF. For fixed t large enough, we choose $r(t)$ as a small solution of

$$r(h(r))^{\frac{1}{2s}} = t^{-\frac{1}{s}}. \tag{1.14}$$

This equation defines a function $r = r(t)$ which, by continuity arguments, goes to zero as t goes to infinity. Indeed, if there exists $t_n \to \infty$ with no solution of (1.14) for $r \in (0,\delta)$, then $h(r) \equiv 0$ in $(0,\delta)$, a contradiction. □

REMARK 1.10. In the case $h(t) = t^\alpha$ with $\alpha > 0$, we can look for a function r of power type, $r(t) = t^\beta$ with $\beta < 0$, and the two conditions read as follows:

$$\beta + 1/s > 0, \quad 1 + \beta s + \alpha\beta < 0. \tag{1.15}$$

This implies that $\beta \in (-1/s, -1/(\alpha+s))$, which is, of course, always possible.

Now we find the decay rate in L^p of solutions of (1.1).

COROLLARY 1.11. *Let $1 < p < \infty$. Under the hypotheses in Theorem* 1.3*, the decay in the L^p-norm of the solution of* (1.1) *is given by*

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le Ct^{-\frac{N}{s}\left(1-\frac{1}{p}\right)}.$$

PROOF. By interpolation (see [**56**]) we have

$$\|u\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}} \|u\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{1}{p}}.$$

Integrating and using Fubini's Theorem it is easy to see that solutions of (1.1) preserve the total mass and consequently the L^1-norm. Hence, the result follows from the above inequalities and the previous results that give the decay in L^∞ of the solutions. ☐

1.2. Refined asymptotics

The goal now is to get refined asymptotic expansions for the solution u of the nonlocal evolution problem (1.1).

For the heat equation a precise asymptotic expansion in terms of the fundamental solution and its derivatives was found in [**96**]. In fact, for the fundamental solution v of the heat equation, under adequate assumptions on the initial condition, we have

$$\left\| v(x,t) - \sum_{|\alpha|\le k+1} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int_{\mathbb{R}^N} u_0(x) x^\alpha \right) \partial^\alpha G_t^2 \right\|_{L^q(\mathbb{R}^N)} \le Ct^{-\theta}, \tag{1.16}$$

where

$$\theta = \frac{N}{2}\left(\frac{k+1}{N} + 1 - \frac{1}{q}\right) = \frac{k+1}{2} + \frac{N}{2}\left(1 - \frac{1}{q}\right),$$

and G_t^2 is given in (1.3).

As pointed out by the authors in [**96**], the same asymptotic expansion can be done in a more general setting, dealing with the equations $v_t = -(-\Delta)^{\frac{s}{2}} v$, $s > 0$.

The main objective here is to study if an expansion analogous to (1.16) holds for the nonlocal problem (1.1). We find a complete expansion for $u(x,t)$ in terms of the derivatives of the regular part of the fundamental solution K_t given in Lemma 1.6.

Concerning the first term, it has been shown that (see Theorem 1.3) if J satisfies $\hat{J}(\xi) = 1 - |\xi|^s + o(|\xi|^s)$ as $\xi \to 0$, then the asymptotic behaviour of the solution $u(x,t)$ of (1.1) is given by

$$\lim_{t\to+\infty} t^{\frac{N}{s}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -(-\Delta)^{\frac{s}{2}} v(x,t)$ with initial condition $v(x,0) = u_0(x)$. As a consequence, the decay rate is given by $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\,t^{-\frac{N}{s}}$ and the asymptotic profile is as follows:

$$\lim_{t\to+\infty} \left\| t^{\frac{N}{s}} u(yt^{\frac{1}{s}}, t) - \left(\int_{\mathbb{R}^N} u_0 \right) G_1^s(y) \right\|_{L^\infty(\mathbb{R}^N)} = 0.$$

In contrast with the analysis done in the previous section, where the long time behaviour is studied in the $L^\infty(\mathbb{R}^N)$-norm, here we also consider $L^q(\mathbb{R}^N)$-norms for $q \geq 1$. In the sequel we denote by $L^1(\mathbb{R}^N, a(x))$ the following weighted space:

$$L^1(\mathbb{R}^N, a(x)) = \left\{ \varphi : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N} a(x)|\varphi(x)|dx < \infty \right\}.$$

THEOREM 1.12. *Let $0 < s \leq 2$, $m > 0$ and $c > 0$ satisfy*

$$\widehat{J}(\xi) = 1 - |\xi|^s + o(|\xi|^s) \quad \text{as } \xi \to 0 \tag{1.17}$$

and

$$|\widehat{J}(\xi)| \leq \frac{c}{|\xi|^m} \quad \text{as } |\xi| \to \infty. \tag{1.18}$$

Then, for any $2 \leq q \leq \infty$ and $k+1 < m - N$, there exists a constant

$$C(q, k, u_0) = C(q, k)\| |x|^{k+1} u_0 \|_{L^1(\mathbb{R}^N)}$$

such that

$$\left\| u(x,t) - \sum_{|\alpha| \leq k+1} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int_{\mathbb{R}^N} u_0(x) x^\alpha \right) \partial^\alpha K_t \right\|_{L^q(\mathbb{R}^N)} \leq C(q, k, u_0) t^{-\theta} \tag{1.19}$$

for all $u_0 \in L^1(\mathbb{R}^N, 1 + |x|^{k+1})$, where

$$\theta = \frac{k+1}{s} + \frac{N}{s}\Big(1 - \frac{1}{q}\Big).$$

REMARK 1.13. The condition $k + 1 < m - N$ guarantees that all the partial derivatives $\partial^\alpha K_t$ of order $|\alpha| = k+1$ make sense. In addition if $\widehat{J}$ decays at infinity faster than any polynomial,

$$\forall\, m \in \mathbb{N} \setminus \{0\}, \ \exists\, c(m) \text{ such that } |\widehat{J}(\xi)| \leq \frac{c(m)}{|\xi|^m}, \quad |\xi| \to \infty, \tag{1.20}$$

then the expansion (1.19) holds for all k.

To deal with L^q-norms for $1 \leq q < 2$ more restrictive assumptions have to be imposed. In the sequel $[s]$ stands for the floor function of s.

THEOREM 1.14. *Let $N \leq 3$. Assume J satisfies* (1.17) *with $[s] > N/2$ and that for any $m \geq 0$ and any index α there exists $C(m, \alpha)$ such that*

$$|\partial^\alpha \widehat{J}(\xi)| \leq \frac{C(m,\alpha)}{|\xi|^m}, \qquad |\xi| \to \infty. \tag{1.21}$$

Then for any $1 \leq q < 2$, the asymptotic expansion (1.19) *holds for any k.*

REMARK 1.15. Recall that (1.17) implies $0 < s \leq 2$, hence the hypothesis $[s] > N/2$ implies that if $N = 1$ then $1 \leq s \leq 2$, and if $N = 2$ or $N = 3$ then $s = 2$ in the previous theorem.

1.2.1. Estimates on the regular part of the fundamental solution. To prove the previous results we need some estimates on the kernel K_t and its derivatives for large t. The strategy is as follows: for $2 \le q \le \infty$ the behaviour in the $L^q(\mathbb{R}^N)$-norms follows from the estimates in the L^∞-norm and the L^2-norm, for which we use Plancherel's identity. The case $1 \le q < 2$ is more tricky. In order to evaluate the $L^1(\mathbb{R}^N)$-norm of K_t we use the Carlson type inequality (see for instance [**31**], [**54**])

$$\|f\|_{L^1(\mathbb{R}^N)} \le \|f\|_{L^2(\mathbb{R}^N)}^{1-\frac{N}{2n}} \||x|^n f\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2n}}, \tag{1.22}$$

which holds for $n > N/2$. The use of the above inequality with $f = \partial^\alpha K_t$ imposes that $|x|^n \partial^\alpha K_t$ belongs to $L^2(\mathbb{R}^N)$. To guarantee that property and to obtain the decay rate in the $L^2(\mathbb{R}^N)$-norm of $|x|^n \partial^\alpha K_t$, the additional hypothesis of Theorem 1.14 will be imposed in Lemma 1.17.

The following lemma gives us the decay rate in the $L^q(\mathbb{R}^N)$-norms of the kernel K_t and its derivatives for $2 \le q \le \infty$.

