CHAPTER 4

Definition of Seifert Fibering

Throughout this chapter, G will be a Lie group, even though it is enough to have G be a locally compact topological group until Lemma 4.2.5.

A Seifert fibering is a map $\tau : X \to B$, which is supposed to generalize the notion of a fiber bundle with a homogeneous space as fiber. The inverse images $\tau^{-1}(b)$ are called fibers. There are typical fibers and singular fibers. The typical fibers are all isomorphic to a fixed homogeneous space of G, and the singular fibers will be quotients of the typical fiber by a compact group. We need to impose sufficient conditions to get an effectively computable theory and one which will yield interesting geometric applications. Our approach will be global rather than local. We shall begin with the underlying idea which when later refined will yield a precise definition.

4.1. Examples

4.1.1. Suppose on a space P, there are two group actions (G, P, φ) and $(\varPi, P, \psi)$ with $\varPi$ normalizing the action of G. That is, for each $f \in \varPi$, there exists a continuous automorphism α_f of G such that $f(au) = \alpha_f(a)f(u)$, for all $a \in G$, $u \in P$. Hence, each $f \in \varPi$ is a weak G-self-equivalence; see Subsection 1.1.4. Note that $\varPi \to \mathrm{Aut}(G)$, given by $f \mapsto \alpha_f$, is a homomorphism. The elements of $\varPi$ thus map the G-orbits to G-orbits. Consequently, there is an action of $\varPi$ induced on the orbit space $G\backslash P = W$. Let $B = \varPi\backslash W$ denote the orbit space of this $\varPi$-action on W, and let $\tilde{\tau} : P \longrightarrow W = G\backslash P$ and $\bar{\nu} : W \longrightarrow B = \varPi\backslash W$ be the orbit mappings.

On the other hand, we could first let $\varPi$ act on P. Denote this orbit space by $X = \varPi\backslash P$ and the orbit mapping by ν. If $u \in \nu^{-1}(x)$, then $f(u)$ and $f(au) = \alpha_f(a)f(u)$ lie on the same G-orbit of P. Therefore, there is induced a map $\tau : X \to B$ which makes the diagram

$$\begin{array}{ccc} P & \xrightarrow[G\backslash]{\tilde{\tau}} & W \\ {\scriptstyle \nu}\big\downarrow{\scriptstyle \varPi\backslash} & & {\scriptstyle \bar{\nu}}\big\downarrow{\scriptstyle \varPi\backslash} \\ X & \xrightarrow{\ \tau\ } & B \end{array}$$

commutative. If the action of $\varPi$ were to centralize the G-action, that is, if $\alpha_f = \mathrm{id}_G$ for each $f \in \varPi$, then the actions commute and there would be *a G-action induced on X whose orbit space is B.*

This setup is the *prototype* of a Seifert fibering $\tau : X \to B$ modeled on the G-action on P. Without further restrictions, the map τ may have undesirable properties. For example, in Example 1.9.5, X is a torus and $\tau^{-1}(b)$ is a line injectively immersed in the torus. $\mathbb{R}$ acts freely on this immersed line, but the action is not proper. While τ is a foliation of the torus by lines, the base space B is not Hausdorff and the induced $\mathbb{Z} \times \mathbb{Z}$-action on $W = \mathbb{R}$ is also not proper.

What is $\tau^{-1}(b)$ in our prototype? This is $\nu(\tilde{\tau}^{-1}(\bar{\nu}^{-1}(b)))$. Suppose $\tau(x) = b$ and $\nu(u) = x$, then $\nu(G(u)) = \tau^{-1}(b)$. If $\tilde{\tau}(u) = w$, then Π_w sends $G(u)$ onto itself since Π_w fixes w. Therefore $\tau^{-1}(b)$ is the ν-image of the action of Π_w on the orbit $G(u)$. If G is a Lie group acting properly on P, then the orbit $G(u)$ is the homogeneous space $G_u\backslash G$ and $\Pi_w\backslash(G_u\backslash G)$ is mapped continuously, one-to-one and onto $\tau^{-1}(b)$.

$$\begin{array}{ccc} u & \xrightarrow{\tilde{\tau}} & w \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} \\ x & \xrightarrow{\tau} & b \end{array} \qquad\qquad \begin{array}{ccc} Gu & \xrightarrow{\tilde{\tau}} & w \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} \\ \Pi_w\backslash Gu & \xrightarrow{\tau} & b \end{array}$$

The example above shows that $\Pi_w\backslash Gu \to \tau^{-1}(b)$ still may fail to be a homeomorphism. We shall impose additional restrictions to remedy this situation and to discern and control the nature of the Π_w-action on $G_u\backslash G$. For each fixed G-action on P, there are obviously many Π-actions that normalize it. We shall study the normalizer in TOP(P), the group of homeomorphisms of P, of the G-actions on P. After this study, we will then be able to determine more precisely how Π_w acts on $G_u\backslash G$. But first we shall treat a few elementary examples.

4.1.2. The map from Example 1.6.3

$$\tau : G \times_H S \to H\backslash S = B$$

is a prototypical fibering. The fibers are the G-orbits of the G-action on $G \times_H S$.

EXERCISE 4.1.3. Let $P = S^1 \times I$. Describe an $S^1 \times \mathbb{Z}_2$-action on P so that $\mathbb{Z}_2\backslash P$ is the Möbius band and the orbit mapping of the induced S^1-action $\mathbb{Z}_2\backslash P \to \mathbb{Z}_2\backslash I$ is a prototype Seifert fibering such that each principal orbit (see Remark 1.7.9) is a free orbit and there is exactly one singular orbit.

EXAMPLE 4.1.4. Let $P = S^1 \times T^2$ and define an action of $S^1 \times \mathbb{Z}_2$ on P by

$$z(z_1, (z_2, z_3)) = (zz_1, (z_2, z_3)),$$
$$\alpha(z_1, (z_2, z_3)) = (-z_1, (\bar{z}_2, \bar{z}_3))$$

for $z, z_1 \in S^1$, $(z_2, z_3) \in T^2$, and α is the generator of $\mathbb{Z}_2$. We obtain the following diagram of orbit mappings.

$$\begin{array}{ccc} (S^1 \times \mathbb{Z}_2, S^1 \times T^2) & \xrightarrow[\tilde{\tau}]{S^1\backslash} & (\mathbb{Z}_2, T^2) \\ {\scriptstyle\mathbb{Z}_2\backslash}\downarrow{\scriptstyle\nu} & & {\scriptstyle\mathbb{Z}_2\backslash}\downarrow{\scriptstyle\bar{\nu}} \\ (S^1, S^1 \times_{\mathbb{Z}_2} T^2) & \xrightarrow[\tau]{S^1\backslash} & \mathbb{Z}_2\backslash T^2 \end{array}$$

The action of $\mathbb{Z}_2$ on $P = T^3$ is by isometries in the usual flat metric, and $X = S^1 \times_{\mathbb{Z}_2} T^2$ is a flat 3-dimensional Riemannian manifold. (In [**Wol77**, Chapter 3], $S^1 \times_{\mathbb{Z}_2} T^2$ is called $\mathfrak{G}_2$ in the Hantzsche-Wendt classification of flat 3-dimensional manifolds.) The action of $\mathbb{Z}_2$ on T^2 has four fixed points $(\pm 1, \pm 1)$ and is free otherwise. The quotient $\mathbb{Z}_2\backslash T^2$ is homeomorphic to the 2-sphere. The induced S^1-action on X has four singular orbits (fibers) with isotropy $\mathbb{Z}_2$ over the image in S^2 of the four fixed points in S^2. The *typical* fibers are the free S^1 orbits.

Let us take the same $\mathbb{Z}_2$-action on $S^1 \times T^2$ but use a translational T^2-action on the T^2-factor. The $\mathbb{Z}_2$-action normalizes (in the homeomorphisms of T^3) the

T^2-action but does not centralize it. By projecting to the S^1-factor, we induce the nonprincipal fiber bundle

$$T^2 \to S^1 \times_{\mathbb{Z}_2} T^2 \to S^1/\mathbb{Z}_2.$$

The typical fiber is now T^2. There are no singular fibers.

One can also think of $T^3 = P$ acting by left translations and $\Pi = \mathbb{Z}_2$ acting on T^3 as before. This induces the map $\mathbb{Z}_2\backslash T^3 \to$ point. Here the typical fiber is T^3 and the fiber is $\mathfrak{G}_2$ itself.

4.1.5. The base space $\mathbb{Z}_2\backslash T^2$ of the first Seifert fibering in the previous example is a Seifert fibering itself again, over an arc with S^1 as a *typical* fiber over the interior points and arcs over the end points. We may think of this as the projection of a "square pillow" (a topological 2-sphere) onto an arc as shown in Figure 1.

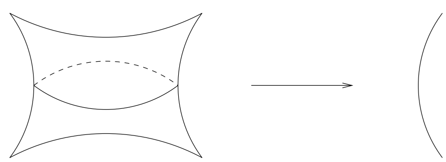

FIGURE 1. Square pillow

Our setup is

$$P = T^2 = S^1 \times S^1, \qquad G = S^1, \qquad \Pi = \mathbb{Z}_2,$$

where G acts on the first factor of P as multiplication.

Take the torus parametrized by (z_1, z_2) and define an action of $S^1 \rtimes \mathbb{Z}_2 = \mathrm{O}(2, \mathbb{R})$ on the torus by

$$(z, \epsilon)(z_1, z_2) = (z\bar{z}_1, \bar{z}_2),$$

where ϵ is the generator of $\mathbb{Z}_2$. Then S^1 is normalized but not centralized by $\mathbb{Z}_2$. Dividing out by the normal S^1 gives the circle as the orbit space $(z_1, z_2) \mapsto z_2$. This, followed by the induced $\mathbb{Z}_2$-action $(z_2 \mapsto \bar{z}_2)$, gives us an arc for orbit space. On the other hand, if we divide out first by the diagonal $\mathbb{Z}_2$ isometric action, we get $S^1 \times_{\mathbb{Z}_2} S^1$.

This is a topological 2-sphere (the surface of a square pillow). There is then a map induced from $S^1 \times_{\mathbb{Z}_2} S^1 \xrightarrow{\tau} S^1/\mathbb{Z}_2$ which is our fibering and τ is not an orbit mapping.

$$\begin{array}{ccc} (S^1 \rtimes \mathbb{Z}_2, S^1 \times S^1) & \xrightarrow[\tilde{\tau}]{S^1\backslash} & (\mathbb{Z}_2, S^1) \\ {\scriptstyle \mathbb{Z}_2\backslash}\Big\downarrow{\scriptstyle \nu} & & {\scriptstyle \mathbb{Z}_2\backslash}\Big\downarrow{\scriptstyle \bar{\nu}} \\ S^1 \times_{\mathbb{Z}_2} S^1 & \xrightarrow[\tau]{} & \mathbb{Z}_2\backslash S^1 \end{array}$$

The two singular fibers are quotients of S^1 by $\mathbb{Z}_2$ where the $\mathbb{Z}_2$ does not act as translations but as automorphisms of S^1.

4.1.6. Note the composite $S^1 \times_{\mathbb{Z}_2} T^2 \to \mathbb{Z}_2\backslash T^2 \to \mathbb{Z}_2\backslash S^1$ can be viewed as a Seifert fibering (2-dimensional fiber, 1-dimensional base space)

$$T^2 \longrightarrow T^2 \times_{\mathbb{Z}_2} S^1 \longrightarrow \mathbb{Z}_2\backslash S^1 (=\text{arc})$$

modeled on $T^2 \times S^1$ with typical fiber T^2 and singular fibers homeomorphic to a Klein bottle over the end points of the arc. (Just think of what lies above each horizontal section of the square pillow in Figure 1.) Now, by examining what lies above the top half of the square pillow, we see that $\mathfrak{G}_2$ is also homeomorphic to the double mapping cylinder of the double covering of the Klein bottle by the torus. Using this identification, one sees that $\mathfrak{G}_2$ fibers without singularities over the Klein bottle with fiber S^1. This S^1-bundle is nontrivial over each orientation reversing curve of the Klein bottle. Thus $\mathfrak{G}_2$ is the total space of two different Seifert fiberings with typical fiber S^1 and also the total space of two distinct Seifert fiberings with typical fiber T^2. Note all the bases are distinct. In Example 4.5.10, we analyze this manifold in great detail. We can view it as a Seifert manifold modeled on principal bundles $\mathbb{R}^i \to \mathbb{R}^3 \to \mathbb{R}^j$ in different ways ($i + j = 3$), yielding different Seifert structures for the same space.

EXERCISE 4.1.7. Define an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $T^3 = S^1 \times T^2$ so that we get a Seifert fibering $S^1 \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} T^2 \to K$ over the Klein bottle with typical fiber S^1 and two singular fibers doubly covered by a typical fiber. (Hint: Use the first $\mathbb{Z}_2$-action in defining $\mathfrak{G}_2$ and a new involution $(z, z_1, z_2) \mapsto (\bar{z}, -z_1, \bar{z}_2)$. Note that $\mathfrak{G}_2$ freely doubly covers this orbit space.)

4.1.8. Let us consider the important example where G is a closed subgroup of a Lie group P and $\mathit{\Pi}$ is another closed subgroup of P. We let $G = \ell(G)$ act on the left as translations and $\mathit{\Pi}$ act on P by

$$\psi(\alpha)(x) = x\alpha^{-1},\ \alpha \in \mathit{\Pi},\ x \in P.$$

This yields a left $\mathit{\Pi}$-action which commutes with $\ell(G)$ on P. We have

$$\psi(\mathit{\Pi}) \cap \ell(G) = (G \cap \mathit{\Pi}) \cap \mathcal{Z}(P) = \Gamma,$$

a closed central subgroup of G. Our fibering diagram becomes

$$\begin{array}{ccc} P & \xrightarrow[\tilde{\tau}]{G\backslash} & G\backslash P \\ {\scriptstyle \mathit{\Pi}\backslash}\big\downarrow{\scriptstyle \nu} & & {\scriptstyle (\mathit{\Pi}/\Gamma)\backslash}\big\downarrow{\scriptstyle \bar{\nu}} \\ P/\mathit{\Pi} & \xrightarrow[\tau]{(G/\Gamma)\backslash} & G\backslash P/\mathit{\Pi}. \end{array}$$

A very interesting case occurs when P has a finite number of connected components and G is a *maximal compact subgroup* of P. (For example, take $P = \mathrm{GL}(n, \mathbb{R})$ with $G = \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$.) Then $G\backslash P$ is diffeomorphic to $\mathbb{R}^n$ for some $n \geq 0$. If $\mathit{\Pi}$ is a discrete and torsion-free subgroup, then $G \cap \mathit{\Pi} = 1$ so $\Gamma = 1$, and $Q = \mathit{\Pi}/\Gamma = \mathit{\Pi}$ acts freely on $\mathbb{R}^n = G\backslash P$. The $G \times \mathit{\Pi}$-action is proper and $G\backslash P/\mathit{\Pi}$ is a $K(\mathit{\Pi}, 1)$-manifold, and $P/\mathit{\Pi}$ is a principal G-bundle over $G\backslash P/\mathit{\Pi}$. However, usually a discrete $\mathit{\Pi}$ in P will not be torsion free. In this case, τ is Seifert fibering where $\tau^{-1}(b) = G/\mathit{\Pi}_w = \Gamma\backslash G/Q_w$, $w \in \bar{\nu}^{-1}(b)$, $1 \to \Gamma \to \mathit{\Pi}_w \to Q_w \to 1$ is exact and the finite Q_w is the isotropy of Q at w. The typical fiber is $\Gamma\backslash G$.

4.1.9. We must exercise caution in the above example. For G and $\mathit{\Pi}$ could both act properly and commute, but the subgroup $G \times \mathit{\Pi} \subset P \times P$, acting on P, may not act properly. For example, in $P = \mathbb{R}$, let $G \cong \mathbb{Z}$ be generated by 1 and $\mathit{\Pi}$ be generated by $\sqrt{2}$. Then G and $\mathit{\Pi}$ act freely, properly and commute with each other, but the free action of $G \times \mathit{\Pi}$ is not proper.

4.2. $\mathrm{TOP}_G(P)$, the group of weak G-equivalences

As mentioned in the previous section, the prototype of a Seifert fibering is too general and we cannot avoid pathological situations. Furthermore, if we want to get as much control as possible on the singular fibers, we need to impose additional conditions. We take the actions of G and Π to be effective on P; i.e., G and Π are subgroups of $\mathrm{TOP}(P)$, the group of homeomorphisms of P. Furthermore, as Π normalizes G, Π is in the normalizer of G in $\mathrm{TOP}(P)$. This normalizer turns out to be the group of all weak G-equivalences of P. There is a natural homomorphism from this normalizer into $\mathrm{Aut}(G) \times \mathrm{TOP}(W)$. To study this homomorphism, we shall examine its kernel. The kernel becomes effectively computable if we assume that the action of G on P is free. Then to remove any pathology on these actions, we assume that G acts locally properly. If we also assume that this G-action has local cross sections, then P becomes a principal G-bundle and the G-action is just given by left translations of G along the fibers. (If G is a Lie group acting locally properly, then the G-action has local cross section because of the existence of a slice.) Under the assumptions, the normalizer, which is the group of weak G-equivalences of P, is the same as the group of weak bundle automorphisms of the principal G-bundle P. Thus for a group Π in $\mathrm{TOP}(P)$ to normalize the free G-action on the bundle P, it must be a subgroup of the weak bundle automorphisms of P.

Recall that (Subsection 1.1.4) a homeomorphism $f : P \to P$ is a weak G-equivalence if there exists a continuous automorphism α_f of G such that

$$f(xu) = \alpha_f(x) f(u),$$

for all $x \in G$, $u \in P$. Note that α_f is *unique* because the G-action on P is effective.

NOTATION 4.2.1. $\mathrm{TOP}_G(P)$ denotes the group of all weak G-equivalences of P.

4.2.2 (Topology of $\mathrm{TOP}_G(P)$). $\mathrm{TOP}_G(P)$ has the compact-open topology. The groups Π and G act locally properly on P, and consequently are embedded as closed subsets of $\mathrm{TOP}_G(P)$ in the point-open topology and hence also as closed subsets in the compact-open topology. Our Lie groups Π and $\ell(G)$ (the principal left G-action) are acting properly on P and so they are embedded as closed subsets in the point-open topology and hence also as closed subsets in the compact-open topology. $\mathrm{TOP}_G(P)$ is not in general a topological group. However, if P is locally compact and either connected or locally connected, then $\mathrm{TOP}_G(P)$ is a topological group under the compact-open topology; see Remark 1.2.6. We will not use this fact unless we specifically mention it.

LEMMA 4.2.3 ([**LR89**]). *$\mathrm{TOP}_G(P)$ is the normalizer of G in $\mathrm{TOP}(P)$, and there exists a natural homomorphism $\mathrm{TOP}_G(P) \to \mathrm{Aut}(G) \times \mathrm{TOP}(G\backslash P)$.*

PROOF. For each $f \in \mathrm{TOP}_G(P)$, we have $f \circ a = \alpha_f(a) \circ f$ for each $a \in G$. Or, in other words, $f \circ a \circ f^{-1} = \alpha_f(a)$. So f normalizes G.

Conversely, suppose $f \in \mathrm{TOP}(P)$ normalizes G; that is, $f \circ G \circ f^{-1} = G$. Then $f \circ a \circ f^{-1} = a_f$, for some $a_f \in G$. Then

$$(ab)_f = f \circ (ab) \circ f^{-1} = (f \circ (a) \circ f^{-1})(f \circ (b) \circ f^{-1}) = a_f \circ b_f$$

shows that f induces, by conjugation, an automorphism of G, by sending $a \mapsto a_f$. If we denote this automorphism by α_f, then we have $f(xu) = \alpha_f(x) f(u)$ so that $f \in \mathrm{TOP}_G(P)$.

Since each G-orbit is mapped homeomorphically onto another G-orbit for $f \in \mathrm{TOP}_G(P)$, there is induced a homeomorphism $\bar{f} : G\backslash P \to G\backslash P$. Furthermore, associated with f, we have the unique automorphism $\alpha_f \in \mathrm{Aut}(G)$. It is easy to check that $f \mapsto (\alpha_f, \bar{f})$ is a homomorphism with the group operation in $\mathrm{TOP}_G(P)$ being composition. Also, if we use the compact-open topology (when P is locally compact), this is a continuous homomorphism of topological groups but we will not need that now. □

The homomorphism in the lemma, as we shall see later, may not be onto even if P is a principal G-bundle; see Subsections 4.2.13 and 4.2.14. On the other hand, if P is a product $G \times W$, the homomorphism will be onto and split.

To get a handle on the kernel of this homomorphism, we now assume the action of G on P is free. On G itself, G also acts by conjugation, giving a homomorphism $G \to \mathrm{Inn}(G)$.

NOTATION 4.2.4. Let $\mathrm{M}_G(P, G)$ be the (continuous) G-equivariant maps from P to G, where the action of G on G is given by conjugation. Therefore, for a map $\eta : P \to G$, $\eta \in \mathrm{M}_G(P, G)$ if and only if $\eta(au) = a\eta(u)a^{-1}$ for all $a \in G$ and $u \in P$.

These maps form a group by $(\eta_1 \cdot \eta_2)(u) = \eta_1(u)\eta_2(u)$, for $\eta_1, \eta_2 \in \mathrm{M}_G(P, G)$ and $u \in P$. Note $\eta^{-1}(u) = (\eta(u))^{-1}$.

LEMMA 4.2.5. *The kernel of the homomorphism* $\mathrm{TOP}_G(P) \to \mathrm{Aut}(G) \times \mathrm{TOP}(G\backslash P)$ *is isomorphic to* $\mathrm{M}_G(P, G)$.

PROOF. We can view $\mathrm{M}_G(P, G)$ as a subgroup of $\mathrm{TOP}_G(P)$; namely, we define a homomorphism

$$\psi : \mathrm{M}_G(P, G) \hookrightarrow \mathrm{TOP}_G(P)$$

by $\psi(\eta)(u) = \eta(u)^{-1}u$, for $\eta \in \mathrm{M}_G(P, G)$, $u \in P$. Now,

$$\psi(\eta)(au) = \eta(au)^{-1}au = a\eta(u)^{-1}a^{-1}(au) = a\eta(u)^{-1}u = a\psi(\eta)(u).$$

So $\psi(\eta) \in \mathrm{TOP}_G(P)$, and is in the kernel. Further, ψ is a homomorphism because

$$\begin{aligned}\psi(\eta_1\eta_2)(u) &= (\eta_1\eta_2)(u)^{-1}u = (\eta_1(u)\eta_2(u))^{-1}u \\ &= (\eta_2(u))^{-1}(\eta_1(u))^{-1}u = (\eta_2(u))^{-1}(\psi(\eta_1)(u)) \\ &= \psi(\eta_1)((\eta_2(u))^{-1}u) = \psi(\eta_1)\psi(\eta_2)(u).\end{aligned}$$

On the other hand, if $f \in$ Kernel, then $\bar{f} = 1_{G\backslash P}$ and $\alpha_f = 1_G$. Therefore, because of freeness, $f(u) = (\eta_f(u))^{-1}u$, for some well-defined function $\eta_f : P \to G$. Define $\theta(f) = \eta_f$. We shall show $\theta : \mathrm{Kernel} \to \mathrm{M}_G(P, G)$ is the inverse of ψ. Since $af(u) = f(au)$, then $\eta_f(au)^{-1}(au) = a(\eta_f(u))^{-1}u$. Because the G-action is free, we have $a\eta_f(u)a^{-1} = \eta_f(au)$. Thus, $\eta_f \in \mathrm{M}_G(P, G)$. Furthermore, $\theta(g \circ f) = \theta(g)\theta(f)$ because $g(f(u)) = (g \circ f)(u) = (\eta_{g\circ f}(u))^{-1}(u) = g(\eta_f(u)^{-1})(u) = (\eta_f(u))^{-1}g(u) = (\eta_f(u))^{-1}(\eta_g(u))^{-1}u = ((\eta_g\eta_f)(u))^{-1}u$.

If $\eta \in \mathrm{M}_G(P, G)$, then define f_η by $f_\eta(u) = \psi(\eta)(u) = \eta(u)^{-1}u$. The formula $(\eta_{f_\eta})^{-1}(u) = f_\eta(u) = (\eta(u))^{-1}(u)$ shows that θ and ψ are inverses to each other, and so the inclusion $\psi : \mathrm{M}_G(P, G) \to \mathrm{TOP}_G(P)$ is the kernel of $\mathrm{TOP}_G(P) \to \mathrm{Aut}(G) \times \mathrm{TOP}(G\backslash P)$. □

REMARK 4.2.6. Since the locally proper G-action on P is free, the orbit mapping $P \to G\backslash P$ is a principal G-bundle provided we also assume that the G-action on P has local cross sections. For the remainder of the section, we assume that G

and Π are Lie groups acting locally properly with the action of G also being free. (The result will also be valid if we do not assume the locally compact G to be Lie but require that the free locally proper G action has local cross sections on P. In other words, that P is a principal G-bundle.) In the parlance of principal G-bundles, the group $\mathrm{TOP}_G(P)$ is the group of *weakly G-equivariant bundle automorphisms.*

NOTATION 4.2.7. Let G be a group. We always denote *conjugation* by μ, so that

$$\mu(a)(x) = axa^{-1}$$

for $a, x \in G$; $\mathrm{Inn}(G)$ denotes the group of inner automorphisms $\mu(a)$ of G; $\mathcal{Z}(G)$ denotes the center of G. Therefore, $\mathrm{Inn}(G) \cong G/\mathcal{Z}(G)$. For a subset K of G, we denote the set of centralizers of K in G by $C_G(K)$.

Since G acts on the left, we shall denote this by $\ell(G)$. Then every $\ell_a = \ell(a) \in \ell(G)$ induces the identity on $W = G\backslash P$. However,

$$\ell_a(xu) = axu = axa^{-1}au = \mu(a)(x) \cdot \ell_a(u)$$

shows that ℓ_a induces $\mu(a)$ under $\mathrm{TOP}_G(P) \to \mathrm{Aut}(G)$. If we put $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, the group of (continuous) outer automorphisms of G, the kernel of

$$\mathrm{TOP}_G(P) \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$$

is generated by $\ell(G)$ and $\mathrm{M}_G(P, G)$. If $\ell_a = \psi(\eta) = f_\eta \in \ell(G) \cap \mathrm{M}_G(P, G)$, then

$$\ell_a(bu) = f_\eta(bu) = (\eta(bu))^{-1}bu = b\eta(u)^{-1}(b^{-1}b)u = b\ell_a u = \ell_b \ell_a(u).$$

Therefore, $\ell(G) \cap \psi(\mathrm{M}_G(P, G)) = \mathcal{Z}(G)$. Thus, $\ell(G) \times_{\mathcal{Z}(G)} \psi(\mathrm{M}_G(P, G))$, the quotient of $\ell(G) \times \psi(\mathrm{M}_G(P, G))$ by the normal subgroup $\{(z, z) | z \in \ell(G) \cap \mathrm{M}_G(P, G)\} = \{(z, z^{-1}) | z \in \ell(G) \cap \psi(\mathrm{M}_G(P, G))\}$ is the kernel of $\mathrm{TOP}_G(P) \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$. We suppress $\psi : \mathrm{M}_G(P, G) \hookrightarrow \mathrm{TOP}_G(P)$ and simply consider $\mathrm{M}_G(P, G)$ as a subgroup of $\mathrm{TOP}_G(P)$.

PROPOSITION 4.2.8. *For a principal G-bundle P with $W = G\backslash P$, we have the exact sequence*

$$1 \to \ell(G) \times_{\mathcal{Z}(G)} \mathrm{M}_G(P, G) \to \mathrm{TOP}_G(P) \to \mathrm{Out}(G) \times \mathrm{TOP}(W). \quad \square$$

COROLLARY 4.2.9. *For the principal fibration $\tilde{\tau} : P \to W = G\backslash P$, the subgroup of $\mathrm{TOP}_G(P)$ which leaves the fiber $\tilde{\tau}^{-1}(w)$ invariant lies in $(\ell(G) \times_{\mathcal{Z}(G)} M_G(\tilde{\tau}^{-1}(w), G)) \cdot \mathrm{Aut}(G) = M_G(\tilde{\tau}^{-1}(w), G) \rtimes \mathrm{Aut}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$.*

The corollary holds because $r(G) \subset \mathrm{M}_G(\tilde{\tau}^{-1}(w), G)$ implies $\ell(G) \cdot r(G) \subset \mathrm{Aut}(G)$. There are two important situations where we can show that $\mathrm{M}_G(P, G)$ only depends upon W: they are $P = G \times W$ (i.e., trivial bundle) and when G is Abelian.

COROLLARY 4.2.10 ($P = G \times W$). *The exact sequence in Proposition 4.2.8 becomes*

$$1 \to \mathrm{M}(W, G) \rtimes \mathrm{Inn}(G) \to \mathrm{TOP}_G(G \times W) \to \mathrm{Out}(G) \times \mathrm{TOP}(W) \to 1.$$

In fact, $\mathrm{M}(W, G) \times_{\mathcal{Z}(G)} \ell(G) = \mathrm{M}(W, G) \rtimes \mathrm{Inn}(G)$*, and*

$$\mathrm{TOP}_G(G \times W) = \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W)),$$

whose group law is given by

$$\begin{aligned}(\lambda_1, \alpha_1, h_1) \cdot (\lambda_2, \alpha_2, h_2) &= (\lambda_1 \cdot^{(\alpha_1, h_1)} \lambda_2, \alpha_1 \circ \alpha_2, h_1 \circ h_2) \\ &= (\lambda_1 \cdot (\alpha_1 \circ \lambda_2 \circ h_1^{-1}), \alpha_1 \circ \alpha_2, h_1 \circ h_2).\end{aligned}$$

The action of $\mathrm{TOP}_G(G \times W)$ *on* $G \times W$ *is given by*

$$\begin{aligned}(\lambda, \alpha, h) \cdot (x, w) &= ((\lambda, 1, 1) \circ (1, \alpha, h))(x, w) \\ &= (\lambda, 1, 1)(\alpha(x), h(w)) \\ &= (\alpha(x) \cdot (\lambda(h(w)))^{-1}, h(w)).\end{aligned}$$

PROOF. Choose a global cross section $W \to P$ once and for all. For $\alpha \times h \in \mathrm{Aut}(G) \times \mathrm{TOP}(W)$, define

$$(\alpha, h)(x, w) = (\alpha(x), h(w)), \text{ for } (x, w) \in G \times W.$$

Then $(\alpha, h) \in \mathrm{TOP}_G(G \times W)$ because $(\alpha, h)(ax, w) = (\alpha(ax), h(w)) = \alpha(a)(\alpha(x), h(w))$, for all $a \in G$. So, $\mathrm{TOP}_G(G \times W) \to \mathrm{Aut}(G) \times \mathrm{TOP}(W)$ is *onto and splits.*

We now examine the kernel $\mathrm{M}_G(G \times W, G)$. For $u = (x, w)$, $\eta \in \mathrm{M}_G(G \times W, G)$, we have $\eta(x, w) = x\eta(1, w)x^{-1}$. Thus $\eta(x, w)$ is determined by $\eta(1, w)$ and x. Put $\lambda(w) = \eta(1, w)$. The assignment $\eta \mapsto \lambda$ is a homomorphism from $\mathrm{M}_G(G \times W, G)$ to $\mathrm{M}(W, G)$. Conversely, given $\lambda : W \to G$, define $\eta \in \mathrm{M}_G(G \times W, G)$ by $\eta(x, w) = x\lambda(w)x^{-1}$. This gives an isomorphism between $\mathrm{M}_G(G \times W, G)$ and $\mathrm{M}(W, G)$. Thus the isomorphisms are given by

$$\begin{array}{ccc} \mathrm{M}(W, G) & \longrightarrow & \mathrm{M}_G(G \times W, G), \\ \lambda & \longrightarrow & \mu \times \lambda, \end{array} \qquad \begin{array}{ccc} \mathrm{M}_G(G \times W, G) & \longrightarrow & \mathrm{M}(W, G), \\ \eta & \longrightarrow & \eta(1, -). \end{array}$$

We have that $\psi : \mathrm{M}(W, G) \hookrightarrow \mathrm{TOP}_G(G \times W)$ injects and λ acts, via $\psi(\lambda)$, on $G \times W$ by $\psi(\lambda)(x, w) = (x(\lambda(w))^{-1}, w)$. We shall suppress the ψ.

We claim the action of $\mathrm{Aut}(G) \times \mathrm{TOP}(W)$ on $\mathrm{M}(W, G)$ is given by

$$(\alpha, h) \times \lambda \mapsto \alpha \circ \lambda \circ h^{-1}.$$

For,

$$\begin{aligned}((\alpha, h) \circ \lambda \circ (\alpha, h)^{-1})(x, w) &= ((\alpha, h) \circ \lambda)(\alpha^{-1}(x), h^{-1}(w)) \\ &= (\alpha, h)(\alpha^{-1}(x)\lambda^{-1}(h^{-1}(w)), h^{-1}(w)) \\ &= (x\alpha(\lambda^{-1}(h^{-1}(w))), w) \\ &= (x\alpha(\lambda(h^{-1}(w)))^{-1}, w), \\ &= (\alpha \circ \lambda \circ h^{-1})(x, w),\end{aligned}$$

which yields the desired formula.

For $a \in G$, the constant map $W \to G$ sending W to a is denoted by $r(a) = r_a$. That is,

$$r_a = (a, 1, 1) \in \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W)),$$

and $r_a(x, w) = (x \cdot a^{-1}, w)$. We may therefore write ℓ_a as

$$\ell_a = r_{a^{-1}} \circ \mu(a) = (a^{-1}, \mu(a), 1) \in \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W)).$$

Finally, the kernel $\mathrm{M}(W, G) \times_{\mathcal{Z}(G)} \ell(G)$ of $\mathrm{TOP}_G(P) \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$ can be written as $\mathrm{M}(W, G) \rtimes \mathrm{Inn}(G)$ with the specific isomorphism given by

$$\lambda \cdot \ell_a \mapsto (\lambda, 1)(r_{a^{-1}}, \mu(a)) = (\lambda r_{a^{-1}}, \mu(a)). \quad \square$$

EXERCISE 4.2.11. Show that $\ell(G)$ and $r(G)$ are subgroups of $\mathrm{M}(W,G)\rtimes\mathrm{Aut}(G)$; and $\ell(G)\rtimes\mathrm{Aut}(G)$ and $r(G)\rtimes\mathrm{Aut}(G)$ are identical subgroups of $\mathrm{M}(W,G)\rtimes\mathrm{Aut}(G)$ with the correspondence given by

$$\ell_a=(a^{-1},\mu(a),1),\ r_a=(a,1,1)\ \in\ \mathrm{M}(W,G)\rtimes(\mathrm{Aut}(G)\times\mathrm{TOP}(W)).$$

COROLLARY 4.2.12 (G is Abelian). *The exact sequence in Proposition* 4.2.8 *becomes*

$$1\to\mathrm{M}(W,G)\to\mathrm{TOP}_G(P)\to\mathrm{Aut}(G)\times\mathrm{TOP}(W).$$

Let $\eta\in\mathrm{M}_G(P,G)$. Then $\eta(au)=a\eta(u)a^{-1}=\eta(u)$. Therefore, η factors through $\mathrm{M}(W,G)$ and we get the exact sequence. Since G is Abelian, $\mathrm{M}(W,G)$ is Abelian so that the conjugation homomorphism $\mathrm{TOP}_G(P)\longrightarrow\mathrm{M}(W,G)$ factors through $\mathrm{Aut}(G)\times\mathrm{TOP}(W)$. In other words, if $f\in\mathrm{TOP}_G(P)$ maps to $(\alpha,h)\in\mathrm{Aut}(G)\times\mathrm{TOP}(W)$, then, for $\eta\in\mathrm{M}(W,G)$,

$$f\circ\psi(\eta)\circ f^{-1}=\alpha\circ\psi(\eta)\circ h^{-1},$$

just as in the product case.

4.2.13. *The map* $\mathrm{TOP}_G(P)\to\mathrm{Aut}(G)\times\mathrm{TOP}(W)$ *is not necessarily onto* even for G Abelian in general. Take the Hopf bundle, that is, the principal S^1-bundle over the 2-sphere whose total space is S^3. Let $h:S^2\to S^2$ be the antipodal map, and take $(1,h)\in\mathrm{Aut}(S^1)\times\mathrm{TOP}(S^2)$. If there exists $f\in\mathrm{TOP}_{S^1}(S^3)$ such that $f\mapsto(1,h)$, then f would have to be orientation reversing. Therefore, the Lefschetz number of f is $1+(-1)^3(-1)=2$, implying that f has a fixed point on S^3. But this is impossible since h is fixed point free.

EXERCISE 4.2.14. Use the argument above to show $(1_{S^1},h)\in\mathrm{Aut}(G)\times\mathrm{TOP}(W)$ does not lift to any element in $\mathrm{TOP}_G(P)$, where P is any principal nontrivial S^1-bundle over the 2-sphere. Also it can be shown [**NR78, KLR86**] that the total space of each nontrivial principal S^1-bundle over an orientable closed surface, different from S^2, admits no orientation reversing self-homeomorphism. Use this fact to show that the map $\mathrm{TOP}_G(P)\to\mathrm{Aut}(G)\times\mathrm{TOP}(W)$ is not onto.

LEMMA 4.2.15. *The subgroup* $id\times\mathrm{TOP}_0(W)$ *is in the image of* $\mathrm{TOP}_G(P)$.

PROOF. Here $\mathrm{TOP}_0(W)$ denotes the subgroup of $\mathrm{TOP}(W)$ which consists of maps isotopic to the identity. Let h be a homeomorphism of W, and $H:W\times I\to W$ be a homotopy so that $H|_{W\times 0}=\mathrm{id}$ and $H|_{W\times 1}$ is h. Then by the covering homotopy theorem for bundles (e.g., Steenrod [**Ste51**]), there exists a homotopy $\tilde H:P\times I\to P$ so that $\tilde H|_{P\times 0}=\mathrm{id}$ and $\tilde H$ covers H. Thus $f=\tilde H|_{P\times 1}$ covers the homeomorphism h. Since f is a bundle map and covers a homeomorphism, it too is a homeomorphism and is homotopic, through bundle maps, to the identity. In particular, if h itself was isotopic to the identity, then f is also isotopic to the identity. □

REMARK 4.2.16. We have actually shown that if h is homotopic to the identity, then there is a G-equivariant map, i.e., a bundle isomorphism $f\in\mathrm{TOP}_G(P)$ which maps onto $\mathrm{id}\times h$. We assumed that our spaces are paracompact Hausdorff spaces to employ the covering homotopy theorem. Assume now that every open subspace of W is paracompact. From the covering homotopy theorem for weakly G-equivariant maps between principal G-bundles (cf. [**Par91**]), we can characterize the entire image of $\mathrm{TOP}_G(P)$ in $\mathrm{Aut}(G)\times\mathrm{TOP}(W)$: Let $f\in\mathrm{TOP}_G(P)$ with $f(x\cdot u)=\alpha_f(x)\cdot$

$f(u)$. Let $\mathrm{TOP}(W)_h$ be the isotopy classes (i.e., path components) of $\mathrm{TOP}(W)$ whose elements are homotopic to h, where $h : W \to W$ is induced by f. (Let $[\alpha_f]$ denote the image of α_f in $\mathrm{Aut}(G)$.) Then $[\alpha_f] \times \mathrm{TOP}(W)_h$ is in the image of $\mathrm{TOP}_G(P)$. The union of all such is the complete image of $\mathrm{TOP}_G(P)$ in $\mathrm{Aut}(G) \times \mathrm{TOP}(W)$.

Remark 4.2.17. The projection $p : \mathrm{TOP}_G(P) \to \mathrm{Aut}(G) \times \mathrm{TOP}(W)$ induces the projections

$$\begin{aligned} p_1 &: \mathrm{TOP}_G(P) \to \mathrm{Aut}(G), \\ p_2 &: \mathrm{TOP}_G(P) \to \mathrm{TOP}(W), \\ \bar{p} &: \mathrm{TOP}_G(P) \to \mathrm{Out}(G) \times \mathrm{TOP}(W). \end{aligned}$$

If P is locally compact, these maps are continuous homomorphisms of topological groups, in the compact-open topology. While this statement should seem plausible to the reader, the details involve careful computations with the compact-open topology and can be found in [**Par89**].

From Subsection 4.2.5 through Proposition 4.2.8, we described $\psi(\mathrm{M}_G(P,G))$ as the kernel of p, and $\ell(G) \times_{\mathcal{Z}(G)} \psi(\mathrm{M}_G(P,G))$ as the kernel of $\bar{p}$. If $f \in \mathrm{TOP}_G(P)$ is in the kernel of p_2, Park defines $\eta_f \in M^w_G(P,G)$ to be an element $\eta \in M(P,G)$ satisfying

$$\eta(au) = a\eta(u)(p_1(f)(a))^{-1} = a\eta(u)(\alpha_f(a))^{-1},$$

for $a \in G$, $u \in P$, $p_1(f) = \alpha_f$. This map is G-equivariant where the G-action on G is given by $(a,x) \mapsto ax\,\alpha_f(a)^{-1}$. (Note, if $\eta_f \in M_G(P,G)$, then $\alpha_f = \mathrm{id}$.) The group operation is $(\eta_f \cdot \eta_g)(u) = \eta_f(u)\alpha_{\eta_f}(\eta_g(u))$ making $M^w_G(P,G)$ into a (topological–if P is locally compact) group. Park defines $M^s_G(P,G) \subset M^w_G(P,G)$ if $\alpha_f \in \mathrm{Inn}(G)$. So we have

$$\mathrm{M}_G(P,G) \subset M^s_G(P,G) \subset M^w_G(P,G).$$

Define $\psi : M^w_G(P,G) \to \mathrm{TOP}_G(P)$ by $\psi(\eta_f)(u) = \eta_f(u)^{-1}u$. Then ψ is an isomorphism onto the kernel of p_2. By restricting ψ to $M^s_G(P,G)$ and $\mathrm{M}_G(P,G)$, we obtain an isomorphism onto the kernels of $\bar{p}$ and p.

