Preface

This book is an exploration of Seifert fiberings. These are mappings which extend the notion of fiber bundle mappings by allowing some of the fibers to be singular. Seifert fiberings are mappings whose typical fibers are homeomorphic to a fixed homogeneous space. The singular fibers are quotients of the homogeneous space by distinguished groups of homeomorphisms.

In a remarkable paper, in 1933, Herbert Seifert, introduced a class of 3-manifolds which became known as Seifert manifolds. They play a very significant role in low dimensional topology and remain under intense scrutiny today. A Seifert manifold maps onto a 2-dimensional surface such that the inverse image of each point on the surface is homeomorphic to a circle. The set of singular fibers are isolated from each other and the typical fibers wind nontrivially around the singular fibers. We will describe in detail the Seifert 3-manifolds as a special case of the general construction of Seifert fiberings. Our major focus, however, is on higher dimensional phenomenon where the typical fiber is a homogeneous space.

A major inspiration for a generalization to higher dimensions comes from transformation groups. Let (G, X) be a proper action of the connected Lie group G on a path-connected X and examine the homomorphism $\mathrm{ev}^x_* : \pi_1(G, e) \to \pi_1(X, x)$ induced by the evaluation map $\mathrm{ev}^x : g \mapsto gx$, $g \in G$. The image H of ev^x_* is a central subgroup of $\pi_1(X, x)$ independent of the base point x.

The G-action on X can be lifted to the covering space X_H of X associated with the subgroup H of $\pi_1(X, x)$ so that $\pi_1(X_H, \widehat{x}) = H$. The covering transformations, $Q = \pi_1(X, x)/H$, commute with the lifted G-action on X_H and so induce a proper Q-action on $W = G\backslash X_H$. We get the following commutative diagram of orbit mappings:

$$\begin{array}{ccc}
(G, X_H) & \xrightarrow[\tau']{G\backslash} & (Q, W) \\
{\scriptstyle \nu'}\Big\downarrow{\scriptstyle Q\backslash} & & {\scriptstyle \nu}\Big\downarrow{\scriptstyle Q\backslash} \\
(G, X) & \xrightarrow[\tau]{G\backslash} & G\backslash X = Q\backslash W
\end{array}$$

What we discover is that the lifted action of G on X_H is usually simpler than on the original X. The discrete action of Q on W can be used to describe the action of G on X locally. In fact, under the appropriate circumstances, the Q-action on W can be used to construct all the possible G-actions on X whose orbit space is $Q\backslash W$.

For example, if $G = T^k$, the k-dimensional torus, and ev^x_* is injective, then (T^k, X) is called an injective torus action. For this, X_H splits into a product

$T^k \times W$, where T^k acts as translations on the first factor. In this case, the elements of $H^2(Q;\mathbb{Z}^k)$ completely determine all the injective torus actions, up to equivalence, where the T^k-orbit space will be $Q\backslash W$.

For a general Seifert fibering, the covering space X_H is replaced by a principal G-bundle P over the base of the bundle W. Acting properly on P is a group, $\mathit{\Pi}$, normalizing $\ell(G)$, the left translational action of G on P.

Put $\Gamma = \ell(G) \cap \mathit{\Pi}$. On W there is induced an action of $Q = \mathit{\Pi}/\Gamma$.

$$\begin{array}{ccccc} (G,\mathit{\Pi},P) & \xrightarrow{G\backslash} & (Q,W) & \qquad & \text{Principal } G\text{-bundle} \\ \big\downarrow{\scriptstyle \mathit{\Pi}\backslash} & & \big\downarrow{\scriptstyle Q\backslash} & & \\ \mathit{\Pi}\backslash P & \xrightarrow{\tau} & Q\backslash W & & \text{Seifert fibering} \end{array}$$

The induced mapping τ is our Seifert fibering. The typical fiber is the homogeneous space $\Gamma\backslash G$ where $\Gamma = \mathit{\Pi} \cap \ell(G)$, and $Q = \mathit{\Pi}/\Gamma$. The singular fibers are quotients of $\Gamma\backslash G$.