LEMMA 1.16. *Let $2 \le q \le \infty$ and J satisfy* (1.17) *and* (1.18). *Then for all multi-indices α such that $|\alpha| < m - N$ there exists a constant $C(q, \alpha)$ such that*

$$\|\partial^\alpha K_t\|_{L^q(\mathbb{R}^N)} \le C(q, \alpha)\, t^{-\frac{N}{s}(1-\frac{1}{q}) - \frac{|\alpha|}{s}}$$

holds for sufficiently large t.

Moreover, if J satisfies (1.20), *then the same result holds with no restriction on α.*

PROOF. We consider the cases $q = 2$ and $q = \infty$. The other cases follow by interpolation. We denote by *e.s.* the exponentially small terms.

First, let us consider the case $q = \infty$. Using the definition of K_t,

$$\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1),$$

we get, for any $x \in \mathbb{R}^N$,

$$|\partial^\alpha K_t(x)| \le Ce^{-t} \int_{\mathbb{R}^N} |\xi|^{|\alpha|} \left| e^{t\widehat{J}(\xi)} - 1 \right| d\xi.$$

Since $|e^y - 1| \le 2|y|$ for $|y|$ small, say $|y| \le c_0$, by (1.18), we obtain

$$\left| e^{t\widehat{J}(\xi)} - 1 \right| \le 2t|\widehat{J}(\xi)| \le \frac{2ct}{|\xi|^m}$$

for all $|\xi| \ge c(t) := \left(\frac{ct}{c_0}\right)^{\frac{1}{m}}$ with t large enough. Then

$$e^{-t} \int_{|\xi| \ge c(t)} |\xi|^{|\alpha|} \left| e^{t\widehat{J}(\xi)} - 1 \right| d\xi \le 2cte^{-t} \int_{|\xi| \ge c(t)} \frac{|\xi|^{|\alpha|}}{|\xi|^m} d\xi = Ct^{\frac{N}{m} + \frac{|\alpha|}{m}} e^{-t}$$

provided that $|\alpha| < m - N$.

It is easy to see that if (1.20) holds, no restriction on the multi-index α has to be assumed.

It remains to estimate

$$e^{-t}\int_{|\xi|\le c(t)}|\xi|^{|\alpha|}|e^{t\widehat{J}(\xi)}-1|\,d\xi.$$

We observe that the term

$$e^{-t}\int_{|\xi|\le c(t)}|\xi|^{|\alpha|}d\xi$$

is exponentially small, so we concentrate on

$$I(t)=e^{-t}\int_{|\xi|\le c(t)}\left|e^{t\widehat{J}(\xi)}\right||\xi|^{|\alpha|}\,d\xi.$$

Now, let us choose a and δ as in Lemma 1.8 iv) for $D=\frac{1}{2}$ (recall that now $A=1$). Taking into account (1.8) and (1.9), and using the change of variables $\eta=\xi t^{1/s}$,

$$|I(t)|=e^{-t}\int_{|\xi|\le a}\left|e^{t\widehat{J}(\xi)}\right||\xi|^{|\alpha|}d\xi+e^{-t}\int_{a\le|\xi|\le c(t)}\left|e^{t\widehat{J}(\xi)}\right||\xi|^{|\alpha|}d\xi$$

$$\le\int_{|\xi|\le a}e^{-\frac{t|\xi|^s}{2}}|\xi|^{|\alpha|}+e^{-t\delta}\int_{a\le|\xi|\le c(t)}|\xi|^{|\alpha|}d\xi$$

$$\le\int_{|\xi|\le a}e^{-\frac{t|\xi|^s}{2}}|\xi|^{|\alpha|}+e.s.$$

$$=t^{-\frac{|\alpha|}{s}-\frac{N}{s}}\int_{|\eta|\le at^{\frac{1}{s}}}e^{-\frac{|\eta|^s}{2}}|\eta|^{|\alpha|}+e.s.\le Ct^{-\frac{|\alpha|}{s}-\frac{N}{s}}.$$

Now, for $q=2$, by Plancherel's identity we have

$$\|\partial^\alpha K_t\|^2_{L^2(\mathbb{R}^N)}\le Ce^{-2t}\int_{\mathbb{R}^N}|e^{t\widehat{J}(\xi)}-1|^2|\xi|^{2|\alpha|}d\xi.$$

Putting out the exponentially small terms, it remains to estimate

$$\int_{|\xi|\le a}\left|e^{t(\widehat{J}(\xi)-1)}\right|^2|\xi|^{2|\alpha|}d\xi,$$

where a is as above. Then, by (1.8), and working as before, we get

$$\int_{|\xi|\le a}\left|e^{t(\widehat{J}(\xi)-1)}\right|^2|\xi|^{2|\alpha|}d\xi\le\int_{|\xi|\le a}e^{-t|\xi|^s}|\xi|^{2|\alpha|}d\xi\le Ct^{-\frac{N}{s}-\frac{2|\alpha|}{s}},$$

which finishes the proof. □

Once the case $2\le q\le\infty$ has been analyzed the next step is to obtain similar decay rates for the L^q-norms with $1\le q<2$. These estimates follow from an L^1-estimate and interpolation.

LEMMA 1.17. *Let* $N\le 3$. *Assume that* J *satisfies* (1.17) *and* (1.21) *with* $[s]>N/2$. *Then, for any multi-index* α *and any* $1\le q<2$,

$$\|\partial^\alpha K_t\|_{L^q(\mathbb{R}^N)}\le Ct^{-\frac{N}{s}(1-\frac{1}{q})-\frac{|\alpha|}{s}}\qquad\textit{for large } t. \tag{1.23}$$

REMARK 1.18. There is no restriction on s if J is such that

$$|\partial^\alpha \widehat{J}(\xi)| \leq \min\{|\xi|^{s-|\alpha|}, 1\}, \quad |\xi| \leq 1.$$

This happens if s is a positive integer and $\widehat{J}(\xi) = 1 - |\xi|^s$ in a neighborhood of the origin.

REMARK 1.19. The case $\alpha = (0, \dots, 0)$ can be easily treated. Let w be the fundamental solution of (1.1) given in Lemma 1.6. As a consequence of the mass conservation,

$$\int_{\mathbb{R}^N} w(x,t)\, dx = 1,$$

we obtain

$$\int_{\mathbb{R}^N} |K_t| \leq 1,$$

and therefore (1.23) follows with $\alpha = (0, \dots, 0)$.

REMARK 1.20. The condition (1.21) is satisfied, for example, for any smooth, compactly supported function J.

PROOF OF LEMMA 1.17. Fix α. The estimates for $1 < q < 2$ follow from the cases $q = 1$ and $q = 2$ (Lemma 1.16) using interpolation. Let us deal with $q = 1$. We use inequality (1.22) with $f = \partial^\alpha K_t$ and n such that $[s] \geq n > N/2$. We take $n = 1$ if $N = 1$ (in this case, $1 \leq s \leq 2$) and $n = 2$ if $N = 2$ or 3 (in this case, $s = 2$). We have

$$\|\partial^\alpha K_t\|_{L^1(\mathbb{R}^N)} \leq \|\partial^\alpha K_t\|_{L^2(\mathbb{R}^N)}^{1-\frac{N}{2n}} \, \||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2n}}.$$

The condition $n \leq [s]$ guarantees that, for $j = 1, \dots, N$, $\partial^n_{\xi_j} \widehat{J}$ makes sense near $\xi = 0$ and thus the derivatives $\partial^n_{\xi_j} \widehat{K_t}$ exist. Observe that the moment of order n of K_t imposes the existence of the partial derivatives $\partial^n_{\xi_j} \widehat{K_t}$, $j = 1, \dots, N$.