4.3. Seifert fiberings modeled on a principal G-bundle

We will now define a Seifert fibering by taking the prototype of Subsection 4.1.1 and imposing appropriate restrictions that eliminate pathology on $\mathit{\Pi}\backslash P$ and B. With these restrictions in place, a useful theory incorporating a conventional geometric interpretation of the fibers can be developed.

4.3.1. Let G be a connected Lie group and P a principal G-bundle over the space W. Assume that P is connected, completely regular, and admits covering space theory. This implies that W is completely regular and also admits covering space theory. Let $\mathit{\Pi} \subset \mathrm{TOP}_G(P)$ be a Lie group ($\mathit{\Pi}$ could be discrete), act effectively and properly on P.

Since $\mathit{\Pi}$ normalizes $\ell(G)$, it acts on W. If in addition, we assume that the effective part of the $\mathit{\Pi}$-action on W (i.e., the image of $\mathit{\Pi} \hookrightarrow \mathrm{TOP}_G(P) \to \mathrm{TOP}(W)$) is proper, then the orbit space $\mathit{\Pi}\backslash P$ is called a *Seifert fibered space modeled on the principal G-bundle P*. The map

$$\tau : \mathit{\Pi}\backslash P \longrightarrow \mathit{\Pi}\backslash W$$

is called a *Seifert fibering* and the space $\Pi\backslash W = B$ is called its *base*. The principal G-bundle P is called the *model space* for the Seifert fibering.

Since Π and $\ell(G)$ act properly on P, they are closed subsets of $\mathrm{TOP}_G(P)$ in the compact-open topology; see Subsection 4.2.2. Let

$$\Gamma_\ell = \Pi \cap \ell(G).$$

This is a closed normal subgroup of Π which acts properly and freely as left translations on each principal G-fiber of P. Thus on W, there is an induced action of

$$Q_\ell = \Pi/\Gamma_\ell$$

on W. (Notice that the action (Q_ℓ, W) may not be effective because it may have an ineffective part, but the image $Q_\ell \to \mathrm{TOP}(W)$ must act properly on W because the image of $\Pi \hookrightarrow \mathrm{TOP}_G(P) \to \mathrm{TOP}(W)$ acts properly on W.) More precisely, we get the following commutative diagram.

$$\begin{array}{ccc} (\ell(G)\cdot\Pi, P) & \xrightarrow[\tilde{\tau}]{\ell(G)\backslash} & (Q_\ell, W) \\ {\scriptstyle \Pi\backslash}\big\downarrow{\scriptstyle \nu} & & {\scriptstyle Q_\ell\backslash}\big\downarrow{\scriptstyle \bar{\nu}} \\ X = \Pi\backslash P & \xrightarrow[\tau]{} & Q_\ell\backslash W = B \end{array} \tag{4.3.1}$$

Then $\tau : \Pi\backslash P \longrightarrow Q_\ell\backslash W$ is a *Seifert fibering modeled on the principal G-bundle P with typical fiber $\Gamma_\ell\backslash G$.*

4.3.2 (The typical and regular fibers). Let $\widehat{\Gamma}$ be the kernel of

$$\Pi \longrightarrow \mathrm{TOP}_G(P) \xrightarrow{p_2} \mathrm{TOP}(W).$$

Note that $\Gamma_\ell \subset \widehat{\Gamma}$ and both are closed normal subgroups of Π which leave each principal G-fiber of P invariant. The action (Π, P) induces an action (Π, W). The quotient group $\Pi/\widehat{\Gamma} = \widehat{Q}$ is the effective part of the induced Π-action on W. Then $\widehat{Q}$, by definition, acts properly on W with $B = \widehat{Q}\backslash W = Q_\ell\backslash W = \Pi\backslash W$.

We have the following commutative diagram of exact sequences of groups.

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \Gamma_\ell & \xrightarrow{=} & \Gamma_\ell & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \widehat{\Gamma} & \longrightarrow & \Pi & \longrightarrow & \widehat{Q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \\ 1 & \longrightarrow & F & \longrightarrow & Q_\ell & \longrightarrow & \widehat{Q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array}$$

This yields

$$\begin{array}{ccccc} (\Gamma_\ell, G) & \longrightarrow & (\mathit{\Pi}, P) & \longrightarrow & (Q_\ell, W) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_\ell\backslash G & \longrightarrow & \mathit{\Pi}\backslash P & \longrightarrow & Q_\ell\backslash W \end{array}$$

and

$$\begin{array}{ccccc} (\widehat{\Gamma}, G) & \longrightarrow & (\mathit{\Pi}, P) & \longrightarrow & (\widehat{Q}, W) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\Gamma}\backslash G & \longrightarrow & \mathit{\Pi}\backslash P & \longrightarrow & \widehat{Q}\backslash W. \end{array}$$

Note that F is ineffective on W and Q_ℓ acts properly on W if and only if F is compact.

Let $\mathit{\Pi}_w$ be the isotropy of the action $(\mathit{\Pi}, W)$ at $w \in W$. Recall that $\mathit{\Pi}_w$ leaves the G-fiber over w invariant and $\tau^{-1}(b) = \mathit{\Pi}_w\backslash G$, where $\bar{\nu}(w) = b$; see Subsection 4.1.1. Recall the following notations

$$\begin{aligned} \Gamma_\ell &= \mathit{\Pi} \cap \ell(G) \\ \widehat{\Gamma} &= \text{the kernel of } \mathit{\Pi} \to \mathrm{TOP}(W) \\ \mathit{\Pi}_w &= \text{the isotropy of the (ineffective) action } (\mathit{\Pi}, W) \text{ at } w \in W \end{aligned}$$

in increasing order by inclusion.

DEFINITION 4.3.3. $\Gamma_\ell\backslash G$ is called the *typical fiber* and $\widehat{\Gamma}\backslash G$ is called the *regular fiber*. If $\mathit{\Pi}_w$ is strictly bigger than $\widehat{\Gamma}$, then $\mathit{\Pi}_w\backslash G$ is a *singular fiber*.

Since Γ_ℓ is a subgroup of $\ell(G)$, the typical fiber $\Gamma_\ell\backslash G$ is a homogeneous space; see Subsection 4.4.3. $\widehat{\Gamma}$ is the ineffective part of the $\mathit{\Pi}$-action on $W = G\backslash P$. By Corollary 4.2.9, $\mathit{\Pi}_w$ lies in $\ell(G) \rtimes \mathrm{Aut}(G)$. Therefore, the singular fiber (also the regular fiber) is an infra-homogeneous space. See Subsection 4.4.3 for a definition. Notice that typical fibers and regular fibers are independent of the point on the base space.

This fiber nomenclature is motivated by a consequence of Smith theory: if Q is discrete and acts effectively and properly on a connected manifold W, then there is an open and dense subset of W for which $Q_w = 1$. We have adopted this notation because most of our applications fit these conditions. In general, observe that $\widehat{Q}_w = 1$ if and only if $\mathit{\Pi}_w = \widehat{\Gamma}$, and $Q_w = 1$ if and only if $\mathit{\Pi}_w = \Gamma_\ell$.

EXAMPLE 4.3.4. In Exercise 4.1.3, $\mathbb{Z}_2 = \mathit{\Pi}$, $G = S^1$, $P = S^1 \times I$, and W is an arc. Double P along its boundary to form $P' = S^1 \times I \bigcup_{\partial(S^1\times I)} S^1 \times I = S^1 \times S^1$. Extend the $(S^1 \times \mathbb{Z}_2)$-action to an action on $S^1 \times S^1$. Then $S^1 \times_{\mathbb{Z}_2} S^1$ is the Klein bottle with an induced S^1-action. The orbit mapping of the S^1-action is a Seifert fibering over an arc. Each interior point of the arc corresponds to principal orbits which are both regular and typical fibers while the orbits over the end points of the arc are singular fibers.

EXAMPLE 4.3.5. In Subsection 4.1.5, $\mathbb{Z}_2 = \mathit{\Pi}$, $G = S^1$, $P = S^1 \times S^1$, and W is an arc. Each fiber over an interior point of the arc is both a regular and a typical fiber. The fibers over the end points are singular fibers.

In Subsection 4.1.8, P is not necessarily connected. (Connectedness of P and G is only convenience and not essential to the notion of Seifert fibering.) $W = G\backslash P$

and $\psi(\varPi)\cap\ell(G) = (G\cap\varPi)\cap\mathcal{Z}(P) = \Gamma$ is the kernel of $\varPi \longrightarrow \mathrm{TOP}(W)$. Hence $\Gamma\backslash G$ is both the regular and typical fiber and τ is the orbit mapping of a $\Gamma\backslash G$-action. For the special case where G is a maximal compact subgroup of P, $\varPi$ is discrete and torsion free, all fibers are both regular and typical and isomorphic to G. If $\varPi$ is discrete but not torsion free, then the principal $\Gamma\backslash G$-orbits will also be typical and regular fibers, and the nonprincipal orbits of the $\Gamma\backslash G$-action on $\varPi\backslash P$ will be the singular fibers.

Remark 4.3.6. As mentioned above, connectedness of P and G is a convenience and is not essential to the notion of Seifert fiberings. Most of the concepts and definitions still make sense without these restrictions. In some of our prototype examples, we did not require that $G\cdot\varPi$ act properly (e.g., Subsection 4.1.9). Without the restriction of properness, orbit spaces fail to be Hausdorff in the usual quotient topology. The injection θ will still exist on the algebraic level but θ will not necessarily be a topological isomorphism onto its image in $\mathrm{TOP}_G(P)$. The fibers then are more likely to resemble the leaves of a foliation as in descent to the torus of the plane field of parallel lines with irrational slope; see Example 1.1.9. Such considerations have much dynamical significance, but exploration of these matters entails different methods.

4.3.7. Let $\theta : \varPi \to \mathrm{TOP}_G(P)$ be an injective homomorphism so that $\varPi\backslash P \to \varPi\backslash W$ is a Seifert fibering. Let Γ be a closed normal subgroup of $\varPi$ so that $\theta(\Gamma) \subset \theta(\varPi)\cap\big(\ell(G)\times_{\mathcal{Z}(G)}\mathrm{M}_G(P,G)\big)$. Denote this restriction by i. The following diagram commutes.

(4.3.2)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \varPi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \varphi\times\rho} & & \\
1 & \longrightarrow & \ell(G)\times_{\mathcal{Z}(G)}\mathrm{M}_G(P,G) & \longrightarrow & \mathrm{TOP}_G(P) & \longrightarrow & \mathrm{Out}(G)\times\mathrm{TOP}(W) & &
\end{array}$$

Here $\rho : Q \to \mathrm{TOP}(W)$ is the homomorphism induced by the $\varPi$-action on W. The effective part of this action is proper on W. The horizontal rows are exact. If $\Gamma_\ell \subset i(\Gamma)$, then $i(\Gamma)$ lies in between $\Gamma_\ell = \theta(\varPi)\cap\ell(G)$ and $\widehat{\Gamma} = \ker\{\varPi \longrightarrow \mathrm{TOP}(W)\}$.

4.4. The topology and geometry of the fibers

4.4.1. Let $1 \to \Gamma \to \varPi \to Q \to 1$ be as in Subsection 4.3.7, and the exact sequence $1 \to \Gamma \to \varPi_w \to Q_w \to 1$ be the pullback induced by the inclusion $Q_w \hookrightarrow Q$. The group $\varPi_w$ acts properly by Proposition 1.2.4(6(c)) on the fiber $\tilde{\tau}^{-1}(w)$ as a subgroup of $\mathrm{TOP}_G(\tilde{\tau}^{-1}(w))$. By Corollary 4.2.9, we have

$$\mathrm{M}_G(\tilde{\tau}^{-1}(w), G)\times_{\mathcal{Z}(G)}\ell(G) = r(G)\times_{\mathcal{Z}(G)}\ell(G) = \ell(G)\rtimes\mathrm{Inn}(G).$$

The diagram (4.3.2) becomes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \varPi_w & \longrightarrow & Q_w & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \varphi} & & \\
1 & \longrightarrow & \ell(G)\rtimes\mathrm{Inn}(G) & \longrightarrow & \ell(G)\rtimes\mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1.
\end{array}$$

Now, $\tau^{-1}(b) = \varPi_w\backslash\tilde{\tau}^{-1}(w) = Q_w\backslash(\Gamma\backslash G)$, where $\varPi_w$ lies in $\ell(G)\rtimes\mathrm{Aut}(G)$; the fiber $\tau^{-1}(b)$ is obtained as the quotient of $\Gamma\backslash G$ by the group Q_w. If $i(\Gamma)\subset\ell(G)$, we can interpolate the exact sequence $1 \to \ell(G) \to \ell(G)\rtimes\mathrm{Aut}(G) \to \mathrm{Aut}(G) \to 1$

between the top and bottom sequence. The homomorphism $Q_w \xrightarrow{\tilde{\varphi}} \mathrm{Aut}(G)$ is a lift of the homomorphism $Q_w \xrightarrow{\varphi} \mathrm{Out}(G)$.

4.4.2. To make more precise the geometry carried by the singular fibers, we endow G with the linear connection defined by the left invariant vector fields. Since the parallel transport is the effect of the left translations on the tangent vectors of G, and hence clearly independent of paths, the connection is flat. A geodesic through the identity element $e \in G$ is a 1-parameter subgroup of G and thus defined for any real value of the affine parameter. All geodesics are translates of geodesics through e and thus the connection is complete. One easily checks that the torsion tensor has vanishing covariant derivative. According to [**KT68**, Proposition 2.1],

$$\mathrm{Aff}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$$

is the group of *affine diffeomorphisms* (connection-preserving diffeomorphisms) of G, and $(a, \alpha) \in G \rtimes \mathrm{Aut}(G)$ acts on G by $(a, \alpha)(x) = a \cdot \alpha(x)$ for all $x \in G$. For example, if $G = \mathbb{R}^n$, $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$, the ordinary affine group of $\mathbb{R}^n$.

The theorem of Kamber and Tondeur is stated for a simply connected Lie group G. It also hold for any connected Lie group G: If $\tilde{G}$ is the universal covering group of G, then the lifting sequence of $\ell(G) \rtimes \mathrm{Aut}(G)$ becomes the exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \ell(\tilde{G}) \rtimes \mathrm{Aut}(\tilde{G}, \pi_1(G)) \longrightarrow \ell(G) \rtimes \mathrm{Aut}(G) \longrightarrow 1.$$

$\pi_1(G)$, the group of covering transformations, is a central subgroup of $\tilde{G}$. An automorphism of G lifts uniquely to an automorphism of $\tilde{G}$ (the other lifts are not automorphisms). Thus, the group $\mathrm{Aut}(\tilde{G}, \pi_1(G))$, the automorphisms of $\tilde{G}$ that leave $\pi_1(G)$ invariant, is naturally isomorphic to $\mathrm{Aut}(G)$.

4.4.3. If H is a closed subgroup of a Lie group G, the quotient space G/H is called a *homogeneous space*. It is a smooth manifold on which G acts transitively, (the principal isotropy group is H). Now suppose Π is a closed subgroup of $\mathrm{Aff}(G)$ acting properly on G. Now suppose Π is a closed subgroup of $\mathrm{Aff}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$ acting properly on G. If $\Gamma = \Pi \cap \ell(G)$, then Γ is normal in Π and $Q = \Pi/\Gamma$ acts properly on the homogeneous space $\Gamma\backslash G$. The quotient $Q\backslash(\Gamma\backslash G) = \Pi\backslash G$ is called an *infra-homogeneous space*. It may not be a manifold. However, if Π acts freely, then $\Pi\backslash G$ is called an *infra-homogeneous manifold*. Anyways, the map $\Gamma\backslash G \to \Pi\backslash G$ is extremely nice because the action of $Q = \Pi/\Gamma$ comes from $\mathrm{Aff}(G)$. If Π is discrete, then $\Pi\backslash G$ is an orbifold (i.e., a V-manifold) with the map $\Gamma\backslash G \to \Pi\backslash G$ being a regular (possibly branched) covering.

For a Seifert fibering modeled on P, take $w \in W$ on the orbit $b \in B$. Then $\tau^{-1}(b)$ is homeomorphic to $\Pi_w\backslash G$, where G is a principal fiber over w and $1 \to \Gamma \to \Pi_w \to Q_w \to 1$ is exact. The group Π_w acts as affine transformations on G. The infra-homogeneous space $\tau^{-1}(b)$ is the quotient of the homogeneous space $\Gamma\backslash G$ by the compact group Q_w of affine transformations on $\Gamma\backslash G$. If Π_w acts freely on G, then $\Pi_w\backslash G$ is an *infra-homogeneous manifold*.

4.4.4. It is of great interest to decide when Π acts freely on P. We have seen that this reduces to showing that Π_w acts freely on each fiber $\tilde{\tau}^{-1}(w)$, $w \in W$. Assume Γ is a discrete subgroup of G via $i : \Gamma \to \ell(G)$, but not necessarily that $\theta(\Pi) \cap \ell(G) = i(\Gamma)$. We have the following

PROPOSITION 4.4.5. *With Π, G and P as in Subsection* 4.4.1*, Π acts freely on P if and only if Π_w acts freely on each $\tilde{\tau}^{-1}(w)$ as elements of* $\mathrm{Aff}(G)$. *In particular,* *if G is simply connected and solvable (so that G is diffeomorphic to $\mathbb{R}^n$), then Π acts freely on P if and only if each Π_w is torsion free. Furthermore, if W is a finite dimensional contractible space, then Π acts freely on $P \approx G \times W$ if and only if Π is torsion free.*

PROOF. If G is simply connected and solvable, it is diffeomorphic to $\mathbb{R}^n$. The group Π_w acts properly on $G \times w$. If Π_w has torsion, then it contains a nontrivial finite p-subgroup H for some prime p. By Lemma 3.1.6, $(G \times w)^H \neq \emptyset$. So Π_w cannot act freely. Of course, if Π_w is torsion free and some element other than $1 \in H$ fixes a point in $G \times w$, this would contradict the properness of the action of Π. $\square$

4.5. Examples with Π discrete

The theory of Seifert fiberings is developed most thoroughly for discrete Π. Theorems 1.9.2 and 1.9.3 offer sufficient easily checked conditions that a discrete Π-action on P will lead to a Seifert fibering. If P is locally compact, then the assumptions of Theorem 1.9.2 frequently hold. If P is not locally compact, one may be able to use Theorem 1.9.3 to check if the Π-action leads to a Seifert fibering by putting the E of this theorem equal to the subgroup in $\mathrm{TOP}_G(P)$, generated by Π and $\ell(G)$. In this case, note that the Q in Theorem 1.9.3 is

$$Q_\ell = \ell(G) \cdot \Pi/\ell(G) = \Pi/(\ell(G) \cap \Pi) = \Pi/\Gamma_\ell.$$

The example below shows that these sufficient conditions may fail to be necessary conditions.

4.5.1. Let Δ be a lattice of G, and let $P = G \times W$. Suppose $\Pi = \Delta \times \widehat{Q}$, where $\Delta \subset r(G)$ and $\widehat{Q}$ acts trivially on the G-factor and properly and effectively on the W-factor. Note $\widehat{Q}$ is the effective part of the Π-action on W. The regular fiber is G/Δ and the typical fiber $\Gamma_\ell \backslash G$ is $(\Delta \cap \ell(G))\backslash G = (\mathcal{Z}(G) \cap \Delta)\backslash G$. If the $(\widehat{Q}, W)$-action is free, each $\tau^{-1}(b)$ is a regular fiber but not a typical fiber unless $\mathcal{Z}(G) \cap \Delta = \Delta$.

4.5.2 (The orbit mapping of a locally injective T^k-action as a Seifert fibering). The notion of a generalization to the classical 3-dimensional Seifert fiberings was inspired, in part, by the analysis of injective and locally injective toral actions as described in Section 2.8. Recall from there that (T^k, X) is locally injective if and only if the lifted action $(T^k, X_{\mathrm{Im}(\mathrm{ev}_*^x)})$ is free. The lifted action commutes with the covering transformations $Q = \pi_1(X, x)/\mathrm{Im}(\mathrm{ev}_*^x)$. Thus, assuming in addition that X is completely regular, the orbit mapping $X_{\mathrm{Im}(\mathrm{ev}_*^x)} = P \xrightarrow{\tilde{\tau}} T^k\backslash P = W$ is a principal T^k-bundle mapping and $\tilde{\tau}$ is $(T^k \times Q)$-equivariant, where the T^k-action on W is trivial. We have a diagram, as in Subsection 4.1.1.

$$\begin{array}{ccc}
(T^k \times Q,\ P) & \xrightarrow[T^k\backslash]{\tilde{\tau}} & (Q,\ T^k\backslash P = W) \\
\nu \big\downarrow Q\backslash & & \bar{\nu} \big\downarrow Q\backslash \\
(T^k,\ X) & \xrightarrow{\tau} & B = Q\backslash W
\end{array}$$

The map τ is our Seifert fibering. It is modeled on the principal T^k-bundle P. In this situation, W is simply connected and the homomorphism $\partial : \pi_2(W) \longrightarrow \pi_1(T^k)$, from the exact homotopy sequence of the principal bundle P (see Theorem

1.3.3), is an element of $\mathrm{Hom}(H_2(W;\mathbb{Z}),\pi_1(T^k)) \subset H^2(W;\mathbb{Z}^k)$. It can be easily seen that $[\partial] \in H^2(W;\mathbb{Z}^k)$ is the characteristic class of the bundle P and classifies the bundle P over W (when W is assumed to be locally semisimply connected and paracompact). Here the role of $\mathit{\Pi}$ in Subsection 4.3.1 is played by the covering Q-action on P.

Let K be the kernel of the T^k-action (i.e., the finite subgroup of T^k that leaves X fixed). This coincides with $\widehat{\Gamma}$, the kernel of the induced Q-action on $T^k\backslash P = W$. Thus $K\backslash T^k$ is the regular fiber. If some T^k-orbit on X has trivial isotropy group, for example if X is a manifold and the action of T^k is effective, then $\widehat{\Gamma} = \Gamma = K = 1$. These principal orbits coincide with the regular fibers.

4.5.3. If (T^k, X) is injective (see Definition 3.1.10), the lifted action (T^k, X_H) splits to $(T^k, T^k \times W)$, with T^k acting just as translations on the first factor. Replace the (T^k, X_H) by the obvious (ineffective) $(\mathbb{R}^k, X_H)$. The $\mathbb{R}^k$-action lifts to $(\mathbb{R}^k, \mathbb{R}^k \times W)$ on the universal covering $\tilde{X_H} = \mathbb{R}^k \times W$, with the $\mathbb{R}^k$-action being the translations along the first factor. The group $\pi_1(X, x)$ of covering transformations centralizes this $\mathbb{R}^k$-action. The group $\pi_1(X, x)$ acts on W with $\pi_1(T^k)$ acting trivially and the quotient Q acting properly. Now, the orbit mapping $\tau : (T^k, X) \to B = Q\backslash W$, is a Seifert fibering modeled on the trivial principal $\mathbb{R}^k$-bundle $\mathbb{R}^k \times W$. The group $\mathit{\Pi} = \pi_1(X, x)$ is a central extension of $\mathbb{Z}^k$ by Q and commutes with the $\mathbb{R}^k$-action on $\mathbb{R}^k \times W$. The Q-action is effective on W if and only if the T^k-action on X is effective. If (T^k, X) is effective, $\pi_1(T^k) = \mathbb{Z}^k = \Gamma = \mathit{\Pi} \cap \mathbb{R}^k$ and $\widehat{\Gamma} = \Gamma$. Then, both the regular and typical fibers are the principal orbits of X. In terms of Subsection 4.3.1, we have

$$\begin{array}{ccccccc}
1 \longrightarrow \pi_1(T^k, e) & \longrightarrow & \pi_1(X, x) & \longrightarrow & Q & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle 1\times\rho} & & \\
1 \to \mathrm{M}(W, \mathbb{R}^k) & \to & \mathrm{M}(W, \mathbb{R}^k) \rtimes (\mathrm{Aut}(\mathbb{R}^k) \times \mathrm{TOP}(W)) & \to & \mathrm{Aut}(\mathbb{R}^k) \times \mathrm{TOP}(W) & \to & 1
\end{array}$$

in the injective case. Suppose the isomorphism of $\mathbb{Z}^k$ into $\mathbb{R}^k$ is the standard inclusion. Then we may write

$$\begin{aligned}
\theta(n, \alpha)(x, w) &= \theta(n)\theta(1, \alpha)(x, w) \\
&= \ell(n)(\lambda_\alpha, 1, \rho(\alpha))(x, w) \\
&= (-n, \mu(n), 1)(\lambda_\alpha, 1, \rho(\alpha))(x, w) \\
&= (-n + \lambda_\alpha, 1, \rho(\alpha))(x, w) \\
&= (x - (-n + \lambda(\rho(\alpha)(w))), \rho(\alpha)(w)) \\
&= (x + n - \lambda(\rho(\alpha)(w)), \rho(\alpha)(w)),
\end{aligned}$$

where $(n, \alpha) \in \pi_1(X, x)$ and $(x, w) \in \mathbb{R}^k \times W$. The homomorphism $1 \times \rho$ maps Q trivially into the first factor and, with kernel, the ineffective part of (T^k, X) into the second factor.

4.5.4. Analysis of a locally injective (T^k, X) (see Definition 2.8.1) is more complicated. We have seen that (T^k, X) lifts to (T^k, X_H) where X_H is a principal T^k-bundle P over the simply connected $W = T^k\backslash P$. The group $H = \mathrm{ev}_*^x(\pi_1(T^k))$ is isomorphic to a group $C \oplus F$, C is a free Abelian group of rank s, with $s \leq k$, and F is finite Abelian. For convenience of notation, we shall assume that (T^k, X)-action is effective. We have the following

LEMMA 4.5.5. *There is a splitting of T^k into $T^s \times T^{k-s}$ so that $\mathrm{ev}_*^x|_{\pi_1(T^s)}$ is injective, $\mathrm{ev}_*^x|_{\pi_1(T^{k-s})}$ has finite image, and $\mathrm{ev}_*^x(\pi_1(T^k)) \cong C \oplus F$.*

PROOF. Let K be the kernel of $\mathrm{ev}_*^x : \pi_1(T^k, 1) \longrightarrow \pi_1(X, x)$. This is a free Abelian group of rank $k - s$ and is contained in a summand B, of $\pi_1(T^k)$ of rank $k - s$. Let A be any summand of $\pi_1(T^k)$ so that $A \times B = \pi_1(T^k)$. Observe, first, that $\mathrm{Im}(B) = F$. For if $b \in B$, then $nb \in K$, for some n. Thus, $\bar{b} = \mathrm{Im}(b)$ has order a divisor of n. Consequently, $\mathrm{Im}(B) \subset F$. If $(a \times b) \subset A \times B$ has image $\bar{a} + \bar{b} = x \in F$, then $n\bar{a} + n\bar{b} = 0$, if $nx = 0$. This implies $n\bar{a} \in F$ and so $ma \in K$, for some m. Therefore, a must be 0, $\mathrm{Im}(B) = F$, and $A \to C \oplus F$ must be injective. Because $\mathrm{Im}(B) = F$, $\mathrm{Im}(A) \cap \mathrm{Im}(B) = 0$, and $\mathrm{Im}(A)$ and $\mathrm{Im}(B)$ generates $C \oplus F$, $\mathrm{Im}(A)$ must be a free summand of $C \oplus F$. We replace the splitting of $C \oplus F$ with another splitting $\mathrm{Im}(A) \oplus \mathrm{Im}(B)$. Thus the lemma now follows by factoring the torus T^k into $T^s \times T^{k-s}$, where $\pi_1(T^s) = A$ and $\pi_1(T^{k-s}) = B$. □

4.5.6. By examining the homotopy exact sequence for P, as we have $\pi_2(W) \xrightarrow{\partial} A \times B \longrightarrow \mathrm{Im}(A) \times \mathrm{Im}(B) \to 1$, where $\mathrm{Im}(\partial) = K \subset B$. Therefore, $[\partial] \in \mathrm{Hom}(\pi_2(W), A \times B) = \mathrm{Hom}(\pi_2(W), A) \times \mathrm{Hom}(\pi_2(W), B)$. The coordinate of $[\partial]$ on the first factor $\mathrm{Hom}(\pi_2(W), A) = H^2(W; \mathbb{Z}^s)$, is trivial. Consequently, the T^s-subbundle of P is trivial and the T^s-action is a product action. Therefore, the free (T^k, P)-action factors in two ways by

$$\begin{array}{ccc}
(T^s \times T^{k-s}, P) & \xrightarrow{T^{k-s}\backslash} & (T^s, T^{k-s}\backslash P) = (T^s, T^s \times W) \\
{\scriptstyle T^s\backslash}\big\downarrow & & \big\downarrow{\scriptstyle T^s\backslash} \\
(T^{k-s}, T^s\backslash P) & \xrightarrow{T^{k-s}\backslash} & W.
\end{array}$$

The horizontal actions are locally injective with no injective part. The vertical actions are injective. Summarizing we have

THEOREM 4.5.7. *If (T^k, X) is a locally injective action, there is a splitting of $T^k = T^s \times T^{k-s}$ so that the action is equivalent to $(T^s \times T^{k-s}, X)$, where the T^s-action is injective and the (T^{k-s}, X)-action is locally injective with the property that no nontrivial torus subgroup of T^{k-s} acts injectively.*

4.5.8. We should mention one technical caveat. We have used $[\partial]$ to classify the principal T^k-bundle P over W. This may not work for an arbitrary and somewhat pathological space W. What is involved is an identification of the Čech cohomology $\check{H}^2(W; \mathbb{Z})$ as a subgroup of the singular cohomology $H^2(W; \mathbb{Z})$. This is no problem if, for each $x \in X$ and neighborhood U of x, there is a neighborhood V with $x \in V \subset U$, such that $H_i(V; \mathbb{Z}) \to H_i(U; \mathbb{Z})$ is trivial for singular homology, $i \leq 1$. Without this assumption, a more complicated cohomology argument is needed to ensure that the theorem holds.

4.5.9. The case where G is a simply connected Abelian or nilpotent Lie group is especially important for us. For $G = \mathbb{R}^n$, $\mathrm{O}(n)$ is a maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$. A *uniform* discrete subgroup (i.e., cocompact discrete subgroup) Π of $\mathbb{R}^n \rtimes \mathrm{O}(n)$ is called a *crystallographic group*. By a theorem of Bieberbach, $\Pi \cap \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$, and is a lattice of $\mathbb{R}^n$. If Π is torsion free, we call Π a *Bieberbach* *group* (torsion-free crystallographic group). Flat manifolds are the orbit spaces $\varPi\backslash\mathbb{R}^n$, where $\varPi$ is a Bieberbach group. Note that each flat manifold is finitely covered by a flat torus $\mathbb{Z}^n\backslash\mathbb{R}^n$; see Theorem 8.1.2 for more details.

EXAMPLE 4.5.10. $E(3) = \mathbb{R}^3 \rtimes \mathrm{O}(3)$ is the group of isometries of $\mathbb{R}^3$ (3-dimensional Euclidean space). As a set, it is the Cartesian product $\mathbb{R}^3 \times \mathrm{O}(3)$, where $\mathrm{O}(3)$ is the orthogonal group. The group operation is given by

$$(a, A)(b, B) = (a + Ab, AB).$$

This acts on $\mathbb{R}^3$ by

$$(a, A) \cdot x = a + Ax$$

for $x \in \mathbb{R}^3$. The matrix A is called the *rotational part*, and a is called the *translational part* of (a, A). It is easy to check that this is actually an action:

$$\begin{aligned}(a, A)((b, B) \cdot x) &= (a, A) \cdot (b + Bx)\\ &= a + A(b + Bx)\\ &= (a + Ab) + ABx\\ &= ((a, A)(b, B)) \cdot x.\end{aligned}$$

Consider the subgroup $\varPi \subset E(3)$ generated by

$$\varPi = \langle t_1 = (e_1, I),\ t_2 = (e_2, I),\ t_3 = (e_3, I),\ \alpha = (a, A)\rangle,$$

where

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},\ e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},\ a = \frac{1}{2}e_1,\ A = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}.$$

Since $t_i \cdot x = x + e_i$, each t_i is a translation by the ith unit vector. On the other hand,

$$\alpha \cdot x = \left(\begin{bmatrix}1/2\\0\\0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\right) \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 + 1/2\\-x_2\\-x_3\end{bmatrix}$$

shows that α rotates the x_2x_3-plane by 180^o, while it advances the x_1 direction by a half unit. Moreover,

$$\alpha^2 = t_1.$$

Since

$$\alpha t_1\alpha^{-1} = t_1,\ \alpha t_2\alpha^{-1} = t_2^{-1},\ \alpha t_3\alpha^{-1} = t_3^{-1},$$

the center of $\varPi$ is generated by t_1, and is isomorphic to $\mathbb{Z}$. The quotient is

$$Q = \varPi/\mathbb{Z} = \langle \bar{t}_2,\ \bar{t}_3, \bar{\alpha}\rangle \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2.$$

Clearly, $\varPi \cap \mathbb{R}^3 = \mathbb{Z}^3$ (the translation part), and $\varPi/\mathbb{Z}^3 = \mathbb{Z}_2$ generated by α. Thus our space is

$$M = \varPi\backslash\mathbb{R}^3 = \mathbb{Z}_2\backslash(\mathbb{Z}^3\backslash\mathbb{R}^3) = \mathbb{Z}_2\backslash T^3.$$

It is exactly the same as the space $X = \mathfrak{G}_2$ of Example 4.1.4. It is $S^1 \times_{\mathbb{Z}_2} (S^1 \times S^1)$, where $\mathbb{Z}_2$ acts on $S^1 \times S^1$ by $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ and on S^1 by $z \mapsto -z$. Then $S^1\backslash X$ is the surface of the square pillow space; see Subsection 4.1.5. There are five different ways of looking at the same space M as Seifert manifolds modeled on various different principal bundles.

(1) Split $\mathbb{R}^3$ as $\mathbb{R} = \mathbb{R}\{e_1\}$, $\mathbb{R}^2 = \mathbb{R}\{e_2, e_3\}$.

(1a) Model space with the principal fibering $\mathbb{R}\{e_1\} \to \mathbb{R}^3 \to \mathbb{R}\{e_2, e_3\}$. Using the subgroup $\ell(\mathbb{R}^1) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^1) \times \mathrm{Isom}(\mathbb{R}^2)\Big) = \ell(\mathbb{R}^1) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^1) \times (\mathbb{R}^2 \rtimes \mathrm{O}(2))\Big)$, the nontrivial element α is represented by

$$\alpha = \left(\tfrac{1}{2}, 1, \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)\right).$$

This α rotates $\mathbb{R}^2$ by 180 degrees while it advances by half a unit on the fiber $\mathbb{R}$. Clearly, $\mathbb{Z}\backslash\mathbb{R}^1 = S^1$, and $S^1 \to M \to \Pi\backslash\mathbb{R}^2$ is a Seifert fibering modeled on the principal $\mathbb{R}^1$-bundle over $\mathbb{R}^2$ and with typical fiber S^1. Since $\Pi/\mathbb{Z}$ acts effectively on $\mathbb{R}^2$, the regular fibers are equal to the typical fiber. There are four singular fibers, all of which are $\mathbb{Z}_2\backslash S^1$.

(1b) Model space with the principal fibering $\mathbb{R}\{e_2, e_3\} \to \mathbb{R}^3 \to \mathbb{R}\{e_1\}$. Using the subgroup $\ell(\mathbb{R}^2) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Isom}(\mathbb{R}^1)\Big) = \ell(\mathbb{R}^2) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^2) \times (\mathbb{R}^1 \rtimes \mathrm{O}(1))\Big)$, the nontrivial element α is represented by

$$\alpha = \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, (\tfrac{1}{2}, 1)\right).$$

This α acts on the base $\mathbb{R}^1$ as a translation by $\frac{1}{2}$ freely yielding a circle $\mathbb{Z}\backslash\mathbb{R} = \mathbb{Z}_2\backslash S^1$ again while it rotates the fiber $\mathbb{R}^2$ by 180 degrees. Thus our space is a T^2-bundle over the circle with structure group $\pm I$. This resulting bundle is not a principal bundle. Every fiber is a typical (so regular) fiber T^2.

(2) Split $\mathbb{R}^3$ as $\mathbb{R} = \mathbb{R}\{e_3\}$, $\mathbb{R}^2 = \mathbb{R}\{e_1, e_2\}$.

(2a) Model space with the principal fibering $\mathbb{R}\{e_3\} \to \mathbb{R}^3 \to \mathbb{R}\{e_1, e_2\}$. Using the subgroup $\ell(\mathbb{R}^1) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^1) \times \mathrm{Isom}(\mathbb{R}^2)\Big) = \ell(\mathbb{R}^1) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^1) \times (\mathbb{R}^2 \rtimes \mathrm{O}(2))\Big)$, the nontrivial element α is represented by

$$\alpha = \left(0, -1, \left(\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)\right).$$

This α acts on T^2 freely yielding the Klein bottle while it also flips the fiber. This is a S^1-bundle over the Klein bottle with structure group $\mathbb{Z}_2 = \langle -1 \rangle$. This resulting bundle is not a principal bundle. Since the induced action of Π on the base $\mathbb{R}^2$ is free, every fiber is a typical (so regular) fiber S^1.

(2b) Model space with the principal fibering $\mathbb{R}\{e_1, e_2\} \to \mathbb{R}^3 \to \mathbb{R}\{e_3\}$. Using the subgroup $\ell(\mathbb{R}^2) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Isom}(\mathbb{R}^1)\Big) = \ell(\mathbb{R}^2) \rtimes \Big(\mathrm{Aut}(\mathbb{R}^2) \times (\mathbb{R}^1 \rtimes \mathrm{O}(1))\Big)$, the nontrivial element α is represented by

$$\alpha = \left(\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, (0, -1)\right).$$

The base space is an arc with two end points singular. Regular fibers (and singular fibers) are T^2, and the two singular fibers are Klein bottles.

(3) On the other hand, X can also be regarded as a Seifert fibering $X \to X \to$ point, modeled on the principal $\mathbb{R}^3$-bundle over a point. There is just one fiber, $X = \Pi\backslash\mathbb{R}^3$, itself. It is a regular fiber. The typical fiber is $(\Pi \cap \mathbb{R}^3)\backslash\mathbb{R}^3 = \mathbb{Z}^3\backslash\mathbb{R}^3$, the 3-torus. Therefore the typical fiber does not appear here as an actual fiber in this Seifert fibering.

4.5.11. Let G be a connected, simply connected nilpotent Lie group; see Subsection 6.1.2. Choose a maximal compact subgroup C of $\mathrm{Aut}(G)$. A uniform (i.e., cocompact) discrete subgroup Π of $G \rtimes C$ is called an *almost crystallographic group*. If it is torsion free, it is called an *almost Bieberbach group*. An almost Bieberbach group Π yields an *infra-nilmanifold* $\Pi\backslash G$. Note here again that any infra-nilmanifold is finitely covered by the nilmanifold $\Gamma\backslash G$, where $\Gamma = \Pi \cap G$.

Thus, nilmanifolds are a generalization of tori, and infra-nilmanifolds are generalizations of flat manifolds. It is also known that a manifold is diffeomorphic to an infra-nilmanifold if and only if it is *almost flat*. This term is due to Gromov. See [**FH83**] for a proof of the above fact. See Chapter 8 for an extensive discussion.

EXAMPLE 4.5.12. We shall now formulate the 3-dimensional nilmanifolds as homogeneous spaces in terms of their lattices and also as classical 3-dimensional Seifert manifolds with typical fiber S^1 and base the 2-torus.

Consider the *Heisenberg group*

$$N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;:\; x, y, z \in \mathbb{R} \right\},$$

which is connected, simply connected, and two-step nilpotent. We denote such a matrix by (z, x, y) so that

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad \longleftrightarrow \quad (z, x, y).$$

Then the group operation is

$$(z', x', y') \cdot (z, x, y) = (z' + z + x'y,\ x' + x,\ y' + y),$$

and the center of N is 1-dimensional $\mathcal{Z} = \mathbb{R}$, consisting of all matrices with $x = y = 0$. The quotient $W = N/\mathcal{Z}$ is isomorphic to $\mathbb{R}^2$ so that

$$1 \to \mathbb{R} \to N \to \mathbb{R}^2 = W \to 1$$

is an exact sequence of Lie groups. As spaces, this is a smooth fibration which is also a product $N = \mathbb{R} \times W$. Let

$$\alpha = (0, 1, 0),\ \beta = (0, 0, 1),\ \text{ and } \gamma = (1/p, 0, 0) \in N,\ \text{ with } p \text{ an integer} \neq 0.$$

These three elements generate a discrete group Π in N. (The coordinates of the group generated are of the form $(n + s/p, q, r)$ where n, s, q, r are all integers and so Π is a discrete subset of $\mathbb{R}^3$ which is diffeomorphic to N.) Moreover Π is uniform (i.e., cocompact) because $\Pi\backslash N$ is compact as we shall now show. Observe that the left translational action of Π on N commutes with the free central $\mathbb{R}$ translational action on $N \cong \mathbb{R} \times \mathbb{R}^2$ and $\mathbb{R} \cap \Pi$ is the center of the subgroup Π generated by $\gamma = (1/p, 0, 0)$. The $\mathbb{R}$-action descends to a free $S^1 = \langle\gamma\rangle\backslash\mathbb{R}$-action on $\Pi\backslash N$. The map $\Pi\backslash(\mathbb{R} \times \mathbb{R}^2) \to \mathbb{Z}^2\backslash\mathbb{R}^2 = T^2$ is the orbit mapping of the S^1-action on $\Pi\backslash N$, and as $\Pi/\langle\gamma\rangle \cong \mathbb{Z}^2$ acts freely on $\mathbb{R}^2$, $\mathbb{Z}^2\backslash\mathbb{R}^2$ has a 2-torus as orbit space.