Even though we formulate Seifert fiberings for general spaces, our interest is directed towards geometric applications. Consequently, almost all of our illustrations and applications are devoted to manifolds or manifolds with singularities. We also focus on the Seifert fiberings where $\mathit{\Pi}$ and Q are discrete, and it is only in Chapter 12 that we consider Seifert fiberings where Q is a compact connected Lie group.

The many different topics covered in this book show the broad range of applicability at many levels. From this, it is clear that a mathematician studying geometric problems will often have to analyze singular fiberings, and we believe that this book provides some good tools for attacking interesting geometric phenomena.

Our interest is to engage the reader who has a modest background in topology, geometry and algebra as found in the second year of graduate school. We set language and notation and add some background material to fill in things that may lie outside the standard courses. Occasionally, we use and quote results that are readily available in good sources elsewhere. But on the whole, we have tried to be reasonably complete in our presentations. Examples are given to familiarize the reader with definitions and to illustrate special cases of the theorems. Similarly, the exercises are designed to enhance understanding of the text. Most of them are not difficult, and the reader is encouraged to do them.

Chapter 1 is an introductory chapter which establishes notation and fills in needed facts about proper actions of noncompact Lie groups.

In Chapter 2, covering spaces and lifting group actions to covering spaces are discussed. While much may be familiar to the reader, our approach is very explicit for ease in later computations.

Chapter 3 continues the theme that lifting a group action to a covering space simplifies the action. For example, locally injective actions lift to free actions and injective torus actions lift to product actions. These lifting techniques are then used to study actions of compact Lie groups G on closed aspherical manifolds and their generalizations, the admissible manifolds. It is shown that the effective action of the connected component of G on an admissible manifold M is a torus T^k which acts injectively and with $k \leq$ the rank of the center of $\pi_1(M)$. If the center of $\pi_1(M)$ is finite, then the finite G injects into the outer automorphism group of $\pi_1(M)$, $\mathrm{Out}(\pi_1(M))$. These results lead to constructions of closed manifolds that admit no effective action of any finite group.

In Chapter 4, a formal definition of a Seifert fibering is given. It is motivated by some new examples and the actions studied in Chapter 3. Let P be a principal G-bundle over a space W. Let $\mathit{\Pi}$ act properly on P and normalize the left principal G-action $\ell(G)$ on P. Then there exists a commutative diagram

$$\begin{array}{ccc} (G, \mathit{\Pi}, P) & \xrightarrow{G\backslash} & (Q, W) \\ \downarrow{\scriptstyle \mathit{\Pi}\backslash} & & \downarrow{\scriptstyle Q\backslash} \\ X = \mathit{\Pi}\backslash P & \xrightarrow{\tau} & B = Q\backslash W \end{array} \quad \text{where } Q = \mathit{\Pi}/\mathit{\Pi} \cap \ell(G), \quad W = G\backslash P.$$

Assume the induced action of Q on W is proper. The map τ, induced by the three other orbit mappings, is by definition a Seifert fibering modeled on the principal G-bundle P. In our definition, G and $\mathit{\Pi}$ are Lie groups with G and P usually connected. The inverse image $\tau^{-1}(b)$, $b \in B$, is called a fiber. If $\tau^{-1}(b)$ is the homogeneous space $\Gamma\backslash G$, where $\Gamma = \mathit{\Pi} \cap \ell(G)$, it is called a typical fiber; otherwise, it is called a singular fiber. The singular fibers turn out to be quotients of the typical fiber by the action of a compact group of affine diffeomorphism of $\Gamma\backslash G$. More precisely, a singular fiber is a quotient of G by a group $\Gamma' \subset \mathrm{Aff}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$, where $\Gamma' \cap \ell(G) = \Gamma$, and $\mathrm{Aut}(G)$ is the group of continuous automorphisms of G.