In view of Lemma 1.16 and the above inequality, we obtain

$$\|\partial^\alpha K_t\|_{L^1(\mathbb{R}^N)} \leq C t^{-\left(\frac{N}{2s} + \frac{|\alpha|}{s}\right)\left(1-\frac{N}{2n}\right)} \||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2n}}.$$

Thus it is sufficient to prove that

$$\||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^N)} \leq C t^{\frac{n}{s} - \frac{N}{2s} - \frac{|\alpha|}{s}}$$

for all sufficiently large t. Observe that by Plancherel's Theorem

$$\int_{\mathbb{R}^N} |x|^{2n} |\partial^\alpha K_t(x)|^2\, dx \leq C \int_{\mathbb{R}^N} (x_1^{2n} + \cdots + x_N^{2n}) |\partial^\alpha K_t(x)|^2\, dx$$

$$\leq C \sum_{j=1}^N \int_{\mathbb{R}^N} |\partial^n_{\xi_j} (\xi^\alpha \widehat{K_t}(\xi))|^2\, d\xi.$$

Therefore, it remains to show that, for any $j = 1, \dots, N$,

$$\int_{\mathbb{R}^N} |\partial^n_{\xi_j} (\xi^\alpha \widehat{K_t}(\xi))|^2\, d\xi \leq C t^{\frac{2n}{s} - \frac{N}{s} - \frac{2|\alpha|}{s}} \qquad \text{for } t \text{ large.} \tag{1.24}$$

We analyze the case $j = 1$; the others follow by the same arguments. Observe that if $N = 1$, $\xi_1 = \xi$, and in this case we have chosen $n = 1$, so

$$\partial_\xi (\xi^\alpha \widehat{K_t}(\xi)) = \alpha \xi^{\alpha-1} \widehat{K_t}(\xi) + \xi^\alpha \partial_\xi \widehat{K_t}(\xi)$$

($\alpha\xi^{\alpha-1}\widehat{K_t}(\xi)=0$ if $\alpha=0$), and for $n=2$ (now, $N=2$ or 3, $s=2$)

$$\begin{aligned}&\partial^2_{\xi_1}(\xi^\alpha\widehat{K_t}(\xi))\\&=\xi_2^{\alpha_2}\cdots\xi_N^{\alpha_N}\left(\alpha_1(\alpha_1-1)\xi_1^{\alpha_1-2}\widehat{K_t}(\xi)+2\alpha_1\xi_1^{\alpha_1-1}\partial^1_{\xi_1}\widehat{K_t}(\xi)+\xi_1^{\alpha_1}\partial^2_{\xi_1}\widehat{K_t}(\xi)\right)\end{aligned}$$

($\alpha_1(\alpha_1-1)\xi_1^{\alpha_1-2}\widehat{K_t}(\xi)=0$ if $\alpha_1=0$ or 1, $\alpha_1\xi_1^{\alpha_1-1}\partial^1_{\xi_1}\widehat{K_t}(\xi)=0$ if $\alpha_1=0$). Therefore, (1.24) is reduced to

$$\int_{\mathbb{R}^N}\xi_1^{2(\alpha_1-k)}\xi_2^{2\alpha_2}\cdots\xi_N^{2\alpha_N}|\partial^{n-k}_{\xi_1}\widehat{K_t}(\xi)|^2\,d\xi\le Ct^{\frac{2n}{s}-\frac{N}{s}-\frac{2|\alpha|}{s}}\qquad\text{for } t \text{ large}$$

for all $0\le k\le\min\{\alpha_1,n\}$, or equivalently to

$$\begin{aligned}I(k,t)&:=\int_{\mathbb{R}^N}\xi_1^{2(\alpha_1+k-n)}\xi_2^{2\alpha_2}\cdots\xi_N^{2\alpha_N}|\partial^{k}_{\xi_1}\widehat{K_t}(\xi)|^2\,d\xi\\&\le Ct^{\frac{2n}{s}-\frac{N}{s}-\frac{2|\alpha|}{s}}\qquad\text{for } t \text{ large,}\end{aligned}\tag{1.25}$$

for all $n-\min\{\alpha_1,n\}\le k\le n$; k can be 0 or 1 if $n=1$, and 0, 1 or 2 if $n=2$. Let us call

$$\beta(k)=(\alpha_1+k-n,\alpha_2,\dots,\alpha_N).$$

First we analyze the case $k=0$ in (1.25) (observe that this case appears only when $\alpha_1>0$ if $N=1$, and when $\alpha_1>1$ if $N=1$ or 2). In this case

$$I(0,t)=\int_{\mathbb{R}^N}\left|\xi^{\beta(0)}\widehat{K_t}(\xi)\right|^2\,d\xi,$$

and in view of Lemma 1.16 we obtain the desired decay property.

Let us now analyze the cases $k=1$ and $k=2$ in (1.25). Taking into account that $\widehat{K_t}(\xi)=e^{t(\widehat{J}(\xi)-1)}-e^{-t}$, we get

$$\partial^1_{\xi_1}\widehat{K_t}(\xi)=e^{t(\widehat{J}(\xi)-1)}t\partial^1_{\xi_1}\widehat{J}(\xi)$$

and

$$\partial^2_{\xi_1}\widehat{K_t}(\xi)=e^{t(\widehat{J}(\xi)-1)}\left(t^2\left(\partial^1_{\xi_1}\widehat{J}(\xi)\right)^2+t\partial^2_{\xi_1}\widehat{J}(\xi)\right).$$

Using that all the partial derivatives of $\widehat{J}$ decay, as $|\xi|\to\infty$, faster than any polynomial in $|\xi|$, we have that

$$\int_{|\xi|>a}|\xi|^{2|\beta(k)|}|\partial^k_{\xi_1}\widehat{K_t}(\xi)|^2d\xi\le C\,e^{-\delta t}t^{2k},$$

where a and δ are chosen as in Lemma 1.8 iv) for $D=\frac{1}{2}$. Having in mind that $n\le[s]$ and $\widehat{J}(\xi)-1+|\xi|^s=o(|\xi|^s)$ as $|\xi|\to0$, we obtain

$$|\partial^j_{\xi_1}\widehat{J}(\xi)|\le C\,|\xi|^{s-j},\ 1\le j\le n,$$

for all $|\xi|\le a$. Then for all $|\xi|\le a$, by (1.8), the following holds:

$$|\partial^1_{\xi_1}\widehat{K_t}(\xi)|^2\le C\,e^{-t|\xi|^s}t^2|\xi|^{2(s-1)}$$

and

$$|\partial^2_{\xi_1}\widehat{K_t}(\xi)|^2\le C\,e^{-t|\xi|^s}\left(t^4|\xi|^{4(s-1)}+t^2|\xi|^{2(s-2)}\right).$$

Observe that the second derivative only appears when $n = 2 = s$, so in this case,

$$|\partial^2_{\xi_1} \widehat{K_t}(\xi)|^2 \le C\, e^{-t|\xi|^2} \left(t^4|\xi|^4 + t^2\right).$$

Using that for any $l \ge 0$,

$$\int_{\mathbb{R}^N} e^{-t|\xi|^s} |\xi|^l d\xi \le Ct^{-\frac{N}{s}-\frac{l}{s}},$$

we get

$$\int_{|\xi|\le a} |\xi|^{2|\beta(1)|} |\partial^1_{\xi_1} K_t(\xi)|^2 d\xi \le Ct^{2-\frac{N}{s}-\frac{2|\beta(1)|+2(s-1)}{s}},$$

from which (1.25) follows for $k = 1$. For $k = 2$ (so $n = 2 = s$),

$$\int_{|\xi|\le a} |\xi|^{2|\beta(2)|} |\partial^2_{\xi_1} K_t(\xi)|^2 d\xi \le C\left(t^{4-\frac{N}{2}-\frac{2|\beta(2)|+4}{2}} + t^{2-\frac{N}{2}-\frac{2|\beta(2)|}{2}}\right)$$

$$= Ct^{2-\frac{N}{2}-|\beta(2)|} = Ct^{2-\frac{N}{2}-|\alpha|},$$

and (1.25) also holds. □

Now we are ready to prove Theorems 1.12 and 1.14.

Proof of Theorems 1.12 and 1.14. Following [**96**] we obtain that the initial condition $u_0 \in L^1(\mathbb{R}^N, 1 + |x|^{k+1})$ has the following decomposition:

$$u_0 = \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0 x^\alpha\, dx\right) D^\alpha \delta_0 + \sum_{|\alpha|=k+1} D^\alpha F_\alpha,$$

where

$$\|F_\alpha\|_{L^1(\mathbb{R}^N)} \le C\|u_0\|_{L^1(\mathbb{R}^N,\, |x|^{k+1})}$$

for all multi-indices α with $|\alpha| = k + 1$.

In view of (1.1) the solution u satisfies

$$u(x, t) = e^{-t}u_0(x) + (K_t * u_0)(x).$$

Since the first term is exponentially small, it suffices to analyze the long time behaviour of $K_t * u_0$. Using the above decomposition, Lemma 1.16 and Lemma 1.17 we get

$$\left\| K_t * u_0 - \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha dx\right) \partial^\alpha K_t \right\|_{L^q(\mathbb{R}^N)}$$

$$\le \sum_{|\alpha|=k+1} \|\partial^\alpha K_t * F_\alpha\|_{L^q(\mathbb{R}^N)}$$

$$\le \sum_{|\alpha|=k+1} \|\partial^\alpha K_t\|_{L^q(\mathbb{R}^N)} \|F_\alpha\|_{L^1(\mathbb{R}^N)}$$

$$\le Ct^{-\frac{N}{s}(1-\frac{1}{q})} t^{-\frac{(k+1)}{s}} \|u_0\|_{L^1(\mathbb{R}^N,\, |x|^{k+1})}.$$

This ends the proof. □

1.2.2. Asymptotics for the higher order terms. Next it is studied if the higher order terms of the asymptotic expansion found in Theorem 1.12 have some relation with the corresponding ones for the heat equation. The results show that the difference between them is of lower order. Again we have to distinguish between $2 \le q \le \infty$ and $1 \le q < 2$.