It is not hard to see that the Euler class of this bundle is $-p$; see Subsection 15.4.1(1). Since $\Pi\backslash N$ is compact, Π is a uniform (i.e., cocompact) discrete subgroup of N, and so $\Pi\backslash N$ is a nilmanifold.

Let us look at $\Pi\backslash N$ from a different point of view. We take $W = \mathbb{R}^2$. A group $Q = \mathbb{Z}^2$ acts on W as translations. Let

$$1 \to \mathbb{Z} \to \Pi \to Q \to 1$$

be a central extension of $\mathbb{Z}$ by Q. Then Π has a presentation

$$\Pi = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma^p, [\alpha, \gamma] = [\beta, \gamma] = 1 \rangle,$$

where γ is a generator of the center $\mathbb{Z}$ and the images of α, β in Q are generators of Q. Suppose $p \neq 0$. Using $\mathbb{Z} \subset \mathbb{R}$, one can obtain an effective action of Π on the product $\mathbb{R} \times W$ as follows: for $(z, x, y) \in \mathbb{R} \times W$,

$$\begin{array}{rcllll} \alpha(z,x,y) & = & (z+y, & x+1, & y), \\ \beta(z,x,y) & = & (z, & x, & y+1), \\ \gamma(z,x,y) & = & (z+\frac{1}{p}, & x, & y). \end{array} \tag{4.5.1}$$

The elements α, β, γ in the description of the Heisenberg group also satisfy the same relations as those satisfied by Π. Furthermore, the actions of Π on N by left translations are the same as the actions of Π on $\mathbb{R} \times W$. Notice that these maps are of the form

$$(z, x, y) \mapsto (\phi(z) - \lambda(h(x, y)), h(x, y)),$$

where ϕ is an automorphism of $\mathbb{R}$ and h is an action of Q on W, and λ is a map $W \to \mathbb{R}$. Consequently, the group Π lies in $\mathrm{TOP}_{\mathbb{R}}(\mathbb{R} \times W)$ as

$$(\lambda, \phi, h) \in \mathrm{M}(W, \mathbb{R}) \rtimes (\mathrm{GL}(1, \mathbb{R}) \times \mathrm{TOP}(W)).$$

This represents $\Pi\backslash N = \Pi\backslash(\mathbb{R} \times W) \longrightarrow \mathbb{Z}^2\backslash W$ as a Seifert fibering modeled on the trivial $\mathbb{R}$-bundle $\mathbb{R} \times \mathbb{R}^2$ with typical fiber $\mathbb{Z}\backslash\mathbb{R} = S^1$ and base the 2-torus. We shall discuss both classical 3-dimensional Seifert fiberings and nilmanifolds in great detail in Chapters 14 and 15. As in Example 4.5.10, the same space $\Pi\backslash N$ also fibers over a point when G is taken to be N itself instead of $\mathbb{R}$ in N, and so here $\Gamma_\ell = \widehat{\Gamma} = \Pi$.

Exercise 4.5.13. Let $\alpha' = (0, p, 0)$, $\beta' = (0, 0, 1)$, $\gamma' = (p, 0, 0) \in N$, with p an integer $\neq 0$. Show that the group generated by $\{\alpha', \beta', \gamma'\}$ is isomorphic to the group generated by $\alpha = (0, 1, 0)$, $\beta = (0, 0, 1)$, $\gamma = (1, 0, 0) \in N$. Also show that the nilmanifolds can cover themselves nontrivially.

4.5.14. Many of the examples that we used for illustrations have featured Π as a *discrete* group and a *trivial* principal bundle $G \times W \to W$. In analogy to injective torus actions (see Definition 3.1.10), we call such a Seifert fibering an *injective Seifert fibering*. If Π is *discrete* and the bundle is *not necessarily trivial*, we call this a *locally injective Seifert fibering*; cf. Definition 2.8.1.

Example 4.5.15. Let us consider SO(3), the group of orientation preserving linear isometries of $\mathbb{R}^3$ fixing the origin. Let $G = \mathrm{SO}(2)$ be the subgroup of rotations about the z-axis, and let $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ be rotations of 180^o about x, y and z-axes together with the identity. Thus G and H are subgroups of SO(3). Let G act freely on SO(3) on the right, and let H act on the left as multiplications. Since these actions commute, the SO(2)-action descends to $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash\mathrm{SO}(3)$. To examine the descended SO(2)-action, we instead look at the induced $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ of the principal SO(2)-bundle $\mathrm{SO}(3) \longrightarrow \mathrm{SO}(3)/\mathrm{SO}(2)$. Recall SO(3) is diffeomorphic to $\mathbb{R}P_3$. This $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on S^2 is equivalent to the restriction of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on the unit sphere. We see that we have three pairs of poles corresponding to the intersections of the axes with the unit sphere which are fixed by different elements of order 2 in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Otherwise, the actions are free. Thus the orbit space of $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash S^2$ is again a 2-sphere and there are exactly three singular orbits with isotropy $\mathbb{Z}_2$. Therefore, the S^1-action on

$$M = (\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash \mathrm{SO}(3)$$

is free off of three singular orbits where the isotropy is $\mathbb{Z}_2 \subset \mathrm{SO}(2)$. The free SO(2)-action on SO(3) generates the fundamental group of SO(3) and so does not lift to the universal covering group $S^3 = \mathrm{Spin}(3)$, the group of unit quaternions. If we take the extended lifting sequence of the free $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action, we get a central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$; this is the quaternion group H generated by $\{1, i, j, k\}$. The lifting sequence for SO(2) gives a connected double covering group $\widehat{\mathrm{SO}}(2)$ of SO(2) and is the maximal torus of Spin(3). Therefore, $M = (\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash \mathrm{SO}(3)$ has as its fundamental group the quaternion group of order 8. M is a locally injective Seifert fibering over the 2-sphere with three singular fibers and modeled on the SO(2)-bundle $\mathrm{SO}(3) \to S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. We have the following commutative diagram of orbit mappings.

$$\begin{array}{ccc}
(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{Spin}(3), \widehat{\mathrm{SO}}(2)) & \xrightarrow{/\widehat{\mathrm{SO}}(2)} & S^2 \\
{\scriptstyle /\mathbb{Z}_2}\downarrow & & {\scriptstyle =}\downarrow \\
(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{SO}(3), \mathrm{SO}(2)) & \xrightarrow{/\mathrm{SO}(2)} & (\mathbb{Z}_2 \times \mathbb{Z}_2, S^2) \\
{\scriptstyle (\mathbb{Z}_2\times\mathbb{Z}_2)\backslash}\downarrow & & \downarrow \\
((\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash \mathrm{SO}(3), \mathrm{SO}(2)) & \xrightarrow{/\mathrm{SO}(2)} & (\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash S^2 \approx S^2
\end{array}$$

The SO(2)-action on $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash \mathrm{SO}(3)$ is locally injective, with three singular orbits of multiplicity 2, by Proposition 2.8.7.

In addition, if we take $\mathbb{Z}_p \subset \mathrm{SO}(2)$, p odd, the SO(2)-action descends to an effective $(\mathrm{SO}(2)/\mathbb{Z}_p) \approx S^1$-action on the 3-manifold $M/\mathbb{Z}_p$ covered by M. The quotient by $\mathrm{SO}(2)/\mathbb{Z}_p$ is still $(\mathbb{Z}_2 \times \mathbb{Z}_2)\backslash S^2 \approx S^2$, with the same multiplicities, and $\mathrm{SO}(2)/\mathbb{Z}_p$ is locally injective on $M/\mathbb{Z}_p$.

$$\begin{array}{ccc}
(M, \mathrm{SO}(2)) & \xrightarrow{/\mathrm{SO}(2)} & M/\mathrm{SO}(2) \approx S^2 \\
{\scriptstyle \mathbb{Z}_p}\downarrow & & {\scriptstyle =}\downarrow \\
(M/\mathbb{Z}_p, \mathrm{SO}(2)/\mathbb{Z}_p) & \xrightarrow{/(\mathrm{SO}(2)/\mathbb{Z}_p)} & (M/\mathbb{Z}_p)/(\mathrm{SO}(2)/\mathbb{Z}_p)
\end{array}$$

The manifolds just constructed are examples of 3-dimensional spherical space forms.

4.5.16. A *spherical space form* is the quotient of a sphere by a finite group of freely acting isometries. That is, if one takes the unit sphere $S^n \subset \mathbb{R}^{n+1}$, then the group of isometries of S^n, with the metric induced from $\mathbb{R}^{n+1}$, is $\mathrm{O}(n+1, \mathbb{R})$ acting on S^n in the usual fashion. If the finite group $\Pi \subset \mathrm{O}(n+1, \mathbb{R})$ acts freely, then the orbit space $\Pi\backslash S^n$ is called a spherical space form. In differential geometric terms, the spherical space forms are the smooth manifolds with a Riemannian metric whose sectional curvature is constant and positive; see [**Wol77**] for details.

EXERCISE 4.5.17. Show that in dimension 2, S^2 and $\mathbb{R}P_2$ are the only spherical space forms. Similarly, in dimension $2n$, only S^{2n} and $\mathbb{R}P_{2n}$ appear as spherical space forms. (Use the Lefschetz fixed point formula to show that Π must be $\mathbb{Z}_2$.) See [**Wol77**].

4.5.18. In each odd dimensions ≥ 3, there are infinitely many topologically distinct spherical space forms with distinct non-Abelian fundamental groups. Those with Abelian fundamental groups are the lens spaces.

A remarkable characterization was given by H. Zassenhaus for a finite solvable group to admit a free linear action on a sphere. In general it states: *For Π to admit a free faithful representation into $U(n)$, (i.e., $\theta : \Pi \to U(n)$ as above) so that Π acts freely on S^{2n-1} (and hence $\Pi \backslash S^{2n-1}$ is a spherical space form), every subgroup of Π of order pq, p and q primes, is a cyclic group. For solvable groups, this necessary condition is also sufficient. To achieve sufficiency for nonsolvable Π, one must also require that the only noncyclic composition factor allowed is the simple group $I \cong A_5 \cong \widetilde{PSL}(2,5)$.* We discuss more aspects of spherical space forms in the next section as well as in Section 11.8, Chapters 14 and 15. In Section 15.3, we analyze the 3-dimensional space forms in the spirit of Example 4.5.15 and 4.6.3(5) when $n = 1$. Then the space forms are characterized by their Seifert invariants and their fundamental groups; see also [**Orl72**, §6.2] for another complete discussion of 3-dimensional spherical space forms. For a general discussion with emphasis on the topological classification of finite groups acting freely and not necessarily linearly on spheres, see [**AD02**]

We claim that each $(2n-1)$-dimensional spherical space form has the structure of a Seifert fibering modeled on a principal S^1-bundle over $\mathbb{C}P_{n-1}$. Let $\Delta_n = \{\lambda I_n : \lambda \in \mathbb{C}, |\lambda| = 1\}$ be the subgroup of $U(n)$. Then Δ_n is isomorphic to $S^1 \cong U(1)$ and is the center of $U(n)$. The quotient group $PU(n) = U(n)/\Delta_n$ is called the *projective unitary group*. It is a simple Lie group in adjoint form. In the linear action of $U(n)$ on S^{2n-1}, the group Δ_n acts freely, and the orbit mapping $S^{2n-1} \to \Delta_n \backslash S^{2n-1} \cong \mathbb{C}P_{n-1}$ is the Hopf fibering. The action of $U(n)$ on S^{2n-1} projects to an effective action of $PU(n)$ on $\mathbb{C}P_{n-1}$. It is known that this induced action is as isometries with respect to the natural Kähler metric on $\mathbb{C}P_{n-1}$.

Let $\varphi : \Pi \to U(n)$ be a free representation and put $\Gamma = \Delta_n \cap \varphi(\Pi)$ where Π is a finite group. The induced action of Δ_n on $\varphi(\Pi)\backslash S^{2n-1} = M$ has no fixed points, for otherwise its lift back to Δ_n on S^{2n-1} would have fixed points. The orbit mapping, $(\Delta_n, \Pi\backslash S^{2n-1}) \to \Pi\backslash\mathbb{C}P_{n-1}$ is a locally injective Seifert fibering modeled on the principal Δ_n/Γ-bundle $(\Delta_n/\Gamma, M) \to \mathbb{C}P_{n-1}$ with typical and regular fiber $\Delta_n/\Gamma \cong S^1$. The bundle has the first Chern class (or Euler class) $-|\Gamma|$. We may also describe this Seifert fibering as being modeled over the principal Δ_n-bundle $S^{2n-1} \to \mathbb{C}P_{n-1}$ with typical fiber Δ_n and regular fiber Δ_n/Γ. The spherical space form M may admit different Seifert fiberings than the one described. However, in the case where $2n-1$ is the minimal dimension for which Π has a free orthogonal representation, the Seifert fibering just described is essentially unique; see Proposition 11.8.7.

4.6. The Seifert Construction

As before, P is a principal G-bundle, $W = G\backslash P$. Recall that $\mathrm{TOP}_G(P)$ is the group of all weakly G-equivariant homeomorphisms of P.

Let $\mathcal{U}$ be a closed subgroup of $\mathrm{TOP}_G(P)$. A Seifert Construction for Π into $\mathcal{U}$ is simply a homomorphism $\theta : \Pi \to \mathcal{U} \hookrightarrow \mathrm{TOP}_G(P)$. This $\mathcal{U}$ is called the *uniformizing group* for the Seifert construction. The smaller $\mathcal{U}$ is, the more restrictive the fiber space structure will be. Therefore, the more the fiber structure is restricted, the more likely the geometric structure is enhanced.

DEFINITION 4.6.1. A *Seifert Construction* for

(1) a group extension $1 \longrightarrow \Gamma \longrightarrow \Pi \xrightarrow{p} Q \longrightarrow 1$, Π discrete,
(2) a homomorphism $i : \Gamma \to \ell(G) \times_{\mathcal{Z}(G)} \mathrm{M}_G(P, G)$,
(3) a proper action $\rho : Q \to \mathrm{TOP}(W)$,

with the *uniformizing group* $\mathcal{U} \subset \mathrm{TOP}_G(P)$ (closed subgroup), is a homomorphism

$$\theta : \Pi \longrightarrow \mathcal{U}$$

such that

$$\theta|_{\Gamma} = i,$$

and the diagram

$$\begin{array}{ccccc} \Pi & \xrightarrow{\theta} & \mathcal{U} & \xrightarrow{\subset} & \mathrm{TOP}_G(P) \\ {\scriptstyle =}\downarrow & & & & \downarrow \\ \Pi & \xrightarrow{p} & Q & \xrightarrow{\rho} & \mathrm{TOP}(W) \end{array}$$

is commutative. (That is, the Π-action on W via $\Pi \xrightarrow{\theta} \mathcal{U} \subset \mathrm{TOP}_G(P) \longrightarrow \mathrm{TOP}(W)$ is the same as $\Pi \xrightarrow{p} Q \xrightarrow{\rho} \mathrm{TOP}(W)$.)

4.6.2. We may require a Seifert construction to satisfy additional natural conditions as more information on Π is given. For example, when $P = G \times W$, the trivial bundle, we may require $i = \ell : \Gamma \hookrightarrow \ell(G)$ to be a cocompact discrete subgroup such that every automorphism of Γ extends to a unique automorphism of G (i.e., (G, Γ) has the Unique Automorphism Extension Property (UAEP), see Definition 5.3.3) and $\mathcal{U} = \mathrm{TOP}_G(P)$. In this case, the Seifert Construction $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$ must make the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi\times\rho} & & \\ 1 & \longrightarrow & \mathrm{M}(W, G) \rtimes \mathrm{Inn}(G) & \longrightarrow & \mathrm{TOP}_G(G \times W) & \longrightarrow & \mathrm{Out}(G) \times \mathrm{TOP}(W) & \longrightarrow & 1 \end{array}$$

commutative, where $\rho : Q \to \mathrm{Out}(G)$ is the homomorphism induced from the abstract kernel $Q \to \mathrm{Out}(\Gamma)$ and $\mathrm{Out}(\Gamma) \to \mathrm{Out}(G)$ from the UAEP.

In Chapters 7 and 9, we show for various classes of discrete groups that every extension, $1 \to \Gamma \to \Pi \to Q \to 1$, admits a Seifert Construction. To what extent such a construction is unique is also examined. The existence and uniqueness of the Seifert Construction are essential for many of our applications.

4.6.3 (Examples of uniformizing groups). (1) Let $P = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$, the universal covering group of $\mathrm{PSL}(2, \mathbb{R})$. It is a principal G-bundle ($G = \mathbb{R}$) over $\mathbf{H} = \mathbb{R}\backslash\widetilde{\mathrm{PSL}}(2, \mathbb{R}) = \mathrm{SO}(2)\backslash\mathrm{PSL}(2, \mathbb{R})$, the 2-dimensional real hyperbolic plane. Take $\mathcal{U} = \mathrm{Isom}_0(P)$, the connected component of the identity of the group of isometries of P. Let $\rho : Q \subset \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group. Then a Seifert construction for $1 \to \mathbb{Z} \to \Pi \to Q \to 1$ with the uniformizing group $\mathrm{Isom}_0(\widetilde{\mathrm{PSL}}(2, \mathbb{R}))$ is a homomorphism $\theta : \Pi \to \mathrm{Isom}_0(\widetilde{\mathrm{PSL}}(2, \mathbb{R}))$ such that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi \times \rho} & & \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Isom}_0(\widetilde{\mathrm{PSL}}(2, \mathbb{R})) & \longrightarrow & \mathrm{PSL}(2, \mathbb{R}) & \longrightarrow & 1 \end{array}$$

is commutative; see Chapter 13.

(2) $G = \mathbb{R}^k$ and $P = \mathbb{R}^k \times \mathbb{R}^n$. The group $\mathrm{GL}(k, \mathbb{R}) \times \mathrm{Aff}(\mathbb{R}^n)$ acts on $L(\mathbb{R}^n, \mathbb{R}^k)$, the group of all maps of the form $x \mapsto Ax + a$, where A is a $(k \times n)$-matrix and $a \in \mathbb{R}^k$. Then $L(\mathbb{R}^n, \mathbb{R}^k)$ is not invariant under $\mathrm{GL}(k, \mathbb{R}) \times \mathrm{TOP}(\mathbb{R}^n)$. One cannot form a semidirect product $L(\mathbb{R}^n, \mathbb{R}^k) \rtimes (\mathrm{GL}(k, \mathbb{R}) \times \mathrm{TOP}(\mathbb{R}^n))$, but can form a semidirect product $\mathcal{U} = L(\mathbb{R}^n, \mathbb{R}^k) \rtimes (\mathrm{GL}(k, \mathbb{R}) \times \mathrm{Aff}(\mathbb{R}^n))$. Clearly, this is a subgroup of $\mathrm{Aff}(k + n)$, and is a closed subgroup of $\mathrm{TOP}_{\mathbb{R}^k}(\mathbb{R}^k \times \mathbb{R}^n)$; see [**Lee83**] and Section 11.4.

(3) If W is a smooth manifold, one can take $\mathcal{U} = \mathrm{Diff}_G(G \times W) = \mathcal{C}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{Diff}(W))$, where $\mathcal{C}(W, G)$ is the group of all smooth maps from W to G.

(4) One can even take $\mathcal{U} = r(G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W)) = \ell(G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W))$ for $G \times W$.

(5) To create a spherical space form in dimension $2n-1$, begin with a representation $\varphi : \Pi \to \mathrm{O}(2n)$ such that no $\varphi(g)$ with $g \neq 1$ has an eigenvalue equal to 1. This means that Π acts on $\mathbb{R}^n$ as isometries fixing the origin and freely elsewhere. The unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ is left invariant, and the orbit space $\varphi(\Pi)\backslash S^{2n-1}$ is a spherical space form. The representation is called a free representation. Suppose $\varphi(\Pi)\backslash S^{2n-1}$ and $\psi(\Pi)\backslash S^{2n-1}$ are spherical space forms. They will be isometric (respectively, diffeomorphic, homeomorphic) if and only if there exists an automorphism $a : \Pi \to \Pi$ and an equivariant homeomorphism

$$h : \ (\varphi(\Pi), S^{2n-1}) \longrightarrow (\psi(a\Pi), S^{2n-1}),$$

where $h \in \mathrm{O}(2n)$ (respectively, $\mathrm{Diff}(S^{2n-1}), \mathrm{TOP}(S^{2n-1})$). In this case, the two representations are said to be orthogonally (respectively, differentiably, topologically) equivalent.

To classify spherical space forms up to isometry (respectively, diffeomorphism, homeomorphism) in dimension $2n - 1$, one must:

1. Find all finite groups Π which admit free orthogonal representations into $\mathrm{O}(n)$;
2. Classify the equivalence classes of these representations of Π into $\mathrm{O}(n)$ (respectively, $\mathrm{Diff}(S^{2n-1})$, $\mathrm{TOP}(S^{2n-1})$).

The classification of spherical space forms, up to isometry, is the work of many group theorists and geometers: Burnside, Zassenhaus, Killing, Vincent, Wolf, and others; see [**Wol77**, Chapters 4, 5, 6]. The answers for the most part are algorithmic. The classification up to diffeomorphism agrees with the isometric classification and is due to Franz, DeRham, and others. The topological classification of spherical space forms agrees with the diffeomorphic classification because of the topological invariance of Whitehead torsion. However, the topological classification of free finite group actions on spheres is only partially solved.

The first part is solved by finding all Π which have free unitary representations $\Pi \to U(n)$. Now, $U(n)$ embeds into $\mathrm{SO}(2n) \subset \mathrm{O}(2n)$ by the realification homomorphism

$$j(z_{r,s}) = \begin{bmatrix} x_{2r-1,2s-1} & -y_{2r-1,2s} \\ y_{2r,2s-1} & x_{2r,2s} \end{bmatrix},$$

a (2×2)-block in the $(2n \times 2n)$-matrix of $\mathrm{O}(2n)$, for each r, s entry with $z = x + iy$. If $\theta : \Pi \to U(n)$ is free (respectively, free and irreducible), $j \circ \theta$ is free (respectively, free and irreducible). Moreover, if $\varphi : \Pi \to \mathrm{O}(2n)$ is free, then there is a free $\theta : \Pi \to U(n)$ such that $j \circ \theta = \hat{\theta}$. $\hat{\theta}$ and φ are orthogonally equivalent. Unitarily equivalent θ's yield orthogonally equivalent $\hat{\theta}$'s. Therefore, no generality is lost in considering only free unitary representations when considering the topological types of odd dimensional spherical space forms. Metrically, however, distinct unitary representations may represent the same spherical space form.

For a $(2n-1)$-dimensional spherical space form with a free representation $\varphi : \mathit{\Pi} \to U(n)$, and the Hopf fibering $\Delta_n \to S^{2n-1} \to \mathbb{C}P_{n-1}$, we get the following embeddings.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle i_\ell} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & S^1 = \Delta_n & \longrightarrow & U(n) & \longrightarrow & PU(n) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \cap} & & \Big\downarrow{\scriptstyle \cap} & & \Big\downarrow{\scriptstyle \cap} & & \\
0 & \longrightarrow & \mathrm{M}(\mathbb{C}P_{n-1}, S^1) & \longrightarrow & \mathrm{TOP}_{S^1}(S^{2n-1}) & \longrightarrow & \mathrm{Aut}(S^1) \times \mathrm{TOP}(\mathbb{C}P_{n-1}) & &
\end{array}$$

This yields a Seifert construction for $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$ with uniformizing group $U(n)$. For an embedding of the top exact sequence into the bottom exact sequence, we get a Seifert construction into the universal uniformizing group $\mathrm{TOP}_{S^1}(S^{2n-1})$. If the image of this $\mathit{\Pi}$ can be conjugated in $\mathrm{TOP}_{S^1}(S^{2n-1})$ to lie in $U(n)$ and act freely on S^{2n-1}, then $\mathit{\Pi}\backslash S^{2n-1}$ is topologically conjugate to a spherical space form.

Exercise 4.6.4. Let $\mathrm{SU}(n) = U(n) \cap \mathrm{SL}(n, \mathbb{C})$. Then $\mathrm{SU}(n)$ is a maximal compact subgroup of $\mathrm{SL}(n, \mathbb{C})$. Show the center of $\mathrm{SU}(n)$ is $\mathbb{Z}_n$ generated by the diagonal matrix $e^{\frac{2\pi i}{n}} I_n$. Prove $\pi_1(\mathrm{PU}(n)) = \pi_1(\mathrm{PSU}(n)) = \mathbb{Z}_n$.

For example, PU(2) is SO(3). $\mathrm{PU}(n)$ is the maximal compact subgroup of the group of holomorphic automorphisms of $\mathbb{C}P_{n-1}$ and is the group of isometries of $\mathbb{C}P_{n-1}$ with respect to the natural Kähler metric on $\mathbb{C}P_{n-1}$. Thus $U(n)$ is a group of unitary isometries of S^{2n-1} which centralizes the principal S^1-action, and so $U(n) \subset \mathrm{TOP}_{S^1}(S^{2n-1})$.

CHAPTER 11

Applications

The injective Seifert Construction, which is a special embedding, $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$, of the group Π into $\mathrm{TOP}_G(G \times W)$ such that Π acts properly on $G \times W$, preserves some of the properties of both G and W on $\theta(\Pi)\backslash(G \times W)$. Furthermore, the action of Π on $G \times W$ *twists* the topology and geometry of G and W to create the orbit space $\theta(\Pi)\backslash(G \times W)$ in the same way that the group structures of Γ and Q *twists* to create the group Π. In other words, this algebraic twisting of Π makes the geometric twisting of the *bundle with singularities*

$$\Gamma\backslash G \to \theta(\Pi)\backslash(G \times W) \to Q\backslash W,$$

where the homogeneous space $\Gamma\backslash G$ is a typical fiber. In the several applications, we have included here these features seem especially prominent.

One of the important topological problems that has motivated the development of Seifert fiberings is the construction of closed aspherical manifolds realizing Poincaré duality groups Π of the form $1 \to \Gamma \to \Pi \to Q \to 1$. The Seifert Construction enables one to find explicit aspherical manifolds $M(\Pi)$ when Q acts on a contractible manifold W and Γ is a torsion-free lattice in a Lie group; see Section 11.1.

The rigidity of the Seifert Construction is important for classification problems. We exhibit, in Section 11.2, several instances where rigidity is useful for classification.

For a topological manifold, the homotopy classes of self-homotopy equivalences can be regarded as algebraic data. We show, in Section 11.3, how the Seifert Construction can often be used to lift finite subgroups of homotopy classes to an action on the manifold. The lifting problem consists of two stages. First, one has an abstract kernel $\psi : G \to \mathrm{Out}(\pi_1(M))$ that must be realized as a group extension. If this fails, no lifting of G is possible. In order for an extension to exist, a certain 3-dimensional cohomology class must be 0. When this fails and $\mathcal{Z}(\pi_1(M))$ is finitely generated free Abelian, we show in Theorem 11.3.30 that there is an *inflation*, $H \xrightarrow{\mathrm{inf}} G \xrightarrow{\psi} \mathrm{Out}(\pi_1(M))$ for which the obstruction for the existence of a group extension realizing the abstract kernel $\mathrm{inf} \circ \psi : H \to \mathrm{Out}(\pi_1(M))$ vanishes. Then the method of Theorem 7.3.2 used for Seifert Constructions can be invoked for completing the second stage for the group H.

The interaction between the fundamental group and geometry is especially strong for aspherical manifolds. A representation $\theta : \Gamma \to \mathrm{Aff}(\mathbb{R}^k)$ which yields a proper action with $\theta(\Gamma)\backslash\mathbb{R}^k$ compact is called an *affine structure* on Γ. Analogously, $\theta : \Gamma \to P(\mathbb{R}^k)$, where $P(\mathbb{R}^k)$ is the group of all polynomial diffeomorphisms of $\mathbb{R}^k$, is called a *polynomial structure* on Γ. The affine diffeomorphisms are polynomial diffeomorphisms of degree less than or equal to 1. Not all torsion-free polycyclic-by-finite groups admit an affine structure but, as sketched in Section 11.4, they do admit a polynomial structure.

The close connection between the Bieberbach theorems and the existence, uniqueness, and rigidity of the Seifert Construction is explored. The rigidity of homeomorphisms between infra-nilmanifolds is extended to continuous maps. This enables us to recapture some recent results of Nielsen fixed point theory on infra-nilmanifolds and infra-solvmanifolds; see Section 11.5.

As mentioned earlier, a torus action (T^k, X) is *homologically injective* if the evaluation map is injective on the first homology group. This is a much stronger concept than being injective on the fundamental group. Homologically injective actions are characterized by being splittable as $(T^k, X) = (T^k, T^k \times_\Delta Y)$, where Δ is a finite Abelian group acting diagonally and freely as translations on the first factor. Consequently, X, which fibers over $G\backslash X = \Delta\backslash Y$, also fibers without singularities over T^k/Δ with fiber Y. There is also induced a T^k-equivariant map $(T^k, X) \to (T^k, T^k/\Delta)$. In Section 11.6, we characterize this splitting and also extend the theorems to Seifert fiberings with cocompact lattices in simply connected nilpotent Lie groups. As one would expect, the strong condition leads to interesting examples in topology and geometry, and some of these are also described in this section. These splitting theorems provide analogue for lattices of type (S1) and (S2) to the splitting theorems in Chapter 9, Corollary 9.4.6.

If M is a closed aspherical manifold and $C = \mathcal{Z}(\pi_1(M))$, it is unknown, in general, if there exists an effective torus action on M for which $\mathrm{Im}(\mathrm{ev}_*^x(\pi_1(T^k))) = C$. Such an action, if it exists, is called a maximal torus action on M. In Section 11.7, we construct maximal torus actions for a large class of aspherical Seifert manifolds, including solvmanifolds and double coset spaces. In Section 11.8, we construct the analogue of maximal torus actions on spherical space forms.

This chapter consists of the following sections:

(1) Existence of closed $K(\mathit{\Pi}, 1)$-manifolds
(2) Rigidity for Seifert fiberings
(3) Lifting problem for homotopy classes of self-homotopy equivalences
(4) Polynomial structures for solvmanifolds
(5) Applications to fixed-point theory
(6) Homologically injective torus operations
(7) Maximal torus actions on solvmanifolds and double coset spaces
(8) Toral rank of spherical space forms

11.1. Existence of closed $K(\mathit{\Pi}, 1)$-manifolds

11.1.1. There are two difficult problems related to the title:

(1) *Which groups can be the fundamental group of a closed aspherical manifold?*
(2) *If $\mathit{\Pi}$ is the fundamental group of an aspherical manifold, can we give an actual explicit construction of an aspherical manifold for the group $\mathit{\Pi}$?*

There are some general criteria for the first problem, such as $\mathit{\Pi}$ must be finitely presented, have finite cohomological dimension, and satisfy the Poincaré duality in that dimension. The Seifert Construction gives answers to both questions for a large class of groups $\mathit{\Pi}$. The idea is that if $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$ is a torsion-free extension where Γ is the fundamental group of a closed aspherical manifold and

Q is a proper action on a contractible manifold W with compact quotient, then Π should be the fundamental group of a closed aspherical manifold. We have the following

THEOREM 11.1.2. *Let* Γ *be a cocompact special lattice in* G*; see Section* 7.3.1*, and let* $\rho : Q \to \mathrm{TOP}(W)$ *be a proper action of a discrete group on a contractible manifold* W *with compact quotient. If* $1 \to \Gamma \to \Pi \to Q \to 1$ *is a torsion-free extension of* Γ *by* Q*, then for any Seifert Construction* $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$*,*

(1) $M(\theta(\Pi)) = \theta(\Pi)\backslash(G \times W)$ *is a closed aspherical manifold if* Γ *is of type* (S3)*, and*
(2) $M(\theta(\Pi)) = \theta(\Pi)\backslash((G/K) \times W)$*, where* K *is a maximal compact subgroup of* G*, is a closed aspherical manifold if* Γ *is of type* (S4)*.*

PROOF. For each extension $1 \to \Gamma \to \Pi \to Q \to 1$, there exists a homomorphism θ of Π into $\mathrm{TOP}_G(G\times W)$, by Theorem 7.3.2 for G of type (S3), (respectively, $\mathrm{TOP}_{(G,K)}(G/K \times W)$ for G of type (S4)).

Since Π acts properly on $G \times W$ (or $G/K \times W$), which is contractible and is torsion free, Π must act freely since any isotropy subgroup must be finite. We need only to check that θ is injective. Suppose Q_0 is the kernel of $\psi \times \rho : Q \to \mathrm{Out}(G) \times \mathrm{TOP}(W)$. Then Q_0 is finite since the Q-action on W is proper. Let $1 \to \Gamma \to \Pi_0 \to Q_0 \to 1$ be the pullback via $Q_0 \subset Q$. By Corollary 7.7.4, θ is injective if and only if Π_0 is torsion free. But the group Π_0 is torsion free since Π is assumed to be torsion free. □

REMARK 11.1.3. (1) If W is a smooth contractible manifold and $\rho : Q \to \mathrm{Diff}(W)$, then the construction can be done smoothly and $M(\theta(\Pi))$ is smooth.

(2) If ρ_1 and ρ_2 are *rigidly related* (i.e., there exists $h \in \mathrm{TOP}(W)$ for which $\rho_2 = \mu(h) \circ \rho_1$) and Γ is characteristic in Π, then $M(\theta_1(\Pi))$ and $M(\theta_2(\Pi))$ are homeomorphic via a Seifert automorphism; see Remark 7.4.4. Moreover, if we fix ℓ and ρ, then the constructed $M(\theta_i(\Pi))$ are all *strictly equivalent*; see Section 7.4, especially Subsection 7.4.2.

(3) When $W = \{p\}$ is a point (a 0-dimensional contractible manifold), then Q must be finite for Q to act properly, and every $\rho : Q \to \mathrm{TOP}(\{p\})$ is rigidly related. The closed aspherical manifolds constructed are infra-G-manifolds; cf. Example 7.4.5.

(4) One important application of these constructions is that they provide model aspherical manifolds with often strong geometric properties. If one wants to study the famous conjecture that two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic via the methods of controlled surgery, then the constructed aspherical Seifert manifolds are excellent model manifolds.

This point of view has been taken by H. Rees in his thesis [**Ree83**]; cf. [**HR83**], [**FH83**], and A. Nicas and C. Stark in [**NS85**]. A consequence of the last reference is that if M^{n+2} is a closed aspherical manifold admitting a codimension-two torus action and $f : N \to M$ is a homotopy equivalence with N a closed manifold, then f is homotopic to a homeomorphism provided that $n \neq 3$ or 4.

(5) In order for $M(\theta(\Pi))$ to be closed aspherical in Theorems 11.1.2 and 11.1.4, W needs only be a contractible manifold factor rather than an actual manifold. By a *manifold factor*, we mean a space W such that $W \times \mathbb{R}^1$ is homeomorphic to a topological manifold.

If W is a noncontractible manifold factor, the Seifert construction still produces a topological manifold $M(\theta(\Pi))$ provided Π acts freely on $G \times W$ or $G/K \times W$. The group Π acts freely if and only if Π_w in the extension

$$1 \to \Gamma \to \Pi_w \to Q_w \to 1$$

is torsion free, for each $w \in W$. This extension is the pullback from $1 \to \Gamma \to \Pi \to Q \to 1$, induced by the inclusion of Q_w into Q.

The above procedure can be extended for even more general extensions. As an example,

THEOREM 11.1.4. *Let Π be a torsion-free extension of a virtually poly-$\mathbb{Z}$ group Γ by Q, where Q acts on a contractible manifold W properly with compact quotient. Then there exists a closed $K(\Pi, 1)$-manifold.*

PROOF. A torsion-free virtually poly-$\mathbb{Z}$ group Γ has a unique maximal normal nilpotent subgroup Δ, which is called the *discrete nilradical* of Γ; see Subsection 8.4.6. Then the quotient Γ/Δ is virtually free Abelian of finite rank. Furthermore, since Δ is a characteristic subgroup of Γ, it is normal in Π. Consider the commuting diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta & \xrightarrow{=} & \Delta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
1 & \longrightarrow & \Gamma/\Delta & \longrightarrow & \Pi/\Delta & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Since Γ/Δ is virtually free Abelian of finite rank (say, of s), it contains a characteristic subgroup $\mathbb{Z}^s$. (This can be seen as follows: Let $\mathbb{Z}^s$ be a normal subgroup of Γ/Δ of finite index q. Let $\Gamma(q)$ be the subgroup of Γ/Δ generated by $\{x^q : x \in \Gamma/\Delta\}$. Then clearly, $\Gamma(q)$ is a free Abelian group and is a characteristic subgroup of Γ/Δ.) Let $Q' = (\Pi/\Delta)/\mathbb{Z}^s$. Then the natural projection $Q' \longrightarrow Q$ has a finite kernel. Therefore, if we let Q' act on W via Q, the action will still be proper.

One can do a Seifert fiber space construction with the exact sequence

$$1 \longrightarrow \mathbb{Z}^s \longrightarrow \Pi/\Delta \longrightarrow Q' \longrightarrow 1,$$

which yields a proper action of Π/Δ on $\mathbb{R}^s \times W$ with compact quotient. Using this action of Π/Δ on $\mathbb{R}^s \times W$, one does a Seifert fiber space construction with the exact sequence

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow \Pi/\Delta \longrightarrow 1.$$

This gives rise to a proper action of Π on $N \times (\mathbb{R}^s \times W)$, where N is the unique simply connected nilpotent Lie group containing Δ as a lattice, with compact quotient.

If the space W is smooth, and the action of Q on W is smooth, both constructions can be done smoothly so that the proper action of Π on $N \times (\mathbb{R}^s \times W)$ is smooth.

In any case, since the group Π is torsion free, the resulting action of Π on $N \times (\mathbb{R}^s \times W)$ is free. Consequently, we get a closed $K(\Pi,1)$-manifold

$$M = \Pi\backslash(N \times \mathbb{R}^s \times W).$$

It has a Seifert fiber structure

$$F \longrightarrow M \longrightarrow Q\backslash W,$$

where the typical fiber F itself has a Seifert fiber structure

$$\Delta\backslash N \longrightarrow F \longrightarrow T^s = \mathbb{Z}^s\backslash\mathbb{R}^s.$$

In fact, since the action of the characteristic subgroup $\mathbb{Z}^s$ on $\mathbb{R}^s$ is free, F is a genuine fiber bundle, with fiber a nilmanifold $\Delta\backslash N$ over the base torus T^s. □

The space W does not have to be aspherical. As long as the action of discrete Q is proper, this iterated construction works. The resulting action of Π is free if and only if the preimage of Q_w (the isotropy of the Q-action at $w \in W$) in Π is torsion free. In this case, the space $\Pi\backslash(G \times W)$ will not be aspherical; see Theorem 7.3.2 and cf. also Theorems 9.5.6 and 9.5.7.

11.1.5. In a slightly different vein, Frank Johnson [**Joh78**] has defined the notion of a poly $\mathbb{L}_+$-group:

• A group is in $\mathbb{L}$ if it is a discrete uniform subgroup in a connected finite covering group of a semisimple Lie group of type (S4).

• A group is in $\mathbb{L}_+$ if either it is in $\mathbb{L}$ or it is virtually poly-$\mathbb{Z}$. A *poly-$\mathbb{L}_+$-group* Π is a group having a filtration $1 = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \cdots \subset \Pi_k = \Pi$, where Π_i is normal in Π_{i+1} and $\Pi_{i+1}/\Pi_i \in \mathbb{L}_+$.

Johnson made two other assumptions about the group Π. In [**LR84**, section 4.6], these assumptions are shown to be redundant, and a proof of the Johnson theorem is given there using the techniques of Seifert fiberings.