Let $\mathrm{TOP}_G(P)$ be the normalizer of $\ell(G)$ in $\mathrm{TOP}(P)$, the group of homeomorphisms of P with the compact-open topology. The former is also the same as both the weak bundle automorphisms of P and the weak $\ell(G)$-equivalences of P. Thus, a Seifert mapping τ arises from an embedding of a group $\mathit{\Pi}$ in $\mathrm{TOP}_G(P)$. The imposed condition of properness excludes pathological situations.

One goal of the later chapters is to emulate fiber bundle theory and determine the various spaces X which fiber over a fixed B with typical fiber $\Gamma\backslash G$. Obviously, we need to find all the $\mathit{\Pi}$ that embed into $\mathrm{TOP}_G(P)$ satisfying the properness conditions. In particular, each $\mathit{\Pi}$ will be a (topological) extension of the Lie group Γ by Q.

Locally injective and injective actions on X and their orbit mappings τ give rise to a large and important class of Seifert fiberings. In most cases, Q is discrete.

If $\mathcal{U}$ is a geometrically interesting subgroup of $\mathrm{TOP}_G(P)$ (for example, $\mathcal{U} = \mathrm{Isom}(P)$, the group of isometries of P) and $\mathit{\Pi} \subset \mathrm{TOP}_G(P)$, then determining when $\mathit{\Pi}$ can be mapped into $\mathcal{U}$, or at least can be deformed into $\mathcal{U}$, is another important goal investigated in later chapters.

Chapter 5 fills in what is needed from the cohomology of groups. The emphasis here is on the low dimensional cohomology and cohomology sets for discrete groups with non-Abelian coefficients.

Chapter 6 discusses facts needed from Lie group theory, especially for nilpotent, completely solvable, and semisimple Lie groups.

Chapter 7 answers the questions concerning the existence and construction of Seifert fiberings. Treated also is the uniqueness and rigidity of the construction for simply connected Abelian, nilpotent, and completely solvable G as well as for semisimple G in adjoint form. They possess the property that an isomorphism of a lattice Γ in a G with another lattice Γ' in G' can be uniquely extended to an isomorphism of G into G'. This is called the ULIEP (Unique Lattice Isomorphism Extension Property).

The theorems of this chapter are crucial for later chapters; Chapters 8, 9, 11, 13, and 14.

In Chapter 8, we investigate the significance and geometric meaning of a very important special case of the existence, uniqueness, and rigidity theorems of Chapter 7. For example, if $P = \mathbb{R}^n \times \text{point}$, then $\mathrm{TOP}_{\mathbb{R}^n}(P) = \ell(\mathbb{R}^n) \rtimes \mathrm{Aut}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R}) = \mathrm{Aff}(\mathbb{R}^n)$, the group of affine diffeomorphisms of Euclidean affine space. Now let Π be any extension $1 \to \mathbb{Z}^n \to \Pi \to Q \to 1$, where Q is a finite group and the action of Q on $\mathbb{Z}^n$, induced by conjugation by elements of Π, is faithful. Then, and only then, does there exist an injection $\theta : \Pi \to \mathrm{Aff}(\mathbb{R}^n)$ such that $\theta(\Pi) \cap \ell(\mathbb{R}^n) = \mathbb{Z}^n$. This, unless Π is $\mathbb{Z}^n$, results in a Seifert fibering $\tau : \theta(\Pi)\backslash\mathbb{R}^n \to \text{point}$, where the only fiber is a singular fiber. The typical fiber $\theta(\mathbb{Z}^n)\backslash\mathbb{R}^n$, an n-torus, only appears as a covering of the singular fiber.