THEOREM 1.21. *Let J be as in Theorem 1.12 and assume in addition that there exists $r > 0$ such that*

$$\widehat{J}(\xi) = 1 - |\xi|^s + B|\xi|^{s+r} + o(|\xi|^{s+r}), \qquad \xi \to 0, \tag{1.26}$$

for some real number B. Then for any $2 \le q \le \infty$ and $|\alpha| \le m - N$ there exists a positive constant $C = C(\alpha, q, N, s, r)$ such that

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^q(\mathbb{R}^N)} \le C t^{-\frac{N}{s}(1-\frac{1}{q}) - \frac{|\alpha|+r}{s}} \qquad \text{for } t \text{ large.} \tag{1.27}$$

THEOREM 1.22. *Let $N \le 3$. Assume J is as in Theorem 1.21 with $[s] > N/2$. Assume also that all the derivatives of $\widehat{J}$ decay at infinity faster than any polynomial, that is,*

$$|\partial^\alpha \widehat{J}(\xi)| \le \frac{c(m, \alpha)}{|\xi|^m}, \qquad \xi \to \infty.$$

Then, for any $1 \le q < 2$ and any multi-index α, the inequality (1.27) holds.

Note that these results do not imply that the asymptotic expansion

$$\sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha \right) \partial^\alpha K_t$$

coincides with the expansion that holds for the equation $v_t = -(-\Delta)^{\frac{s}{2}} v$,

$$\sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha \right) \partial^\alpha G_t^s.$$

They only say that the corresponding terms agree up to a better order. When J is compactly supported or rapidly decaying at infinity, then $s = 2$ and we obtain an expansion analogous to the one that holds for the heat equation.

PROOF OF THEOREM 1.21. We consider the case $q = \infty$; the case $q = 2$ can be handled similarly, and the rest of the cases, $2 < q < \infty$, follow again by interpolation.

Writing each of the two terms in Fourier variables we obtain

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^\infty(\mathbb{R}^N)} \le C \int_{\mathbb{R}^N} |\xi|^{|\alpha|} \left| e^{-t}(e^{t\widehat{J}(\xi)} - 1) - e^{-t|\xi|^s} \right| d\xi.$$

Let us choose $a > 0$ such that

$$\left| \widehat{J}(\xi) - 1 + |\xi|^s \right| \le C|\xi|^{r+s}, \qquad \text{for } |\xi| \le a. \tag{1.28}$$

And let $\delta > 0$ such that

$$|\hat{J}(\xi)| \le 1 - \delta, \qquad \text{for } |\xi| \ge a.$$

Then, for $|\xi| \geq a$ all the terms are exponentially small as $t \to \infty$. Thus the behaviour of the difference $\partial^\alpha K_t - \partial^\alpha G_t$ is given by the following integral:

$$I(t) = \int_{|\xi|\leq a} |\xi|^{|\alpha|} \Big| e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s} \Big| d\xi.$$

In view of the elementary inequality $|e^y - 1| \leq C|y|$ for all $|y|$ bounded, we have

$$\begin{aligned} I(t) &= \int_{|\xi|\leq a} |\xi|^{|\alpha|} e^{-t|\xi|^s} \Big| e^{t(\widehat{J}(\xi)-1+|\xi|^s)} - 1 \Big| d\xi \\ &\leq C \int_{|\xi|\leq a} |\xi|^{|\alpha|} e^{-t|\xi|^s} \big| t(\widehat{J}(\xi) - 1 + |\xi|^s) \big| d\xi \\ &\leq Ct \int_{|\xi|\leq a} |\xi|^{|\alpha|} e^{-t|\xi|^s} |\xi|^{s+r} d\xi \\ &\leq Ct^{-\frac{N}{s} - \frac{r}{s} - \frac{|\alpha|}{s}}. \end{aligned}$$

This finishes the proof. □

Proof of Theorem 1.22. Using the same ideas as in the proof of Lemma 1.17 it remains to prove that for some $N/2 < n \leq [s]$ the following holds:

$$\| |x|^n (\partial^\alpha K_t - \partial^\alpha G_t^s) \|_{L^2(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2s} + \frac{n-(|\alpha|+r)}{s}}.$$

Like there, we chose $n = 1$ if $N = 1$ (in this case $1 \leq s \leq 2$), and necessarily $n = 2$ if $N = 2$, 3 (in this case $s = 2$).

Applying Plancherel's identity the proof of the last inequality is reduced to the proof of the following one:

$$\int_{\mathbb{R}^N} |\partial^n_{\xi_j} [\xi^\alpha (\widehat{K_t}(\xi) - \widehat{G^s_t}(\xi))]|^2 \, d\xi \leq Ct^{-\frac{N}{2s} + \frac{n-(|\alpha|+r)}{s}}, \quad j = 1, \dots, N,$$

provided that all the above terms make sense. This means that all the partial derivatives $\partial^k_{\xi_j} \widehat{K_t}$ and $\partial^k_{\xi_j} \widehat{G^s_t}$, $j = 1, \dots, N$, $k = 0, \dots, n$ must be defined. Thus we need $n \leq [s]$.

As in Lemma 1.17 we must see that

$$\int_{\mathbb{R}^N} \xi^{2\beta(k)} |\partial^k_{\xi_1} (\widehat{K_t}(\xi) - \widehat{G^s_t}(\xi))|^2 \, d\xi \leq Ct^{-\frac{N}{s} + \frac{2(k-|\beta(k)|-r)}{s}}$$

for all $n - \min\{n, \alpha_1\} \leq k \leq n$, where

$$\beta(k) = (\alpha_1 + k - n, \alpha_2, \dots, \alpha_N).$$

Again, k can be 0 or 1 if $n = 1$, and 0, 1 or 2 if $n = 2$.

The case $k = 0$ follows easily from Theorem 1.21. Let us deal with the cases $k = 1$ and $k = 2$. Using that the integral outside of a ball of radius a decays exponentially, it remains to analyze the decay of the following integral

$$I(k,t) := \int_{|\xi|\leq a} |\xi|^{2|\beta(k)|} |\partial^k_{\xi_1} (\widehat{K_t}(\xi) - \widehat{G^s_t}(\xi))|^2 d\xi,$$

where a is as in the proof of Theorem 1.21. Using the definition of $\widehat{K_t}$ and G_t^s we obtain that

$$\partial^1_{\xi_1}\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} t\partial^1_{\xi_1}\widehat{J}(\xi)$$

and

$$\partial^1_{\xi_1}\widehat{G^s_t}(\xi) = e^{-t|\xi|^s} t\partial^1_{\xi_1}\left(-|\xi|^s\right);$$

and for $k = 2$, which appears only when $n = 2 = s$,

$$\partial^2_{\xi_1}\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)}\left(t^2\left(\partial^1_{\xi_1}\widehat{J}(\xi)\right)^2 + t\partial^2_{\xi_1}\widehat{J}(\xi)\right)$$

and

$$\partial^2_{\xi_1}\widehat{G^2_t}(\xi) = e^{-t|\xi|^2}\left(t^2\left(2\xi_1\right)^2 - 2t\right).$$

Let us first analyze $I(1,t)$. We can write

$$|\partial^1_{\xi_1}\widehat{K_t}(\xi) - \partial^1_{\xi_1}\widehat{G^s_t}(\xi)|^2 \le I_1(\xi,t) + I_2(\xi,t),$$

where

$$I_1(\xi,t) = 2\left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}\right|^2 t^2|\partial^1_{\xi_1}\widehat{J}(\xi)|^2$$

and

$$I_2(\xi,t) = 2e^{-2t|\xi|^s}t^2\left|\partial^1_{\xi_1}\widehat{J}(\xi) - \partial^1_{\xi_1}\left(-|\xi|^s\right)\right|^2.$$