THEOREM 11.1.6. *If Π is a torsion-free poly-$\mathbb{L}_+$-group, then there exists a closed smooth $K(\Pi,1)$-manifold.*

EXAMPLE 11.1.7. (Codimension-2 injective Seifert fiberings with the torus T^k as typical fiber). As an illustration of the foregoing sections, let Q act properly and effectively on $\mathbb{R}^2 = W$ with compact quotient. Then Q can be topologically conjugated into the group of Euclidean motions if Q is solvable and to the group of hyperbolic isometries if Q is not solvable. For each extension $1 \to \mathbb{Z}^k \to \Pi \to Q \to 1$, there is a Seifert Construction:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi\times\rho} & & \\
1 & \longrightarrow & \mathrm{M}(\mathbb{R}^2,\mathbb{R}^k) & \longrightarrow & \mathrm{TOP}_{\mathbb{R}^k}(\mathbb{R}^k\times\mathbb{R}^2) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^k)\times\mathrm{TOP}(\mathbb{R}^2) & \longrightarrow & 1.
\end{array}$$

Let $M(\Pi)$ denote the space $\theta(\Pi)\backslash\mathbb{R}^k \times \mathbb{R}^2$. Since ρ is injective, θ is injective. The mapping $M(\Pi) \to Q\backslash\mathbb{R}^2$ is a Seifert fibering with typical (and also regular) fiber $T^k = \mathbb{Z}^k\backslash\mathbb{R}^k$. The base of the fibering $B = Q\backslash\mathbb{R}^2$ is a 2-dimensional Euclidean or hyperbolic orbifold. If $\varphi : Q \to \mathrm{Aut}(\mathbb{R}^k)$ is trivial, the fibering is the orbit mapping of an injective T^k-action on $M(\Pi)$. Furthermore, Π is torsion free if and only if Π acts freely on $\mathbb{R}^k \times \mathbb{R}^2$, in which case $M(\Pi)$ is an aspherical manifold. If Π is not torsion free, then $M(\Pi)$ is an orbifold.

The group Q contains no nontrivial normal Abelian subgroups when Q is not solvable. Therefore, $\mathbb{Z}^k \subset \Pi$ is the characteristic maximal normal Abelian subgroup of Π. Since Q is topologically rigid, the Seifert Construction, θ, is unique and rigid in the sense of Theorem 7.3.2(3). Thus, any isomorphism $\theta(\Pi_1) \to \theta(\Pi_2)$ can be realized by a Seifert isomorphism. That is, $M(\Pi_1)$ is homeomorphic to $M(\Pi_2)$ via a Seifert fiber preserving homeomorphism induced by a conjugation of $\theta(\Pi_1)$ to $\theta(\Pi_2)$ by an element of $\mathrm{TOP}_{\mathbb{R}^k}(\mathbb{R}^k \times \mathbb{R}^2)$.

If the fibering is the orbit mapping of a T^k-action (i.e., $\varphi : Q \to \mathrm{Aut}(\mathbb{R}^k)$ is trivial), then the Seifert isomorphism is a weak T^k-equivalence. Similar results hold for Q solvable and centerless because $\mathrm{ev}_*^x(\pi_1(T^k)) = \mathbb{Z}^k$ is the center and is a characteristic subgroup of $\pi_1(M)$.

11.1.8. In Example 11.1.7, if $k = 1$ and Π is torsion free, the Seifert Construction will produce all the possible closed aspherical Seifert 3-manifolds with typical fiber S^1. We call all of the Seifert 3-manifolds listed by Seifert in [**Sei33**], (see [**ST80**] for a translation into English), the *classical* Seifert 3-manifolds. For Seifert, each fiber is an S^1 with a tubular neighborhood a Seifert fibered solid torus $(S^1, S^1 \times_{\mathbb{Z}_n} D^2)$ where $\mathbb{Z}_n$ acts diagonally by translating freely on the first factor and by rotation on the second factor. The base is a closed surface with a finite number of *cone* singularities. Seifert does not consider *special exceptional* fibers which are S^1-fibers with a tubular neighborhood $(S^1, S^1 \times_{\mathbb{Z}_2} D^2)$, which is a solid Klein bottle (that is, a Möbius band $\times I$), where $\mathbb{Z}_2$ acts freely on S^1 and by reflection on D^2. With special exceptional fibers, the base is a surface with boundary, with possible cone singularities in the interior of the base, where the inverse image of the boundary points are all special exceptional fibers. Any Seifert 3-manifold, not necessarily classical, having a normal $\mathbb{Z}$ in its fundamental group Π and quotient group $G = \Pi/\mathbb{Z}$ infinite, will be aspherical.

A thorough investigation of Seifert 3-manifolds can be found in Chapters 14 and 15. There the classification in terms of explicit numerical invariants is exploited to derive significant connections with other lower dimensional phenomena.

11.1.9. In line with the methods of this section, we state two problems where a complete answer is unknown.

(1). If M is a closed aspherical manifold with $1 \to \Gamma \to \pi_1(M) \to Q \to 1$ exact and Γ is a cocompact lattice in a group of type (S3) or (S4), does there exist a contractible manifold W on which Q acts properly with compact quotient?

(2) If Q is a discrete group which acts properly and cocompactly on a contractible manifold, does Q have a torsion-free subgroup of finite index?

11.1.10. A large and important class of closed aspherical manifolds has been constructed, using Coxeter groups, by M. Davis [**Dav83**]; see also Remark 3.1.20. They exhibit properties quite different from the aspherical manifolds constructed with the aid of Lie groups. Some of these manifolds are not smoothable nor even admit a PL structure, and their universal coverings are not homeomorphic to Euclidean space. Davis's methods, when combined with the methods of this section, lead to an even larger class of closed aspherical manifolds.

11.2. Rigidity of Seifert fibering

11.2.1. The observant reader will have noticed that in Theorem 11.1.2, we did not require that $\rho : Q \to \mathrm{TOP}(W)$ be injective as we did in our illustrations with

$W = \mathbb{R}^2$. For $k = 1$, no torsion-free Π occurs that was not already detected when ρ is injective. This is not the case when $k > 1$. In general (see Subsection 7.7.1), for θ to be injective and $\Pi \cap \mathbb{R}^k = \mathbb{Z}^k$, we can assume, without loss of generality, that $\varphi \times \rho$ is injective.

Consider an example in dimension 4: for the extension $1 \to \mathbb{Z}^2 \to \Pi \to \mathbb{Z}_2 \times Q \to 1$ with Q a surface group, where $\varphi \times \rho : \mathbb{Z}_2 \times Q \to \mathrm{Aut}(\mathbb{Z}^2) \times \mathrm{TOP}(\mathbb{R}^2)$, we assume φ is injective on $\mathbb{Z}_2$ and trivial on Q, and ρ is trivial on $\mathbb{Z}_2$ and injective on Q. (We can choose Π so that $\Pi = \pi_1(\text{Klein bottle}) \times Q$ and $\theta(\Pi)\backslash\mathbb{R}^4 =$ Klein bottle $\times\, Q\backslash\mathbb{R}^2$. The typical fiber is the 2-torus, but each fiber is a regular fiber, the Klein bottle.)

Let Γ, Π, Q be as in Theorem 11.1.2. Let $\Pi_L \subset \mathrm{Aff}(G)$ be an extension of a lattice $\Gamma = \Pi_L \cap \ell(G)$ in the completely solvable Lie group G by a finite group L. Let $1 \to \Pi_L \to \Pi \xrightarrow{j'} Q' \to 1$ be an extension where $\rho' : Q' \to \mathrm{TOP}(W)$ is an effective proper action.

PROPOSITION 11.2.2. *There exists an injection* $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$ *such that* $\theta|_{\Pi_L} : \Pi_L \to \mathrm{Aff}(G) \subset \mathrm{TOP}_G(G \times W)$ *and* $\Pi_L\backslash G$ *is a regular fiber for the Seifert Construction* θ.

PROOF. The group Γ is a characteristic subgroup of Π_L and so $1 \to \Gamma \to \Pi \to \Pi/\Gamma = Q \to 1$ is exact. The natural map $Q = \Pi/\Gamma \to \Pi/\Pi_L = Q' \subset \mathrm{TOP}(W)$ has kernel L. We have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Gamma & = & \Gamma & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_L & \longrightarrow & \Pi & \longrightarrow & Q' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & L & \longrightarrow & \Pi/\Gamma = Q & \longrightarrow & Q' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

We use the middle vertical sequence to make the Seifert Construction. □

The proposition shows that replacing the lattice Γ by the larger Π_L does not lead to any new Seifert fiberings.

11.2.3. Let $1 \to \Gamma \to \Pi \to Q \to 1$ be an extension of discrete groups where Γ is isomorphic to a special lattice in a connected Lie group G of type (S3); see Subsection 7.3.1. Let Q act properly on a space W via the homomorphism ρ, and let ℓ be an isomorphism of Γ onto a lattice of G. Then the existence part of Theorem 7.3.2 asserts that there exists a homomorphism $\theta : \Pi \to \mathrm{TOP}_G(G \times W)$ such that $\theta|_\Gamma = \ell$ and we have the commutative diagram (7.3.1). This yields a Seifert fibering $M = \theta(\Pi)\backslash(G \times W) \to Q\backslash W = B$ with typical fiber $\Gamma\backslash G$. The theorem also says the construction is unique. That is, congruent extensions are conjugate in $\mathrm{M}(W, G) \subset \mathrm{TOP}_G(G \times W)$, Theorem 7.3.2(2); i.e, M_1 and M_2 are strictly equivalent. Furthermore, for $i = 1, 2$, if there are extensions fitting into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_i & \longrightarrow & \mathit{\Pi}_i & \longrightarrow & Q_i & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell_i} & & \downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \varphi_i\times\rho_i} & & \\ 1 & \longrightarrow & \ell(G)\rtimes \mathrm{Inn}(G) & \longrightarrow & \mathrm{TOP}_G(G\times W) & \longrightarrow & \mathrm{Out}(G)\times \mathrm{TOP}(W) & \longrightarrow & 1 \end{array}$$

and an isomorphism $\eta : \mathit{\Pi}_1 \to \mathit{\Pi}_2$ so that $\eta|_{\Gamma_1} : \Gamma_1 \to \Gamma_2$ is an isomorphism, inducing an isomorphism $\overline{\eta} : Q_1 \to Q_2$, and a homeomorphism $h \in \mathrm{TOP}(W)$ so that $\mu(h) \circ \rho_1 = \rho_2 \circ \overline{\eta}$, then there exists a conjugation $\mu(\lambda, a, h)$ so that $\theta_2 \circ \eta = \mu(\lambda, a, h) \circ \theta_1$.

This is rigidity of the Seifert Construction and implies that there is induced a Seifert isomorphism $M_1 \to M_2$. Obviously, in classifying Seifert fiberings, rigidity can play an important role and so we shall give some practical conditions that allow us to verify that the hypothesis of rigidity holds.

For our problem it suffices to check when the θ_i are injective. Corollary 7.7.4 tells how to recognize the kernels of θ_i. So we assume the θ_i are injective and without loss of generality, $\varphi_i \times \rho_i$ are injective. This implies $\theta_i(\Gamma_i) = \theta(\mathit{\Pi}_i) \cap \ell(G)$.

THEOREM 11.2.4 (cf. [**Ray79**, Section 3]). *Let* $1 \to \Gamma_i \to \mathit{\Pi}_i \to Q_i \to 1$ $(i = 1, 2)$*, be extensions of discrete groups where* Γ_i *are isomorphic to special lattices in a connected Lie group* G *of type* (S3)*. Let* Q_i *act properly on a space* W *via the homomorphisms* ρ_i *with* $\rho_i(Q)\backslash W$ *compact, and* ℓ_i *isomorphisms of* Γ_i *onto lattices of* G*. Let* $\theta_i : \mathit{\Pi}_i \to \mathrm{TOP}_G(G \times W)$ *be Seifert Constructions. Suppose one of the following conditions on* (Q_i, W) *holds:*

(i) $W = \mathbb{R}^2$ *and* $\rho_i(Q_i)$ *is not solvable.*
(ii) W *is a Riemannian symmetric space of noncompact type (Section* 9.4*) with no compact factors and no* 1*-dimensional factors,* $\rho_i(Q_i) \subset \mathrm{Isom}(W)$*,* $Q_i\backslash W$ *compact. If* W *has* 2*-dimensional factors, then the projection of* $\rho_i(Q_i)$ *to those factors are dense.*

Then any isomorphism $\eta : \mathit{\Pi}_1 \to \mathit{\Pi}_2$ *induces a Seifert isomorphism*

$$\theta_1(\mathit{\Pi}_1)\backslash(G \times W) \to \theta_2(\mathit{\Pi}_2)\backslash(G \times W).$$

For the proof of the theorem, we need to show that the isomorphism η restricts to an isomorphism of Γ_1 onto Γ_2. This will induce an isomorphism of Q_1 to Q_2. Furthermore, we want this isomorphism to induce an isomorphism of $\rho_1(Q_1)$ to $\rho_2(Q_2)$. We then check that $\rho_1(Q_1)$ is conjugate to $\rho_2(Q_2)$ in $\mathrm{TOP}(W)$. Actually we prove the conjugation to be in $\mathrm{Isom}(W)$.

In (i), if $\rho_i(Q_i)$ is not solvable, then it is topologically hyperbolic. That is, if $\overline{\eta} : \rho_1(Q_1) \to \rho_2(Q_2)$ is an isomorphism, there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ so that $\rho_2(\overline{\eta}(q)) = h \circ \rho_1(q) \circ h^{-1}$, by a theorem of Macbeath [**Mac67**, Theorem 3]. Furthermore, $\rho_i(Q_i)$ can be conjugated, within $\mathrm{TOP}(\mathbb{R}^2)$, into the subgroup of full isometries of the hyperbolic plane, $\mathrm{PSL}(2, \mathbb{R}) \rtimes \mathbb{Z}_2$. Therefore, according to Lemmas 11.2.5 and 11.2.6, $\rho_i(Q_i)$ contains no normal finite subgroups and no normal solvable subgroups other than the trivial subgroup. This latter fact will be used in showing that η restricts to an isomorphism of Γ_1 to Γ_2.

A symmetric space as in (ii) is a homogeneous space G/K, where G is a semisimple Lie group with no compact factors in adjoint form and K is a maximal compact subgroup. This is exploited to show Γ_1 is mapped isomorphically onto Γ_2, and $\rho_1(Q_1)$ is conjugate to $\rho_2(Q_2)$ in $\mathrm{Isom}(G/K)$.

We shall use the following two lemmas.

LEMMA 11.2.5. *Let Q be a discrete group acting effectively and properly on a contractible manifold W so that $Q\backslash W$ is compact. Assume that Q contains a torsion-free subgroup of finite index. Then a normal finite subgroup of Q is the trivial group.*

PROOF. Let H be the intersection of the torsion-free subgroup with all of its conjugates. Then H is torsion free and normal in Q. Let F be a normal finite subgroup of Q. Then, $F \cap H = \{1\}$. Let $F \cdot H$ be the group generated by F and H. Then $(F \cdot H)/H = F/(F \cap H) = F$. Therefore, F splits back to $F \cdot H$. Then as F is normal, and the group $F \cdot H$ is congruent to $F \times H$. $F \times H$ acts effectively on W which is the universal cover of the closed manifold $H\backslash W$. Since $H\backslash W$ is an admissible manifold (Subsection 3.2.1), F must be trivial. □

LEMMA 11.2.6. *Let Γ be a cocompact lattice in a semisimple Lie group G without compact factors and in adjoint form. Then any nontrivial normal subgroup N of Γ is not solvable. In fact, no subgroup of N of finite index is solvable.*

PROOF. If N is finite, then it is central in G. Therefore, N is $\{1\}$ because G is in adjoint form. The lattice Γ is dense in the Zariski topology by Borel's density theorem. So the Zariski closure of N is a normal, and hence a semisimple subgroup of G. If N were solvable, its Zariski closure would be solvable. This also implies that no subgroup of N of finite index in N can be solvable. □

A symmetric space W of noncompact type with no compact and no 1-dimensional factors is the quotient space G/K of a connected semisimple Lie group G in adjoint form and without compact factors, where K is a maximal compact subgroup. A cocompact lattice Γ in G acts on G/K as a group of isometries. The full group of isometries on G/K is $\mathrm{Aut}(G)$ where the connected component of the identity is G itself and $W = G/K$ can be identified with $\mathrm{Aut}(G)/\overline{K} = \overline{G}/\overline{K}$, $\overline{K}$ being the maximal compact subgroup of $\overline{G} = \mathrm{Aut}(G)$. As seen in Section 9.4, there is an exact sequence of Lie groups $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$, and $\mathrm{Out}(G)$ is finite.

If $\rho(Q)$ is now a discrete, cocompact subgroup of isometries of W, then $\rho(Q)$ is a cocompact subgroup of $\overline{G}$. $\overline{G}$ has a faithful representation into $\mathrm{GL}(n, \mathbb{R})$ for some n. (Actually we need only that $\rho(Q) \cap G$ has a characteristic torsion free subgroup of finite index to get a torsion-free normal subgroup of finite index in $\rho(Q)$.) Therefore, $\rho(Q)$ has a torsion-free subgroup of finite index by Selberg's Lemma in Subsection 6.1.8. Consequently, Lemma 11.2.5 applies and $\rho(Q)$ contains no nontrivial, finite normal subgroups. For, if H is a normal solvable subgroup of $\rho(Q)$, then $H \cap (\rho(Q) \cap G)$ is solvable in $\rho(Q) \cap G$. By Lemma 11.2.6, $H \cap (\rho(Q) \cap G)$ is trivial, hence H is finite and, consequently by Lemma 11.2.5, it is trivial. Therefore, $\rho(Q)$ contains no normal nontrivial solvable subgroups.

In Remark 9.4.9, we explained the connections between the earlier part of section 9.4 and [**RW77**] where it is observed that the Mostow rigidity theorem in G extends to lattices in the larger group $\mathrm{Aut}(G)$.

Let Π be a lattice in $\mathrm{Aut}(G)$, with G of type S(4). Then $\Pi \cap G = \Gamma$ is a lattice in G. We may write Π as an extension of Γ by the finite group F. We may assume that $\psi : F \to \mathrm{Out}(\Gamma)$ injects. For, if not, let L be the kernel of ψ. Then $\Gamma \times L$ is in Π. This would lead to a contradiction by choosing a torsion-free sublattice Γ' in Γ and we would get an effective action of $\Gamma' \times L$ on $\Gamma' \backslash G/K$'s universal covering (K is maximal compact in G).

The homomorphism $\psi : F \to \mathrm{Out}(\Gamma)$ with UAEP of (Γ, G) induces a homomorphism $\tilde{\psi} : F \to \mathrm{Out}(G)$. We may also assume that $\tilde{\psi}$ is injective. For otherwise, we would get an extension, Π', of Γ by a subgroup of F, with $\Gamma \subset \Pi' \subset G \cap \Pi$.

Now we apply Theorem 9.4.5 with W a point. We have this embedding of Π into $\mathrm{Aut}(G) = \overline{\mathrm{Aff}}(G, K)$ which carries Γ into $\mathrm{Inn}(G) = \overline{\mathrm{Aff}}_0(G, K)$ (this embedding forces $F \to \mathrm{Aut}(G)$ to be injective).

LEMMA 11.2.7. *Suppose G is centerless and has* ULIEP*. Then, any finite extension $\bar{G} \subset \mathrm{Aut}(G)$ of G has* ULIEP*.*

PROOF. Let $Q_1, Q_2 \subset \bar{G}$ be cocompact discrete subgroups, and let $\theta : Q_1 \to Q_2$ be an isomorphism. Since $\bar{G}/G$ is finite, the images $Q_i \to \bar{G} \to \bar{G}/G$ are finite. Let m be the product of the orders of the images of Q_i's in $\bar{G}/G$. Let

$$\Gamma_1 = \langle x^m : \ x \in Q_1 \rangle,$$
$$\Gamma_2 = \langle y^m : \ y \in Q_2 \rangle.$$

Then Γ_i is a characteristic subgroup of Q_i so that the isomorphism θ maps Γ_1 onto Γ_2 isomorphically. Clearly, Γ_i's are lattices of G by the choice of m. Then ULIEP of G yields an automorphism $\omega : G \to G$ extending $\theta : \Gamma_1 \to \Gamma_2$. Now $\omega \in \mathrm{Aut}(G)$ can be interpreted in two different ways as an automorphism of G. The first is as an automorphism of G as it was. The second way is via conjugation in $\mathrm{Aut}(G)$ as $\mu(\omega)$ is conjugation by ω. With $g \mapsto \mu(g)$ via $G \subset \mathrm{Aut}(G)$, we have

$$\mu(\omega(g)) = \omega \circ \mu(g) \circ \omega^{-1}$$

because

$$\begin{aligned}(\omega \circ \mu(g) \circ \omega^{-1})(x) &= \omega\left(\mu(g)(\omega^{-1}(x))\right) \\ &= \omega(g \cdot \omega^{-1}(x) \cdot g^{-1}) \\ &= \omega(g) \cdot x \cdot \omega(g)^{-1} \\ &= \mu(\omega(g))(x).\end{aligned}$$

Suppressing all the μ's, we have

$$\omega(g) = \omega \circ g \circ \omega^{-1}$$

for $\omega \in \mathrm{Aut}(G)$ and $g \in G$. The second interpretation, ω as $\mu(\omega)$, enables us to look at $\mu(\omega)$ as an automorphism of $\mathrm{Aut}(G)$ (not just as a automorphism of G). Let $Q_3 = \mu(\omega^{-1})(Q_2) \subset \mathrm{Aut}(G)$, and let

$$\theta' = \mu(\omega^{-1}) \circ \theta : \ Q_1 \to Q_2 \to Q_3.$$

Notice that Q_3 may not be in $\bar{G}$, but it is in $\mathrm{Aut}(G)$. In any case, $\theta' : \ Q_1 \to Q_3$ is an isomorphism which is the identity map on $Q_1 \cap G$. (In fact, it was the identity map on Γ_1, but the ULIEP implies that it is the identity on the bigger group $Q_1 \cap G$. Even more is true. Clearly, $\theta'|_{Q_1 \cap G}$ extends to a unique automorphism of G, which is the identity map on G.)

We have two groups $Q_1, Q_3 \subset \mathrm{Aut}(G)$ and an isomorphism between them such that $Q_1 \cap G = Q_3 \cap G$ and $\theta'|_{Q_1 \cap G}$ is the identity map. We claim that $Q_1 = Q_3$

and θ' is the identity map. Suppose there is $\alpha \in Q_1$ such that $\theta'(\alpha) \neq \alpha$. Then there exists $g \in G$ such that $\theta'(\alpha)(g) \neq \alpha(g)$. However,

$$\begin{aligned} \theta'(\alpha)(g) &= \theta'(\alpha) \cdot g \cdot \theta'(\alpha)^{-1} \\ &= \theta'(\alpha) \cdot \theta'(g) \cdot \theta'(\alpha)^{-1} \quad (\text{since } \theta'|_G \text{ is identity}) \\ &= \theta'(\alpha g \alpha^{-1}), \\ \alpha(g) &= \alpha g \alpha^{-1}. \end{aligned}$$

But, since $\alpha g \alpha^{-1} \in G$, we have $\theta'(\alpha g \alpha^{-1}) = \alpha g \alpha^{-1}$. This implies $\theta'(\alpha)(g) = \alpha(g)$, a contradiction. □

The argument proves also the following

COROLLARY 11.2.8. *Suppose G is centerless, Γ a lattice of G, and (Γ, G) has* UAEP. *Then, for any finite extension $\bar{G} \subset \mathrm{Aut}(G)$ of G and a lattice $\bar{\Gamma}$ of $\bar{G}$, $(\bar{\Gamma}, \bar{G})$ has* UAEP.

COROLLARY 11.2.9. *If $\rho(Q_1)$ and $\rho(Q_2)$ are groups as in Theorem* 11.2.4 *and are isomorphic and torsion free, then the locally symmetric Riemannian manifolds $\rho(Q_i)\backslash G/K$ are isometric.*

Thus, if $\overline{\eta} : \rho_1(Q_1) \to \rho_2(Q_2)$ is an isomorphism between cocompact lattices in $\mathrm{Aut}(G)$, then there exists an automorphism $\widehat{\eta} : \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ such that $\widehat{\eta}|_{Q_1} = \overline{\eta}$. This allows us to conclude that if $\overline{\eta} : \rho_1(Q_1) \to \rho_2(Q_2)$ is an isomorphism of discrete groups of isometries of W with $\rho_i(Q_i)\backslash W$ compact, then there is an isometry $h : G/K \to G/K$ such that $h \circ \rho_1 \circ h^{-1} = \rho_2 \circ \overline{\eta}$ provided W has no 1-dimensional and no compact factors and any projection to a 2-dimensional factor is dense.

All that remains in proving our theorem is to show that η restricts to an isomorphism between Γ_1 and Γ_2 and induces an isomorphism of $\rho_1(Q_1)$ to $\rho_2(Q_2)$. So, let $L_i = \ker(\rho_i) \subset Q_i$. Then L_i is a finite normal subgroup of Q_i. Let $1 \to \Gamma_i \to \Pi_{L_i} \to L_i \to 1$ be the pullback of $1 \to \Gamma_i \to \Pi_i \to Q_i \to 1$ via $L_i \hookrightarrow Q_i$. Since $\rho_2(\eta(\Gamma_1))$ is a normal subgroup of $\rho_2(Q_2)$, and $\rho_2(Q_2)$ has no normal solvable subgroup, $\rho_2(\eta(\Gamma_1))$ must be trivial. That is, $\eta(\Gamma_1) \subset \Pi_{L_2}$. Now Γ_1 has finite index in Π_{L_1}. Therefore, $\rho_2(\eta(\Pi_{L_1}))$ is a finite normal subgroup of $\rho_2(Q_2)$, which must be trivial again. Thus, we have $\eta(\Pi_{L_1}) \subset \Pi_{L_2}$. By symmetry,

$$\eta(\Pi_{L_1}) = \Pi_{L_2}.$$

Since $\varphi_i \times \rho_i$ is injective, $\varphi_1 : Q_1 \to \mathrm{Out}(\Gamma_1) \subset \mathrm{Out}(G)$ is injective on L_1. Then by Theorem 8.4.3 and its proof (assuming G is of type (S3)), we know that Γ_1 maps onto Γ_2 isomorphically by φ. Consequently, η induces an isomorphism $\rho_1(Q_1) \to \rho_2(Q_2)$. Since η maps Γ_1 onto Γ_2 isomorphically, we can apply the rigidity part of Theorem 7.3.2 to complete the proof of Theorem 11.2.4.

REMARK 11.2.10. (1) The theorem with $\Gamma_1 \cong \Gamma_2 \cong \mathbb{Z}^k$ in (i) and (ii) is a far-reaching extension of Corollary 14.12.5. This classification is not only for manifolds (Π_i's are torsion free) but also for all the possible orbifolds (Q, of course, being topologically hyperbolic). With $k = 2$ and Π_i torsion free, these 4-dimensional Seifert manifolds include all the Kodaira elliptic surfaces with only multiple fibers as singularities and whose fundamental groups are nonsolvable. In [**OR70b**], explicit canonical normal forms of the Seifert fiberings which arise as orbit mappings of T^2-actions on 4-manifolds are given. The descriptions in terms of orbit invariants yield an equivariant classification in the spirit of Chapter 14. Presentations of torsion-free Π when $k \geq 2$, $\varphi : Q \to \mathrm{Aut}(\mathbb{Z}^k)$, can be found in [**Zie69**]. However, in this case, it is not possible to find an algorithm that decides, in a finite number of steps, whether two presentations of the fundamental groups for these Seifert manifolds are isomorphic if $k > 2$; see [**Zim85**].

(2) For $W = \mathbb{R}^2$, $Q \subset \mathrm{TOP}(\mathbb{R}^2)$, topologically hyperbolic, and $\Gamma = \mathbb{Z}^k$ with $\varphi : Q \to \mathrm{Aut}(\mathbb{R}^k)$ trivial (i.e., admitting an injective T^k-action), Nicas and Stark [**NS85**] show that a homotopy equivalence from $M = \theta(\Pi)\backslash(\mathbb{R}^k \times \mathbb{R}^2)$, ($\Pi$ torsion free) to a closed manifold N is homotopic to a homeomorphism provided $k \geq 3$. They use M as a model manifold and then apply surgery procedures to obtain the result. We conjecture that their theorem is also valid when $\varphi : Q \to \mathrm{Out}(G)$ is not necessarily trivial and Γ is completely solvable.

(3) Locally symmetric spaces of the noncompact type are Riemannian manifolds whose sectional curvature is less than or equal to 0. Farrell and Jones [**FJ89**] have shown if N^n, $n > 4$, is a closed Riemannian manifold whose sectional curvature is less than or equal to 0, then any homotopy equivalence $f : M \to N$ from a closed manifold M can be deformed to a homeomorphism. This important work has many applications; see, for example, [**FJ98**]. A consequence to our Seifert fiberings is the following. Let θ be a Seifert Construction for the extension $1 \to \Gamma \to \Pi \to Q \to 1$ with Γ a special lattice and Q abstractly isomorphic to a cocompact group of isometries $\overline{Q}$ on a nonpositively curved Riemannian manifold diffeomorphic to $\mathbb{R}^n$. Then, Q contains a torsion-free normal subgroup Q' of finite index and $M = \theta(\Pi)\backslash(G \times \mathbb{R}^n)$ is the quotient of $M' = \theta(\Pi')\backslash(G \times \mathbb{R}^n)$ by a finite group of Seifert automorphisms Q/Q'. M' is homeomorphic to a fiber bundle over the nonpositively curved manifold $\overline{Q}'\backslash\mathbb{R}^n$ with fiber homeomorphic to $\Gamma\backslash G$. Here we applied the rigidity for the locally symmetric manifolds of negative curvature and the rigidity in $\mathrm{Aut}(\mathbb{R}^n)$ for the flat factors.

11.3. Lifting problem for homotopy classes

Let M be a reasonable path-connected space, say a connected ANR, and $\mathcal{E}(M)$ be the H-space of homotopy equivalences of M into itself. Any $f \in \mathcal{E}(M)$ induces an isomorphism $f_* : \pi_1(M, x) \to \pi_1(M, f(x))$. By choosing a path ω from x to $f(x)$, we have an automorphism f_*^ω of $\pi_1(M, x)$, defined by $f_*^\omega([\tau]) = [\omega^{-1} \cdot (f \circ \tau) \cdot \omega]$. A different choice of ω alters f_*^ω only by an inner automorphism. Therefore, we obtain a map

$$\gamma : \mathcal{E}(M) \to \mathrm{Out}(\Pi),$$

where $\Pi = \pi_1(M, x)$. Let $\mathcal{E}_0(M)$ be the space of self-homotopy equivalences which are homotopic to the identity. Then γ maps $\mathcal{E}_0(M)$ to the identity in $\mathrm{Out}(\Pi)$. The homotopy class of homotopy equivalences (i.e., $\mathcal{E}(M) \bmod \mathcal{E}_0(M)$), forms a group, under composition. The map γ induces a homomorphism $\pi_0(\mathcal{E}(M)) = \mathcal{E}(M)/\mathcal{E}_0(M) \to \mathrm{Out}(\Pi)$. If M is aspherical, the map γ induces an isomorphism. There is a natural map $i : \mathrm{TOP}(M) \to \mathcal{E}(M)$ induced by inclusion (we use the compact-open topology). Composing i with γ is a homomorphism. The connected component of the identity maps to the identity in $\pi_0(\mathcal{E}(M))$ and $\mathrm{Out}(\Pi)$. There is induced a sequence of natural homomorphisms $\mathrm{TOP}(M) \to \pi_0(\mathrm{TOP}(M)) \xrightarrow{i_*} \pi_0(\mathcal{E}(M)) \xrightarrow{\gamma} \mathrm{Out}(\Pi)$.

DEFINITION 11.3.1. A homomorphism $\psi : F \to \mathrm{Out}(\Pi)$ is called an *abstract kernel*. A *lifting* of ψ as a group of homeomorphisms is a homomorphism $\hat{\psi} : F \to \mathrm{TOP}(M)$ which makes

$$\begin{array}{ccccc} F & & \xrightarrow{\quad = \quad} & & F \\ \downarrow{\scriptstyle \hat{\psi}} & & & & \downarrow{\scriptstyle \psi} \\ \mathrm{TOP}(M) & \longrightarrow & \mathcal{E}(M) & \longrightarrow & \mathrm{Out}(\pi_1(M)) \end{array}$$

commutative. The abstract kernel $F \to \mathrm{Out}(\Pi)$ is *topologically realizable* if it can be realized as an action of F on M (i.e., a lifting as a group of homeomorphisms exists).

J. Nielsen [**Nie43**] had shown that every cyclic group of outer automorphisms on a closed surface could be *topologically* realized. Others had shown, by sometimes different methods, that finite p-groups and solvable Lie groups could be topologically realized on compact surfaces ([**Mac62**], [**Zie81**]). In 1983, S. Kerckhoff [**Ker83**] showed that all finite subgroups of $\mathrm{Out}(\pi_1(M))$, where M is a closed surface, can be topologically realized. In 1977, the first examples showing the failure of topological realization on closed aspherical n-manifolds $n \geq 3$, were constructed [**RS77**]. These examples were nilmanifolds. Many other examples of failure on closed aspherical manifolds soon followed, e.g., [**ZZ79**], [**LR82**]. The problem of when one can lift finite groups F from $\pi_0(\mathrm{TOP}(M))$, $\pi_0(\mathcal{E}(M))$ or $\mathrm{Out}(\pi_1(M))$ to $\mathrm{TOP}(M)$ or $\mathrm{Diff}(M)$ became known as the *Nielsen realization problem*. Of course, obvious restrictions must be assumed for the problem to have relevance. The problem is certainly relevant for a closed aspherical manifold because homomorphism $\pi_0(\mathcal{E}(M)) \to \mathrm{Out}(\pi_1(M))$ is an isomorphism. In Chapter 3, we saw that admissible manifolds are good generalizations of closed aspherical manifolds and they enjoy some of the same properties as closed aspherical manifolds. The finite groups that may act effectively on admissible manifolds are essentially determined by the finite subgroups of $\mathrm{Aut}(\pi_1(M))$ and $\mathrm{Out}(\pi_1(M))$, see Theorem 3.2.2. We shall produce a simple strong necessary algebraic condition for the existence of a topological realization of an abstract kernel $\psi : F \to \mathrm{Out}(\Pi)$ on an admissible manifold. Unfortunately, on some nonaspherical admissible manifolds, this condition is not sufficient. Whether or not this condition is also sufficient for closed aspherical manifolds is unknown at this time. However, as we shall see, the Seifert Construction permits us to verify that the necessary condition is also sufficient for a topological realization, as a group of Seifert automorphisms, for large classes of aspherical manifolds.

DEFINITION 11.3.2. An extension $1 \to \Pi \to E \to F \to 1$ is called *admissible* [**LR81**] if each torsion element of $C_E(\Pi)$, the centralizer of Π in E, is an element of Π (so, of $\mathcal{Z}(\Pi)$). That is, $\mathcal{Z}(\Pi)$ and $C_E(\Pi)$ have the same torsion elements so that the inclusion $\mathcal{Z}(\Pi) \to C_E(\Pi)$ is an isomorphism when restricted to torsions.

The definition agrees with the definition introduced in [**LR81**] where it is additionally assumed that $\mathcal{Z}(\Pi)$ is torsion free. Recall that (Definition 3.2.1) a closed manifold M is called an *admissible manifold* if the only periodic self-homeomorphisms of $\widetilde{M}$ commuting with the deck transformation group $\pi_1(M)$ are elements of the center of $\pi_1(M)$. This means that, a manifold M is an admissible manifold if and only if, for every finite effective group action (F, M), the lifting exact sequence (see Subsection 2.2.2) of the action is admissible. Theorem 3.2.8 asserts that all closed aspherical, hyper-aspherical, and K-manifolds are admissible manifolds.

COROLLARY 11.3.3. *Let (F, M) be an effective action of a finite group on an admissible manifold M with $\pi_1(M) = \varPi$. Then the induced extension $1 \to \varPi \to F^* \to F \to 1$, where F^* denotes the group of all liftings of F to homeomorphisms of $\widetilde{M}$, is admissible.*

Notice that, in the pullback diagram (see Subsection 5.3.1 for pullback),

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varPi & \longrightarrow & F^* & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \varPi & \longrightarrow & N_{\mathrm{TOP}(\widetilde{M})}\varPi & \longrightarrow & \mathrm{TOP}(M) & \longrightarrow & 1 \end{array}$$

the bottom sequence is admissible, so is the top one.

REMARK 11.3.4. Let (F, M) be an action (not necessarily effective) of a finite group on an admissible manifold M with $\pi_1(M) = \varPi$. Then there exists an extension (not necessarily admissible) $1 \to \varPi \to E \to F \to 1$ realizing the abstract kernel $\psi : F \xrightarrow{\hat{\psi}} \mathrm{TOP}(M) \xrightarrow{\psi'} \mathrm{Out}(\pi_1(M))$.

PROOF. Since $(\hat{\psi}(F), M)$ is effective, there exists an admissible extension E' of $\varPi$ by $\hat{\psi}(F)$, $1 \to \varPi \to E' \to \hat{\psi}(F) \to 1$. We can pullback this short exact sequence via $F \xrightarrow{\hat{\psi}} \hat{\psi}(F)$ to get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varPi & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \hat{\psi}} & & \\ 2 & \longrightarrow & \varPi & \longrightarrow & E' & \longrightarrow & \hat{\psi}(F) & \longrightarrow & 1. \end{array}$$

Certainly the top row is an extension of $\varPi$ by F realizing $(\varPi, F, \psi = \psi' \circ \hat{\psi})$. □

Thus we have a necessary condition for the existence of a lifting of an abstract kernel $\psi : F \to \mathrm{Out}(\varPi)$, as an (effective, respectively,) group action: the existence of an (admissible, respectively,) group extension of $\varPi$ by F realizing the abstract kernel. For finite groups, this necessary condition is also sufficient for some tractable manifolds. However, for some admissible manifolds, as the next examples show, this necessary condition is not always sufficient.

EXAMPLE 11.3.5. The extension $0 \to \mathbb{Z}^4 \to \mathbb{Z}^4 \to \mathbb{Z}_p \to 0$ is admissible when $\psi : \mathbb{Z}_p \to \mathrm{Out}(\mathbb{Z}^4)$ is trivial, and p is prime. However, there is no realization of this abstract kernel as an effective group action on the admissible manifold $M = T^4 \# \mathbb{C}P_2$. We note that the Euler characteristic of M is 1. If $\mathbb{Z}_p$ acts on M, $\chi(M^{\mathbb{Z}^p}) \equiv \chi(M) \bmod p$, by the Smith theorems. Therefore, $M^{\mathbb{Z}^p} \neq \emptyset$. Then $\psi = \theta : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}^4) = \mathrm{Out}(\mathbb{Z}^4)$ must be injective by Theorem 3.2.2. In this case, the existence of an admissible extension for an abstract kernel on this admissible manifold does not yield a lifting of $\mathbb{Z}_p$ to $\mathrm{TOP}(M)$.

EXERCISE 11.3.6. Show if F acts effectively on $M = T^4 \# \mathbb{C}P^2$, then $\theta : F \to \mathrm{GL}(4, \mathbb{Z})$ is injective; cf. Exercise 3.4.6.

EXAMPLE 11.3.7. Let M be a Seifert 3-manifold that fibers over the 2-sphere with three exceptional fibers of multiplicity $\{p, q, r\}$, see Chapter 14. Assume that the p, q, r are all odd, distinct primes. Therefore, M admits an injective S^1-action, is aspherical and admits no orientation reversing self-homotopy equivalence. The sequence $0 \to \mathbb{Z} \to \pi_1(M) \to Q \to 1$ is exact, where $\mathbb{Z}$ is the center of $\pi_1(M)$ and Q is a centerless Fuchsian group normally generated by $\mathbb{Z}_p, \mathbb{Z}_q$ and $\mathbb{Z}_r$. Any finite subgroup of Q is a subgroup of a conjugate of $\mathbb{Z}_p, \mathbb{Z}_q$, or $\mathbb{Z}_r$. It is known that $1 \to \mathrm{Inn}(Q) = Q \to \mathrm{Aut}(Q) \to \mathrm{Out}(Q) = \mathbb{Z}_2 \to 1$ is exact and the $\mathbb{Z}_2$ splits back. ($Q \backslash \mathbb{R}^2$, as an orbifold, is the 2-sphere with three distinct branch points x, y, z. $\mathrm{Out}(Q)$ is isomorphic to $\pi_0(\mathrm{TOP}(S^2; x, y, z))$. This latter group is π_0 of the homeomorphisms of S^2 which fix three distinct points and is isomorphic to $\mathbb{Z}_2$.) From [**CR77**, §6, especially Corollary 1 and 2 on page 65] (note in the notation there, that $\Gamma(a) = 1$, $H^1(Q; \mathbb{Z}) = 0$ and $\mathrm{Inn}(Q) = \mathrm{Inn}(\pi_1(M))$), it follows that

$$\mathrm{Aut}(\pi_1(M)) = \mathrm{Inn}(\pi_1(M)) \rtimes \mathrm{Out}(\pi_1(M)) = Q \rtimes \mathrm{Out}(Q) = Q \rtimes \mathbb{Z}_2.$$

LEMMA 11.3.8. *For M as in Example 11.3.7, the finite groups that act effectively on M are the subgroups of the finite dihedral groups.*

PROOF. Clearly any subgroup of S^1 acts on M. In fact, $\mathrm{O}(2)$ acts on M. The orbit space of a circle action is S^2 with three singular orbits. Arrange these singular orbits along the equator. Then it is easy to see that there is an involution on M which reflects the orbits across the equator. It reverses the orientation of each of the fibers and reverses the orientation of the base. The involution is orientation preserving and compatible with the S^1-action giving an $\mathrm{O}(2)$-action on M. Therefore, any finite dihedral group acts effectively on M.