Because Q is finite and is mapped injectively into $\mathrm{GL}(n,\mathbb{R})$, the uniqueness theorem of Chapter 7 implies that $\theta(\Pi)$ can be conjugated in $\mathrm{Aff}(\mathbb{R}^n)$ so that the image $\theta'(\Pi)$ now lies in $E(n) = \ell(\mathbb{R}^n) \rtimes \mathrm{O}(n,\mathbb{R})$. Since $E(n)$ is the full group of isometries of Euclidean space, $\theta'(\Pi)$ is a Euclidean crystallographic group, and $\theta'(\Pi)\backslash\mathbb{R}^n$ is a Euclidean crystal. If Π is torsion free, $\theta'(\Pi)\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold. The so-called first Bieberbach theorem states that if Δ is a discrete subgroup of $E(n)$ with $\Delta\backslash E(n)$ compact, then $\Delta \cap \ell(\mathbb{R}^n) \cong \mathbb{Z}^n$, a lattice of $\ell(\mathbb{R}^n)$, with $\mathbb{Z}^n$ having finite index in Δ. Thus Δ must be one of the crystallographic groups $\theta'(\Pi)$ constructed above. Now the uniqueness and rigidity theorem of Chapter 7 asserts if $\theta'(\Pi)$ and $\theta''(\Pi)$ are two embeddings in $E(n)$, then they are conjugate in $\mathrm{Aff}(\mathbb{R}^n)$. Consequently, if Π is torsion free, then the flat Riemannian manifold $\theta'(\Pi)\backslash\mathbb{R}^n$ is diffeomorphic to $\theta''(\Pi)\backslash\mathbb{R}^n$ by an affine diffeomorphism. This becomes the content of the second theorem of Bieberbach. (Furthermore, the classification of all crystallographic groups up to conjugacy in $\mathrm{Aff}(\mathbb{R}^n)$ can now be determined by cohomological means.)

These results of Bieberbach are also extended, in Chapter 8, to simply connected nilpotent Lie groups and certain completely solvable Lie groups as well as to the corresponding infra-nilmanifolds and infra-solvmanifolds resulting from the construction.

Chapter 9 extends the definition of a Seifert fibering to one modeled on the product fiber bundle $G/K \times W$ over W. The group K is a closed subgroup of the Lie group G, the typical fiber is the double coset space $\Gamma\backslash G/K$, and singular fibers are finite quotients of the typical fiber. When G has finitely many connected components and K is a maximal compact subgroup, then G/K is diffeomorphic to $\mathbb{R}^n$. In this circumstance, if Γ is a torsion-free lattice in G with $\Gamma\backslash G$ compact, then the double coset space $\Gamma\backslash G/K$ is a closed aspherical manifold. Especially interesting are the Riemannian symmetric spaces of noncompact type G/K and the corresponding locally symmetric spaces $\Gamma\backslash G/K$. The existence, uniqueness, and rigidity theorems have analogues for these new types of fiberings.

Chapter 10 turns to Seifert fiberings modeled on nontrivial principal G-bundles P over W with G a k-dimensional torus. This chapter is independent of the previous three chapters and also recaptures the results of Chapter 7 for $G = \mathbb{R}^k$. Let $\mathrm{M}(W, T^k)$ denote the space of continuous maps from W into T^k. It is an Abelian group. The structure of $\mathrm{TOP}_{T^k}(P)$ (weakly T^k-equivariant homeomorphisms of P) is given by the exact sequence

$$1 \longrightarrow \mathrm{M}(W, T^k) \xrightarrow{\ \psi\ } \mathrm{TOP}_{T^k}(P) \xrightarrow{\ j\ } \mathrm{Aut}(T^k) \times \mathrm{TOP}(W).$$