Let us consider $I_1(\xi,t)$. Taking into account that $n \le [s]$ and $|\widehat{J}(\xi) - 1 + |\xi|^s| \le o(|\xi|^s)$ as $|\xi| \to 0$ we obtain

$$|\partial^j_{\xi_1}\widehat{J}(\xi)| \le C\,|\xi|^{s-j},\ 1 \le j \le n,$$

for all $|\xi| \le a$. On the other hand (see (1.28))

$$\begin{aligned}
&\left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}\right|^2 = e^{-2t|\xi|^s}\left|e^{t(\widehat{J}(\xi)-1+|\xi|^s)} - 1\right|^2 \\
&\le C\,e^{-2t|\xi|^s}\left|t(\widehat{J}(\xi) - 1 + |\xi|^s)\right|^2 \\
&\le C\,t^2 e^{-2t|\xi|^s}|\xi|^{2(r+s)}.
\end{aligned}$$

Therefore,

$$I_1(\xi,t) \le Ct^4 e^{-2t|\xi|^s}|\xi|^{2(r+s)}.$$

Let us now deal with $I_2(\xi,t)$. Choosing eventually a smaller a we can guarantee that for $|\xi| \le a$, since $k = 1 \le [s]$, the following inequality holds:

$$\left|\partial^1_{\xi_1}\widehat{J}(\xi) - \partial^1_{\xi_1}\left(-|\xi|^s\right)\right| \le C|\xi|^{s+r-1}.$$

Consequently

$$I_2(\xi,t) \le Ct^2 e^{-2t|\xi|^s}|\xi|^{2(r+s-1)}.$$

Using that for any $l \ge 0$,

$$\int_{\mathbb{R}^N} e^{-t|\xi|^s}|\xi|^l d\xi \le Ct^{-\frac{N}{s}-\frac{l}{s}},$$

we get the desired decay result for $I(1,t)$.

Finally, to study the decay of $I(2,t)$ we do the following decomposition:

$$
\begin{aligned}
&|\partial^2_{\xi_1}\widehat{K_t}(\xi) - \partial^2_{\xi_1}\widehat{G^2_t}(\xi)|^2 \\
&\leq 2t^4 \left| e^{t(\widehat{J}(\xi)-1)} \left(\partial^1_{\xi_1}\widehat{J}(\xi)\right)^2 - e^{-t|\xi|^2} (2\xi_1)^2 \right|^2 \\
&\quad +2t^2 \left| e^{t(\widehat{J}(\xi)-1)} \partial^2_{\xi_1}\widehat{J}(\xi) - e^{-t|\xi|^2}(-2)) \right|^2 . \\
&\leq 4t^4 \left| e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^2} \right|^2 \left| \partial^1_{\xi_1}\widehat{J}(\xi) \right|^4 \\
&\quad +4t^4 e^{-2t|\xi|^2} \left| \partial^1_{\xi_1}\widehat{J}(\xi) - (-2\xi_1) \right|^2 \left(\left| \partial^1_{\xi_1}\widehat{J}(\xi) \right| + 2|\xi_1| \right) \\
&\quad +4t^2 \left| e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^2} \right|^2 \left| \partial^2_{\xi_1}\widehat{J}(\xi) \right|^2 \\
&\quad +4t^2 e^{-2t|\xi|^2} \left| \partial^2_{\xi_1}\widehat{J}(\xi) - (-2) \right|^2 ,
\end{aligned}
$$

and proceeding similarly to the previous case we finish the proof. □

1.2.3. A different approach. In this subsection we obtain the first two terms in the asymptotic expansion of the solution under less restrictive hypotheses on J.

THEOREM 1.23. *Let $u_0 \in L^1(\mathbb{R}^N)$ with $\widehat{u_0} \in L^1(\mathbb{R}^N)$ and $s < l$ be two positive numbers such that*

$$\widehat{J}(\xi) = 1 - |\xi|^s + B|\xi|^l + o(|\xi|^l), \quad \xi \to 0,$$

for some real number B.

Then for any $2 \leq q \leq \infty$

$$\lim_{t\to\infty} t^{\frac{N}{s}(1-\frac{1}{q})+\frac{l-s}{s}} \|u(t) - v(t) - Bt[(-\Delta)^{\frac{l}{2}} v](t)\|_{L^q(\mathbb{R}^N)} \to 0, \tag{1.29}$$

where v is the solution of $v_t = -(-\Delta)^{\frac{s}{2}} v$ with $v(x,0) = u_0(x)$.

Moreover, if we set h_0 such that $\widehat{h_0}(\xi) = e^{-|\xi|^s}|\xi|^l$, then

$$\lim_{t\to\infty} \left\| t^{\frac{N}{s}+\frac{l}{s}-1} \left(u(yt^{\frac{1}{s}},t) - v(yt^{\frac{1}{s}},t) \right) - Bh_0(y) \left(\int_{\mathbb{R}^N} u_0 \right) \right\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{1.30}$$

Let us point out that the asymptotic expansion given by (1.19) involves K_t (and its derivatives) which is not explicit. On the other hand, the two-terms of the asymptotic expansion (1.29) involves G^s_t, a well known explicit kernel (v is just the convolution of G^s_t and u_0). However, the ideas and methods employed here allow us to find only two terms in the latter expansion. The case $1 \leq q < 2$ in (1.29) can be also treated, but additional hypothesis on J must be imposed.

PROOF OF THEOREM 1.23. The method used here is just to estimate the difference $\|u(t) - v(t) - Bt(-\Delta)^{\frac{l}{2}} v(t)\|_{L^q(\mathbb{R}^N)}$ using Fourier variables.

As before, it is enough to consider the cases $q = 2$ and $q = \infty$. We analyze the case $q = \infty$; the case $q = 2$ follows in the same manner by applying Plancherel's identity.

For $q = \infty$ we have

$$\begin{aligned}
&\|u(t) - v(t) - tB(-\Delta)^{\frac{l}{2}} v(t)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq (2\pi)^{-N} \int_{\mathbb{R}^N} \left|\widehat{u}(\xi,t) - \widehat{v}(\xi,t) - tB\widehat{(-\Delta)^{\frac{l}{2}} v(\xi,t)}\right| d\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}(1 + tB|\xi|^l)||\widehat{u_0}(\xi)\right| d\xi.
\end{aligned}$$

Let a and δ be as in Lemma 1.8 iv) for $D = \frac{1}{2}$. Hence,

$$\int_{|\xi|\geq a} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)| d\xi \leq Ce^{-\delta t}\|\widehat{u_0}\|_{L^1(\mathbb{R}^N)}$$

and

$$\begin{aligned}
&\int_{t^{-\frac{1}{l}}\leq|\xi|\leq a} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)| d\xi \leq C\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{t^{-\frac{1}{l}}\leq|\xi|\leq a} e^{-t|\xi|^s/2}\, d\xi \\
&\leq Ct^{-\frac{N}{s}} \int_{t^{\frac{1}{s}-\frac{1}{l}}\leq|\eta|\leq t^{\frac{1}{s}}a} e^{-|\eta|^s/2}\, d\eta \leq Ct^{-\frac{N}{s}} e^{-\frac{1}{4}t^{1-\frac{s}{l}}}.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s}(1 + tB|\xi|^l)|\widehat{u_0}(\xi)| d\xi \leq C\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s} t|\xi|^l\, d\xi \\
&\leq Ct^{1-\frac{N}{s}-\frac{l}{s}} \int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s}|\eta|^l\, d\eta \\
&\leq Ct^{1-\frac{N}{s}-\frac{l}{s}} e^{-\frac{1}{2}t^{1-\frac{s}{l}}} \int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s/2}|\eta|^l\, d\eta.
\end{aligned}$$

Therefore, we have to analyze

$$I(t) = \int_{|\xi|\leq t^{-\frac{1}{l}}} \left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}(1 + tB|\xi|^l)\right| |\widehat{u_0}(\xi)|\, d\xi.$$

We write $\widehat{J}(\xi) = 1 - |\xi|^s + B|\xi|^l + |\xi|^l f(\xi)$, where $f(\xi) \to 0$ as $|\xi| \to 0$. Thus

$$I(t) \leq I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_{|\xi|\leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} \left|e^{Bt|\xi|^l + t|\xi|^l f(\xi)} - (1 + Bt|\xi|^l + t|\xi|^l f(\xi))\right| |\widehat{u_0}(\xi)|\, d\xi$$

and

$$I_2(t) = \int_{|\xi|\leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} t|\xi|^l |f(\xi)||\widehat{u_0}(\xi)|\, d\xi.$$

For I_1 we have, for t large, that

$$I_1(t) \le C\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\le t^{-\frac{1}{l}}} e^{-t|\xi|^s}(t|\xi|^l + t|\xi|^l|f(\xi)|)^2\,d\xi$$
$$\le C\int_{|\xi|\le t^{-\frac{1}{l}}} e^{-t|\xi|^s} t^2|\xi|^{2l}\,d\xi \le Ct^{-\frac{N}{s}+2-\frac{2l}{s}}$$

and then

$$t^{\frac{N}{s}+\frac{l}{s}-1} I_1(t) \le Ct^{1-\frac{l}{s}} \to 0, \quad t\to\infty.$$

It remains to prove that

$$t^{\frac{N}{s}+\frac{l}{s}-1} I_2(t) \to 0, \quad t\to\infty.$$

Making a change of variables we obtain

$$t^{\frac{N}{s}-1+\frac{l}{s}} I_2(t) \le C\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|\le t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s}|\eta|^l|f(\eta t^{-\frac{1}{s}})|\,d\eta.$$

Note that the integrand is dominated by $\|f\|_{L^\infty(\mathbb{R}^N)}|\eta|^l e^{-|\eta|^s}$, which belongs to $L^1(\mathbb{R}^N)$. Hence, as $f(\eta t^{-\frac{1}{s}}) \to 0$ when $t\to\infty$,

$$t^{\frac{N}{s}+\frac{l}{s}-1} I_2(t) \to 0,$$

and the proof of (1.29) is complete.