Now, suppose H is an effective action of a finite group on M. Let $1 \to \pi_1(M) = \Pi \to E \to H \to 1$ be the lifting exact sequence. Let $\varphi : H \to \mathrm{Out}(\pi_1(M)) \cong \mathbb{Z}_2$ be the abstract kernel and K the kernel of the homomorphism φ. This extension is admissible and $\mathcal{Z}(\Pi) \cong \mathbb{Z}$. Therefore, $C_E(\Pi)$ is also $\mathbb{Z}$ and K is a finite cyclic group of index at most 2 in H. The action of $\mathbb{Z}_2$ in $\mathrm{Out}(\Pi)$ is nontrivial on the center and therefore also on K. Then H is dihedral if $K \neq H$, otherwise it is cyclic. □

11.3.9. Suppose now that A is a finite subgroup of E above. Then A maps isomorphically into a subgroup of H since Π is torsion free. The map from E to $\mathrm{Aut}(\Pi)$ is injective on torsion, since $C_E(\Pi) \cong \mathbb{Z}$. Therefore, A maps isomorphically into a subgroup of $\mathrm{Aut}(\Pi)$ isomorphic to one of the $\mathbb{Z}_{p_i} \rtimes \mathbb{Z}_2$, $i = 1, 2$ or 3. Furthermore, each A must fix some point of M. Since the p_i are prime, a subgroup of A isomorphic to one of the $\mathbb{Z}_{p_i}$ cannot act freely. For if it acted freely, the lifting sequence of the subgroup would have to be torsion free as M is aspherical. Similarly for an element of order 2 which reverses orientation of the base. We have shown the following

COROLLARY 11.3.10. *Any finite subgroup A of the lifting sequence in E acts with fixed points on M and is isomorphic to a subgroup of $\mathbb{Z}_{p_i} \rtimes \mathbb{Z}_2$, for $i = 1, 2$, or 3.*

Let $p_1, q_1, r_1, p_2, q_2, r_2$ be distinct odd primes. Let N be the oriented connected sum $M_1 \# M_2$ where M_i are as above with p_i, q_i, r_i being the orders of the multiplicities of the exceptional fibers of M_i, $i = 1, 2$. The manifold N is hyper-aspherical.

THEOREM 11.3.11. *Every abstract kernel* $\psi : F \to \mathrm{Out}(\pi_1(N))$ *has an algebraic realization as a group extension but only* $\mathbb{Z}_2$ *can act effectively and smoothly on* N.

PROOF. In general, the obstruction to the existence of a group extension realizing this abstract kernel is a cohomology class in $H^3(F; \mathcal{Z}(\pi_1(N)))$. Since $\pi_1(N)$ has trivial center, the obstruction class vanishes and so an extension realizing the abstract kernel always exists.

Meeks and Yau [**MY80**] have shown that any finite group F that acts effectively and smoothly on $M_1 \# M_2$, where M_i are closed, irreducible and aspherical 3-manifolds, has an F-invariant 2-sphere $S \subset N$ along which the connected sum is made. Furthermore, the action of F smoothly extends to an action of F on both M_1 and M_2 by coning over the invariant sphere. Then F must fix the vertices of the cones in M_1 and M_2. Consequently, F must be a subgroup of $\mathrm{Aut}(\pi_1(M_1))$ and also of $\mathrm{Aut}(\pi_1(M_2))$. Since the odd primes for $\mathrm{Aut}(\pi_1(M_1))$ and $\mathrm{Aut}(\pi_1(M_2))$ are different, F must be $\mathbb{Z}_2$.

By Bloomberg (Subsection 3.4.4), we have that $\mathrm{Out}(\pi_1(N)) = \mathrm{Aut}(\pi_1(M_1)) \times \mathrm{Aut}(\pi_1(M_2))$. A torsion subgroup of $\mathrm{Out}(\pi_1(N))$ is a torsion subgroup of $(A \rtimes \mathbb{Z}_2) \times (B \rtimes \mathbb{Z}_2)$, where A is $\mathbb{Z}_{p_1}$, $\mathbb{Z}_{q_1}$, or $\mathbb{Z}_{r_1}$ and B is $\mathbb{Z}_{p_2}$, $\mathbb{Z}_{q_2}$, or $\mathbb{Z}_{r_2}$. For example, if we choose $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and map it into $\mathrm{Out}(\pi_1(N))$ injectively, we will have an admissible extension. The group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, however, is not realizable as a smooth action on N. An action of $\mathbb{Z}_2$ on N can easily be constructed. □

REMARK 11.3.12. Any smooth action of a compact Lie group on M as in Example 11.3.7 can be smoothly conjugated into the O(2)-action described in the theorem. For the connected component, this follows from Chapter 14, and for the dihedral part, it follows from [**MS86**]. Finite topological actions cannot be always be conjugated into the O(2)-action.

11.3.13. While the existence of an admissible extension is a very strong necessary condition for effective topological realization of an abstract kernel on an admissible manifold, the fundamental group does not always capture the homotopy type of the manifold. Moreover, the relationship between homotopy type of an admissible manifold and its homeomorphism type is weaker than it is for aspherical manifolds. Therefore, in seeking to show that the existence of an admissible extension is sufficient for a topological realization of an abstract kernel, it is advisable to confine one's self to the subject of aspherical manifolds. In fact, we have the following

Unsolved Problem: Does there exist a closed aspherical manifold M such that there is an extension $1 \to \pi_1(M) \to E \to F \to 1$ with F finite, but F cannot be topologically realized as a group action on M?

DEFINITION 11.3.14. Let Q act properly on a space W, and let B be the quotient $Q\backslash W$. Suppose for each extension $1 \to Q \to E \to F \to 1$ by a finite group F, the action of Q extends to a proper action of E on W. Then we say that the Q-action on W is *finitely extendable*. In particular, then F acts on B preserving the orbit structure.

If Γ is normal in Π, recall $\mathrm{Aut}(\Pi,\Gamma)$ denotes the automorphisms of Π that leave Γ invariant. Since $\mathrm{Inn}(\Pi)$ leaves Γ invariant, we can put $\mathrm{Aut}(\Pi,\Gamma)/\mathrm{Inn}(\Pi) = \mathrm{Out}(\Pi,\Gamma)$. It is a subgroup of $\mathrm{Out}(\Pi)$.

We are interested in realizing a finite abstract kernel $F \to \mathrm{Out}(\Pi)$ as a group action on a model Seifert fiber space $M(\Pi)$ with a typical fiber $\Gamma\backslash G$. Ideally, we want the F-action to be fiber preserving maps; in fact, Seifert automorphisms. This means that, on the group level, the extension must leave the lattice Γ invariant. In other words, we consider only those abstract kernels which have images in $\mathrm{Out}(\Pi,\Gamma)$.

Theorem 11.3.15. *Let $M(\theta(\Pi)) = \theta(\Pi)\backslash(G \times W)$ be a Seifert fiber space with typical fiber $\Gamma\backslash G$, where G is a Lie group of type* (S3)*; see Subsection* 7.3.1. *Suppose (Q,W) is finitely extendable, where $Q = \Pi/\Gamma$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\Pi,\Gamma)$ of a finite group F can be topologically realized as a group of Seifert automorphisms on $M(\theta(\Pi))$ if and only if the abstract kernel ψ admits some extension.*

Remark 11.3.16. This theorem proves that if (Q,W) is finitely extendable, then $(\theta(\Pi), G \times W)$ becomes finitely extendable itself, provided that Γ is characteristic in Π. Thus, we can enlarge the class of extendable pairs more and more. Here is a list of finitely extendable pairs:

(1) $W = \{p\}$ a point and (Q,W) any finite group;
(2) hyperbolic space and a cocompact lattice;
(3) $\mathbb{R}^n$ and a crystallographic group;
(4) connected, simply connected nilpotent Lie group and its finitely extended lattice;
(5) connected, simply connected completely solvable Lie group and its almost crystallographic group;
(6) a Riemannian symmetric space with no compact, 1- and 2-dimensional factors and Q a group of isometries with compact quotient.

Proof. Let $1 \to \Pi \to E \to F \to 1$ be an extension realizing the abstract kernel ψ. Since $\psi(F) \subset \mathrm{Out}(\Pi,\Gamma)$, Γ is normal in E. We have the commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & E & \longrightarrow & E/\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Consider the induced extension

$$1 \longrightarrow Q \longrightarrow E/\Gamma \longrightarrow F \longrightarrow 1.$$

Since $\rho : Q \to \mathrm{TOP}(W)$ is finitely extendable, there exists $\rho' : E/\Gamma \to \mathrm{TOP}(W)$ extending $\rho : Q \to \mathrm{TOP}(W)$. Again by the existence part of Theorem 7.3.2 for special lattices, there exists $\theta' : E \to \mathrm{TOP}_G(G\times W)$, where $\theta'|_\Gamma = \theta|_\Gamma = i : \Gamma \hookrightarrow G$, and $\rho'|_Q = \rho$. Put $\theta'|_\Pi = \theta'$. Of course, θ' may be different from θ, but as θ and θ' agree on Γ and Q, we can apply Theorem 7.3.2(2) to conjugate $\mathrm{TOP}_G(G\times W)$ by an element of $\mathrm{M}(W,G)\rtimes\mathrm{Inn}(G)$ which carries $\theta'|_\Pi$ to θ so that the new homomorphism $\theta' : E \to \mathrm{TOP}_G(G\times W)$ is an extension of $\theta : \Pi \to \mathrm{TOP}_G(G\times W)$. This yields an action of F on $\theta(\Pi)\backslash(G\times W)$ as a group of Seifert automorphisms as desired. □

COROLLARY 11.3.17. *Let $M = \Pi\backslash G$ be an infra-G-manifold, where G is one of the special Lie groups (see Subsection* 7.3.1*) and $\Pi \subset G \rtimes \mathrm{Aut}(G)$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\Pi,\Gamma)$ of a finite group F can be topologically realized as an (effective, respectively) group of affine diffeomorphisms on M if and only if the abstract kernel ψ admits an (admissible, respectively) extension.*

PROOF. The infra-G-manifold $M(\Pi)$ is modeled on $G\times\{p\}$ (p=point), ($G/K\times\{p\}$ for a convenient form of G in the semisimple case), and $\mathrm{TOP}_G(G\times\{p\}) = \mathrm{Aff}(G)$ (respectively, $\overline{\mathrm{Aff}}(G,K)$); see Chapter 9 for this notation. Trivially, every $Q \to \mathrm{TOP}(\{p\})$ extends to $E/\Gamma \to \mathrm{TOP}(\{p\})$. The above theorem then immediately applies, and F acts on $M(\Pi)$ by Seifert automorphisms which are affine diffeomorphisms. □

EXERCISE 11.3.18. Let $\Pi = \mathbb{Z}^2$ and $M = \Pi\backslash\mathbb{R}^2 = T^2$. Realize the abstract kernel $\psi : \mathbb{Z}_2 \to \mathrm{GL}(2,\mathbb{Z})$. How many distinct actions $(\mathbb{Z}_2, T^2)$ (up to equivalence) do you get?

EXERCISE 11.3.19. Verify the claims for (1)–(5) made in Remark 11.3.16.

REMARK 11.3.20. 1. Since we may introduce a metric structure in the corollary from a left invariant metric on G, $M(\Pi)$ has the structure of a flat, almost flat, Riemannian infra-solvmanifold or a locally symmetric spaces. We may also further conjugate $\theta(\Pi)$ in $\mathrm{Aff}(G)$ so that F now acts on the conjugated manifold by isometries preserving the flat, etc., structures.

2. The proper action of Π in the theorem is not necessarily free nor effective. Thus $M(\Pi)$ could very well be a Seifert orbifold. The corollary then works for such orbifolds, i.e., infra-G-spaces. In the Euclidean case, $M(\Pi)$ would then be a Euclidean *crystal* and Π a Euclidean crystallographic group. In Theorem 11.3.15, F sends fibers (which could be G-crystals instead of infra-G-spaces) to fibers.

For a torsion-free poly-{cyclic or finite} group Π, we can always find a (characteristic) *predivisible* subgroup Γ of finite index in Π; see Section 9.5. Let Q be the finite quotient Π/Γ, and choose $W = \{\text{point}\}$. Then the Seifert Construction of Theorem 9.5.7 produces an embedding $\theta(\Pi) \subset \overline{\mathrm{Aff}}(G,K)$ and the Seifert manifold $M(\Pi) = \theta(\Pi)\backslash G/K$ is a closed smooth $K(\Pi,1)$ manifold.

COROLLARY 11.3.21 (Smooth realization of group actions from homotopy data). *Under the conditions of Theorem* 9.5.7*, let $M(\Pi) = \theta(\Pi)\backslash G/K$ be a Seifert manifold. Suppose now $\psi : F \to \mathrm{Out}(\Pi) = \pi_0\mathcal{E}(M(\Pi))$ is a homomorphism of a finite* *group F into the homotopy classes of self-homotopy equivalences of $M(\Pi)$. Then, F acts on $M(\Pi)$ if and only if there exists an extension,*

$$1 \to \Pi \to E \to F \to 1,$$

realizing the abstract kernel ψ. Moreover, the action can be chosen to be smooth, induced from smooth Seifert automorphisms contained in $\overline{\mathrm{Aff}}(G,K)$. The action of F is effective if and only if $C_E(\Pi)$ is torsion free .

Proof. In order to have an action, we must have a lifting sequence and hence an extension, $1 \to \Pi \to E \to F \to 1$, that realizes the abstract kernel ψ. Since Γ is characteristic in Π, it is normal in E and and $1 \to Q = \Pi/\Gamma \to E/\Gamma \to F \to 1$ is exact. Because of the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & E & \longrightarrow & E/\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & &
\end{array}$$

we can find a Seifert construction $\theta' : E \to \overline{\mathrm{Aff}}(G,K)$ which extends $\theta : \Pi \to \overline{\mathrm{Aff}}(G,K)$. Therefore the group F acts on $M(\Pi)$ smoothly as diffeomorphisms preserving the Seifert structure. The action of F, since M is aspherical, is effective if and only if, $C_E(\Pi)$ is torsion free. In any case, we have a lift $\widetilde{\psi}$,

$$\begin{array}{ccc}
F & \xrightarrow{\widetilde{\psi}} & \mathrm{Diff}(M(\Pi)) \\
{\scriptstyle\psi}\downarrow & & {\scriptstyle j}\downarrow \\
\mathrm{Out}(\Pi) & = & \mathcal{E}(M(\Pi))
\end{array}$$

where j sends a self-diffeomorphism to its homotopy class. In case there exists one extension realizing the abstract kernel ψ, then for each element of $H^2(F, \mathcal{Z}(\Pi))$ there is a congruence class of extensions E, realizing the abstract kernel ψ. Each of these extensions gives rise to a (not necessarily effective) action of F on $M(\Pi)$. $\square$

If we combine this corollary with surgery results, we can get much stronger statements.

Theorem 11.3.22. *Let $1 \to \Pi \to E \to F \to 1$ be an extension of a torsion-free poly{cyclic or finite} group Π by a finite group F with abstract kernel $\psi : F \to \mathrm{Out}(\Pi)$. Let M be a closed aspherical n-manifold, $n > 4$, with $\pi_1(M) = \Pi$. Then there exists an action of F on M which realizes the abstract kernel $\psi : F \to \mathrm{Out}(\Pi) = \pi_0(\mathcal{E}(M))$, the group of homotopy classes of self-homotopy equivalences of M. The action is effective if and only if $C_E(\Pi)$ is torsion free, and is free if and only if E is torsion free. In the latter case, any two such actions are weakly equivalent.*

Proof. Pick Γ a characteristic predivisible subgroup in Π. Then we have a commutative diagram as in the above corollary. With the notation in Definition 9.5.1, we get an action of E/Γ on $\Gamma\backslash G/K = M(\Gamma)$ and an action of F on

$M' = \theta(\Pi)\backslash G/K$, realizing ψ on M'. Since the torsion-free Π is poly-$\mathbb{Z}$ (respectively, poly (cyclic or finite)), a theorem of Wall [**Wal70**] (respectively, Farrell-Jones [**FJ98**]) says that any homotopy equivalence between M and M' is homotopic to a homeomorphism.

Both Wall's and Farrell and Jones's theorems were proven for dimension greater than 4. Their theorems are valid in dimension 4 by surgery results of Freedman and Quinn, and in dimension 3 by the solutions of the Geometric Conjecture. Therefore, we need only to pull back the action of F on M' to obtain the desired action on M. By uniqueness, we have that any two free actions on M' will be weakly equivalent. □

Theorem 11.3.23. *Let G be a semisimple centerless Lie group without any normal compact factors and if G contains any 3-dimensional factors* (i.e., $\mathrm{PSL}(2,\mathbb{R})$), *then the projection of the lattice to each of these factors is dense. Let $M(\theta(\Pi)) = \theta(\Pi)\backslash(G/K\times W)$ be a Seifert manifold with typical fiber $\Gamma\backslash G/K$. Suppose $(\Pi/\Gamma, W)$ is finitely extendable. Then each abstract kernel $\psi : F \to \mathrm{Out}(\Pi,\Gamma)$ of a finite group F can be topologically realized as an (effective, respectively) group of Seifert automorphisms on $M(\theta(\Pi))$ if and only if the abstract kernel ψ admits an (admissible, respectively) extension.*

Proof. Same argument as Theorem 11.3.15. □

Corollary 11.3.24. *Let G be as in the above theorem, and let $M = \theta(\Pi)\backslash G/K$. Then each abstract kernel $\psi : F \to \mathrm{Out}(\Pi,\Gamma)$ of a finite group F can be topologically realized as an (effective, respectively) group action on M if and only if the abstract kernel ψ admits an (admissible, respectively) extension.*

Example 11.3.25 ([**LR82**, (1.1)]). We shall exhibit an abstract kernel on the fundamental group of a flat manifold which has no realization as a group extension. The technique is similar to that employed in [**RS77**].

Let $\mathfrak{G}_2$ (see Example 4.5.10) be the 3-dimensional flat manifold defined by $\mathfrak{G}_2 = (S^1, S^1 \times_{\mathbb{Z}_2} T^2)$. The generator ω of $\mathbb{Z}_2$ acts on T^2 by $\omega(z_1,z_2) = (z_1^{-1}, z_2^{-1})$ and on S^1 by $z \mapsto -z$. Denote points of $\mathfrak{G}_2$ by $\langle r, z_1, z_2\rangle$. Then $\langle r,z_1,z_2\rangle = \langle r-1, z_1^{-1}, z_2^{-1}\rangle$ for $r \in \mathbb{R}$, $(z_1,z_2) \in T^2$. Define $D : \mathfrak{G}_2 \to \mathfrak{G}_2$ by $D\langle r,z_1,z_2\rangle = \langle r-1, z_2^{-1}, z_1\rangle$. Think of D as given by the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Note that D is well defined, $D\langle 0,1,1\rangle = \langle 0,1,1\rangle$ and $D^2\langle r,z_1,z_2\rangle = \langle r, z_1^{-1}, z_2^{-1}\rangle$. Define $D_t^2\langle r,z_1,z_2\rangle = \langle r-t, z_1^{-1}, z_2^{-1}\rangle$. Then $D_0^2 = D^2$, $D_1^2 = \mathrm{id}_{\mathfrak{G}_2}$. Hence D^2 is isotopic, through isometries, to the identity. Now D^2 restricted to $T^2 = \{\langle 0,z_1,z_2\rangle\} \subset \mathfrak{G}_2$ is ω, and $\pi_1(T^2, \langle 0,1,1\rangle)$ is a characteristic subgroup of $\pi_1(\mathfrak{G}_2)$ since it is the kernel of $\pi_1(\mathfrak{G}_2) \to H_1(\mathfrak{G}_2)\otimes\mathbb{Q}$.

If there is H homotopic to D such that $H^2 = \mathrm{id}$, then we have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & E & \longrightarrow & \langle H\rangle \cong \mathbb{Z}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \psi & & \\ 1 & \longrightarrow & \mathrm{Inn}(\pi_1(\mathfrak{G}_2)) & \longrightarrow & \mathrm{Aut}(\pi_1(\mathfrak{G}_2)) & \longrightarrow & \mathrm{Out}(\pi_1(\mathfrak{G}_2)) & \longrightarrow & 1. \end{array}$$

Choose $e \in E$ such that conjugation by e, $\mu(e)$ is precisely $D_* \in \mathrm{Aut}(\pi_1(\mathfrak{G}_2))$. $\mathfrak{G}_2$ fibers over S^1 by $\langle r, z_1, z_2\rangle \mapsto e^{2\pi i r}$, and let η be the generator of π_1 of the section $\langle r, 1, 1\rangle$ in $\pi_1(\mathfrak{G}_2)$. Then $\mu_\eta = \mu_{e^2}$. Therefore $e^2 = c\cdot\eta$ for some $c \in \mathcal{Z}(\pi_1(\mathfrak{G}_2)) \cong \mathbb{Z}$. Note $\mathcal{Z}(\pi_1(\mathfrak{G}_2)) \cong \mathbb{Z}$ is generated by η^2. Since $D_*(\eta) = \eta^{-1}$, $D_*(e^2) = e^{-2}$. On the other hand, $D_*(e^2) = \mu_e(e^2) = e^2$. Therefore, $e^4 = 1$. But as $e^2 \in \pi_1(\mathfrak{G}_2)$, which is torsion free, we have a contradiction. Therefore, no extension exists and so no involution H topologically realizing the abstract kernel ψ exists.

COROLLARY 11.3.26. *There exists an isometry D on $\mathfrak{G}_2$ such that D^2 is isotopic, through isometries, to the identity but D is not homotopic to any involution H.*

11.3.27. Observe that $\mathbb{Z}_4 \to \mathbb{Z}_2 \xrightarrow{\psi} \mathrm{Out}(\pi_1(\mathfrak{G}_2))$, in Example 11.3.25, is realized by the group of isometries $(D) \cong \mathbb{Z}_4$. We call $\mathbb{Z}_4$ an *inflation* of the abstract kernel ψ. Note, in this case, the new abstract kernel has an admissible extension.

In general, whenever the abstract kernel $\psi : F \to \mathrm{Out}(\Pi)$ fails to have an extension realizing the abstract kernel, we may find a larger finite group H which maps homomorphically onto F so that the composite with ψ, has an extension realizing this new abstract kernel, $H \to F \xrightarrow{\psi} \mathrm{Out}(\Pi)$. When we couple this with some of the previous theorems which guarantee topological realization by Seifert automorphisms, we have particular solutions to the Nielsen realization problem.

11.3.28. Let Π be a group whose center is $\mathcal{Z}(\Pi) = C$. If $\psi : F \to \mathrm{Out}(\Pi)$ is a homomorphism, we may choose a map $\widetilde{\psi} : F \to \mathrm{Aut}(\Pi)$ such that the composite $F \xrightarrow{\widetilde{\psi}} \mathrm{Aut}(\Pi) \to \mathrm{Out}(\Pi)$ is ψ. The map $\widetilde{\psi}$ induces a homomorphism of F to $\mathrm{Aut}(C)$ independent of the choice of lift $\widetilde{\psi}$. If we put $E = \Pi \times F$, we can attempt, using ψ, to construct a group structure so that $1 \to \Pi \to E \to F \to 1$ is an extension realizing the abstract kernel. If we follow the procedure of Section 5.2, we find that this is frustrated by the possible failure of the associative law of the group operation. This failure is measured by a cocycle in $Z^3_\psi(F; C)$. The cocycle depends upon choices made in the attempt of constructing a product structure. Varying the suitable choices alters the cocycle by a coboundary. Therefore we obtain an obstruction element $o(\psi) \in H^3_\psi(F; C)$ which vanishes if and only if there is an extension E realizing this abstract kernel. In particular, if $C = 0$, then $o(\psi) = 0$. In this case, there is an extension induced by the pullback from

$$1 \to \mathrm{Inn}(\Pi) = \Pi \to \mathrm{Aut}(\Pi) \to \mathrm{Out}(\Pi) \to 1.$$

For details, see [**ML75**, Chapter 4, §8 and §9].

11.3.29. Now suppose F is finite and $C = \mathbb{Z}^k$, $k > 0$, and $\psi : F \to \mathrm{Out}(\Pi)$, a given abstract kernel. Then the obstruction $o(\psi)$ has finite order, say n. Embed $\alpha : \mathbb{Z}^k \to (\frac{1}{n}\mathbb{Z})^k$. Since any automorphism of $\mathbb{Z}^k$ extends uniquely to an automorphism of $(\frac{1}{n}\mathbb{Z})^k$, there is a homomorphism

$$\alpha_* : H^3_\psi(F; \mathbb{Z}^k) \to H^3_\psi\big(F; (\tfrac{1}{n}\mathbb{Z})^k\big)$$

under which $\alpha_*(o(\psi)) = 0$.

We have a commutative diagram of F-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathbb{R}^k & \longrightarrow & T^k & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & (\frac{1}{n}\mathbb{Z})^k & \longrightarrow & \mathbb{R}^k & \longrightarrow & T^k/(\mathbb{Z}_n)^k & \longrightarrow & 1.
\end{array}$$

Since $H^i(F;\mathbb{R}^k) = 0$, $i > 0$, we have $H^i_\psi(F;T^k) \overset{d^i}{\cong} H^{i+1}_\psi(F;\mathbb{Z}^k)$, $i \geq 0$. Let $\gamma = (d^2)^{-1}(o(\psi)) \in H^2(F;T^k)$. Since $\alpha_*(o(\psi)) = 0$,

$$(d^2)^{-1} \circ \alpha_* \circ d^2 = \alpha'_* : H^2(F;T^k) \to H^2(F;T^k/(\mathbb{Z}_n)^k)$$

maps γ to 0. Consequently, there is a class $\delta \in H^2(F;(\mathbb{Z}_n)^k)$ for which $i_*(\delta) = \gamma$, where $i : (\mathbb{Z}_n)^k \to (S^1)^k = T^k$ is the natural inclusion. The element δ determines an extension $1 \to (\mathbb{Z}_n)^k \xrightarrow{j} G \xrightarrow{\eta} F \to 1$. Consider now the new abstract kernel $\psi \circ \eta : G \to \mathrm{Out}(\Pi)$.

From the commutative diagram

$$\begin{array}{rcccc}
\delta \in & H^2(F;(\mathbb{Z}_n)^k) & \xrightarrow{\eta^*} & H^2(G;(\mathbb{Z}_n)^k) \\
{\scriptstyle i_*}\big\downarrow & {\scriptstyle i_*}\big\downarrow & & {\scriptstyle i_*}\big\downarrow \\
(d^2)^{-1} \circ (\alpha\psi) = \gamma \in & H^2(F;T^k) & \xrightarrow{\eta^*} & H^2(G;T^k) \\
 & {\scriptstyle \alpha'_*}\big\downarrow & & {\scriptstyle \alpha'_*}\big\downarrow \\
 & H^2(F;T^k/(\mathbb{Z}_n)^k) & \xrightarrow{\eta^*} & H^2(G;T^k/(\mathbb{Z}_n)^k)
\end{array}$$

we see that $\eta^*((d^2)^{-1} \cdot o(\psi)) \in H^2(G;T^k)$ is 0 if $\eta^*\delta = 0$. But $\eta^*\delta$ is the pullback of

$$\begin{array}{ccccccc}
1 & \longrightarrow & (\mathbb{Z}_n)^k & \longrightarrow & \eta^*(G) & \longrightarrow & G \\
 & & \big\Vert & & \big\downarrow & & \big\downarrow{\scriptstyle \eta} \\
1 & \longrightarrow & (\mathbb{Z}_n)^k & \longrightarrow & G & \longrightarrow & F.
\end{array}$$

The top sequence splits, and so $\eta^*\delta = 0$ and $o(\eta \circ \psi) = \eta^*(o(\psi)) \in H^3(G;\mathbb{Z}^3) = 0$. Therefore we have shown

Theorem 11.3.30 ([**Zim80**]). *If $\mathcal{Z}(\Pi) = C \cong \mathbb{Z}^k$, $k > 0$, and $o(\psi) \in H^3_\psi(F;C)$ has order n, then there is an extension $1 \to \Pi \to E \to G \to 1$ realizing the abstract kernel $\eta \circ \psi = \psi'$, where $1 \to (\mathbb{Z}_n)^k \to G \to F \to 1$ is an inflation of the abstract kernel ψ as constructed above.*

There are many extensions for each $\psi \circ \eta : G \to \mathrm{Out}(\Pi)$. The set of congruence classes of extensions realizing the kernel ψ' is in one-to-one correspondence with $H^2_{\psi'}(G;\mathcal{Z}(\Pi))$.

Theorem 11.3.31 ([**Lee82a**], [**Lee82b**], [**Zim80**]). *Let M be a flat manifold. Given a finite subgroup F of $\pi_0(\mathcal{E}(M))$, there always exists a group F^*, together with a surjective homomorphism $F^* \to F$ with a finite Abelian kernel such that it can be realized as a group of affine diffeomorphisms of M. Furthermore, the finite Abelian kernels are uniformly bounded by $H^1(M;Z)/\mathrm{Center}(\pi_1(M))$.*

11.3.32. The argument in Subsection 11.3.29 is purely algebraic with no assumption on Π other than $\mathcal{Z}(\Pi)$ is $\mathbb{Z}^k$. In any case, for an abstract kernel $\psi : F \to \mathrm{Out}(\Pi)$ with $o(\psi) \neq 0$ and of order n, there is always an inflation $\eta : G \to F$ of F with kernel $(\mathbb{Z}_n)^k$ and an extension $1 \to \Pi \to E \to G \to 1$ realizing the abstract kernel $\psi' = \psi \circ \eta$. Furthermore, if $M(G(\Pi))$ is a Seifert manifold as in Subsections 11.3.15–11.3.24, with (Q, W) finitely extendable, we can apply the method of proof of Theorem 11.3.15 to topologically realize the abstract kernel ψ' as a group of Seifert automorphisms. For example, in Corollary 11.3.17, the realization will be by affine diffeomorphisms.

The realization may not be effective. Suppose $\psi : F \to \mathrm{Out}(\Pi)$ is injective and $1 \to (\mathbb{Z}_n)^k \to G \xrightarrow{\eta} F \to 1$ is the inflation of F. Let $1 \to \Pi \to E \to G \to 1$ be an extension realizing ψ'. Then $\theta : E \to \mathrm{TOP}_G(G \times W)$ (or $\mathrm{TOP}_{G,K}(G \times W)$) is injective (i.e., G will be effective) if and only if $C_E(\Pi)$ is torsion free. If not torsion free, then the torsion T of $C_E(\Pi)$, a subgroup of $(\mathbb{Z}_n)^k$, is precisely the kernel of θ. Then G/T acts effectively on $M(\theta(\Pi))$ as Seifert automorphisms realizing the abstract kernel $G/T \to F \to \mathrm{Out}(\Pi)$ whose admissible extension is $1 \to \Pi \to E/T \to G/T \to 1$. The kernel of $G/T \to F$ is $(\mathbb{Z}_n)^k/T$.

Theorem 11.3.31 formulates Theorem 11.3.30 more sharply for the special case of flat manifolds and uses somewhat different arguments than given in Subsection 11.3.29.

For more about the realizations up to strict equivalences and finding examples where F does not lift because there are no extensions realizing the abstract kernels, the reader is referred to [**KLR83**], [**LR96**], [**LR81**], [**ZZ79**], [**LR82**], [**RS77**], [**Lee82a**], [**Lee82b**], [**Igo84**], [**SY79**], and [**Ray79**].

11.4. Polynomial structures for solvmanifolds

11.4.1. John Milnor [**Mil77**] asked if every torsion-free polycyclic-by-finite group Γ occurs as the fundamental group of a compact, complete affinely flat manifold. This is equivalent to asking if Γ can act on $\mathbb{R}^K$ properly as affine motions with $\Gamma\backslash\mathbb{R}^K$ compact.

However, Benoist ([**Ben92**], [**Ben95**]) constructed an example of a 10-step nilpotent group Γ of Hirsch length 11 which does not admit an affine structure. This example was generalized to a family of examples by Burde and Grunewald ([**BG95**]). In [**Bur96**], Burde constructs counterexamples of nilpotency class 9 and Hirsch length 10.

11.4.2. A polynomial diffeomorphism f of $\mathbb{R}^n$ is a bijective polynomial transformation of $\mathbb{R}^n$ for which the inverse mapping is again polynomial. Let us write $\mathrm{P}(\mathbb{R}^n)$ for the group consisting of all polynomial diffeomorphisms. Affine diffeomorphisms clearly are polynomial diffeomorphisms of degree less than or equal to 1; smooth actions could be considered as being *polynomial of infinite degree*.

A representation $\theta : \Gamma \to \mathrm{Aff}(\mathbb{R}^K)$ which yields a proper action with $\theta(\Gamma)\backslash\mathbb{R}^K$ compact is called an *affine structure on* Γ. It is also common to call $\theta(\Gamma)$ an *affine crystallographic group* (ACG) ([**FG83**], [**GS94**]). Analogously to the affine structure, a representation $\theta : \Gamma \to \mathrm{P}(\mathbb{R}^K)$ which yields a proper action with $\theta(\Gamma)\backslash\mathbb{R}^K$ compact is called a *polynomial structure on* Γ; $\theta(\Gamma)$ is called a *polynomial crystallographic group*.

THEOREM 11.4.3 ([**DI97**]). *Every polycyclic-by-finite group* Γ *admits a polynomial structure of bounded degree. That is,* Γ *can act on* $\mathbb{R}^K$ *properly as polynomial diffeomorphisms so that* $\Gamma\backslash\mathbb{R}^K$ *is compact. Moreover, all polynomials involved consist entirely of a bounded degree.*

The case when Γ is nilpotent was proved in [**DIL96**]. The construction of this polynomial structure is a special case of an iterated Seifert fiber space construction, which can be achieved here because of a very strong cohomology vanishing theorem, Theorem 11.4.11.

11.4.4 (Polynomial diffeomorphisms). Write $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ for the real vector space of polynomial mappings from $\mathbb{R}^K$ to $\mathbb{R}^k$. An element $p(x_1, \ldots, x_K)$ of $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ consists of k polynomials in K variables:

$$p(x_1, \ldots, x_K) = \begin{pmatrix} p_1(x_1, x_2, \ldots, x_K) \\ p_2(x_1, x_2, \ldots, x_K) \\ \vdots \\ p_k(x_1, x_2, \ldots, x_K) \end{pmatrix}, \text{ with } p_i(x_1, \ldots, x_K) \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}).$$

By the degree of p, denoted by $\deg(p)$, we mean the maximum of the degrees of the p_i $(1 \le i \le k)$. Note in particular, that $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ contains $\mathbb{R}^k$ as the subgroup of constant mappings (degree-0 mappings).

We denote by $\mathrm{P}(\mathbb{R}^K)$ the group of polynomial diffeomorphisms of $\mathbb{R}^K$. Here, the group-law is composition of mappings (so $\mathrm{P}(\mathbb{R}^K)$ is a subset of $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^K)$, but not a subgroup, because the latter has the addition as the group operation). Elements of $\mathrm{P}(\mathbb{R}^K)$ are polynomial bijections whose inverse mappings are again polynomials.

EXAMPLE 11.4.5. Let $p, q : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$p(x, y) = (y + 1, x + y^2) \text{ and } q(x, y) = (y - x^2 + 2x - 1, x - 1).$$

Clearly, they are inverse to each other in $\mathrm{P}(\mathbb{R}^2)$.

11.4.6. The vector space $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k)$ has $\mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K)$-module structure, via

$$\forall (g, h) \in \mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K),\ \forall p \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) :\ \ {}^{(g,h)}p = g \circ p \circ h^{-1}.$$

The resulting semidirect product $\mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{GL}(\mathbb{R}^k) \times \mathrm{P}(\mathbb{R}^K))$ embeds into $\mathrm{P}(\mathbb{R}^{k+K})$ as follows: $\forall p \in \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k),\ \forall g \in \mathrm{GL}(\mathbb{R}^k),\ \forall h \in \mathrm{P}(\mathbb{R}^K) :$

$$\forall x \in \mathbb{R}^k,\ \forall y \in \mathbb{R}^K :\ \ (p, g, h)(x, y) = (g(x) - p(h(y)), h(y)).$$

11.4.7. The crux of the construction is the iteration of the following procedure. Let

$$1 \to \mathbb{Z}^k \to \Pi \to Q \to 1$$

be an exact sequence with abstract kernel $\varphi : Q \to \mathrm{GL}(k, \mathbb{R})$. Let

$$\rho : Q \to \mathrm{P}(\mathbb{R}^K)$$

be a representation which yields a proper action of Q on $\mathbb{R}^K$ with $Q\backslash\mathbb{R}^K$ compact. We try to find a homomorphism $\theta : \Pi \to \mathrm{P}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{GL}(k, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^K))$ so that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \varphi\times\rho} & & \\
1 & \longrightarrow & \mathrm{P}(\mathbb{R}^K,\mathbb{R}^k) & \longrightarrow & \mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K)) & \longrightarrow & \mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K) & \longrightarrow & 1,
\end{array}$$

where $i:\mathbb{Z}^k\to\mathbb{R}^k\subset\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)$ is the standard translations, is commutative.
Note that

$$\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\subset\mathrm{M}(\mathbb{R}^K,\mathbb{R}^k)\quad\text{and}\quad\mathrm{P}(\mathbb{R}^K)\subset\mathrm{TOP}(\mathbb{R}^K),$$

and therefore,

$$\begin{array}{ccc}
\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{P}(\mathbb{R}^K)) & \xrightarrow{\subset} & \mathrm{P}(\mathbb{R}^{K+k}) \\
\cap\Big\downarrow & & \cap\Big\downarrow \\
\mathrm{M}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{GL}(k,\mathbb{R})\times\mathrm{TOP}(\mathbb{R}^K)) & \xrightarrow{\subset} & \mathrm{TOP}(\mathbb{R}^{K+k});
\end{array}$$

see Corollary 4.2.10.

11.4.8 (Canonical type polynomial representations). It is well known ([**Seg83**, lemma 6, pp.16]) that, if Γ is a polycyclic-by-finite group, then there exists an ascending sequence (or filtration) of normal subgroups Γ_i $(0\le i\le c+1)$ of Γ

$$\Gamma_*:\quad \Gamma_0=1\subseteq\Gamma_1\subseteq\Gamma_2\subseteq\cdots\subseteq\Gamma_{c-1}\subseteq\Gamma_c\subseteq\Gamma_{c+1}=\Gamma \tag{11.4.1}$$

for which

$$\Gamma_i/\Gamma_{i-1}\cong\mathbb{Z}^{k_i}\text{ for }1\le i\le c\text{ and some }k_i\in\mathbb{N}_0\text{ and }\Gamma/\Gamma_c\text{ is finite.}$$

Let us call such a filtration of Γ a *torsion-free filtration* (of length c). We will also use $K_i=k_i+k_{i+1}+\cdots+k_c$ and $K_{c+1}=0$. It follows that $h(\Gamma)=K_1$, the *Hirsh number* (or rank) of Γ.

Definition 11.4.9. For every i, write $\varphi_i:\Gamma/\Gamma_i\to\mathrm{Aut}(\mathbb{Z}^{k_i})$ for the morphism induced by the short exact sequence

$$1\to\mathbb{Z}^{k_i}\ (\cong\Gamma_i/\Gamma_{i-1})\to\Gamma/\Gamma_{i-1}\to\Gamma/\Gamma_i\to 1.$$

A polynomial representation $\rho=\rho_0:\Gamma\to\mathrm{P}(\mathbb{R}^{h(\Gamma)})$ will be called *of canonical type* with respect to Γ_* (or simply of canonical type) if and only if it induces a sequence of representations

$$\rho_i:\Gamma/\Gamma_i\to\mathrm{P}(\mathbb{R}^{K_{i+1}})\ \ (1\le i\le c)$$

and a sequence of morphisms

$$j_i:\mathbb{Z}^{k_i}\hookrightarrow\mathbb{R}^{k_i}\to\mathrm{P}(\mathbb{R}^{K_{i+1}},\mathbb{R}^{k_i})\ \ (1\le i\le c)$$

such that for all i the following diagram commutes:
(11.4.2)

$$\begin{array}{ccccccccc}
1\to\mathbb{Z}^{k_i}\approx & \Gamma_i/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_i & \longrightarrow & 1 \\
 & \Big\downarrow{\scriptstyle j_i} & & \Big\downarrow{\scriptstyle \rho_{i-1}} & & \Big\downarrow{\scriptstyle \psi_i\times\rho_i} & & \\
1\to & \mathrm{P}(\mathbb{R}^{K_{i+1}},\mathbb{R}^{k_i}) & \to & \mathrm{P}(\mathbb{R}^{K_{i+1}},\mathbb{R}^{k_i})\rtimes(\mathrm{GL}(\mathbb{R}^{k_i})\times\mathrm{P}(\mathbb{R}^{K_{i+1}})) & \to & \mathrm{GL}(\mathbb{R}^{k_i})\times P(\mathbb{R}^{K_{i+1}}) & \to & 1,
\end{array}$$

where ψ_i is the unique morphism $\psi_i:\Gamma/\Gamma_i\to\mathrm{GL}(\mathbb{R}^{k_i})$ satisfying

$$\forall\bar\gamma\in\Gamma/\Gamma_i,\ \forall z\in\mathbb{Z}^{k_i}:\psi_i(\bar\gamma)(j_i(z))=j_i(\varphi_i(\bar\gamma)z);$$

i.e., ψ_i is the abstract kernel for the top extension sequence.