To construct a Seifert fibering modeled on the principal fibering $T^k \to P \to W$, we begin with a proper action $\rho : Q \to \mathrm{TOP}(W)$ with Q discrete, and a homomorphism $\varphi : Q \to \mathrm{Aut}(T^k)$. We seek extensions $1 \to F \to \mathit{\Pi} \to Q \to 1$, and injective homomorphisms $\theta : \mathit{\Pi} \to \mathrm{TOP}_{T^k}(P)$ such that $\theta(\mathit{\Pi}) \cap \ell(G) = F \subset \mathrm{M}(W, T^k)$ and $(\varphi \times \rho)(Q) \subset \mathrm{Aut}(T^k) \times \mathrm{TOP}(W)$. To ensure $(\varphi \times \rho)(Q) \subset \mathrm{Im}(j)$, we show this holds if and only if P is invariant under the action of Q. This translates, in cohomological terms, to $[P] \in H^2(W; \mathbb{Z}^k)^Q$, where $[P]$ is the cohomology class representing the principal T^k-bundle P over W.

The Borel space, $EQ \times_Q W = W_Q$, associated to the Q-action on W, plays an important role. For example, the Q-action on W lifts to a group of weak bundle automorphisms of P if and only if the bundle P is the pullback of a T^k-bundle $\widetilde{P}$ over W_Q via the inclusion $W \xrightarrow{i} EQ \times W \xrightarrow{\pi} EQ \times_Q W$. If W is simply connected, we obtain an exact sequence

$$\begin{aligned} 0 \to H^2(Q; \mathbb{Z}^k) \xrightarrow{e_1} H^2(W_Q; \mathbb{Z}^k) \xrightarrow{e_2} H^2(W; \mathbb{Z}^k)^Q \\ \xrightarrow{\delta} H^2(Q; \mathrm{M}(W, T^k)) \to H^3(W_Q; \mathbb{Z}^k). \end{aligned}$$

The classification of the Seifert fiberings $\tau : X \to B$ reduces to an analysis of the terms of this exact sequence. For $[P] \in H^2(W; \mathbb{Z}^k)^Q$, $\delta[P]$ represents an extension of $\mathrm{M}(W, T^k)$ by Q. This extension, $E(P, Q)$, is the group of all weak bundle automorphisms of P that project onto the image of Q in $\mathrm{Aut}(T^k) \times \mathrm{TOP}(W)$. The group $\mathit{\Pi}$ must map into $E(P, Q)$ before it can map into $\mathrm{TOP}_{T^k}(P)$. The group $E(P, Q)$ splits (that is, the Q-action on W lifts to a group of weak bundle automorphisms on P) if and only if $\delta[P] = 0$. If $e_2[\widetilde{P}] = P$, then P is the pullback of the bundle $\widetilde{P}$ over the Borel space. The cohomology $H^*(W_Q; \mathbb{Z}^k)$ is called the Q-equivariant cohomology of W, often written as $H^*_Q(W; \mathbb{Z}^k)$. The elements of $H^2(Q; \mathbb{Z}^k)$ then classify all the distinct $\theta : Q \to \mathrm{TOP}_{T^k}(P)$, for a fixed (Q, W) that lifts to P.

In Chapter 11, a large group of applications capitalizing on the theorems of Chapter 7 are presented.

- We use the Seifert Construction to create a wide class of closed aspherical manifolds and the theorems of Chapter 7 to topologically classify some of them. The rigidity of some Seifert fiberings is used to topologically classify Seifert fiberings.

- We show how homotopy and algebraic data lead to fiber preserving group actions on Seifert manifolds.

- A torsion-free polycyclic-by-finite group Γ sometimes fails to be the fundamental group of a complete affinely flat manifold. However, by using an iteration of the

Seifert fiber construction, it is shown that Γ is the fundamental group of a compact solvmanifold with a polynomial structure that generalizes a complete affinely flat structure.

• A generalization of the second Bieberbach theorem for nilpotent groups in Chapter 8 is proved and used to show that the Nielsen number equals the Lefschetz number for homotopically periodic self-homeomorphisms on infra-solvmanifolds.