Thanks to (1.29), the proof of (1.30) is reduced to showing that

$$\lim_{t\to\infty}\left\| t^{\frac{N}{s}+\frac{l}{s}}[(-\Delta)^{\frac{l}{2}}v](yt^{\frac{1}{s}},t) - h_0(y)\left(\int_{\mathbb{R}^N} u_0\right)\right\|_{L^\infty(\mathbb{R}^N)} = 0.$$

For any $y\in\mathbb{R}^N$, by making a change of variables,

$$I(y,t) = t^{\frac{N}{s}+\frac{l}{s}}[(-\Delta)^{\frac{l}{2}}v](yt^{\frac{1}{s}},t) = \int_{\mathbb{R}^N} e^{-|\xi|^s}|\xi|^l e^{iy\xi}\widehat{u_0}(\xi/t^{\frac{1}{s}}).$$

Thus, using the dominated convergence theorem,

$$\left\| I(y,t) - h_0(y)\int_{\mathbb{R}^N} u_0\right\|_{L^\infty(\mathbb{R}^N)} \le C\int_{\mathbb{R}^N} e^{-|\xi|^s}|\xi|^l\left|\widehat{u_0}(\xi t^{-\frac{1}{s}}) - \widehat{u_0}(0)\right|\,d\xi \to 0$$

as $t\to\infty$. □

1.3. Rescaling the kernel. A nonlocal approximation of the heat equation

In this section it is shown that the problem $v_t(x,t) = \Delta v(x,t)$ can be approximated by nonlocal problems like the ones presented here when they are rescaled in an appropriate way. Concretely, we will rescale the problem

$$\begin{cases} u_t(x,t) = (J*u)(x,t) - u(x,t), \\ u(x,0) = u_0(x), \end{cases}$$

for $x\in\mathbb{R}^N$ and $t>0$, with J satisfying condition (H) and such that

$$\widehat{J}(\xi) = 1 - |\xi|^2 + o(|\xi|^2) \quad \text{as } \xi\to 0. \tag{1.31}$$

THEOREM 1.24. *Assume* (1.31). *Let* u_ε *be the unique solution to*

$$(1.32)\qquad \begin{cases} (u_\varepsilon)_t(x,t) = \dfrac{J_\varepsilon * u_\varepsilon(x,t) - u_\varepsilon(x,t)}{\varepsilon^2}, & x \in \mathbb{R}^N,\ t>0,\\ u_\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where the kernel J *is rescaled according to*

$$J_\varepsilon(x) = \varepsilon^{-N} J\left(\frac{x}{\varepsilon}\right).$$

Then, for every $T>0$, *we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - v\|_{L^\infty(\mathbb{R}^N\times(0,T))} = 0,$$

v *being the solution of the local problem* $v_t(x,t) = \Delta v(x,t)$ *with the same initial condition* $v(x,0) = u_0(x)$.

REMARK 1.25. Note that J_ε satisfies

$$\int_{\mathbb{R}^N} J_\varepsilon(x)\,dx = 1.$$

PROOF OF THEOREM 1.24. The proof uses once more the explicit formula for the solutions in Fourier variables. We have, arguing exactly as before,

$$\widehat{u_\varepsilon}(\xi,t) = e^{\frac{\widehat{J_\varepsilon}(\xi)-1}{\varepsilon^2}t}\,\widehat{u_0}(\xi)$$

and

$$\hat{v}(\xi,t) = e^{-|\xi|^2 t}\,\widehat{u_0}(\xi).$$

Now, we just observe that $\widehat{J_\varepsilon}(\xi) = \hat{J}(\varepsilon\xi)$ and therefore we obtain

$$\int_{\mathbb{R}^N} \left|\widehat{u_\varepsilon} - \hat{v}\right|(\xi,t)\,d\xi = \int_{\mathbb{R}^N} \left|\left(e^{\frac{\hat{J}(\varepsilon\xi)-1}{\varepsilon^2}t} - e^{-|\xi|^2 t}\right)\widehat{u_0}(\xi)\right| d\xi$$

$$\le \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \left(\int_{|\xi|\ge r(\varepsilon)} \left|e^{\frac{\hat{J}(\varepsilon\xi)-1}{\varepsilon^2}t} - e^{-|\xi|^2 t}\right| d\xi \right.$$

$$\left. + \int_{|\xi|< r(\varepsilon)} \left|e^{\frac{\hat{J}(\varepsilon\xi)-1}{\varepsilon^2}t} - e^{-|\xi|^2 t}\right| d\xi\right).$$

For $t \in [0,T]$ we can proceed as in the proof of the asymptotic behaviour (Theorem 1.3) to obtain that

$$\max_x |u_\varepsilon(x,t) - v(x,t)| \le (2\pi)^{-N} \int_{\mathbb{R}^N} \left|\widehat{u_\varepsilon} - \hat{v}\right|(\xi,t)\,d\xi \to 0, \quad \varepsilon\to 0. \qquad \square$$

1.4. Higher order problems

This section deals with the asymptotic behaviour of solutions of a nonlocal diffusion operator of higher order in the whole $\mathbb{R}^N$.

Let us consider the following nonlocal evolution problem:

$$\left\{\begin{aligned} u_t(x,t) &= (-1)^{n-1}\left(J*Id-1\right)^n\left(u(x,t)\right) \\ &= (-1)^{n-1}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(J*)^k(u)\right)(x,t), \\ u(x,0) &= u_0(x), \end{aligned}\right. \tag{1.33}$$

for $x\in\mathbb{R}^N$ and $t>0$, where the hypothesis on the convolution kernel J is again (H), and $(J*)^k(u)=J*\overset{k}{\cdots}*J*u$.

Note that in the problem (1.33) we just have the iteration (n times) of the nonlocal operator $J*u-u$ as right hand side of the equation. This can be seen as a nonlocal generalization of higher order equations of the form

$$v_t(x,t)=-A^n(-\Delta)^{\frac{\alpha n}{2}}v(x,t), \tag{1.34}$$

where A and α are positive constants specified later in this section. Observe that when $\alpha=2$, (1.34) is just $v_t(x,t)=-A^n(-\Delta)^n v(x,t)$. Nonlocal higher order problems have been, for instance, proposed as models for periodic phase separation. Here the nonlocal character of the problem is associated with long-range interactions of "particles" in the system. An example is the nonlocal Cahn-Hilliard equation (cf. e.g. [**112**], [**135**], [**136**]).

Here (1.33) is proposed as a model for higher order nonlocal evolution. Existence and uniqueness of solutions of (1.33) are shown first, but the main aim is to study the asymptotic behaviour as $t\to\infty$ of such solutions.

1.4.1. Existence and uniqueness. As it was done for (1.1), we first prove the existence and uniqueness of a solution, which is understood in the integral sense.

THEOREM 1.26. *Let $u_0\in L^1(\mathbb{R}^N)$ such that $\widehat{u_0}\in L^1(\mathbb{R}^N)$. There exists a unique solution $u\in C^0([0,\infty);L^1(\mathbb{R})^N)$ of (1.33) that, in Fourier variables, is given by the explicit formula,*

$$\hat{u}(\xi,t)=e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\widehat{u_0}(\xi).$$

PROOF. We have formally

$$u_t(x,t)=(-1)^{n-1}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(J*)^k(u)\right)(x,t).$$

Applying the Fourier transform to this equation we obtain

$$\hat{u}_t(\xi,t)=(-1)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(\hat{J}(\xi))^k\hat{u}(\xi,t)$$

$$=(-1)^{n-1}(\hat{J}(\xi)-1)^n\hat{u}(\xi,t).$$

Hence

$$\hat{u}(\xi,t)=e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\widehat{u_0}(\xi).$$

Since $\widehat{u_0}(\xi)\in L^1(\mathbb{R}^N)$ and $e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}$ is continuous and bounded, $\hat{u}(\cdot,t)\in L^1(\mathbb{R}^N)$ and the result follows by taking the inverse Fourier transform. □

The next result is a lemma concerning the fundamental solution of (1.33).