11.4.10. Iterating this procedure, we will have found a desired homomorphism $\Gamma \to \mathrm{P}(\mathbb{R}_K)$. The existence of ρ_{i-1} is guaranteed by

$$H^2(\Gamma/\Gamma_i; \mathrm{P}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^k_i)) = 0$$

as the proof of the general construction shows; see Theorem 7.3.2. Also,

$$H^1(\Gamma/\Gamma_i; \mathrm{P}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^k_i)) = 0$$

guarantees the uniqueness of such ρ_{i-1} (with fixed j_i and $\psi_i \times \rho_i$). These are achieved by the following theorem. In fact, the major work of the paper [**DI97**] is a proof of the following.

THEOREM 11.4.11 (Main cohomology vanishing theorem). *If* Γ *is a polycyclic-by-finite group admitting a canonical type polynomial representation* $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^m)$, *then, for every representation* $\varphi : \Gamma \to \mathrm{GL}(\mathbb{R}^n)$ *and for all* $i > 0$, $H^i_{\varphi\times\rho}(\Gamma; \mathrm{P}(\mathbb{R}^m, \mathbb{R}^n)) = 0$.

EXAMPLE 11.4.12. Take N the discrete Heisenberg group

$$N = \langle a, b, c \mid [b, a] = c,\ [c, a] = [c, b] = 1\rangle$$

equipped with the torsion-free filtration

$$N_* \ :\ N_0 = 1 \subseteq N_1 = Z(N) = \langle c\rangle \subseteq N_2 = N \subseteq N_3 = N.$$

In this case $k_1 = 1$ and $k_2 = 2$. Let $q(y)$ be any polynomial over the field of real numbers. Then the morphism $\rho_q : N \to \mathrm{P}(\mathbb{R}^3)$ with

$$\rho_q(a)(x, y, z) = (x, y, z + 1),\ \rho(b)(x, y, z) = (x + q(y) + z, y + 1, z),$$
$$\rho(c)(x, y, z) = (x + 1, y, z)$$

is a canonical type polynomial representation of N with respect to N_*. The upper bound on the degrees of the polynomials involved in ρ can be as big as one wants by choosing a different $q(y)$.

11.4.13 (Rigidity of polynomial structures). Two polynomial actions ρ_1 and ρ_2 of Γ are said to be *polynomially conjugated* if there exists p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all g in Γ, one has $p \circ \rho_1(g) = \rho_2(g) \circ p$.

THEOREM 11.4.14 ([**BD02**, Theorem 1.1]). *Let* Γ *be a polycyclic-by-finite group. Then any two polynomial crystallographic actions of* Γ *of bounded degree on some* $\mathbb{R}^n$ *are polynomially conjugated.*

Note that this statement does not assume any *canonical type* embeddings. The main tool in the proof is the notion of algebraic hull of a polynomial action of bounded degree. Let $\mathrm{P}(\mathbb{R}^n)$ be the group of polynomial bijections of $\mathbb{R}^n$ with polynomial inverse, and let $\mathrm{P}^d(\mathbb{R}^n)$ be the subset of polynomial bijections p such that the degrees of p and p^{-1} are bounded by d.

A regular map $i : X \to Y$ between two real algebraic varieties is said to be a *closed immersion* if the image $i(X)$ is Zariski closed and if the map $i^* : \mathbb{R}[Y] \to \mathbb{R}[X] : \phi \mapsto \phi \circ i$ is surjective.

Let $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ be a polynomial action of Γ on $\mathbb{R}^n$. The Zariski closure $G := A(\rho(\Gamma))$ of $\rho(\Gamma)$ in $\mathrm{P}^d(\mathbb{R}^n)$ is a subgroup of $\mathrm{P}(\mathbb{R}^n)$ which does not depend on the choice of d. The real algebraic group $A(\rho(\Gamma))$ is called the *algebraic hull* of $\rho(\Gamma)$. A subgroup of $\mathrm{P}(\mathbb{R}^n)$ is said to be *Zariski closed* if it is of bounded degree and equal to its algebraic hull.

PROPOSITION 11.4.15. *Let $\Gamma \subseteq \mathrm{P}(\mathbb{R}^n)$ be a polycyclic-by-finite crystallographic subgroup of bounded degree. Then the unipotent radical $U(\Gamma)$ of $A(\Gamma)$ acts simply transitively on $\mathbb{R}^n$.*

PROPOSITION 11.4.16. *For $i = 1, 2$, let G_i be a Zariski closed subgroup of $\mathrm{P}(\mathbb{R}^n)$, suppose that the Zariski connected component of G_i is solvable, and that the unipotent radical U_i of G_i acts simply transitively on $\mathbb{R}^n$. Then, for any isomorphism of algebraic groups $F : G_1 \to G_2$, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all g_1 in G_1, $p \circ g_1 = F(g_1) \circ p$.*

11.4.17 (Proof of Theorem 11.4.14). Let us denote by $\rho_1, \rho_2 : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ these two actions. First of all recall that the kernel of ρ_i $(i = 1, 2)$ is the unique maximal finite normal subgroup F_Γ of Γ. Therefore, we can assume, without loss of generality, that $F_\Gamma = 1$ and that ρ_1 and ρ_2 are injective. Moreover, by [**Rag72**, Lemma 4.41], we know that the isomorphism $\rho_2 \circ \rho_1^{-1} : \rho_1(\Gamma) \to \rho_2(\Gamma)$ extends to an isomorphism of algebraic groups $F : A(\rho_1(\Gamma)) \to A(\rho_2(\Gamma))$. By Propositions 11.4.15 and 11.4.16, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all a in $A(\rho_1(\Gamma))$, one has $F(a) \circ p = p \circ a$. This map p is again the one we are looking for.

11.5. Applications to fixed-point theory

We show that Bieberbach's rigidity theorem for flat manifolds still holds true for any continuous maps on infra-nilmanifolds. Namely, every endomorphism of an almost crystallographic group is semiconjugate to an affine endomorphism. Applying this result to fixed-point theory, we obtain a criterion for the Lefschetz number and Nielsen number for a map on infra-nilmanifolds to be equal. Some material is taken from [**Lee95b**].

11.5.1. Let G be a connected Lie group. Consider the semigroup $\mathrm{Endo}(G)$, the set of all endomorphisms of G, with the composition as operation. We form the semidirect product $G \rtimes \mathrm{Endo}(G)$ and call it $\mathrm{aff}(G)$. With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set $\mathrm{aff}(G)$ forms a semigroup with identity (e, I), where $e \in G$ and $I \in \mathrm{Endo}(G)$ are the identity elements. The semigroup $\mathrm{aff}(G)$ "acts" on G by

$$(a, A) \cdot x = a \cdot Ax.$$

Note that (a, A) is not a homeomorphism unless $A \in \mathrm{Aut}(G)$. Clearly, $\mathrm{aff}(G)$ is a subsemigroup of the semigroup of all continuous maps of G into itself, for $((a, A)(b, B))x = (a, A)((b, B)x)$ for all $x \in G$. We call elements of $\mathrm{aff}(G)$ *affine endomorphisms*.

11.5.2 (Generalization of the Second Bieberbach Theorem). Let G be a connected and simply connected nilpotent Lie group. In Section 8.4, we have seen that, for any isomorphism between two almost crystallographic groups, is a conjugation by an element of $\mathrm{Aff}(G)$. We shall generalize this result to all homomorphisms (not necessarily isomorphisms). Topologically, this implies that every continuous map on an infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. It can be stated as: every endomorphism of an almost crystallographic group is *semiconjugate* to an affine endomorphism.

THEOREM 11.5.3. *Let $\Pi, \Pi' \subset \mathrm{Aff}(G)$ be two almost crystallographic groups. Then for any homomorphism $\theta : \Pi \to \Pi'$, there exists $g = (d, D) \in \mathrm{aff}(G)$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \Pi$.*

EXAMPLE 11.5.4. The subgroup $\Gamma = \Pi \cap G$ of an almost crystallographic group Π is characteristic, but not fully invariant. The homomorphism θ in Theorem 11.5.3 may not map the maximal normal nilpotent subgroup Γ of Π into that of Π'. This causes a lot of trouble. Let Π be an orientable 4-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\Pi \subset \mathbb{R}^4 \rtimes O(4) = E(4) \subset \mathrm{Aff}(\mathbb{R}^4)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I), (e_4, I), and (a, A), where $a = (1/2, 0, 0, 0)^t$, and A is diagonal matrix with diagonal entries $1, -1, -1$, and 1. Note that $(a, A)^2 = (e_1, I)$. The subgroup generated by (e_1, I), (e_2, I), (e_3, I), and (a, A) forms a 3-dimensional Bieberbach group $\mathcal{G}_2$, and $\Pi = \mathcal{G}_2 \times \mathbb{Z}$. Consider the endomorphism $\theta : \Pi \to \Pi$ which is the composite $\Pi \to \mathbb{Z} \to \Pi$, where the first map is the projection onto $\mathbb{Z} = \langle (e_4, I) \rangle$ and the second map sends (e_4, I) to (a, A). Thus the homomorphism θ does not map the maximal normal Abelian subgroup $\mathbb{Z}^4$ (generated by the four translations) into itself. Such a $\mathbb{Z}^4$ is characteristic but not fully invariant in Π. Let

$$d = \begin{bmatrix} x \\ 0 \\ 0 \\ y \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and let $g = (d, D)$. Then it is easy to see that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \Pi$.

It turns out that the element $g = (d, D)$ is the most general form. The matrix D is uniquely determined and the translation part d can vary only in two dimensions.

11.5.5 (Proof of Theorem). Let $\Gamma = \Pi \cap G$, $\Gamma' = \Pi' \cap G$. As the example shows, the characteristic subgroup Γ may not go into Γ' by the homomorphism θ. Let p be the product of the orders of Π/Γ and Π'/Γ'. Let Λ, Λ' be the normal subgroups of Π, Π' generated by

$$\{x^p : \ x \in \Pi\} \text{ and } \{y^p : \ y \in \Pi'\}.$$

Then Λ and Λ' are fully invariant subgroups of Π and Π', and they have finite indices, both lying in G. Clearly, θ maps Λ into Λ'. Let $Q = \Pi/\Lambda$.

Consider the homomorphism $\Lambda \xrightarrow{\theta} \Lambda' \hookrightarrow G$. Since Λ is a lattice of G, by Malćev's work, any such a homomorphism extends uniquely to a continuous homomorphism $C : G \to G$; cf. [**Lee92**, 2.11]. Thus, $\theta|_\Lambda = C|_\Lambda$, where $C \in \mathrm{Endo}(G)$, and hence, $\theta(z, 1) = (Cz, 1)$ for all $z \in \Lambda$ (more precisely, $(z, 1) \in \Lambda$).

Let us denote the composite homomorphism $\Pi \xrightarrow{\theta} \Pi' \hookrightarrow G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ by $\overline{\theta}$ and define a map $f : \Pi \to G$ by

$$\theta(w, K) = (Cw \cdot f(w, K), \overline{\theta}(w, K)). \tag{11.5.1}$$

For any $(z, 1) \in \Lambda$ and $(w, K) \in \Pi$, apply θ to both sides of $(w, K)(z, 1)(w, K)^{-1} = (w \cdot Kz \cdot w^{-1}, 1)$ to get $Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)(Cz) \cdot f(w, K)^{-1} \cdot (Cw)^{-1} = \theta(w \cdot Kz \cdot w^{-1})$. However, $w \cdot Kz \cdot w^{-1} \in \Lambda$ since Λ is normal in Π, and the latter term equals to $C(w \cdot Kz \cdot w^{-1}) = Cw \cdot CKz \cdot (Cw)^{-1}$ since $C : G \to G$ is a homomorphism. From this we have

$$\overline{\theta}(w, K)(Cz) = f(w, K)^{-1} \cdot CKz \cdot f(w, K). \tag{11.5.2}$$

This is true for all $z \in \Lambda$. Note that $\overline{\theta}(w, K)$ and K are automorphisms of the Lie group G and $C : G \to G$ is an endomorphism. By the uniqueness of extension of a homomorphism $\Lambda \to G$ to an endomorphism $G \to G$, as mentioned above, the equality (2) holds true for all $z \in G$. It is also easy to see that $f(zw, K) = f(w, K)$ for all $z \in \Lambda$ so that $f : \Pi \to G$ does not depend on Λ. Thus, f factors through $Q = \Pi/\Lambda$. Moreover, $\overline{\theta} : \Pi \to \mathrm{Aut}(G)$ also factors through Q since Λ maps trivially into $\mathrm{Aut}(G)$. We still use the notation (w, K) to denote elements of Q and $\overline{\theta}$ to denote the induced map $Q \to \mathrm{Aut}(G)$.

We claim that: *with the Q-structure on G via $\overline{\theta} : Q \to \mathrm{Aut}(G)$, $f \in Z^1(Q; G)$; i.e., $f : Q \to G$ is a crossed homomorphism.*

We shall show $f((w, K) \cdot (w', K')) = f(w, K) \cdot \overline{\theta}(w, K)f(w', K')$ for all (w, K), $(w', K') \in \Pi$. (Note that we are using the elements of Π to denote the elements of Q.) Apply θ to both sides of $(w, K)(w', K') = (w \cdot Kw', KK')$ to get $Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)[Cw' \cdot f(w', K')] = C(w \cdot Kw') \cdot f((w, K)(w', K'))$. From this it follows that

$$f((w, K)(w', K')) = (CKw')^{-1} \cdot f(w, K) \cdot \overline{\theta}(w, K)(Cw') \cdot \overline{\theta}(w, K)f(w', K').$$

From (2) we have $\overline{\theta}(w, K)Cw' = f(w, K)^{-1} \cdot CKw' \cdot f(w, K)$ so that $f((w, K) \cdot (w', K')) = f(w, K) \cdot \overline{\theta}(w, K)f(w', K')$.

According to Theorem 8.4.3 (and its proof), it was proved that $H^1(Q; G) = 0$ whenever Q is a finite group and G is a connected and simply connected nilpotent Lie group. The proof uses induction on the nilpotency of G together with the fact that $H^1(Q; G) = 0$ for a finite group Q and a real vector group G. This means that any crossed homomorphism is *principal*. In other words, there exists $d \in G$ such that

$$f(w, K) = d \cdot \overline{\theta}(w, K)(d^{-1}). \tag{11.5.3}$$

Let $D = \mu(d^{-1}) \circ C$ and $g = (d, D) \in \mathrm{aff}(G)$, and we check that θ is *conjugation* by g. Using equalities (11.5.1), (11.5.2), and (11.5.3), one can show $\overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C = \mu(d^{-1}) \circ C \circ K$. Thus, for any $(w, K) \in \Pi$,

$$\begin{aligned}
\theta(w, K) \cdot (d, D) &= (Cw \cdot f(w, K),\ \overline{\theta}(w, K)) \cdot (d,\ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot f(w, K) \cdot \overline{\theta}(w, K)(d),\ \overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot d \cdot \overline{\theta}(w, K)(d^{-1}) \cdot \overline{\theta}(w, K)(d),\ \overline{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\
&= (Cw \cdot d,\ \mu(d^{-1}) \circ C \circ K) \\
&= (d, D) \cdot (w, K).
\end{aligned}$$

This finishes the proof of theorem. □

Corollary 11.5.6. *Let $M = \Pi\backslash G$ be an infra-nilmanifold, and $h : M \to M$ be any map. Then h is homotopic to a map induced from an affine endomorphism $G \to G$.*

Proof. Since M is a manifold, h can be homotoped to a map with a fixed point, say x. We start with the homomorphism $h_* : \pi_1(M, x) \to \pi_1(M, x)$, induced from h, as our θ in Theorem 11.5.3, and obtain $\widetilde{g} = (d, D)$ satisfying

$$h_*(\alpha) \circ \widetilde{g} = \widetilde{g} \circ \alpha.$$

Let $g : M \to M$ be the induced map. Then $h_* = g_*$. Since any two continuous maps on a closed aspherical manifold inducing the same homomorphism on the fundamental group (up to conjugation by an element of the fundamental group) are homotopic to each other, h is homotopic to g. This completes the proof of the corollary. □

Corollary 11.5.7 (Corollary 8.4.4). *Homotopy equivalent infra-nilmanifolds are affinely diffeomorphic.*

Now we consider the uniqueness problem in Theorem 11.5.3: How many g's are there? Let $\Phi = \Pi/(G\cap\Pi) \subset \mathrm{Aut}(G)$ and $\Phi' = \Pi'/(G\cap\Pi') \subset \mathrm{Aut}(G)$ be the holonomy groups of Π and Π'. Let Ψ' be the image of $\theta(\Pi)$ in Φ'. So $\Psi' \subset \Phi' \subset \mathrm{Aut}(G)$. Let $G^{\Psi'}$ denote the fixed-point set of the action. Recall the notation: For $c \in G$, $\mu(c)$ denotes conjugation by c. Therefore, $\mu(c)(x) = cxc^{-1}$ for all $x \in G$.

Proposition 11.5.8 (Uniqueness). *With the same notation as above, suppose $\theta(\alpha)\cdot g = g\cdot\alpha$ for all $\alpha\in\Pi$. Then $\theta(\alpha)\cdot\gamma = \gamma\cdot\alpha$ for all $\alpha\in\Pi$ if and only if $\gamma = \xi\cdot g$, where $\xi = (c,\mu(c^{-1}))$, for $c\in G^{\Psi'}$. Therefore, D is unique up to* $\mathrm{Inn}(G)$. *If θ is an isomorphism, then $c\in G^{\Phi'}$. In particular, if Π is a Bieberbach group with $H^1(\Pi;\mathbb{R}) = 0$ and θ is an isomorphism, then such a g is unique.*

Proof. Let $g = (d,D)$, $\gamma = (c,C)$. Since $\theta(\alpha)\cdot g = g\cdot\alpha$ holds when $\alpha = (z,1)\in\Lambda$, we have $Dz = d^{-1}z'd$, where $\theta(z,1) = (z',1)$. Similarly, $Cz = c^{-1}z'c$. Thus $Cz = \mu(c^{-1}d)Dz$ for all $z\in\Lambda$. Since Λ is a lattice, this equality holds on G. Consequently, $C = \mu(c^{-1}d)D$. Now $\gamma = (c,C) = (c,\ \mu(c^{-1}d)D) = (d^{-1}c,\ \mu(c^{-1}d))(d,D)$ $= (h,\mu(h^{-1}))(d,D)$, if we let $h = d^{-1}c$. Set $\xi = (h,\mu(h^{-1}))$. Then $\gamma = \xi\cdot g$. Now we shall observe that $h\in G^{\Psi'}$. Let $\theta(\alpha) = (b,B)$. Then $\theta(\alpha)\xi g = \theta(\alpha)\gamma = \gamma\alpha = \xi g\alpha = \xi\theta(\alpha)g$ yields $Bh = h$ for all $(b,B) = \theta(\alpha)$. Clearly then $B\in\Psi'$ by definition. For a Bieberbach group Π, note that rank $H^1(\Pi;\mathbb{Z}) = \dim G^{\Phi}$. $\square$

11.5.9 (Application to fixed-point theory). Let M be a closed manifold, and let $f : M\to M$ be a continuous map. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) := \sum_k (-1)^k \mathrm{Trace}\{(f_*)_k : H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})\}$$

To define the *Nielsen number* $N(f)$ of f, we define an equivalence relation on $\mathrm{Fix}(f)$ as follows: For $x_0, x_1\in\mathrm{Fix}(f)$, $x_0\sim x_1$ if and only if there exists a path c from x_0 to x_1 such that c is homotopic to $f\circ c$ relative to the end points. An equivalence class of this relation is called a *fixed-point class* (FPC) of f. To each FPC F, one can assign an integer $\mathrm{ind}(f,F)$. An FPC F is called *essential* if $\mathrm{ind}(f,F)\neq 0$. Now,

$$N(f) := \text{the number of essential fixed-point classes.}$$

These two numbers give information on the existence of fixed-point sets. If $L(f)\neq 0$, every self-map of M homotopic to f has a nonempty fixed-point set. The Nielsen number is a lower bound for the number of components of the fixed-point set of all maps homotopic to f. Even though $N(f)$ gives more information than $L(f)$ does, it is harder to calculate. If M is an infra-nilmanifold, and f is homotopically periodic, then it will be shown that $L(f) = N(f)$.

Lemma 11.5.10. *Let $B\in\mathrm{GL}(n,\mathbb{R})$ with a finite order. Then* $\det(I-B)\geq 0$.

Proof. Since B has finite order, it can be conjugated into the orthogonal group $O(n)$. Since all eigenvalues are roots of unity, there exists $P\in\mathrm{GL}(n,\mathbb{R})$ such that PBP^{-1} is a block diagonal matrix, with each block being a (1×1)- or a (2×2)-matrix. All (1×1)-blocks must be $D = [\pm 1]$, and hence $\det(I-D) = 0$ or 2. For a (2×2)-block, it is of the form $\left[\begin{smallmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{smallmatrix}\right]$. Consequently, each (2×2)-block D has the property that $\det(I-D) = (1-\cos t)^2 + \sin^2 t = 2(1-\cos t)\geq 0$. $\square$

THEOREM 11.5.11. *Let $f: M \to M$ be a continuous map on an infra-nilmanifold $M = \mathit{\Pi}\backslash G$. Let $g = (d, D) \in \mathrm{aff}(G)$ be a homotopy lift of f by Corollary* 11.5.6. *Then $L(f) = N(f)$ (respectively, $L(f) = -N(f)$) if and only if* $\det(I - D_* A_*) \geq 0$ *(respectively,* $\det(I - D_* A_*) \leq 0$*) for all $A \in \Phi$, the holonomy group of M.*

PROOF. Since $L(f)$ and $N(f)$ are homotopy invariants, we may assume that f is the map induced from g. Let $\Gamma = \mathit{\Pi} \cap G$. Then Γ is a normal subgroup of $\mathit{\Pi}$, of finite index, say p. Let Λ be the normal subgroups of $\mathit{\Pi}$ generated by $\{x^p : x \in \mathit{\Pi}\}$. Then $f_* : \mathit{\Pi} \to \mathit{\Pi}$ maps Λ into itself. Therefore, f induces a map on the finite-sheeted regular covering space $\Lambda\backslash G$ of $\mathit{\Pi}\backslash G$.

Let $\widetilde{f}$ be a lift of f to $\Lambda\backslash G$. Then

$$L(f) = \frac{1}{[\mathit{\Pi} : \Lambda]} \sum L(\alpha\widetilde{f}) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \frac{\det(A_* - D_*)}{\det A_*},$$

$$N(f) = \frac{1}{[\mathit{\Pi} : \Lambda]} \sum |N(\alpha\widetilde{f})| = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - D_*)|,$$

where the sum ranges over all $\alpha \in \mathit{\Pi}/\Lambda$; see, [**Jia83**, III 2.12] and [**KLL05**, Theorem 3.5] and [**LL06**, Theorem 3.4]. Since $|\det A_*| = 1$ and $\det(A_* - D_*)/\det A_* = \det(I - D_* A_*^{-1})$, it is easy to see from the formula above that the theorem is proved. □

COROLLARY 11.5.12 ([**KL88b**]). *Let $f: M \to M$ be a homotopically periodic map on an infra-nilmanifold. Then $N(f) = L(f)$.*

PROOF. Here is an argument which is completely different from the one in [**KL88b**]. Let $\Gamma = \mathit{\Pi} \cap G$ and $\Phi = \mathit{\Pi}/\Gamma$, the holonomy group. Let $g = (d, D) \in G \rtimes \mathrm{Aut}(G)$ be a homotopy lift of f to G. Let E be the lifting group of the action of $\langle g \rangle$ to G. That is, E is generated by $\mathit{\Pi}$ and g. Then E/Γ is a finite group generated by Φ and D. For every $A \in \Phi$, DA lies in E/Γ, and has a finite order. By Lemma 11.5.10, $\det(I - DA) \geq 0$ for all $A \in \Phi$. By Theorem 11.5.11, $L(f) = N(f)$. □

COROLLARY 11.5.13 ([**McC94**]). *Let $f: M \to M$ be a homotopically periodic map on an infra-solvmanifold. Then $N(f) = L(f)$.*

PROOF. In [**Lee92**], the statement for solvmanifolds was proved. We needed a subgroup invariant under f_*. To achieve this, a new model space M' which is homotopy equivalent to M, together with a map $f' : M' \to M'$ corresponding to f was constructed. The new space M' is a fiber bundle over a torus with fiber a nilmanifold; and f' is fiber preserving. Moreover, we found a fully invariant subgroup Λ of $\mathit{\Pi}$ of finite index (so, it is invariant under f'_*). Now we can apply the same argument as in the proof of Theorem 11.5.11. □

EXAMPLE 11.5.14. Let $\mathit{\Pi}$ be an orientable 3-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\mathit{\Pi} \subset \mathbb{R}^3 \rtimes O(3) = E(3)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I), and (a, A), where $a = (1/2, 0, 0)^t$, A is a diagonal matrix with diagonal entries $1, -1$, and -1. Note that $(a, A)^2 = (e_1, I)$. Let $M = \mathbb{R}^3/\mathit{\Pi}$ be the flat manifold. Consider the endomorphism $\theta : \mathit{\Pi} \to \mathit{\Pi}$ which is defined by the conjugation by $g = (d, D)$, where

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Let $f : M \to M$ be the map induced from g. There are only two conjugacy classes of g; namely, g and αg. $\mathrm{Fix}(g) = (0,0,0)^t$ and $\mathrm{Fix}(\alpha g) = (1/4, 0, 0)^t$. Since $\det(I - D) = \det(I - AD) = +2$, $L(f) = N(f) = 2$.

The Lefschetz number can be calculated from homology groups also.

(1) $H_0(M; \mathbb{R}) = \mathbb{R}$; f_* is the identity map.
(2) $H_1(M; \mathbb{R}) = \mathbb{R}$, which is generated by the element (e_1, I).
f_* is multiplication by 3 (the $(1,1)$-entry of D).
(3) $H_2(M; \mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det \left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right] = -2$.
(4) $H_3(M; \mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det(D) = -6$.

Therefore, $L(f) = \Sigma(-1)^i \mathrm{Trace} f_* = 1 - 3 + (-2) - (-6) = 2$. Note that f has infinite period, and this example is not covered by Corollary 11.5.13.

Example 11.5.15. Let Π be same as in Example 11.5.14. This time $g = (d, D)$, is given by

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Let $f : M \to M$ be the map induced from g. There are six conjugacy classes of g; namely, g and αg, $\alpha t_1 g$, $\alpha t_1^2 g$, $\alpha t_1^3 g$, and $\alpha t_1^4 g$. Each class has exactly one fixed point. Clearly, $\det(I - D) = +2$ and $\det(I - AD) = -10$. Therefore, the first fixed point has index $+1$ and the rest have index -1. Consequently, $L(f) = -4$, while $N(f) = 6$.

11.6. Homologically injective torus operations

11.6.1. There have been many efforts trying to split a manifold as a product of two manifolds. Let M be a flat Riemannian manifold whose fundamental group contains a nontrivial center. Calabi has shown that such an M almost splits. More precisely, there exists a compact flat manifold N and a finite Abelian group Φ such that $M = T^k \times_\Phi N$, the quotient space of $T^k \times N$ by a free diagonal action of Φ, where Φ acts freely as translations on the first factor and as isometries on the second factor; see [**Wol77**]. Lawson and Yau [**LY72**] and Eberlein [**Ebe82**] have shown the same fact for closed manifolds M of nonpositive sectional curvature: If $\pi_1(M)$ has nontrivial center $\mathbb{Z}^k$, then M splits as $M = T^k \times_\Phi N$, where N is a closed manifold of nonpositive sectional curvature and Φ is a finite Abelian group acting diagonally and freely on the T^k-factor as translations.

Prior to Lawson and Yau's and Eberlein's work, Conner and Raymond [**CR71**] generalized Calabi's results to homologically injective torus actions. Let (T^k, M) be a torus action on a topological space. For a base point $x_0 \in M$, consider the evaluation map $\mathrm{ev} : (T^k, e) \to (M, x_0)$ sending $t \mapsto t x_0$. Recall that the action is *injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_* : \pi_1(T^k, e) \to \pi_1(M, x_0)$. It is *homologically injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_* : H_1(T^k, \mathbb{Z}) \to H_1(M; \mathbb{Z})$. By Theorem 2.4.2, the injectiveness condition is independent of choice of the base point and the image is a central subgroup of $\pi_1(X)$. For a Riemannian manifold of nonpositive sectional curvature, the existence of a nontrivial center $\mathbb{Z}^k$ of $\pi_1(M)$ guarantees that the manifold has an action of torus T^k, and all such actions are homologically injective.

In this section, topological spaces are always assumed to be paracompact, path-connected and locally path-connected, and either (1) locally compact and semisimply connected, or (2) has the homotopy type of the CW-complex. Therefore, our topological spaces admit covering space theory.

THEOREM 11.6.2 (Splitting Theorem) [**CR71**]). *If a topological space X with $H_1(X; \mathbb{Z})$ finitely generated admits a homologically injective* (*topological*) *torus action (T^k, X), then X splits as $T^k \times_\Phi N$ for some N, where Φ is a finite Abelian group acting diagonally and acting freely on the T^k-factor as translations.*

The *splitting* $X = T^k \times_\Phi N$ as above implies that X has a Seifert fiber space structure with typical fiber T^k and base space N/Φ. All the singular fibers are again tori which are finitely covered by T^k. It also gives rise to another genuine fiber structure; namely, X fibers over the torus T^k/Φ with the fiber N and a finite structure group. The above theorem does not require that the space X be aspherical. On the other hand, the only compact connected Lie group that can act on aspherical manifolds are tori. Therefore, splitting a manifold using a group action for an aspherical manifold forces the group to be a torus. In other words, for aspherical manifolds, there can be no generalization of splitting using compact Lie group actions other than tori.

The condition for a torus action (T^k, X) to be homologically injective is equivalent to the element $[\pi_1(X)]$ in $H^2(Q; \mathbb{Z}^k)$ having finite order [**CR71**]. Keep in mind that the cohomology class $[\pi_1(X)]$ is represented by the extension sequence $1 \to \mathbb{Z}^k \to \pi_1(X) \to Q \to 1$.

DEFINITION 11.6.3. An extension $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$ is called *inner* if the abstract kernel, $Q \to \mathrm{Out}(\Gamma)$, is trivial. Suppose Γ is a subgroup of G. The extension $\mathit{\Pi}$ is *G-inner* if, for every $\sigma \in \mathit{\Pi}$, $\mu(\sigma) \in \mathrm{Aut}(\Gamma)$ is equal to conjugation by an element of G. If $\mathit{\Pi}$ is inner, then it is G-inner.

A normal subgroup A of C is said to be *homologically injective* in C if the inclusion induces an injective homomorphism on the first homology, $H_1(A; \mathbb{Z}) \to H_1(C; \mathbb{Z})$, or equivalently, $A \cap [C, C] = \{1\}$.

The Splitting Theorem 11.6.2 can now be generalized to injective Seifert fiberings with typical fiber a compact solvmanifold of type (R).

THEOREM 11.6.4. *Let X be an injective Seifert fibering with typical fiber a compact solvmanifold $\Gamma\backslash G$ of type* (R). *Assume that $\mathit{\Pi}$, with finitely generated center, acts freely on $\widetilde{X}$. Then the following are equivalent:*

(1) *The abstract kernel, $Q \to \mathrm{Out}(\Gamma)$, of the associated exact sequence $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$, $\mathit{\Pi} = \pi_1(X)$, has finite image in $\mathrm{Out}(\Gamma)$ and the center of Γ, $\mathcal{Z}(\Gamma)$, homologically injects into $C_{\mathit{\Pi}}(\Gamma)$;*
(2) *$X = (\Gamma\backslash G) \times_\Phi X'$, where Φ is a finite group which acts diagonally, as affine maps on the first factor.*

For a proof, we need the following lemmas. Some part of the first lemma is essentially proved in [**CR71**].

LEMMA 11.6.5. *Let $\mathit{\Pi}$ be a finitely generated group, and let $0 \to \mathbb{Z}^k \to \mathit{\Pi} \to Q \to 1$ be a central extension. Then the following are equivalent:*

(1) *$[\mathit{\Pi}] \in H^2(Q; \mathbb{Z}^k)$ has finite order;*
(2) *$\mathbb{Z}^k$ is homologically injective;*
(3) *$\mathit{\Pi}$ contains a normal subgroup $\widehat{Q}$ such that $\mathbb{Z}^k \cap \widehat{Q} = 1$ and $\mathit{\Pi}/(\mathbb{Z}^k \times \widehat{Q})$ is a finite Abelian group.*

PROOF. $(1 \Rightarrow 2)$. Suppose $m[\mathit{\Pi}] = 0 \in H^2(Q; \mathbb{Z}^k)$. Let $\mathbb{Z}^k = A$, $(\frac{1}{m}\mathbb{Z})^k = B$, and $C = B/A$ so that $0 \to A \to B \to C \to 0$ is a short exact sequences of trivial Q-modules. Consider the Bockstein cohomology sequence

$$\cdots \to H^1(Q; C) \xrightarrow{\delta^*} H^2(Q; A) \xrightarrow{\iota} H^2(Q; B) \to \cdots .$$

The fact $m[\mathit{\Pi}] = 0 \in H^2(Q; A)$ implies that $\iota[\mathit{\Pi}] = 0 \in H^2(Q; B)$. Thus there exists $[\eta] \in H^1(Q; C) = \text{Homomorphism}(Q, C)$ such that $\delta^*[\eta] = [\mathit{\Pi}]$. Let $\widehat{Q}$ be the kernel of the homomorphism $\eta : Q \to C$. Then the subgroup of $\mathit{\Pi}$ corresponding to $\widehat{Q}$ splits (consider $H^1(\widehat{Q}; C) \to H^2(\widehat{Q}; A)$). Therefore, we have a subgroup $A \times \widehat{Q}$ of $\mathit{\Pi}$ with $\mathit{\Pi}/(A \times \widehat{Q}) = Q/\widehat{Q} \subset C$. Let $\Delta = Q/\widehat{Q}$, a finite Abelian group. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \widehat{Q} & = & \widehat{Q} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & A \times \widehat{Q} & \to & \mathit{\Pi} & \to & \Delta & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & A & \to & \mathit{\Pi}/\widehat{Q} & \to & \Delta & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The group $\mathit{\Pi}/\widehat{Q}$ is a central extension of A by a finite Abelian group Δ. Since A is torsion free, $\mathit{\Pi}/\widehat{Q}$ is Abelian; see Corollary 5.5.3. Therefore, $[\mathit{\Pi}, \mathit{\Pi}] \subset \widehat{Q}$, which is disjoint from $A = \mathbb{Z}^k$. This implies $\mathbb{Z}^k$ injects into $\mathit{\Pi}/[\mathit{\Pi}, \mathit{\Pi}] = H_1(\mathit{\Pi}; \mathbb{Z})$.

$(2 \Rightarrow 3)$. Let $A = H_1(\mathit{\Pi}; \mathbb{Z})/\text{Torsion}$. Then $\mathbb{Z}^k \xrightarrow{i} H_1(\mathit{\Pi}; \mathbb{Z}) \xrightarrow{p} A$ (p projection) is still injective. Now let $A/(p \circ i)(\mathbb{Z}^k) = \Delta \oplus \mathbb{Z}^m$, where Δ is a finite group. Set $\overline{Q}$ be the preimage of Δ under the composite

$$\mathit{\Pi} \to H_1(\mathit{\Pi}; \mathbb{Z}) \to H_1(\mathit{\Pi}; \mathbb{Z})/\text{Torsion} = A \to \Delta \oplus \mathbb{Z}^m.$$

Then there is a direct summand to $\overline{Q}$, and hence a homomorphism of $H_1(\mathit{\Pi}; \mathbb{Z}) \to \overline{Q}$. Let $\widehat{Q}$ be the kernel of $\mathit{\Pi} \to H_1(\mathit{\Pi}; \mathbb{Z}) \to \overline{Q}$. Then clearly $\mathbb{Z}^k \cap \widehat{Q} = 1$ and $\mathit{\Pi}/(\mathbb{Z}^k \times \widehat{Q}) = \overline{Q}/\mathbb{Z}^k$ is Abelian.

$(3 \Rightarrow 1)$. The commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^k \times \widehat{Q} & \longrightarrow & \widehat{Q} & \longrightarrow & 1 \\
 & & {\scriptstyle =}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \overline{\theta}} & & \\
1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathit{\Pi} & \longrightarrow & Q & \longrightarrow & 1
\end{array}$$

shows that the natural homomorphism $\overline{\theta}^* : H^2(Q;\mathbb{Z}^k) \to H^2(\widehat{Q};\mathbb{Z}^k)$ maps $[\Pi] \in H^2(Q;\mathbb{Z}^k)$ to $[\mathbb{Z}^k \times \widehat{Q}] = 0 \in H^2(\widehat{Q};\mathbb{Z}^k)$. Let $\gamma; H^2(\widehat{Q};\mathbb{Z}^k) \to H^2(Q;\mathbb{Z}^k)$ be the transfer homomorphism. Then $\gamma \circ \overline{\theta}^* =$ multiplication by $m =$ the order of $Q/\widehat{Q}$. Thus, $m[\Pi] = \gamma \circ \overline{\theta}^*([\Pi]) = \gamma(0) = 0$. □

LEMMA 11.6.6. *Let Γ be a group whose center $\mathcal{Z}(\Gamma)$ is a free Abelian group of finite rank. Let $1 \to \Gamma \to \Pi \to Q \to 1$ be an extension whose abstract kernel has finite image. Then the following are equivalent:*

(1) *$[\Pi]$ has finite order in $H^2(Q;\mathcal{Z}(\Gamma))$;*
(2) *Π contains a normal subgroup $\Gamma \times Q'$ such that $\Phi = \Pi/(\Gamma \times Q')$ is a finite group;*
(3) *$\mathcal{Z}(\Gamma)$ homologically injects into $C_\Pi(\Gamma)$.*

PROOF. (1) $\Longleftrightarrow$ (3). Let $P \subset Q$ be the kernel of $Q \to \mathrm{Out}(\Gamma)$, and let $\Pi' \subset \Pi$ be the preimage of P. Since Q/P is finite, the homomorphism $i^* : H^2(Q,\mathcal{Z}(\Gamma)) \to H^2(P,\mathcal{Z}(\Gamma))$, induced by the inclusion $i : P \hookrightarrow Q$, has finite kernel.

To see this, consider the Lyndon spectral sequence for $i : P \hookrightarrow Q$. The $E_2^{p,q}$ terms are, $H^p(Q/P; H^q(P;\mathcal{Z}(\Gamma)))$. We seek kernel of i^*. This is given by the exact sequence

$$0 \to E_\infty^{2,0} \to \ker(i^*) \to E_\infty^{1,1} \to 0.$$

As both $E_\infty^{2,0}$ and $E_\infty^{1,1}$ are finite, $\ker(i^*)$ is finite. (As an alternative argument, we can apply the transfer homomorphism provided that homology of P and Q are finitely generated in low dimensions.) Therefore, $[\Pi] \in H^2(Q,\mathcal{Z}(\Gamma))$ has finite order if and only if $[\Pi'] \in H^2(P,\mathcal{Z}(\Gamma))$ has finite order. Also, for the statement (3), note that $C_\Pi(\Gamma) = C_{\Pi'}(\Gamma)$. Therefore, in proving (1) $\Longleftrightarrow$ (3), it is enough to work with Π' instead of Π. Hence, we assume that the extension $1 \to \Gamma \to \Pi \to Q \to 1$ has trivial abstract kernel. Then

$$1 \to \mathcal{Z}(\Gamma) \to C_\Pi(\Gamma) \to Q \to 1$$

is a central extension. The extensions $[\Pi]$ and $[C_\Pi(\Gamma)]$ are both classified by the same cohomology group $H^2(Q,\mathcal{Z}(\Gamma))$. Furthermore, since the abstract kernels are trivial, there exist direct products, which correspond to each other naturally. This proves the equivalence of (1) and (3), using Lemma 11.6.5.

(1) $\Longrightarrow$ (2). The condition (1) implies that $[C_\Pi(\Gamma)]$ has finite order. By Lemma 11.6.5, $C_\Pi(\Gamma)$ contains a normal subgroup $\mathcal{Z}(\Gamma) \times Q'$ such that $C_\Pi(\Gamma)/(\mathcal{Z}(\Gamma) \times Q')$ is a finite group. However, $\mathcal{Z}(\Gamma) \times Q'$ may not be normal in Π. Let C' be the intersection of all conjugates of $\mathcal{Z}(\Gamma) \times Q'$ by elements of Π. Since $C_\Pi(\Gamma)$ is normal in Π, and $\mathcal{Z}(\Gamma) \times Q'$ has finite index in $C_\Pi(\Gamma)$, there are only finitely many conjugacy (by elements of Π) classes of $\mathcal{Z}(\Gamma) \times Q'$. Therefore C' is normal in Π and has finite index in $C_\Pi(\Gamma)$. Moreover C' splits also, which we denote by $\mathcal{Z}(\Gamma) \times Q'$ again. Let $\Pi' = \Gamma \cdot Q'$ so that $1 \to \Gamma \to \Pi' \to Q' \to 1$ is exact. Clearly, this splits as $\Pi' = \Gamma \times Q'$, is normal in Π, and $[\Pi : \Pi'] = [\Pi : \Gamma \cdot C_\Pi(\Gamma)][\Gamma \cdot C_\Pi(\Gamma) : \Pi']$ is finite.