• A torus action (T^k, X) is homologically injective if the evaluation homomorphism $\mathrm{ev}_*^x : H_1(T^k, \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is injective. We show this type of torus action can be written as $(T^k, T^k \times_\Delta Y)$. That is, X is finitely covered by the product $T^k \times Y$, the left translation of T^k on $T^k \times Y$ descends to X via the commuting covering transformations. The finite Abelian covering group Δ acts freely as translations on T^k while acting diagonally on the product. These splittings, which are not necessarily unique, are classified. With the appropriate definition of homologically injective Seifert fiberings, a similar splitting theorem for completely solvable G is also obtained.

• An effective torus action (T^k, M) on a closed aspherical manifold M with center of $\pi_1(M)$ isomorphic to $\mathbb{Z}^k$ is called a maximal torus action of M. Smooth maximal torus actions are shown to exist for infra-nilmanifolds as well as for many other Seifert fiberings. For an infra-nilmanifold, the connected component of the affine diffeomorphisms contains a maximal torus action. In particular, for a compact flat Riemannian manifold, a maximal torus action is the connected component of the full isometry group of M.

• This Chapter concludes with a determination of the dimension of the largest torus that acts effectively on most spherical space forms.

Chapter 12 investigates the Seifert Construction for connected Q. If the group Q acting on W is a *compact connected* Lie group, then the Seifert Construction to produce Seifert fiberings $\tau : X \to Q\backslash W = B$ modeled on principal T^k-bundles P over W requires that we classify the liftings of Q to groups of bundle automorphisms of P. While requiring additional techniques than those used so far for discrete Q, the reader will find that the first two sections of Chapter 10 provides a good introduction to what must be overcome to accomplish this classification. This classification with applications is presented in Chapter 12. Some of these results closely resemble those obtained in Chapter 10. Chapter 12, however, is formally independent of the other chapters and can be studied independently of the other chapters.

Chapter 13 studies deformation spaces for 3-dimensional Seifert spaces. There are three classical 2-dimensional Riemannian geometries: spherical, Euclidean, and hyperbolic. For each closed 2-manifold M, one can impose metrics so that the metric universal covering of M is one of these classical 2-dimensional geometries, the standard sphere, the Euclidean plane, or the hyperbolic plane. That is, one can embed the fundamental group Π of M into the full isometry group of the sphere, the Euclidean plane, or the hyperbolic plane so that Π acts as covering transformations to get a 2-dimensional geometric structure on M.

The Thurston geometrization conjecture, now a theorem due to Perelman, states that a closed 3-manifold can be split into pieces such that each piece has a 3-dimensional geometric structure. There are eight so-called 3-dimensional geometries to consider.

As it turns out, each closed Seifert 3-manifold admits exactly one of six of the eight Riemannian geometries.

Any closed Seifert 3-manifold M not covered by the 3-sphere has a finite covering M' which is a principal circle bundle over a closed surface B'. Since B' admits a 2-dimensional geometric structure, one expects M' will admit either a product geometric structure or a 1-dimensional structure coming from the fiber twisted by a 2-dimensional structure coming from the base.

We classify up to isometry the Seifert 3-manifolds for the two twisted geometries. For the Seifert manifold M with B a hyperbolic orbifold and with twisted geometry, the moduli space of geometric structures on M up to isometry is itself a Seifert fibering over the moduli space of the hyperbolic orbifold $B = Q\backslash W$ with typical fiber a torus T^{2g} where g is the genus of B. We will use the term orbifold to simply mean the orbit space of a locally proper action of a discrete group on a space X. This usage does not conform with the accepted meaning and usage of the term "orbifold" as, say in [**Thu97**], but when we use the term in our discussions concerning manifolds, the two meanings will usually coincide.