LEMMA 1.27. *Let $J \in \mathcal{S}(\mathbb{R}^N)$. The fundamental solution w of* (1.33)*, that is, the solution of the equation with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x,t) = e^{-t}\delta_0(x) + V(x,t), \tag{1.35}$$

with $V(x,t)$ smooth. Moreover, if u is a solution of (1.33)*, u can be written as*

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^N} w(x-z,t)u_0(z)\,dz.$$

PROOF. By the previous result we have

$$\hat{w}_t(\xi,t) = (-1)^{n-1}(\hat{J}(\xi)-1)^n \hat{w}(\xi,t).$$

Hence, as the initial datum satisfies $\widehat{w_0} = \widehat{\delta_0} = 1$, we get

$$\hat{w}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} = e^{-t} + e^{-t}\left(e^{((-1)^{n-1}(\hat{J}(\xi)-1)^n+1)t} - 1\right).$$

The first part of the lemma follows by applying the inverse Fourier transform.

To finish the proof we just observe that $w * u_0$ is a solution of (1.33) with $(w * u_0)(x,0) = u_0(x)$. □

1.4.2. Asymptotic behaviour. Next, we deal with the asymptotic behaviour as $t \to \infty$.

THEOREM 1.28. *Assume*

$$\hat{J}(\xi) = 1 - A|\xi|^s + o(|\xi|^s) \quad \text{as } \xi \to 0, \tag{1.36}$$

$A, s > 0$. Let u be a solution of (1.33) *with $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$, $u_0 \ge 0$. Then the asymptotic behaviour of $u(x,t)$ is given by*

$$\lim_{t\to+\infty} t^{\frac{N}{sn}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of

$$v_t(x,t) = -A^n(-\Delta)^{\frac{sn}{2}} v(x,t)$$

with initial condition $v(x,0) = u_0(x)$.

Moreover, there exists a constant $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-\frac{N}{sn}},$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{sn}} u(y t^{\frac{1}{sn}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)}\, G_{A^n}^{sn}(y)\right| = 0.$$

PROOF. By Theorem 1.26, $\hat{u}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\widehat{u_0}(\xi)$. On the other hand, $v(x,t)$ is given by

$$\hat{v}(\xi,t) = e^{-A^n|\xi|^{sn}t}\widehat{u_0}(\xi).$$

Hence in Fourier variables

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\hat{u} - \hat{v}| (\xi, t) \, d\xi \\
&\leq \int_{|\xi| \geq r(t)} \left| \left(e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n |\xi|^{sn} t} \right) \widehat{u_0}(\xi) \right| \, d\xi \\
&+ \int_{|\xi| < r(t)} \left| \left(e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n |\xi|^{sn} t} \right) \widehat{u_0}(\xi) \right| \, d\xi \\
&= [I] + [II],
\end{aligned}$$

where $[I]$ and $[II]$ denote the first and the second integral respectively, and $r(t)$, with $r(t) \to 0$, will be determined later. To deal with $[I]$ we decompose it into two parts,

$$\begin{aligned}
[I] &\leq \int_{|\xi| \geq r(t)} \left| e^{-A^n |\xi|^{sn} t} \widehat{u_0}(\xi) \right| \, d\xi + \int_{|\xi| \geq r(t)} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \widehat{u_0}(\xi) \right| \, d\xi \\
&= [I_1] + [I_2].
\end{aligned}$$

Consider $[I_1]$. Setting $\eta = \xi t^{\frac{1}{sn}}$ and writing $[I_1]$ in the new variable η we get,

$$t^{\frac{N}{sn}} [I_1] \leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| \geq r(t) t^{\frac{1}{sn}}} e^{-A^n |\eta|^{sn}} \, d\eta \overset{t\to\infty}{\longrightarrow} 0$$

if we impose that

$$r(t) t^{\frac{1}{sn}} \overset{t\to\infty}{\longrightarrow} \infty. \tag{1.37}$$

Let a and δ be as in Lemma 1.8 iv) for D such that

$$d := 2 - \left(1 + \frac{D}{A}\right)^n > 0.$$

Therefore

$$\begin{aligned}
[I_2] &\leq \int_{a \geq |\xi| \geq r(t)} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \widehat{u_0}(\xi) \right| \, d\xi \\
&\quad + \int_{|\xi| \geq a} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \widehat{u_0}(\xi) \right| \, d\xi \\
&\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{a \geq |\xi| \geq r(t)} e^{-d^n |\xi|^{sn} t} \, d\xi + C e^{-\delta^n t}.
\end{aligned}$$

Changing variables as before, $\eta = \xi t^{\frac{1}{sn}}$, we get

$$\begin{aligned}
t^{\frac{N}{sn}} [I_2] &\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{a t^{\frac{1}{sn}} \geq |\eta| \geq r(t) t^{\frac{1}{sn}}} e^{-d^n |\eta|^{sn}} \, d\eta + C t^{\frac{N}{sn}} e^{-\delta^n t} \\
&\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| \geq r(t) t^{\frac{1}{sn}}} e^{-d^n |\eta|^{sn}} \, d\eta + C t^{\frac{N}{sn}} e^{-\delta^n t} \to 0
\end{aligned}$$

as $t \to \infty$ if (1.37) holds.

It remains only to estimate $[II]$. We proceed as follows:

$$[II] = \int_{|\xi|<r(t)} e^{-A^n|\xi|^{sn}t} \left| e^{t\left((-1)^{n-1}(\hat{J}(\xi)-1)^n + A^n|\xi|^{sn}\right)} - 1 \right| |\widehat{u_0}(\xi)| \ d\xi.$$

Applying the binomial formula,

$$(-1)^{n-1}(\hat{J}(\xi)-1)^n + A^n|\xi|^{sn}$$

$$= (-1)^{n-1} \left(\sum_{k=1}^{n} \binom{n}{k} \left(\hat{J}(\xi) - 1 + A|\xi|^s \right)^k (-A|\xi|^s)^{n-k} \right).$$

Consequently, since $|e^y - 1| \le C|y|$ for y small, and using (1.7),

$$t^{\frac{N}{sn}}[II] \le Ct^{\frac{N}{sn}} \int_{|\xi|<r(t)} e^{-A^n|\xi|^{sn}t} t \left(\sum_{k=1}^{n} \binom{n}{k} A^{n-k} h(\xi)^k \right) |\xi|^{sn} \, d\xi$$

$$\le Ct^{\frac{N}{sn}} \int_{|\xi|<r(t)} e^{-A^n|\xi|^{sn}t} t h(\xi) |\xi|^{sn} \, d\xi,$$

provided that we impose

$$t(r(t))^{sn} h(r(t)) \to 0 \quad \text{as } t \to \infty. \tag{1.38}$$

In this case, changing variables, $\eta = \xi t^{\frac{1}{sn}}$, we have

$$t^{\frac{N}{sn}}[II] \le C \int_{|\eta|<r(t)t^{\frac{1}{sn}}} e^{-A^n|\eta|^{sn}} \, |\eta|^{sn} \, h(\eta/t^{\frac{1}{sn}}) \, d\eta.$$

Since $h(\eta/t^{\frac{1}{sn}}) \to 0$ as $t \to \infty$, and in the above inequality the integrand is dominated by

$$\|h\|_{L^\infty(\mathbb{R}^N)} \, e^{-A^n|\eta|^{sn}} \, |\eta|^{sn},$$

which belongs to $L^1(\mathbb{R}^N)$, by the dominated convergence theorem,

$$t^{\frac{N}{sn}}[II] \to 0 \quad \text{as } t \to \infty.$$

Combining this with the previous results we have that

$$t^{\frac{N}{sn}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}| \, (\xi, t) \, d\xi \le t^{\frac{N}{sn}}([I] + [II]) \to 0 \quad \text{as } t \to \infty, \tag{1.39}$$

provided we can find an $r(t) \to 0$ as $t \to \infty$ that fulfils both conditions (1.37) and (1.38), which is true by Lemma 1.9 changing there s by sn. To conclude we only have to observe that from the convergence of the Fourier transforms

$$\|\hat{u}(\cdot, t) - \hat{v}(\cdot, t)\|_{L^1(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty,$$

the convergence of $u - v$ in L^∞ follows. Indeed, from (1.39) we obtain

$$t^{\frac{N}{sn}} \max_x |u(x,t) - v(x,t)| \le (2\pi)^{-N} t^{\frac{N}{sn}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}| \, (\xi, t) \, d\xi \to 0, \ \ t \to \infty,$$

which ends the first part of the proof of the theorem.