(2) $\Longrightarrow$ (3). Since $\Pi/(\Gamma \times Q')$ is finite, $C_\Pi(\Gamma)/(\mathcal{Z}(\Gamma) \times Q')$ is finite. Now apply Lemma 11.6.5. □

11.6.7 (Proof of Theorem 11.6.4). (2)$\Rightarrow$(1). Suppose X is of the form $X = (\Gamma\backslash G) \times_\Phi (X')$. Let $Q' = \pi_1(X')$ and $C = C_\Pi(\Gamma)$. Since G is simply connected, $\pi_1((\Gamma\backslash G) \times X') = \Gamma \times Q'$ and $\Pi/(\Gamma \times Q') = \Phi$, a finite group. Since C contains Q', and the image of Q' in Q has finite index, the abstract kernel $Q \to \mathrm{Out}(\Gamma)$ factors through the finite group Φ. Moreover, $\mathcal{Z}(\Gamma) \times Q'$ has finite index in C. By Lemma 11.6.6, this implies that $\mathcal{Z}(\Gamma)$ homologically injects to C.

(1)⇒(2). Assume X satisfies (1). Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & G & \longrightarrow & S^* & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

be the diagram associated with $(G, \widetilde{X})$. Here all the vertical maps are inclusions, and S^* is the subgroup of $\mathrm{TOP}(\widetilde{X})$ generated by the subgroups Π and G.

The abstract kernel $Q \to \mathrm{Out}(\Gamma)$ induces an abstract kernel $Q \to \mathrm{Out}(G)$ since every automorphism of Γ uniquely extends to an automorphism of G. In fact, this abstract kernel is the abstract kernel for the bottom exact sequence of the diagram above. Since the abstract kernel $\phi : Q \to \mathrm{Out}(G)$ has finite image, it lifts to a homomorphism $Q \to \mathrm{Aut}(G)$ which we denote by ϕ again. This is a result of the statement that any finite extension of $\mathrm{Inn}(G)$ splits.

Here is an argument: Let $1 \to G_1 \to H \to F \to 1$ be exact, where $G_1 = \mathrm{Inn}(G)$ above and F is finite. Since F is finite, $[H] \in H^2(F; \mathcal{Z}(G_1))$ has a finite order. If $f : F \times F \to \mathcal{Z}(G_1)$ is a cocycle, $mf = 0$, where m is the order of the cohomology group. If $\lambda : F \to \mathcal{Z}(G_1)$ is such that $\delta\lambda = mf$, since G is simply connected solvable of type (R), so is G_1, and hence G_1 is *divisible*. We have $f(x, y) = (\delta \frac{1}{m} \lambda)(x, y)$. Thus f is a coboundary.

We claim that $S^* = G \rtimes Q$. Since the abstract kernel lifts to $\phi : Q \to \mathrm{Aut}(G)$, there is a canonical one-to-one correspondence between $\mathrm{Opext}(Q, Z(G), \phi)$ and $\mathrm{Opext}(Q, G, \phi)$ by sending the semidirect product $Z(G) \rtimes Q$ to the semidirect product $G \rtimes Q$; see Definition 5.2.4. If $[E] \in \mathrm{Opext}(Q, Z(G), \phi)$ corresponds to $[S^*] \in \mathrm{Opext}(Q, G, \phi)$, there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z}(G) & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & G & \longrightarrow & S^* & \longrightarrow & Q & \longrightarrow & 1, \end{array}$$

where all the vertical maps are inclusions. To see that S^* splits, it will be enough to show that E splits.

Certainly $C_{S^*}(G) \subset E$ and $C_{S^*}(G)/\mathcal{Z}(G)$ has finite index in Q. The hypothesis that $\mathcal{Z}(\Gamma)$ homologically injects into $C = C_{\Pi}(\Gamma)$ implies that $[C]$ has finite order in $H^2(C/\mathcal{Z}(\Gamma); \mathcal{Z}(\Gamma))$. Therefore $\epsilon_*[C] = 0$ in $H^2(C/\mathcal{Z}(\Gamma); \mathcal{Z}(G))$, where ϵ_* is induced by $\epsilon : \mathcal{Z}(\Gamma) \to \mathcal{Z}(G)$. Thus we have $[E] = 0$ in $H^2(Q; \mathcal{Z}(G))$ because $[E] = i^*(\epsilon_*[C])$ from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z}(G) & \longrightarrow & C & \longrightarrow & C/\mathcal{Z}(G) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow i & & \\ 1 & \longrightarrow & \mathcal{Z}(G) & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 1\,. \end{array}$$

This proves that S^* splits into $G \rtimes Q$.

Clearly, by Lemma 11.6.6, there exists a subgroup Q' of Π such that $\Gamma \cap Q' = 1$, $Q' \subset C_\Pi(\Gamma)$, and $\Gamma \times Q'$ is normal in Π with $\phi = \Pi/\Gamma \times Q'$ finite. Without loss of generality, we may assume that Q' is in $1 \times Q \subset G \rtimes Q$.

The action $(G, \widetilde{X})$ is free and $(G, \widetilde{X}) = (G, G \times W)$ equivariantly. For the group operation of $\mathrm{TOP}_G(G \times W)$, and the action of $\mathrm{TOP}_G(G \times W)$ on $G \times W$ are as in Corollary 4.2.10:

$$\begin{aligned}(\lambda_1, \alpha_1, h_1) \cdot (\lambda_2, \alpha_2, h_2) &= (\lambda_1 \cdot (\alpha_1 \circ \lambda_2 \circ h_1^{-1}), \alpha_1 \circ \alpha_2, h_1 \circ h_2),\\ (\lambda, \alpha, h) \cdot (x, w) &= (\alpha(x) \cdot (\lambda(h(w)))^{-1}, h(w)).\end{aligned}$$

The group $G \subset G \rtimes Q$ acts on $G \times W$ as left translations on the first factor and trivially on the second. In general, Q factor mixes the second factor into the first. We would like to alter W to W' so that $\widetilde{X} = G \times W'$ on which Q' acts as a subgroup of $\mathrm{TOP}(W) \subset \mathrm{M}(W, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W))$. For $\alpha \in Q \subset G \rtimes Q$, let $(\lambda(\alpha), \tau(\alpha), \alpha)$ denote its representation as an element of $\mathrm{TOP}_G(W \times G)$. Then we check that

$$\lambda(\alpha\beta) = \lambda(\alpha) \cdot \tau(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}.$$

If we introduce the notation $\alpha_* : \mathrm{Maps}(W, G) \to \mathrm{Maps}(W, G)$ by

$$\alpha_*(s) = \tau(\alpha) \circ s \circ \alpha^{-1},$$

the above becomes

$$\lambda(\alpha\beta) = \lambda(\alpha) \cdot \alpha_*(\lambda(\beta)).$$

Let $s \in \mathrm{Maps}(W, G)$, and put $W' = \{(s(w), w) \mid w \in W\} \subset G \times W$. Suppose, under the action of Q on $G \times W$, W' is invariant. That is, $\alpha(s(w), w) = (s(\alpha w), \alpha w)$ for all $\alpha \in Q$. (Notice that, for $w' = (s(w), w)$, this equality means that

$$\alpha(1, w') = (1, (\alpha w)')$$

for $w' \in W'$ so that Q acts on $G \times W'$ as a subgroup of $\mathrm{TOP}(W')$.) Comparing the first slots, we get $s(\alpha w) = \tau(\alpha)(sw) \cdot (\lambda(\alpha)(\alpha w))^{-1}$. Thus,

$$\begin{aligned}\lambda(\alpha)(\alpha w) &= (s(\alpha w))^{-1} \cdot \tau(\alpha)(sw)\\ &= (s(\alpha w))^{-1} \cdot \tau(\alpha)\ s(\alpha^{-1}\alpha)w\\ &= (s^{-1} \cdot \tau(\alpha) \circ s \circ \alpha^{-1})(\alpha w)\\ &= (s^{-1} \cdot \alpha_*(s))(\alpha w).\end{aligned}$$

This implies $\lambda(\alpha) = s^{-1} \cdot \alpha_*(s)$ for all $\alpha \in Q$.

Conversely, if there is $s \in \mathrm{M}(W, G)$ such that $\lambda(\alpha) = s^{-1} \cdot \alpha_*(s)$ for all $\alpha \in Q$, then W' defined as above will be Q-invariant. Therefore, if we show that the non-Abelian group cohomology $H^1(Q; \mathrm{Maps}(W, G))$ vanishes, the invariant W' will have been shown to exist; see Subsection 5.7.1. We have shown that this cohomology set is trivial in Subsection 7.6.6 and so W' exists and is Q-invariant.

It is easy to see that

$$\begin{array}{ccc} G \times W & \xrightarrow{\ \zeta\ } & G \times W\\ (x, w) & \longrightarrow & (x \cdot s(w), w)\end{array}$$

is G-equivariant and weakly Q-equivariant. More precisely, for $a \in G$ and $\alpha \in Q$,

$$\zeta(a(x, w)) = (ax \cdot s(w), w) = a\zeta(x, w),$$

$$\zeta(\alpha(x, w)) = (\tau(\alpha)(x) \cdot (\lambda(a)(\alpha w))^{-1} \cdot s(\alpha w), \alpha w) = \mu(s^{-1})(\alpha)(\zeta(x, w)),$$

where $\mu(s^{-1}) = \mu(s^{-1},1,1)$ for $(s^{-1},1,1) \in \mathrm{M}(W,G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(W))$. We summarize the facts that we have proved as follows.

(1) Π contains a subgroup Q' such that $\Gamma \cap Q' = 1$ and $\Gamma \times Q'$ is normal, $\Phi = \Pi/\Gamma \times Q'$ is finite.
(2) $G \subset G \rtimes Q$ acts only on the first factor of $\widetilde{X} = G \times W'$ as left translations and trivially on W'-factor.
(3) $Q \subset G \rtimes Q$ acts on $\widetilde{X} = G \times W'$ as a subgroup of $\mathrm{Aut}(G) \rtimes \mathrm{TOP}(W) \subset \mathrm{Maps}(W,G) \rtimes (\mathrm{Aut}(G) \rtimes \mathrm{TOP}(W))$; i.e., $\alpha \mapsto (1, \tau(\alpha), \alpha)$.
(4) Since $\tau(\alpha)$ is conjugation by α, for $\alpha \in Q'$, $\tau(\alpha) = \mathrm{id}$. Therefore, for $\alpha \in Q'$, $\alpha \mapsto (1,1,\alpha)$.
(5) From (2) and (4), $(a,\alpha) \in G \times Q'$ acts on $G \times W'$ by
$$(a,\alpha)(x,w) = (ax, \alpha w).$$

Consequently,
$$\begin{aligned} X &= \Pi\backslash(G \times W) \\ &= (\Gamma \times Q'\backslash G \times W')/\Phi \\ &= (\Gamma\backslash G) \times_\Phi (Q'\backslash W'). \end{aligned}$$

Clearly, the Φ-action on $\Gamma\backslash G$ is as a subgroup of $\mathrm{Aff}(\Gamma\backslash G)$ because the action of $G \rtimes Q$ on G is through $G \rtimes \mathrm{Aut}(G)$. This completes the proof of the theorem. □

We may give an explicit Seifert homeomorphism between $\Pi\backslash(G \times W)$ and $(\Gamma\backslash G) \times_\Phi (Q'\backslash W)$ as follows: On $G \times W$, let $\alpha \in Q$ and $s \in \mathrm{M}(W,G)$ be such that $\lambda(\alpha) = s^{-1} \cdot \alpha_*(s)$, for all $\alpha \in Q$. We have the following commutative diagram

$$\begin{array}{ccccc} (x,w) & \longrightarrow & (x \cdot s(w), w) & \longrightarrow & (x, (s(w), w)) \in G \times W' \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ (\tau(\alpha)(x) \cdot (\lambda(\alpha)(\alpha w))^{-1}, \alpha w) & \to & (\tau(\alpha)(x) \cdot s(\alpha w), \alpha w) & \to & (\tau(\alpha)(x), s\alpha(w), \alpha w). \end{array}$$

Notice that the bottom middle term is
$$(\tau(\alpha)(x), s\alpha(w), \alpha w) = (\tau(\alpha)(x) \cdot \tau(\alpha)s(w) \cdot (\lambda(\alpha)(\alpha w))^{-1}, \alpha w).$$

In the left-hand square, there is a Seifert homeomorphism between $\Pi\backslash(G \times W)$ and $\Pi\backslash(G \times W)$ achieved by conjugation on $G \times W$ by $s \in \mathrm{M}(W,G)$. The second square is a homeomorphism between $G \times W$ and $G \times W'$. The actions of Q are equivariant. We note that in the last column that when $\alpha \in Q'$, $\tau(\alpha)(x) = x$.

Corollary 11.6.8. *Let $(\Gamma\backslash G, X)$ be an injective Seifert fiber space where G is solvable of type* (R) *and $\Pi = \pi_1(X)$. Then the following are equivalent when the center of Π is finitely generated:*

(1) *The associated exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$ has trivial abstract kernel and $\mathcal{Z}(\Gamma)$ homologically injects to $C_\Pi(\Gamma)$;*
(2) *$X = (\Gamma\backslash G) \times_\Delta X'$, where Δ is a finite Abelian group which acts diagonally, freely as translations along the torus $\mathcal{Z}(\Gamma)\backslash\mathcal{Z}(G)$.*

The *splitting* $X = (\Gamma\backslash G) \times_\Phi X'$ in the theorem implies that X has a Seifert fiber structure with the typical fiber $\Gamma\backslash G$ and the base space X'/Φ. The singular fibers are *orbifolds* finitely covered by $\Gamma\backslash G$. In the case of $1 \to \Gamma \to \Pi \to Q \to 1$ with trivial abstract kernel, the singular fibers are again homogeneous spaces finitely covered by $\Gamma\backslash G$. In this case, X also has a genuine fiber structure over the homogeneous space $(\Gamma\backslash G)/\Delta$ with fiber X' and finite Abelian structure group Δ.

Let $(\Gamma\backslash G, X)$ be as in the corollary and satisfy condition (1). Let X^* be the covering of X with $\pi_1(X^*) = C$. Then the torus $T = \mathcal{Z}(\Gamma)\backslash\mathcal{Z}(G)$ acts on X^*. This torus action is homologically injective. In fact, T acts on the space $(\mathcal{Z}(G)\times W)/C$ homologically injectively. Therefore, $(\mathcal{Z}(G)\times W)/C = T\times_\Delta X'$. Let $G_1 = G/\mathcal{Z}(G)$ and $\Gamma_1 = \Gamma/\mathcal{Z}(\Gamma)$. Then Γ_1 is a lattice in G_1, and X fibers over $\Gamma_1\backslash G_1$ with fiber $T\times_\Delta X'$. This is explained by the following commutative diagram easily:

$$\begin{array}{ccc}
\mathcal{Z}(G)\times W & \xrightarrow{C\backslash} & T^k\times_\Delta X' \\
\downarrow & & \downarrow \\
G\times W & \xrightarrow{\Pi\backslash} & X \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\Gamma_1\backslash} & \Gamma_1\backslash G_1\,.
\end{array}$$

All the horizontal maps are coverings and the vertical rows are fibrations. The homotopy exact sequence of the second fibration is $1\to C\to \Pi\to\Gamma_1\to 1$.

Remark 11.6.9. We cite some instances where conditions (1) or (2) of the theorem or the corollary can be verified.

(1) If $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, then Corollary 11.6.8 becomes a restatement of Theorem 11.6.2.

(2) Consider 3-dimensional manifolds M which are injective Seifert fiberings with $G = \mathbb{R}^1$ and typical fiber S^1. They are modeled on $(\mathbb{R}^1, \mathbb{R}^1\times\mathbb{R}^2)$ or $(\mathbb{R}^1, \mathbb{R}^1\times S^2)$. If M is noncompact and $\Pi = \pi_1(M)$ is inner (see Definition 11.6.3), then (S^1, M) is homologically injective. If Π is not inner, then it has a double cover which is inner and so Theorem 11.6.4 always holds. The same thing holds for M compact with nonempty boundary.

If M is compact with empty boundary, then M satisfies the conditions of Theorem 11.6.4 except for those classical Seifert fiberings where M is of type $\mathfrak{o}_1$ or n_2 ($\mathfrak{o}_0$, $\mathfrak{o}_n$ in Seifert's notation, see Subsection 14.13.1) and $e(M)$, the Euler number of M, is different from 0. The easiest way to verify this assertion is to examine the diagram in Proposition 10.5.10 from where it can easily be deduced that the conditions of Lemma 11.6.6 is valid. (We also point out that if $\pi_1(M)$ is finite, the Seifert 3-manifold is closed, covered by S^3, and is of type $\mathfrak{o}_1$ with Euler number $\neq 0$).

(3) For a specific example, consider the nonprincipal S^1-bundles over $\mathbb{R}P_2$. Each bundle is determined by a characteristic class $a\in H^2_\varphi(\mathbb{R}P_2;\mathbb{Z})\cong\mathbb{Z}$, where $\mathbb{Z}$ is a nontrivial local coefficient system. In these bundles, the orientation of a circle fiber is reversed while traversing an orientation reversing curve in the base $\mathbb{R}P_2$. If $a\neq 0$, the bundle is covered by S^3 and $\pi_1(M(a))$ is finite. If $a = 0$, then there is a cross section and $M(a=0)$ can be identified with $S^1\times_{\mathbb{Z}_2} S^2$, where $\mathbb{Z}_2$ acts freely on S^2 and by reflection on S^1. The extension exact sequence for this manifold is $1\to\mathbb{Z}\to\mathbb{Z}\rtimes\mathbb{Z}_2\to\mathbb{Z}_2\to 1$, and it is the noncentral extension associated with this injective Seifert fibering which is modeled on $(\mathbb{R}^1, \mathbb{R}^1\times S^2)$. There is a projection onto $I = \mathbb{Z}_2\backslash S^1$ with the inverse image over the interior points being S^2 and over the end points being $\mathbb{R}P_2$. Thus, $S^1\times_{\mathbb{Z}_2} S^2$ is homeomorphic to $\mathbb{R}P_3\#\mathbb{R}P_3$.

$M(a = 0)$ satisfies Theorem 11.6.4, but not the corollary as the extension exact sequence is not inner.

Similarly, take $M = S^1 \times_{\mathbb{Z}_2} \Sigma_g$, where Σ_g is the orientable double cover of the nonorientable surface Σ'_k with genera satisfying $g = k - 1$. Again, M satisfies Theorem 11.6.4, but not Corollary 11.6.8. The Euler characteristic class of M is $0 \in H^2_\varphi(\Sigma'_k; \mathbb{Z}) \cong \mathbb{Z}$, where each fiber over each orientation reversing curve is reversed. All the other nonprincipal bundles are injective Seifert fiberings with characteristic class of infinite order. These are classical Seifert manifolds of type $\mathfrak{n}_2$ in the notation of Subsection 14.13.1.

(4) The Seifert 3-manifolds modeled on $(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{R}^1)$ with typical fiber T^2 and holonomy infinite never satisfy Theorem 11.6.4. In fact, these are 3-manifolds with the Sol-geometry.

(5) Consider the closed 4-dimensional manifolds M modeled on $(\mathbb{R}^2, \mathbb{R}^2 \times W)$, with $W = \mathbb{R}^2$, and typical fiber T^2. The associated exact sequence to the Seifert structure is

$$1 \to \mathbb{Z}^2 \to \Pi \to Q \to 1$$

with $\Phi : Q \to \operatorname{Aut}(\mathbb{Z}^2)$ and $\mathit{\Pi}$ acting freely. If the image of Φ is finite, then these 4-manifolds are finitely fiberwise covered by an orientable Seifert fibering with an injective torus action. These coverings all admit a complex structure with a complex 1-torus as typical fiber. These complex manifolds are elliptic surfaces with possible multiple fibers (i.e., singular fibers) but no exceptional fibers. The complex structure is Kähler if and only if $[\mathit{\Pi}] \in H^2(Q; \mathbb{Z}^2)$ has finite order and hence the torus action is homologically injective. In this finite case, the finite central covering can be holomorphically deformed to an algebraic surface $(T^2, T^2 \times_\Delta Y)$. If the order $[\mathit{\Pi}]$ is not finite, the complex structure is never Kähler; see [**CR72b**] for more details. For an extensive discussion as to when holomorphic T^k-actions on complex manifolds can holomorphically fibered over the torus, see [**Car72**].

(6) We also have the following:

Theorem ([**CR72b**, §12.6]). *Let (T^k, M) be an action on a homologically Kählerian manifold. Then all isotropy groups are finite if and only if the action is homologically injective.*

A closed (connected) manifold M is homologically Kählerian [**Bor60**, Chap XII §6] if there exists a class $a \in H^2(M; \mathbb{Q})$ such that $H^s(M; \mathbb{Q}) \xrightarrow{\cup a^{n-s}} H^{2n-s}(M; \mathbb{Q})$ is an isomorphism for $s = 0, 1, \dots n$ with $2n$ equal to the dimension of M. A compact connected complex manifold which admits a Kähler metric is homologically Kählerian.

(7) Let (T^k, X) be an injective torus action whose extension sequence is given by

$$0 \to \mathbb{Z}^k \to \pi_1(X) \to Q \to 1.$$

We have

Theorem ([**CR72c**, §2.1]). *If $k >$ rank $H^2(Q; \mathbb{Z})$, then for any $\mathit{\Pi}$ as above, there is an integer $j \geq k -$ rank $H^2(Q; \mathbb{Z})$ and a direct product decomposition $T^j \times T^{k-j}$ so that $H_1(T^j; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ injects and the image $H_1(T^{k-j}; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is finite. Consequently, we may write (T^k, X) as $(T^j \times T^{k-j}; T^j \times_\Delta Y)$, where T^j acts only as translations on the first factor, Δ acts diagonally on $T^j \times Y$ and as translations on the first factor. The group T^{k-j} acts on $T^j \times Y$ and injectively on Y.*

For example, in the case of elliptic surfaces treated in (5), which are not homologically injective, we have $(T^2, X) = (S^1_1 \times S^1_2, S^1_1 \times_\Delta Y)$. Therefore, Y is a Seifert 3-manifold with an injective but not homologically injective S^1_2-action on Y. Thus, any elliptic surface of the kind treated in (5) can be topologically or smoothly analyzed by this method.

(8) Consider all closed M which admit a complete Riemannian metric whose curvature is nonpositive and for which $\pi_1(M)$ contains a nontrivial normal Abelian subgroup. For, the centralizer of the maximal normal Abelian subgroup will be of finite index in $\pi_1(M)$. One then applies the center theorem of [**LY72**] yielding a homologically injective torus action on a finite regular covering. This implies that condition (1) of Theorem 11.6.4 will hold.

Example 11.6.10. Let G be a 3-dimensional Heisenberg group, i.e., the group of all upper triangular matrices with diagonal entries 1. Consider

$$x = I + E_{1,2}, \quad y = I + E_{2,3}, \quad z = I + E_{1,3} \in G,$$

where I is the identity matrix, and $E_{i,j}$ a 3×3 matrix whose (i,j)-entry is 1, and 0 elsewhere. Let Γ be the lattice generated by x^2, y, and z. Let $N = Q' \backslash \mathbf{H}$ be a hyperbolic surface of genus 2, so $Q' \subset \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group. Let $\Phi = \mathbb{Z}_2$ act on $\Gamma\backslash G$ and on N as follows: Let the nontrivial generator $\tau \in \Phi$ act on the universal covering group level as (right) translation by x. It also acts on the surface N by a rotation by 180° with two fixed points. The quotient $\Phi\backslash N$ is a torus with two singular points. Now the manifold $M = (\Gamma\backslash G) \times_{\mathbb{Z}_2} N$ has associated homotopy exact sequence $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$, where $Q = Q' \rtimes \mathbb{Z}_2$. Clearly, $\Gamma \times Q'$ is normal in $\mathit{\Pi}$ and has index 2. The only torus action on M is the circle action of $\mathcal{Z}(G)/\mathcal{Z}(\Gamma)$. Clearly,

$$\mathcal{Z}(\Gamma) \cong \mathbb{Z}.$$

The circle action on M is not homologically injective. This is obvious because the center $\mathbb{Z}$ cannot be separated even in Γ. In other words, $1 \to \mathcal{Z}(\Gamma) \to \Gamma \to \mathbb{Z}^2 \to 1$ represents an element of infinite order in $H^2(\mathbb{Z}^2;\mathbb{Z})$.

Consequently, $[\mathit{\Pi}] \in H^2(Q;\mathcal{Z}(\Gamma))$ has infinite order. This shows that there is no way of splitting off this circle using the action of $\mathcal{Z}(G)/\mathcal{Z}(\Gamma)$. From the construction of the manifold M, there is a splitting of M as $(\Gamma\backslash G) \times_{\mathbb{Z}_2} N$. The injective Seifert fiber space with $\Gamma\backslash G$-fiber

$$\Gamma\backslash G \to M \to \Phi\backslash N$$

has two singular points which are the fixed points of the action of Φ on N. The singular fibers are nilmanifolds $(\Gamma\backslash G)/\Phi$. Note that the extension $1 \to \Gamma \to \mathit{\Pi} \to Q \to 1$ is not inner, but just G-inner, and hence the $\mathbb{Z}_2$-action on $\Gamma\backslash G$ is not in the S^1-action.

Also the action of $\mathbb{Z}_2$ on $\Gamma\backslash G$ lifts to a new lattice $\Gamma' = \langle x, y, z\rangle$, and M has a genuine fibration structure

$$N \to M \to \Gamma\backslash G',$$

where $\Gamma\backslash G'$ is a nilmanifold doubly covered by $\Gamma\backslash G$.

11.6.11 (Nonuniqueness of fibers). This example is due to Tollefson [**Tol69**]. For any closed orientable surface Σ_g, of genus g, there exists a closed oriented surface Σ_m, $m = k(g-1)+1$, with a free $\mathbb{Z}_k$-action whose orbit space is Σ_g. (For example, take $m = 3$, $k = 2$, and $g = 2$.) Form $(S^1, S^1 \times_{\mathbb{Z}_k} \Sigma_m) = (S^1, M)$. M fibers over $S^1 = \mathbb{Z}_k\backslash S^1$, with fiber Σ_m. At the same time, $(S^1, M) \xrightarrow{S^1\backslash} \mathbb{Z}_k\backslash\Sigma_m = \Sigma_g$ is a principal S^1-bundle. It has a characteristic class $a \in H^2(\Sigma_g;\mathbb{Z}) = \mathbb{Z}$, which determines the extension exact sequence

$$0 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(\Sigma_g) \to 1.$$

(The class a is the same as $e(M)$, the Euler class of this principal S^1-bundle.) Since M has a $\mathbb{Z}_k$ central covering $S^1 \times \Sigma_m$, the order of a is finite. In fact, its order must be 1. Therefore, $(S^1, M) = (S^1, S^1 \times \Sigma_g)$. So the same Seifert fibering $(S^1, M) \to \Sigma_g$ has many different fiberings over S^1 (just vary k).

The nonuniqueness of fibers can be rather subtle. Charlap [**Cha65**] constructed closed flat manifolds $M_1 = S^1 \times N_1$ and $M_2 = S^1 \times N_2$ such that M_1 and M_2 are diffeomorphic but N_1 and N_2 have nonisomorphic fundamental groups.

Somewhat simpler examples can be constructed with algebraic surfaces as discussed in [**CR72a**]. Let Y be a closed orientable 2-manifold with an action of $\mathbb{Z}_n$ such that no nontrivial subgroup of $\mathbb{Z}_n$ acts freely on Y. Form $(T^2 \times_{\mathbb{Z}_n\times\mathbb{Z}_n} Y)$. This admits a complex algebraic structure. It can be shown to be homeomorphic to $S^1 \times M_1$ and to $S^1 \times M_2$, where M_1 and M_2 are Seifert 3-manifolds with a homologically injective S^1-action. With a judicious choice of the action of $\mathbb{Z}_n \times \mathbb{Z}_n$ on $T^2 \times Y$, $\pi_1(M_1) \neq \pi_1(M_2)$; see [**CR72a**] for a complete description including the Charlap examples.

11.6.12. From the examples in Subsection 11.6.11, it is clear that a homologically injective Seifert fiber space (T^k, X) can have vastly different splittings $T^k \times_\Delta Y$. It is of interest to classify these splittings. However, to achieve this, one needs to narrow the type of splittings to be considered. Such a classification can be found in [**CR72a**]. Other discussions can also be found in [**Sch81b**] and [**Sad91b**]. Below, we shall treat a special case of this classification.

11.6.13. Let G be a compact connected Lie group acting on a space X and H a closed subgroup. Suppose $f : (G, X) \to (G, G/H)$ is an equivariant map. Then the trivial coset $\{H\}$ in G/H is an H-slice in $(G, G/H)$. Put $Y = f^{-1}(\{H\}) \subset X$, then Y is the pullback of the H-slice $\{H\}$ and so is a global H-slice in (G, X) (Proposition 1.6.4). Hence (G, X) can be written as fiber bundle with fiber Y, structure group H, and base space G/H. In terms of equivariant maps, we have

$$\begin{array}{ccccc}
(G\times H, G) & \longleftarrow & (G\times H, G\times Y) & \xrightarrow{G\backslash} & (H, Y) \\
\downarrow{\scriptstyle H\backslash} & & \downarrow{\scriptstyle H\backslash} & & \downarrow{\scriptstyle H\backslash} \\
(G, G/H) & \longleftarrow & (G, G\times_H Y) & \xrightarrow{G\backslash} & H\backslash Y = G\backslash X.
\end{array}$$

If $(G, X) = (G, G \times_H Y)$, then $f_0 : (G, X) \to (G, G/H)$ defined by $f_0(((g, y)) = gH$, where $((g, y)) = ((gh^{-1}, hy))$ for $h \in H$, is an equivariant map. (Here $((g, y))$ represents the H-orbit in (G, X) of the point $(g, y) \in G \times Y$.)

Of course, there may be other G-equivariant maps $f : (G, X) \to (G, G/H)$. Then $f^{-1}\{H\} = Y_f$ and we get another splitting of (G, X) as $(G, G\times_H, Y_f)$. Here $H\backslash Y_f = G\backslash X = H\backslash Y_{f_0}$, where Y_{f_0} is the original splitting Y. To classify these splittings of (G, X) with *fixed* subgroup H, we say that two splittings $(G, G\times_H, Y_{f_1})$ and $(G, G\times_H, Y_{f_2})$ of (G, X) are *strongly equivalent* if there is an H-equivariant homeomorphism $\theta : Y_{f_1} \to Y_{f_2}$ such that

$$\begin{array}{ccc} Y_{f_1} & \xrightarrow{\theta} & Y_{f_2} \\ & {\scriptstyle H\backslash}\searrow \quad \swarrow{\scriptstyle H\backslash} & \\ & G\backslash X & \end{array}$$

commutes.

11.6.14. As a special case, consider (G, X) of the form $(G, (G \times_H Y)) = (G, G \times_H Y_{f_0})$ where H acts *freely* on Y. We look for all the strong equivalence classes $(G, G \times_H, Y_f)$, G-equivariantly homeomorphic to (G, X), where H acts freely on Y_f. Now, strongly equivalent free actions of H on Y_f must yield equivalent principal G-bundles over $G\backslash X$. Since the principal G-bundle fibers over $(G, G/H)$ with fiber Y_f and structure group H, the structure group of the principal G-fibering $(G, X) \to G\backslash X = H\backslash Y_f$ is reducible to the closed subgroup H.

Let $a \in H^1(H\backslash Y; \mathfrak{G})$ be a sheaf of germs of continuous functions of $H\backslash Y$ into G representing the principal G-bundles (G, X) over $G\backslash X$. Let $b \in H^1(H\backslash Y; \mathfrak{H})$ be the principal H-bundles over $H\backslash Y$ representing (H, Y). Since a is represented by $(G, G \times_H Y)$, $j^*(b) = a$ is a reduction of the structure group, where $j^* : H^1(H\backslash Y; \mathfrak{H}) \to H^1(H\backslash Y; \mathfrak{G})$.

THEOREM ([**CR72a**, 2.7]). *The set of strong equivalence classes (relative to the choice a), is the set of all bundle reductions of a (to the subgroup H); that is, all elements $b \in H^1(H\backslash Y; \mathfrak{H})$ such that $j^*(b) = a$.*

11.6.15. In particular, if G is a torus T^k and H is a closed finite subgroup, then the theorem reduces to the Bockstein sequence

$$0 \longrightarrow H^1(H\backslash Y; \mathbb{Z}^k) \xrightarrow{i} H^1(H\backslash Y; \mathbb{Z}^k) \longrightarrow H^1(H\backslash Y; \mathfrak{H}) \xrightarrow{\beta = j^*} H^2(H\backslash Y; \mathbb{Z}^k).$$

For example, the choice of a is an element of $H^2(H\backslash Y; \mathbb{Z}^k)$. The strong equivalence classes are those elements $b \in H^1(H\backslash Y; H)$ which are carried into a by the Bockstein map β. The set is identified with the elements of

$$H^1(H\backslash Y; \mathbb{Z}^k)/i(H^1(H\backslash Y; \mathbb{Z}^k)).$$

If we turn to the example in Subsection 11.6.11, we see that the bundle over Σ_g is trivial. Then, the set of strongly inequivalent splittings is in one-to-one correspondence with $H^1(\Sigma_g; \mathbb{Z}_k)$ since the Bockstein map is trivial. Note also that many of the inequivalent splittings will have Y_f disconnected.

For another illustration, consider the nonorientable closed 2-manifolds X of nonorientable genus k, $k \geq 1$. There are exactly two principal S^1-bundles over each X, given by an element $a \in H^2(X; \mathbb{Z}) \cong \mathbb{Z}_2$. Let $H = \mathbb{Z}_m$. The homomorphism $m : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is trivial if m is even and an isomorphism if m is odd. Therefore, the Bockstein map $\beta : H^1(X; \mathbb{Z}_m) \to H^2(X; \mathbb{Z})$ is onto for m even and trivial for m odd.

For example, if $X = \mathbb{R}P_2$, $H^1(\mathbb{R}P_2; \mathbb{Z}_m) = \mathbb{Z}_2$ m even, and 0 m odd. Consequently, if m is even, then for each of the two principal S^1-bundles, there is just one strong equivalence class of splittings for a fixed subgroup, $\mathbb{Z}_m$. For the trivial bundle, the fibers are disconnected and for the nontrivial bundle, the fibers are disconnected for $m > 2$. If m is odd, the nontrivial bundle fibers over $\mathbb{Z}_2\backslash S^1$ but every S^1-equivariant map to S^1 must have even degree. In other words, any twisted covering of $\mathbb{R}P_2$, necessary for X to be the nontrivial S^1-bundle, must be even order. For the trivial bundle $X = S^1 \times \mathbb{R}P_2$, there is exactly one strong equivalence class of splitting for each odd m. The fibers in this case are m disjoint copies of $\mathbb{R}P_2$.

The reader may wish to investigate the principal S^1-bundle X over the Klein bottle or surfaces of nonorientable even higher genus. In all these cases, the number of strongly inequivalent splittings is given by the theorem.

11.7. Maximal torus actions

Definition 11.7.1. Any compact, connected Lie group which acts effectively on a closed admissible manifold (see Definition 3.2.1) (e.g., aspherical manifolds) is a torus T^k with $k \leq$ rank of $\mathcal{Z}(\pi_1(M))$, the center of $\pi_1(M)$; see Theorem 3.2.2. When $k = \text{rank}\ \ \mathcal{Z}(\pi_1(M))$, the torus action is called a *maximal torus action*.

Lemma 11.7.2 ([**CR75**, Lemma 1]). *If (T^k, M) is an effective action of a torus on a closed aspherical manifold and if $H \subset \pi_1(M,x)$ is a central subgroup which contains $\text{Im}(\text{ev}_*^x)$, then $H/\text{Im}(\text{ev}_*^x)$ contains no elements of finite order.*

Proof. Let M_H be the covering space associated to the subgroup H (i.e., with $\pi_1(M_H) = H$). By Theorem 2.5.1, the T^k-action lifts to M_H. We will show first that (T^k, M_H) is free. Let $b \in M_H$, and suppose T_b be the isotropy of the action (T^k, M_H). Let $p : (M_H, b) \to (M, p(b))$ be the covering projection. If $T_b \neq 1$, then there exists a finite cyclic subgroup $F \subset T_b \subset T_{p(b)}$. Let

$$g : (I, 0, 1) \longrightarrow (T^k, e, f),$$

where f is a generator of F. The path $p(g(t) \cdot b)$ is the projection of a loop based at $b \in M_H$ to a loop based at $p(b)$ in M. The homotopy class of $p(g(t) \cdot b)$ is an element of H which is in the center of $\pi_1(M)$. Let

$$\alpha : (I, 0, 1) \longrightarrow (M, p(b), p(b))$$

be a loop in M based at $p(b)$. By Lemma 2.7.1,

$$[f \cdot \alpha(s)] = [\overline{p(g(t) \cdot b)} * \alpha(s) * p(g(t) \cdot b)].$$

But as $p(g(t) \cdot b)$ is in the center of $\pi_1(M, p(b))$, we have $f_*(\alpha) = \alpha$. In other words, $F \longrightarrow \text{Aut}(\pi_1(M, p(b)))$ is trivial. This contradicts that $F \longrightarrow \text{Aut}(\pi_1(M, p(b)))$ must be injective since F fixes $p(b)$; see Theorem 3.2.2(2). So $T_b = 1$ for each $b \in M_H$.

Note, $\text{Im}(\text{ev}_*^{p(b)}) = \text{Im}(\text{ev}_*^b) \subset H = \pi_1(M, p(b))$. Put $Q = H/\text{Im}(\text{ev}_*^b)$. Lift the free (T^k, M_H)-action to the splitting action $(T^k, T^k \times W)$, where W is contractible. The group Q acts freely on W because the T^k-action on M_H is free; see Theorem 3.5.2. The group Q is torsion free, for if it contained some p-torsion for some prime p, then there is $w \in W$ such that $\mathbb{Z}_p \subset Q_w$, by the Smith theorem, contradicting the freeness of the Q-action on W.

If H is finitely generated, then $H \cong \mathbb{Z}^n$ for some n. $\text{Im}(\text{ev}_*^{p(b)}) \cong \mathbb{Z}^k$ and $H/\text{Im}(\text{ev}_*^{p(b)}) = Q$ torsion free. Therefore, $\text{Im}(\text{ev}_*^{p(b)})$ is a direct summand of H. □

Corollary 11.7.3. *If (T^k, M) is an effective action of a torus on a closed aspherical manifold and if $H \subset \pi_1(M,x)$ is a finitely generated central subgroup for which $\text{Im}(\text{ev}_*^x) \subset H$, then $\text{Im}(\text{ev}_*^x)$ is a direct summand of H.*

COROLLARY 11.7.4. *Let* (T^k, M) *is a torus action on a closed aspherical manifold* M *for which* $\mathcal{Z}(\pi_1(M))$ *is finitely generated. If it is a maximal torus action, then* $\mathrm{Im}(\mathrm{ev}_*^x) = \mathrm{Center}\ \pi_1(M, x)$. *Conversely, if* $\mathrm{Im}(\mathrm{ev}_*^x) = \mathrm{Center}\ \pi_1(M, x)$, *then* (T^k, M) *is a maximal torus action on* M.

REMARK 11.7.5. Let M be a closed aspherical manifold for which $\mathcal{Z}(\pi_1(M)) \cong \mathbb{Z}^k$. Does M admit a maximal torus action? No examples of closed aspherical manifolds that do not admit maximal torus actions are known to us; see Remark 3.1.19.

This question has a negative answer, in general, if we replace "aspherical" by "admissible". Take $M = T^4 \# \mathbb{C}P_2$. This manifold is hyper-aspherical. $\pi_1(M) = \mathbb{Z}^4$ and no T^k, $k > 0$, acts effectively on it. Since $\chi(M) = 1$, $M^{T^k} \neq \emptyset$. In fact, $M^{T^k} = M$, $k > 0$. Also no nontrivial finite group can act effectively and homotopically trivially on it. In fact, if a finite G acts on M, then $\theta : G \to \mathrm{Aut}(\mathbb{Z}^k)$ is injective; see Exercise 3.4.6.

11.7.6 (Maximal torus action on infra-nilmanifolds). Let M be an infra-nilmanifold. So

$$M = \Pi \backslash G$$

where G is a simply connected nilpotent Lie group,

$$\Pi \subset G \rtimes C \subset G \rtimes \mathrm{Aut}(G) = \mathrm{Aff}(G),$$

Π is a lattice of $G \rtimes C$, C a compact subgroup of $\mathrm{Aut}(G)$. Let

$$N_{\mathrm{Aff}(G)}(\Pi)/\Pi = \mathrm{Aff}(M).$$

Then $\mathrm{Aff}(M)$ is a Lie group, the group of *affine* self-diffeomorphisms of M.