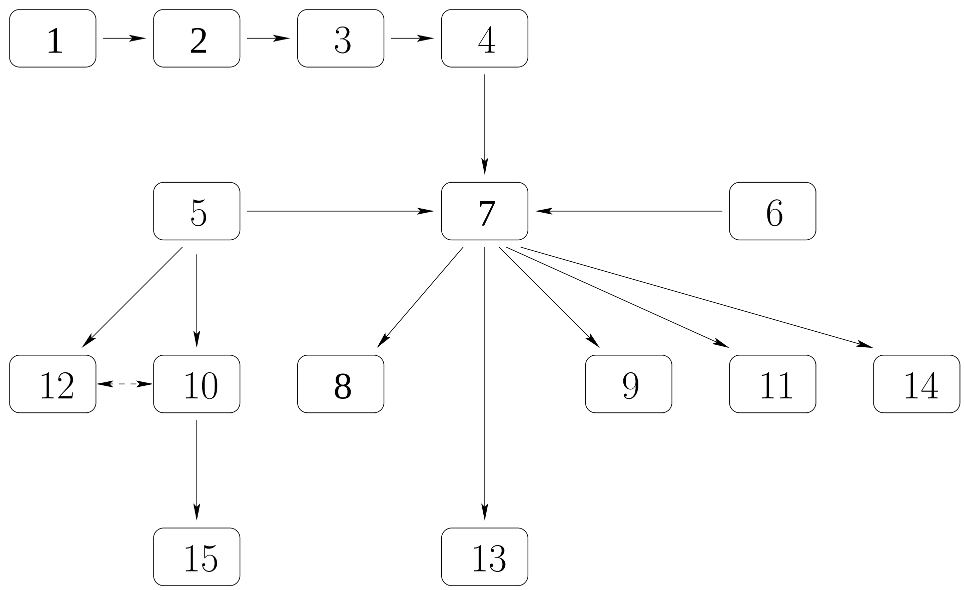
It also turns out that the manifolds with the most interesting of these geometries also has a complete Lorentz structure whose classification is very similar to the Riemannian case.

Chapter 14 begins the analysis of closed 3-dimensional manifolds that admit an effective S^1-action. The classification is given up to S^1-equivalence and for those with infinite $\pi_1(M)$ up to topological equivalence. For the latter case, the rigidity result of Chapter 7 is employed.

When the S^1-action has no fixed points and the fundamental group is not finite Abelian, the orbit mapping is always a Seifert fibering modeled on a principal $\mathbb{R}^1$ or S^1-bundle over $W = S^2$ or $\mathbb{R}^2$. Any 3-dimensional Seifert fibering always has a two-fold Seifert covering which is the orbit mapping of an S^1-action. A large number of algebraic, topological and geometric properties for the orientable Seifert fiberings over an orientable base are derived. A short excursion into holomorphic Seifert fiberings is also given. The chapter concludes with the classification of all the 3-dimensional Seifert fiberings up to fiber preserving homeomorphism in the spirit of Seifert's original methods as well as the topological classification using the rigidity theorem of Chapter 7. Actually, Chapter 14 can be read independently of Chapter 7 provided that one just accepts the rigidity result of Chapter 7 for $G = \mathbb{R}^1$.

Chapter 15 is a continuation of Chapter 10 in that we use the methods and results of Chapter 10 to classify the 3-dimensional Seifert fiberings purely in terms of the equivariant cohomology $H^2(W_Q; \mathbb{Z})$, where $W = S^2$ or $\mathbb{R}^2$. When $W = \mathbb{R}^2$, $H^2(Q; \mathbb{Z})$ is isomorphic to $H^2(W_Q; \mathbb{Z})$. However, the spherical space form case ($W = S^2$) turns out to be especially interesting, and we give a full classification of the 3-dimensional space forms up to topological equivalence by this method. These results and methods provide an alternative to those employed in Chapter 14.

If interest is mainly in later chapters, then the introductory Chapters 1, 2, 3 can be quickly reviewed and Chapter 4 should be understood on a conceptual level—not all the proofs being necessary. The examples of Chapter 7 should be studied and the statement of Theorem 7.3.2 clearly understood. The proofs in Chapter 7 are not needed for the applications. An abbreviated version of some of the topics in this book appeared in [**LR02**].



Flow diagram in chapters