As a consequence of this first part we obtain that the asymptotic behaviour is the same as the one for solutions of the evolution given by a power n of the fractional Laplacian. It is now easy to check that the asymptotic behaviour is in fact the one described in the statement of the second part of the theorem. Indeed, in Fourier variables we have

$$\lim_{t\to\infty} \hat{v}(\eta t^{-\frac{1}{sn}}, t) = \lim_{t\to\infty} e^{-A^n|\eta|^{sn}} \widehat{u_0}(\eta t^{-\frac{1}{sn}})$$

$$= e^{-A^n|\eta|^{sn}} \widehat{u_0}(0) = e^{-A^n|\eta|^{sn}} \left\| u_0 \right\|_{L^1(\mathbb{R}^N)} .$$

Therefore

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{sn}} v(yt^{\frac{1}{sn}}, t) - \left\| u_0 \right\|_{L^1(\mathbb{R}^N)} G^{ns}_{A^n}(y) \right| = 0. \qquad \square$$

With similar arguments one can prove that also the asymptotic behaviour of the derivatives of solutions u of (1.33) is the same as the one for derivatives of solutions v of the evolution of a power n of the fractional Laplacian, assuming sufficient regularity of the solutions u of (1.33).

THEOREM 1.29. *Let u be a solution of* (1.33) *with $u_0 \in W^{k,1}(\mathbb{R}^N)$, $k \le sn$ and $\widehat{u_0} \in L^1(\mathbb{R}^N)$. Then the asymptotic behaviour of $D^k u(x,t)$ is given by*

$$\lim_{t\to+\infty} t^{\frac{N+k}{sn}} \max_x \left| D^k u(x,t) - D^k v(x,t) \right| = 0,$$

where v is the solution of

$$v_t(x,t) = -A^n(-\Delta)^{\frac{sn}{2}} v(x,t)$$

with initial condition $v(x,0) = u_0(x)$.

PROOF. We begin again by transforming our problem for u and v into a problem for the corresponding Fourier transforms $\hat{u}$ and $\hat{v}$. To this end we consider

$$\max_x \left| D^k u(x,t) - D^k v(x,t) \right|$$

$$\le (2\pi)^{-N} \int_{\mathbb{R}^N} \left| \widehat{D^k u(\xi,t)} - \widehat{D^k v(\xi,t)} \right| \, d\xi$$

$$= (2\pi)^{-N} \int_{\mathbb{R}^N} |\xi|^k \left| \hat{u}(\xi,t) - \hat{v}(\xi,t) \right| \, d\xi.$$

The proof of

$$\int_{\mathbb{R}^N} |\xi|^k \left| \hat{u}(\xi,t) - \hat{v}(\xi,t) \right| \, d\xi \to 0$$

as $t \to \infty$ works in an analogous way to the proof of the first part of the previous theorem since the additional term $|\xi|^k$ is always dominated by the exponential terms. $\square$

1.4.3. Rescaling the kernel in a higher order problem. Now we show that the problem $v_t(x,t) = -A^n(-\Delta)^{\frac{sn}{2}} v(x,t)$ can be approximated by nonlocal problems like the ones presented in this section when they are rescaled in an appropriate way.

THEOREM 1.30. *Assume* (1.36)*. Let u_ε be the unique solution to*

$$\begin{cases} (u_\varepsilon)_t(x,t) = (-1)^{n-1} \dfrac{(J_\varepsilon * \mathrm{Id} - 1))^n}{\varepsilon^{sn}} (u_\varepsilon(x,t)), \\ u_\varepsilon(x,0) = u_0(x), \end{cases} \qquad (1.40)$$

where

$$J_\varepsilon(x) = \varepsilon^{-N} J\left(\frac{x}{\varepsilon}\right).$$

Then, for every $T > 0$, we have

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - v\|_{L^\infty(\mathbb{R}^N\times(0,T))} = 0,$$

where v is the solution of the local problem $v_t(x,t) = -A^n(-\Delta)^{\frac{sn}{2}} v(x,t)$ with the same initial condition $v(x,0) = u_0(x)$.

Proof. As for the case of the heat equation, the proof uses once more the explicit formula for the solutions in Fourier variables. We have, arguing exactly as before,

$$\widehat{u_\varepsilon}(\xi,t) = e^{(-1)^{n-1}\frac{(\widehat{J_\varepsilon}(\xi)-1)^n}{\varepsilon^{sn}}t}\widehat{u_0}(\xi)$$

and

$$\hat{v}(\xi,t) = e^{-A^n|\xi|^{sn}t}\widehat{u_0}(\xi).$$

Now, since $\widehat{J_\varepsilon}(\xi) = \hat{J}(\varepsilon\xi)$,

$$\int_{\mathbb{R}^N} |\widehat{u_\varepsilon} - \hat{v}|(\xi,t)\,d\xi = \int_{\mathbb{R}^N} \left|(e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{sn}}t} - e^{-A^n|\xi|^{sn}t})\widehat{u_0}(\xi)\right|\,d\xi$$

$$\le \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)}\left(\int_{|\xi|\ge r(\varepsilon)} \left|e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{sn}}t} - e^{-A^n|\xi|^{sn}t}\right|\,d\xi\right.$$

$$\left.+ \int_{|\xi|< r(\varepsilon)} \left|e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{sn}}t} - e^{-A^n|\xi|^{sn}t}\right|\,d\xi\right).$$

For $t \in [0,T]$ we can proceed as in the proof of Theorem 1.28 to obtain that

$$\max_x |u_\varepsilon(x,t) - v(x,t)| \le (2\pi)^{-N}\int_{\mathbb{R}^N} |\widehat{u_\varepsilon} - \hat{v}|(\xi,t)\,d\xi \to 0, \quad \varepsilon \to 0. \qquad \square$$

Bibliographical notes

The results included in this chapter come from [**68**], [**121**] and [**140**].

There is a large amount of literature dealing with related problems such as the ones considered in this chapter. We refer to equations with a linear nonlocal diffusion operator of the form $J * u - u$. See [**106**] for a general survey. Next, let us describe briefly some of the existing bibliography.

In [**37**], [**38**], [**39**] and [**40**] nonlocal diffusion is used to describe phase transition, and in [**42**] and [**65**] some biological models are studied. Concerning travelling front type solutions to the parabolic problem a great deal of work has been done. We quote [**33**], [**34**], [**35**], [**36**], [**39**], [**69**], [**70**], [**72**], [**74**], [**81**], [**82**], [**85**], [**86**], [**105**], [**145**] and [**155**]. In those papers the authors study the existence and stability of travelling waves for different source terms. Note that the equation for the profile of the travelling wave gives an integro-differential equation. The stability of such profiles is a delicate issue. See also [**71**], [**73**], [**83**], [**107**], [**114**] and [**137**], which deal with a source term of logistic type, bistable or power-like nonlinearity. With respect to a singular perturbation problem for a nonlocal equation see [**131**].

For anisotropic problems we refer to [**84**]. We mention [**115**], where some logistic equations and systems of Lotka-Volterra type are studied. See [**75**] and [**77**] for interesting features in other related nonlocal problems. Concerning large deviations for nonlocal problems (that can be used to gain some intuition on the underlying probabilistic background) we refer to [**53**].

There is also an increasing interest in free boundary problems and regularity issues for nonlocal problems such as the obstacle problem for the fractional Laplacian (that can be regarded as a nonlocal operator with a singular kernel), including a theory of viscosity solutions for nonlocal problems, but we are not dealing with such issues in the present work. We refer to [**24**], [**28**], [**29**], [**62**], [**63**], [**64**], [**67**], [**123**] and [**146**]. For initial traces and well-posedness of an evolution governed by the fractional Laplacian we refer to [**3**], and for fractional mean curvature flows, to [**124**]. These evolution problems governed by the fractional Laplacian are related to Levy processes in probability theory; see [**21**].

For nonlocal equations on spacial lattices (note that in this case we deal with an infinite system of ODEs) see, for example, [**32**], [**38**], [**41**] and [**122**].