We claim the following sequence is exact:

$$1 \to \mathrm{Aff}_0(M) \to \mathrm{Aff}(M) \to \mathrm{Out}(\pi_1(M)) \to 1.$$

Every self-homotopy equivalence induces an automorphism of Π, unique up to an inner automorphism. The homotopy classes of self-homotopy equivalences are in one-to-one correspondence with elements of $\mathrm{Out}(\Pi)$. We think of $M = \theta(\Pi)\backslash(G \times \{\mathrm{pt}\})$ as a Seifert Construction, where $\theta : \Pi = \pi_1(M) \to \mathrm{TOP}_G(G \times \{\mathrm{pt}\}) = \mathrm{Aff}(G \times \{\mathrm{pt}\})$. We use uniqueness and rigidity for this setup. Then every isomorphism $\Pi \to \Pi$ is given by conjugation by a homeomorphism in $\mathrm{Aff}(G)$ and an automorphism of Π will have a realization in $N_{\mathrm{Aff}(G)}(\Pi)$. Thus $\mathrm{Aff}(M) = N_{\mathrm{Aff}(G)}(\Pi)/\Pi \to \mathrm{Out}(\Pi)$ is onto. We are interested in the kernel of this homomorphism. Look at the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}(\Pi) & \longrightarrow & C_{\mathrm{Aff}(G)}(\Pi) & \longrightarrow & C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi & \longrightarrow & N_{\mathrm{Aff}(G)}(\Pi) & \longrightarrow & \mathrm{Aff}(M) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

We see the kernel of $\mathrm{Aff}(M) \to \mathrm{Out}(\Pi)$ is $C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi)$. There are two different ways of expressing $\mathrm{Aff}(G)$. Namely,

$$\begin{aligned} \mathrm{Aff}(G) &= r(G) \rtimes \mathrm{Aut}(G) \\ &= \ell(G) \rtimes \mathrm{Aut}(G). \end{aligned}$$

The actions on $x \in G$ are as follows:

$$(r(a), \alpha) \cdot x = \alpha(x) \cdot a^{-1}, \qquad (\ell(a), \alpha) \cdot x = a \cdot \alpha(x).$$

Clearly, the correspondence

$$(r(a), \alpha) \longleftrightarrow (\ell(a^{-1}), \mu(a) \circ \alpha)$$

is a bijective map between $r(G) \rtimes \mathrm{Aut}(G)$ and $\ell(G) \rtimes \mathrm{Aut}(G)$. The first expression is more natural with respect to the Seifert Construction because

$$\begin{aligned} \mathrm{TOP}_G(G \times \{w\}) &= \mathrm{M}(\{w\}, G) \rtimes (\mathrm{Aut}(G) \times \mathrm{TOP}(\{w\})) \\ &= r(G) \rtimes \mathrm{Aut}(G). \end{aligned}$$

But, in this section, we shall use $\mathrm{Aff}(G) = \ell(G) \rtimes \mathrm{Aut}(G)$.

THEOREM 11.7.7. *Let $M = \Pi\backslash G$ be an infra-nilmanifold, where G is a simply connected nilpotent Lie group, $\Pi \subset G \rtimes \mathrm{Aut}(G)$ an almost Bieberbach group. Then $\mathrm{Aff}_0(M) = C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi) = G^Q/\mathcal{Z}(\Pi)$ (Q the holonomy group), and it contains a* maximal torus action $(\mathcal{Z}(G))^Q/\mathcal{Z}(\Pi)$.

PROOF. We want to show $C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi)$ is connected (as a Lie subgroup). Let $(a, \alpha) \in C_{\mathrm{Aff}(G)}(\Pi)$. Then (a, α) must centralize $\Gamma \subset \Pi$ and so also all of G. Thus,

$$(a, \alpha)(g, 1) = (g, 1)(a, \alpha),$$

which implies $\alpha(g) = a^{-1}ga$ for all $g \in G$. Thus $\alpha = \mu(a^{-1})$. Therefore each element of $C_{\mathrm{Aff}(G)}(\Pi)$ must be of the form $(a, \mu(a^{-1})) = r(a^{-1})$. Now it must also centralize Π. Let $\Pi \cap G = \Gamma$ and $\Pi/\Gamma = Q$. Then the holonomy group Q injects into $\mathrm{Aut}(G)$ naturally and we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G & \longrightarrow & G \rtimes \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & 1.
\end{array}$$

Let $(y,\beta) \in \Pi$. Then $(a,\mu(a^{-1}))(y,\beta) = (y,\beta)(a,\mu(a^{-1}))$ implies $ya = y\beta(a)$ for all $y \in \Gamma$. Thus $a = \beta(a)$. That is, as we run through all the $\beta \in Q$, $\beta(a) = a$ so that $a \in G^Q$, a closed subgroup of G. Clearly G^Q is a simply connected nilpotent subgroup, best seen from the Lie algebra. We have

$$\begin{aligned}\mathrm{Aff}_0(M) &= C_{\mathrm{Aff}(G)}(\Pi)/\mathcal{Z}(\Pi)\\ &= \{(a,\mu(a^{-1})):\ a \in G^Q\}/\mathcal{Z}(\Pi)\\ &\cong r(G^Q)/\mathcal{Z}(\Pi).\end{aligned}$$

Since $\mathcal{Z}(\Pi) \subset \mathcal{Z}(G)$, we have $\mathcal{Z}(\Pi) \subset (\mathcal{Z}(G))^Q$. It is easy to see that $\mathcal{Z}(\Pi)$ is a uniform lattice of $(\mathcal{Z}(G))^Q$, and so $(\mathcal{Z}(G))^Q/\mathcal{Z}(\Pi)$ is a torus, acting effectively on M. Since $\pi_1((\mathcal{Z}(G))^Q/\mathcal{Z}(\Pi)) = \mathcal{Z}(\Pi)$, this torus action is a maximal torus action by Corollary 11.7.4. Note $G^Q/\mathcal{Z}(\Pi)$ may not be compact. □

Example 11.7.8 (Klein bottle). The fundamental group has presentation $\Pi = \{a, b \mid a^2b^2 = 1\}$. $\mathcal{Z}(\Pi) = \{a^2\}$, the maximal normal Abelian subgroup is

$$\Gamma = \mathbb{R}^2 \cap \Pi = \{a^2, b | [a^2, b] = 1\},$$

and the *holonomy group* is $Q = \mathbb{Z}/2$. The universal covering space is the Abelian Lie group $G = \mathbb{R}^2$, and $G^Q = \mathbb{R}^1$ with $G^Q \cap \Pi = \{a^2\} \approx \mathbb{Z}$. Therefore, $\mathrm{Aff}_0(M) \cong G^Q/\mathcal{Z}(\Pi) = \mathbb{R}/\mathbb{Z} = S^1$, a circle. This is the maximal torus action by affine diffeomorphisms (in fact, isometries).

Exercise 11.7.9. Find $\mathrm{Aut}(\pi_1(\text{Klein bottle}))$ and $\mathrm{Out}(\pi_1(\text{Klein bottle}))$. (Answer: $\mathbb{Z}_2 \times \mathbb{Z}_2$).

Example 11.7.10. Let G be a simply connected nilpotent Lie group, and let $\Pi \subset G \rtimes \mathrm{Aut}(G)$ be an almost Bieberbach group. Suppose $\Gamma = \Pi$. That is, Π has trivial holonomy group Q. Then $G^Q = G$ since $Q = 1$. Therefore,

$$\mathrm{Aff}_0(M) = G^Q/\mathcal{Z}(\Gamma) = G/\mathcal{Z}(\Gamma).$$

Topologically, this is a product $\mathcal{Z}(G)/\mathcal{Z}(\Gamma) \times G/\mathcal{Z}(G)$, of a torus with a simply connected nilpotent Lie group. This is the covering space of M corresponding to the image of the evaluation homomorphism of the maximal torus action. Note that $\Gamma/\mathcal{Z}(\Gamma)$ is a lattice in $G/\mathcal{Z}(G)$. The maximal torus action on a nilmanifold is free, since $\Gamma/\mathcal{Z}(\Gamma)$ is torsion free. In fact, any effective torus action on a nilmanifold is free. This follows from Corollary 11.7.3.

Corollary 11.7.11. *Let G be a simply connected nilpotent Lie group, $\Pi \subset G \rtimes \mathrm{Aut}(G)$ an almost Bieberbach group. Then*

$$\mathrm{Aff}_0(M) = r(G^Q)/\mathcal{Z}(\Pi)$$

contains a torus subgroup $(\mathcal{Z}(G))^Q/\mathcal{Z}(\Pi)$, with quotient group a simply connected nilpotent Lie group $G^Q/\mathcal{Z}(G)^Q$. Therefore, if $G = \mathbb{R}^n$ (i.e., Π is a Bieberbach group), then $\mathrm{Aff}_0(M)$ is a torus $G^Q/\mathcal{Z}(\Pi)$. □

Remark 11.7.12. The torus is a maximal torus action and represents also the connected component of the full isometry group.

11.7.13. In [**LR91**], a smooth maximal torus action is constructed on each solvmanifold (quotient of a connected simply connected solvable Lie group by its lattice). Now it is easy to see that every infra-solvmanifold $M = \Pi\backslash G$ of type (R) admits an affine maximal torus action: The center $\mathcal{Z}(\Pi)$ lies in the center of G. Since $\mathcal{Z}(G)$ is connected, $\mathcal{Z}(\Pi)\backslash(\mathcal{Z}(G))^Q$ is a torus. The action is via left translations on the universal covering G, and hence is affine. If the solvmanifold has a metric coming from a left invariant metric of the universal covering group, this torus lies in the group of isometries of M.

11.7.14 (Counterexamples). Most statements which we proved are not true for general solvable Lie groups. Let $H = \mathbb{R}^2 \rtimes \mathbb{R}$, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by $\left[\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right]$. This is the universal covering group of the Euclidean group $E(2)_0$. Note that H is not of type (R). Take $\Gamma = \mathbb{Z}^2 \times \mathbb{Z} \subset H$, where $\mathbb{Z}^2$ is the standard lattice of $\mathbb{R}^2$ and $\mathbb{Z}$ is the center of H. Let Π_1 be a torsion-free extension of Γ by $\Psi = \mathbb{Z}_2 \times \mathbb{Z}_2$, where Ψ acts on $\mathbb{Z}^3$ as diagonal matrices

$$I, \;\; \alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \;\; \beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \;\; \gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This is the 3-dimensional abstract Bieberbach group $\mathfrak{G}_6$ in [**Wol77**]. There is no such a group Ψ in $\mathrm{Aut}(H)$. (The matrices β and γ map the generator of $\mathbb{Z}$ to its negative. That implies they must act on $\mathbb{R}^2$ as $-I$. But the first (2×2)-block of these matrices are not $-I$.) and Π_1 does not embed into $\mathrm{Aff}(H)$. Thus, Theorem 8.4.2 is not true for this example. The discrete nilradical of Π_1 is $\Gamma \cong \mathbb{Z}^3$; nilradical of H is $\mathbb{R}^2$. Therefore, $\mathbb{R}^2 \cap \Pi_1$ is not the discrete nilradical of Π_1. Thus Proposition 8.4.8 also fails. The rigidity theorem, Theorem 8.4.3, holds for this H.

It is easy to find a solvable Lie group G on which the rigidity fails. Let $G = H \times \mathbb{R}^3$ be the direct product of the above H with $\mathbb{R}^3$. Let Π be a torsion-free extension of $\mathbb{Z}^3$ by $\mathbb{Z}_2$ generated by α, an abstract Bieberbach group $\mathfrak{G}_2$ in [**Wol77**]. Then Π embeds into $E(3) \subset \mathrm{Aff}(\mathbb{R}^3)$ and also into $H \rtimes \mathrm{Aut}(H)$ so that $H \cap \Pi = \Gamma \cong \mathbb{Z}^3$. Also $\mathbb{Z}^3$ embeds into H as Γ above as well as into $\mathbb{R}^3$ as the standard lattice. Take $\Pi' = \mathbb{Z}^3 \times \Pi \subset \mathrm{Aff}(G)$ and $\Pi'' = \Pi \times \mathbb{Z}^3 \subset \mathrm{Aff}(G)$. Then, clearly, $\Pi' \cong \Pi''$ by interchanging the components, but the solvmanifold $\Pi'\backslash G$ is not affinely diffeomorphic to $\Pi''\backslash G$, since there is no automorphism of G which interchanges the two factors of G. Therefore the rigidity theorem, Theorem 8.4.3, fails. On $\Pi'\backslash G$, there is an affine maximal torus action T^4, but not all of it comes from left translation, since G does not have $\mathcal{Z}(G) \cong \mathbb{R}^4$.

THEOREM 11.7.15. *Suppose H is a compact Lie group of homeomorphisms acting freely and locally smoothly (see Subsection* 1.8.4*) on an infra-nilmanifold M. Suppose* $\dim(H\backslash M) \neq 3$. *Then the action can be conjugated into* $\mathrm{Aff}(M)$ *so that the subgroup $H_1 = \{h \in H : \; h \simeq \text{identity}\}$ is contained in the standard maximal torus action on M.*

PROOF. The connected component of the identity H_0 in H is a torus of dimension, say s, (because M is aspherical, see Theorem 3.2.2), and is contained in H_1. Let $\pi_1(M) = \Pi$. Then by Corollary 11.7.3, $\pi_1(H_0) \cong \mathbb{Z}^s$ is a direct summand of $\mathcal{Z}(\pi_1(M))$.

Let E be the group of all lifts of the action of H on M so that $1 \to \Pi \to E \to H \to 1$ is exact. Let Q, Q' and F be defined by

$$Q = \Pi/\mathbb{Z}^s, \quad Q' = E/\mathbb{R}^s, \quad F = H/H_0$$

so that the following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathbb{R}^s & \longrightarrow & H_0 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi & \longrightarrow & E & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & F & \longrightarrow & 1.
\end{array}$$

(All vertical lines are also short exact sequences.)

We claim that Q' is torsion free. (Then Q' is a torsion-free, finitely generated, virtually nilpotent group. Thus, it is an *almost Bieberbach group.*) Suppose it has nontrivial torsion. Pick a subgroup $P = \mathbb{Z}_p$ (p prime) of Q'. The restriction of the middle vertical exact sequence gives a short exact sequence $1 \to \mathbb{R}^s \to E_P \to P \to 1$. Note that E_P is sitting in E. Any such sequence splits so that $E_P = \mathbb{R}^s \rtimes P$. Thus

$$\mathbb{Z}_p = P \ \subset\ E_P \ \subset\ E.$$

The group $\mathbb{Z}_p$ acts on the universal covering space $\widetilde{M}$ of M. Since $\widetilde{M}$ is contractible, this action has a nonempty fixed point; see Subsection 1.8.1. Then this $P \subset E \subset \mathrm{TOP}(\widetilde{M})$ projects to $P \subset H \subset \mathrm{TOP}(M)$ which is acting freely, a contradiction.

We may lift the T^s-action on M to $M_{\mathrm{Im}(\mathrm{ev}_*^x)} = M_{\mathbb{Z}^s}$ (the covering space of M with fundamental group $\mathbb{Z}^s$) to get a splitting

$$(T^s, M_{\mathbb{Z}^s}) = (T^s, T^s \times W),$$

and the action of Q' on $M_{\mathbb{Z}^s}$ projects down to (Q', W).

If we lift all of the action of Q' to $\widetilde{M}$, we get an action of E on $\widetilde{M} = \mathbb{R}^s \times W$. As the short exact sequence

$$1 \to \mathbb{R}^s \to E \to Q' \to 1$$

shows, E acts on $\mathbb{R}^s \times W$ as a fiber preserving homeomorphism group. In other words, E is acts as a subgroup of $\mathrm{TOP}_{\mathbb{R}^s}(\mathbb{R}^s \times W) = \mathrm{M}(W, \mathbb{R}^s) \rtimes (\mathrm{GL}(n, \mathbb{R}) \times \mathrm{TOP}(W))$ as follows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^s & \longrightarrow & E & \longrightarrow & Q' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \Theta_1} & & \downarrow & & \\
1 & \longrightarrow & \mathrm{M}(W, \mathbb{R}^s) & \longrightarrow & \mathrm{TOP}_{\mathbb{R}^s}(\mathbb{R}^s \times W) & \longrightarrow & \mathrm{GL}(n, \mathbb{R}) \times \mathrm{TOP}(W) & \longrightarrow & 1;
\end{array}$$

see Corollary 4.2.10.

Since $H_1 \to \mathrm{Out}(\Pi)$ is trivial, we have induced

$$1 \to \mathcal{Z}(\Pi) \to C_E(\Pi) \to H_1 \to 1$$

with $C_E(\Pi)$, the centralizer of Π in E, being the kernel of $E \to \mathrm{Aut}(\Pi)$. Thus it is a central extension.

We analyze $C_E(\Pi)$ more carefully. Since $\pi_1(\mathrm{H}_0) \cong \mathbb{Z}^s$ is a direct summand of $\mathcal{Z}(\pi_1(M))$, we have $\mathcal{Z}(\Pi) = \mathbb{Z}^s \oplus \mathbb{Z}^{k-s}$. Denote $C_E(\Pi)/\mathbb{R}^s$ by Z. Then we have a commutative diagram of exact rows and columns:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathbb{R}^s & \longrightarrow & H_0 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}^s \oplus \mathbb{Z}^{k-s} & \longrightarrow & C_E(\Pi) & \longrightarrow & H_1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}^{k-s} & \longrightarrow & Z & \longrightarrow & F_1 & \longrightarrow & 1.
\end{array}$$

All vertical lines are also short exact sequences.

As shown in the last horizontal line, Z is a torsion-free central extension of $\mathbb{Z}^{k-s}$ by a finite group. Such a group is itself free Abelian so that $Z \approx \mathbb{Z}^{k-s}$; see Corollary 5.5.3. Consequently, the middle vertical short exact sequence splits, and we have

$$C_E(\Pi) = \mathbb{R}^s \oplus Z, \quad Z \approx \mathbb{Z}^{k-s}$$

and

$$H_1 = H_0 \oplus F_1, \quad F_1 \text{ finite}$$

is an Abelian group.

On the other hand, M was an infra-nilmanifold. So $\widetilde{M} = G$, a connected, simply connected nilpotent Lie group, and $M = \Pi\backslash G$ with $\Pi \subset \mathrm{Aff}(G)$ as an almost Bieberbach group. There is a canonical maximal torus action A^k on M. This comes from the left translations $\ell(G)$. More precisely, the centralizer of Π in $\mathrm{Aff}(G)$ is $\mathbb{R}^k$ which lies in the center of $\ell(G)$. We can reparametrize $A^k = A^s \times A^{k-s}$ so that the standard action of A^s has the same evaluation homomorphism as $H_0 = T^s$. That is,

$$\mathrm{ev}_*(\pi_1(T^s)) = \mathrm{ev}_*(\pi_1(A^s)) \ \subset \mathcal{Z}(\Pi).$$

The action of A^s is free because $\pi_1(M)/\mathrm{ev}_*^x(\pi_1(T^s)) = Q$ is torsion free, for otherwise the T^s-action could not be free. We write the centralizer $\mathbb{R}^k$ by

$$\mathbb{R}^k = \widetilde{A}^s \times \widetilde{A}^{k-s}$$

so that $\widetilde{A}^s \approx \mathbb{R}^s$, $\widetilde{A}^{k-s} \approx \mathbb{R}^{k-s}$, and $A^k = (\Pi \cap \mathbb{R}^k)\backslash\mathbb{R}^k$. Clearly $\widetilde{A}^s$ is a normal subgroup of G. Let us denote the quotient by $\overline{G}$ so that

$$1 \longrightarrow \widetilde{A}^s \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1$$

is exact. Then $\overline{G}$ is again a connected, simply connected nilpotent Lie group.

Consider the group Q'. Its subgroup Q acts on the quotient group $\overline{G}$ as a subgroup of $\mathrm{Aff}(\overline{G})$. Since Q' is a torsion-free extension of Q by a finite group F, there is a unique way of embedding Q' into $\mathrm{Aff}(\overline{G})$ so that the original $Q \subset \mathrm{Aff}(\overline{G})$ is untouched (which uses UAEP of the pair $(Q, \overline{G})$).

The extended lift of Q to G is exactly Π. Therefore, the group $\widetilde{\Pi} = \Pi \cdot \widetilde{A}^s$ naturally lies in $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ already. Our goal is to embed H into $\mathrm{Aff}(G)$.

Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & \widetilde{A}^s & = & \widetilde{A}^s & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Pi} = \Pi \cdot \widetilde{A}^s & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & F & \longrightarrow & 1.
\end{array}$$

(All vertical lines are also short exact sequences.)

Since Π is normal in E, and the lattice $\Pi \cap G$ is characteristic in Π, the lattice is normalized by E. Thus we have a homomorphism

$$\theta : E \to \mathrm{Aut}(\Pi \cap G) \to \mathrm{Aut}(G)$$

by UAEP of the lattice. Define a map

$$\lambda : E \to G$$

in such a way that:

(1) for $\alpha \in \widetilde{\Pi}$, $\alpha \mapsto (\lambda(\alpha), \theta(\alpha))$ is the embedding of $\widetilde{\Pi} \subset \mathrm{Aff}(G)$;
(2) for all $\alpha \in E$, $\alpha \mapsto (\lambda(\alpha), \theta(\alpha)) \mapsto \mathrm{Aff}(\overline{G})$ is a homomorphism, where the second map is the natural $\mathrm{Aff}(G) \to \mathrm{Aff}(\overline{G})$.

Our goal is finding a λ such that

$$\begin{array}{ccl}
E & \longrightarrow & G \rtimes \mathrm{Aut}(G) = \mathrm{Aff}(G) \\
\alpha & \longrightarrow & (\lambda(\alpha), \theta(\alpha))
\end{array}$$

is a homomorphism extending $\widetilde{\Pi} \longrightarrow \mathrm{Aff}(G)$. It will be a homomorphism if and only if

$$\lambda(\alpha) \cdot \theta(\alpha)(\lambda(\beta)) = \lambda(\alpha\beta)$$

for all $\alpha, \beta \in E$. By the second condition that $\alpha \mapsto (\lambda(\alpha), \theta(\alpha)) \mapsto \mathrm{Aff}(\overline{G})$ is a homomorphism, this equality holds on $\overline{G}$ level. Therefore, $\lambda(\alpha)\cdot\theta(\alpha)(\lambda(\beta))\cdot\lambda(\alpha\beta)^{-1}$ lies in $\widetilde{A}^s$ for all for all $\alpha, \beta \in E$. Let $f : E \times E \to \widetilde{A}^s$ be defined by

$$f(\alpha, \beta) = \lambda(\alpha) \cdot \theta(\alpha)(\lambda(\beta)) \cdot \lambda(\alpha\beta)^{-1}.$$

We quickly see that f is really a 2-cocycle

$$f : F \times F \to \widetilde{A}^s$$

because $f(\alpha, \beta) = 1$ for $\alpha, \beta \in \widetilde{\Pi}$ by the first condition. As we know very well, $H^2(F; \widetilde{A}^s) = 0$ (recall $\widetilde{A}^s = \mathbb{R}^s$). Consequently, we can find such a desired λ.

Now we compare the two actions of E on $\widetilde{M} = \mathbb{R}^s \times W$ and $\widetilde{M} = G = \widetilde{A}^s \times \overline{G}$. The first one is topological (even though the part Π is affine), and the second one is affine by our construction. The E-actions have corresponding Q'-actions which we denote by

$$\rho_1 : Q' \to \mathrm{TOP}(W),$$
$$\rho_2 : Q' \to \mathrm{TOP}(\overline{G}).$$

These yield the manifolds $Q'\backslash W$ and $Q'\backslash\overline{G}$. Of course, the latter is an infra-nilmanifold. These two spaces are homeomorphic by virtue of the theorem of Farrell and Hsiang [**FH83**] for dimension not equal to 4, 5 and for dimension 4 by Freedman, Farrell and Jones [**FJ98**]. Therefore, the free Q'-actions on W and $\overline{G}$ are weakly equivalent. That is, there is a homeomorphism $h : W \to \overline{G}$ and an automorphism $\omega : Q' \to Q'$ so that

$$\begin{array}{ccc} W & \xrightarrow{\;h\;} & \overline{G} \\ {\scriptstyle\rho_1(\alpha)}\downarrow & & \downarrow{\scriptstyle\rho_2(\omega(\alpha))} \\ W & \xrightarrow{\;h\;} & \overline{G} \end{array}$$

for all $\alpha \in Q'$. But, the Second Generalized Bieberbach Theorem 8.4.3 ensures that ω is a conjugation. More precisely, there exists $(a, A) \in \mathrm{Aff}(\overline{G})$ such that

$$\rho_2(\omega(\alpha)) = (a, A)\rho_2(\alpha)(a, A)^{-1}$$

for all $\alpha \in Q'$. Thus we have a commuting diagram

$$\begin{array}{ccccc} W & \xrightarrow{\;h\;} & \overline{G} & \xrightarrow{(a,A)} & \overline{G} \\ {\scriptstyle\rho_1(\alpha)}\downarrow & & \downarrow{\scriptstyle\rho_2(\omega(\alpha))} & & \downarrow{\scriptstyle\rho_2(\alpha)} \\ W & \xrightarrow{\;h\;} & \overline{G} & \xrightarrow{(a,A)} & \overline{G} \end{array}$$

for all $\alpha \in Q'$.

The fibers $\mathbb{R}^s$ and $\widetilde{A}^s$ are isomorphic by extending the natural isomorphism of the lattices $\mathbb{Z}^s$. (Recall we reparametrized the maximal torus action on the infra-nilmanifold so that the evaluation of subtorus A^s is the same as that of the topological torus action T^s.) So we identify $\mathbb{R}^s$ with $\widetilde{A}^s$.

Now pick a lift of $W \xrightarrow{h} \overline{G} \xrightarrow{(a,A)} \overline{G}$ to a map

$$\mathfrak{F} : \mathbb{R}^s \times W \longrightarrow \widetilde{A}^s \times \overline{G}$$

so that the fiber map is just the identification. Then the action of E on $\mathbb{R}^s \times W$ given by

$$\alpha \mapsto \mathfrak{F} \circ (\lambda(\alpha), \theta(\alpha)) \circ \mathfrak{F}^{-1}$$

is a new Seifert Construction with the homomorphism $\rho_1 : Q' \to \mathrm{TOP}(W)$, which we denote by

$$\Theta_2 : E \longrightarrow \mathrm{TOP}_{\mathbb{R}^s}(\mathbb{R}^s \times W).$$

Thus we have two Seifert Constructions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^s & \longrightarrow & E & \longrightarrow & Q' & \longrightarrow & 1 \\ & & \downarrow & & {\scriptstyle\Theta_1}\downarrow{\scriptstyle\Theta_2} & & \downarrow{\scriptstyle\rho_1} & & \\ 1 & \longrightarrow & \mathrm{M}(W, \mathbb{R}^s) & \longrightarrow & \mathrm{TOP}_{\mathbb{R}^s}(\mathbb{R}^s \times W) & \longrightarrow & \mathrm{GL}(n, \mathbb{R}) \times \mathrm{TOP}(W) & \longrightarrow & 1. \end{array}$$

Now the uniqueness part of Theorem 7.3.2 applies. The two constructions are conjugate by an element of $\mathrm{M}(W, \mathbb{R}^s)$. We conclude that the original topological action $(E, \mathbb{R}^s \times W)$ and the constructed affine action (E, G) are related by a topological conjugation. By the affine action (E, G), the centralizer $C_E(\mathit{\Pi})$ maps into the maximal torus action. □

11.7.16. In our analysis of infra-nilmanifolds, we saw that much of the information about the manifolds is encoded in the Lie group $\text{Aff}(M) = N_{\text{Aff}(M)}(\Phi)/\Phi$. (This is a special case of the geometric information found in $N_{\text{TOP}_G(G\times W)}(\Phi)/\Phi$ for an injective Seifert fibering $M = \Phi\backslash G \times W$.) We have seen that $\pi_0(\text{Aff}(M)) = \text{Aff}(M)/(C_{\text{Aff}(M)}(\Pi)/\mathcal{Z}(\Pi)) = \text{Out}(\Pi)$. $\text{Out}(\Pi)$ can also be identified with $\text{TOP}(M)$ or $\text{Diff}(M)$ modulo those homotopic to the identity in the space of self homotopy equivalences, but $\pi_0(\text{Diff}(M))$ and $\pi_0(\text{TOP}(M))$ may be much larger. In the following commuting diagram, $\mathcal{E}(M)$ is the H-space of all self-homotopy equivalences of M.

$$\begin{array}{ccccc} \text{Aff}(M) & \xrightarrow{\text{inc}} & \text{Diff}(M) & \xrightarrow{\text{inc}} & \mathcal{E}(M) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \pi_0(\text{Aff}(M)) & \xrightarrow{\text{inj}} & \pi_0(\text{Diff}(M)) & \xrightarrow{\text{surj}} & \pi_0(\mathcal{E}(M)) \end{array}$$

Note that the composite homomorphism on the bottom $\pi_0(\text{Aff}(M)) \longrightarrow \pi_0(\mathcal{E}(M))$ is an isomorphism.

The kernel, $\pi_0(\text{TOP}(M)) \to \text{Out}(\Phi) = \pi_0(\mathcal{E}(M))$, denoted by K is *exotic* (the indications are that K consists solely of 2-torsions). Farrell and Jones [**FJ93**] have shown that K is isomorphic to $\sum_{i=1}^{\infty}(\mathbb{Z}_2)_i = \mathbb{Z}_2^{\infty}$ when $\dim(M) > 10$ and M is closed and homeomorphic to a nonpositively curved manifold.

Proposition 11.7.17 (cf. [**LR82**, (4.1) Corollary 2]). *If M is a closed manifold homeomorphic to an infra-nilmanifold, then no nontrivial finite subgroup of K (the kernel of $\pi_0(\text{TOP}(M)) \to \text{Out}(\Phi)$) can act freely on M.*

Proof. Let G be a finite subgroup of K and suppose it acts freely on M. Each element $g \in G$ is homotopic to the identity but not isotopic to the identity, $g \neq 1$. Since the group G acts freely, it also acts locally smoothly by default. Therefore, G can be embedded in the maximal torus action of M. That is, the action can be conjugated in $\text{TOP}(M)$ so that its image is in the standard maximal torus action on M. Each such conjugated g is isotopic to the identity inside the maximal torus action. This is a contradiction. □

Remark 11.7.18. For some admissible manifolds, we can guarantee that any homotopically trivial action is free.

1. Any compact Lie group G that acts effectively and homotopically trivially on an admissible manifold M is Abelian. If the center of $\pi_1(M)$ is finitely generated, then G is isomorphic to a direct product of torus group with a finite Abelian group. Moreover, if $\text{Inn}(\pi_1(M))$ is torsion free, then G acts freely.

2. If M is a nilmanifold, then any homotopically trivial action of a compact Lie group is a free action. If the action is locally smooth, then the action can be conjugated into the standard free torus action. Therefore no nontrivial subgroup of K can act on M.

3. ([**LR82**, Corollary 2]) If $\mathcal{Z}(\pi_1(M))$ is a direct product factor of $\pi_1(M)$, where M is a closed infra-nilmanifold, then any locally smooth homotopically trivial action is free and embeds in the maximal torus action. In particular, no nontrivial subgroup of K acts effectively on M.

4. If M is a closed flat manifold and the holonomy has odd order, then no nontrivial subgroup of K acts effectively on M, assuming $\dim(M) > 10$.

5. If M is an admissible manifold and $\mathcal{Z}(\pi_1(M)) = 1$, then no nontrivial compact Lie group acts homotopically trivially.

6. Suppose M and M' are homotopy equivalent infra-nilmanifolds; $M = \Pi\backslash G$ and $M' = \Pi'\backslash G$. Then by the Generalized Bieberbach Theorem (Theorem 8.4.3), Π' is a conjugate of Π, say by $\alpha \in \mathrm{Aff}(G)$. Then α induces an affine diffeomorphism from M to M'. Of course, this induced diffeomorphism sends the standard maximal torus action of M to the standard maximal torus action of M'. However, in general, maximal torus actions cannot be compared easily.

PROOF. 1. First we note that a finite subgroup F of G acting on M is Abelian by Theorem 3.2.2(3). Secondly, if the center of $\pi_1(M)$ is finitely generated of rank k, say, then F can be embedded in a torus of rank k by Theorem 3.2.2(4). Then by Proposition 5.5.5, G itself is Abelian and G_0, the connected component of the identity, is a torus T^s with $0 \le s \le k$. Finally, the universal covering $\widetilde{G}$ of G is split by Lemma 8.4.1, and so G is split. Thus, as G_0 is a torus and G is Abelian, G is a product of T^s with a finite Abelian group.

Now suppose $\mathrm{Inn}(\pi_1(M))$ is torsion free. Then $G_x \subset G$ is finite and injects into $\mathrm{Aut}(\pi_1(M))$, and hence lies in $\mathrm{Inn}(\pi_1(M)) \subset \mathrm{Aut}(\pi_1(M))$. As $\mathrm{Inn}(\pi_1(M))$ is torsion free, $G_x = 1$.

2. Since $\mathrm{Inn}(\pi_1(M))$ is torsion free, G acts freely, and if it acts locally smoothly, G can be conjugated into the maximal torus action. As each element of the maximal torus action is isotopic to the identity, no subgroup of K can act on M.

3. $\pi_1(M) = \mathcal{Z}(\pi_1(M)) \times \pi_1(M)/\mathcal{Z}(\pi_1(M))$. The group $\pi_1(M)/\mathcal{Z}(\pi_1(M)) = \mathrm{Inn}(\pi_1(M))$ and is torsion free. Therefore the previous argument applies.

4. Suppose G is a finite subgroup of K that acts effectively on M. Let E be the extended lifting to $\widetilde{M} = \mathbb{R}^n$. Let $e \in E$ be such that its image in G is g where g is not isotopic to the identity on M. The image of e in $\mathrm{Aut}(\Pi)$ lies in $\mathrm{Inn}(\Pi)$. Therefore there is $\sigma \in \Pi$ such that $\mu(\sigma) = \mathrm{Im}(e)$. Then $(\mu(\sigma))^2 = \mu(\sigma^2) = 1$ implies $\sigma^2 \in \mathcal{Z}(\Pi)$. Denote σ by $(a, A) \in \mathrm{Aff}(\mathbb{R}^n)$. Then $((I + A)a, A^2) = \sigma^2 \in \mathcal{Z}(\Pi)$ implies $A^2 = I$. So M has even order holonomy, a contradiction.

5. If G is a finite group acting on an admissible M and $\mathcal{Z}(\pi_1(M)) = 1$, then $\psi : G \to \mathrm{Out}(\pi_1(M))$ must be faithful. So G cannot act homotopically trivially.

6. A maximal torus action on a closed aspherical manifold is not necessarily unique up to topological equivalence. (Of course it is understood that we do not distinguish between two maximal torus actions if they differ only by an automorphism of the tori.) Without local smoothness assumptions, even a free maximal torus action may have a nonlocally Euclidean orbit space. (For example: $S^1 \times \mathfrak{G}_6$ is a flat manifold with a free S^1 maximal torus action. However, $S^1 \times \mathfrak{G}_6$ is homeomorphic to $S^1 \times \mathfrak{G}_6^*$, where $\mathfrak{G}_6^*$ is not locally Euclidean (collapse a badly embedded arc—$\mathfrak{G}_6^*$ cannot have an S^1-action on it). Therefore, the maximal torus action with S^1 acting as translation on the first factor of $S^1 \times \mathfrak{G}_6^*$ cannot be conjugated in $\mathrm{TOP}(S^1 \times \mathfrak{G}_6^*)$ to the standard maximal torus action.) □

The rest of this section is devoted to the study of maximal torus action on solv-manifolds. In general, a solv-manifold is a quotient of a simply connected solvable Lie group by a closed subgroup. The main reference for this is [**LR91**].

11.7.19. Let S be a connected, simply connected solvable Lie group, and let H be a closed subgroup of S. The coset space $H\backslash S$ is called a *solvmanifold*. In this section, we consider only the case when $H = \Gamma$ is discrete so that Γ is a uniform lattice of G. More generally, let Π be a subgroup of $\mathrm{Aff}(S) = S \rtimes \mathrm{Aut}(S)$ acting freely on S such that $\Gamma = \Pi \cap S$ is a lattice of S and Π/Γ is finite. We call the orbit space $\Pi\backslash S$ an *infra-solvmanifold*. Therefore, a compact infra-solvmanifold is finitely covered by a solvmanifold.

It is a theorem of Mostow that two compact solvmanifolds of the same fundamental group are diffeomorphic. The significance of this statement is seen from the fact, different from the nilpotent theory, that the group Γ does not determine the Lie group S. In other words, given a group Γ, there may exist two distinct connected, simply connected solvable Lie groups S_1, S_2 both containing a copy of Γ as a lattice. Mostow's theorem says that S_1 is diffeomorphic to S_2, and $\Gamma\backslash S_1$ is diffeomorphic to $\Gamma\backslash S_2$.

For many closed $K(\Pi, 1)$-manifolds such as those with virtually poly-$\mathbb{Z}$ fundamental groups, it is verified in [**LR84**] that $\mathcal{Z}(\Pi)$ is finitely generated and the manifold admits a maximal torus action. In fact, one may find a topological version of Theorem 11.7.20 (see below) in [**LR84**]. It uses surgery results of Wall and does not explain how the torus action arises explicitly from the solvable group S. The following theorem does not rely on surgery theory, and the solution is given explicitly in terms of Lie theory.

THEOREM 11.7.20. *Let S be a simply connected solvable Lie group and Π be a lattice of S. Then the solvmanifold $\Pi\backslash S$ admits a smooth maximal torus action.*

The plan based upon Lie theory is to construct a new connected and simply connected solvable Lie group $S(\Gamma)$ and an embedding of Π into $\mathrm{Aff}(S(\Gamma)) = S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma))$ with the following properties: (a) the infra-solvmanifold $\Pi\backslash S(\Gamma)$ is diffeomorphic to the solvmanifold $\Pi\backslash S$; and (b) $\Pi\backslash S(\Gamma)$ admits a smooth maximal torus action. The torus action constructed on $\Pi\backslash S(\Gamma)$ descends from a central vector group of $S(\Gamma)$ which commutes with the affine action of Π on $S(\Gamma)$. Then we can pull back the torus action on $\Pi\backslash S(\Gamma)$ to $\Pi\backslash S$ obtaining a smooth maximal torus action on the solvmanifold $\Pi\backslash S$. The reader should find the elementary Example 11.7.27 instructive. It illustrates, in an explicit fashion, some of steps to be taken in the proof of the theorem. The proof starts in Subsection 11.7.25.

11.7.21. Since Π is a lattice of S, it is a *strongly torsion-free $\mathcal{S}$ group*; that is, Π contains a finitely generated, torsion-free nilpotent normal subgroup D with the quotient Π/D free Abelian of finite rank. Such a group Φ contains a unique *maximal normal nilpotent subgroup* M which automatically contains $[\Pi, \Pi]$. The group Π also contains a characteristic subgroup Γ of finite index such that Γ is *strongly torsion-free $\mathcal{S}$ group of type I* (*predivisible group*), and $\Gamma \supset M$. This means that:

(1) $\Gamma\backslash M$ is torsion free.
(2) Let $\mu(\gamma)$ be the automorphism of the real nilpotent Lie group $M_{\mathbb{R}}$ (see below for notation) containing M as a lattice, induced from the conjugation by an element $\gamma \in \Gamma$. If θ is an eigenvalue of the derivative of $\mu(\gamma)$, then

$$\theta|\theta|^{-1} = \cos 2\pi\rho + i \sin 2\pi\rho,$$

where ρ is either 0 or irrational.

NOTATION 11.7.22. For a finitely generated, torsion-free nilpotent group D, the unique connected and simply connected nilpotent Lie group is denoted by $D_{\mathbb{R}}$. This is the *Mal'cev completion*.

11.7.23. The short exact sequence of groups $1 \to M \to \Gamma \to \mathbb{Z}^k \to 1$ induces an exact sequence $1 \to M_{\mathbb{R}} \to \Gamma M_{\mathbb{R}} \to \mathbb{Z}^k \to 1$. One may think of $\Gamma M_{\mathbb{R}}$ as the pushout of $M \to \Gamma$ with $M \hookrightarrow M_{\mathbb{R}}$ since $(M, M_{\mathbb{R}})$ has the unique automorphism extension property; see Definition 5.3.3. In other words, $\Gamma M_{\mathbb{R}}$ is the unique group fitting into the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & \Gamma M_{\mathbb{R}} & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1.
\end{array}$$

Does there exist a connected and simply connected solvable Lie group $S(\Gamma)$ containing $\Gamma M_{\mathbb{R}}$? Using Wang's construction, Auslander constructed such a group $S(\Gamma)$ which fits into the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & \Gamma M_{\mathbb{R}} & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & M_{\mathbb{R}} & \longrightarrow & S(\Gamma) & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1,
\end{array}$$

where $\mathbb{Z}^k \subset \mathbb{R}^k$ as a lattice; see [**Wan56**] and [**Aus61b**]. Moreover, $S(\Gamma)$ has the property that there exists $\gamma_1, \gamma_2, \ldots, \gamma_k$ whose images form a set of generators for Γ/M which lie on 1-parameter groups in $S(\Gamma)$.

11.7.24 (More properties of $S(\Gamma)$). (1) $\Gamma \subset S(\Gamma)$ as a lattice.

(2) There exists a torus subgroup T^* of $\mathrm{Aut}(S(\Gamma))$ such that $S \subset S(\Gamma) \rtimes T^*$. Moreover, the composite $S \hookrightarrow S(\Gamma) \rtimes T^* \to T^*$ is surjective.
(3) Let N be the *nilradical* of S; that is, the maximal normal nilpotent connected Lie subgroup of S. Then ΓN can be naturally identified with $\Gamma M_{\mathbb{R}}$. With this identification, we have $[S(\Gamma), S(\Gamma)] \subset \Gamma N$, and hence ΓN is normal in $S(\Gamma)$.
(4) Any automorphism θ of $\Gamma M_{\mathbb{R}}$ which is trivial on $\Gamma M_{\mathbb{R}}/M_{\mathbb{R}}$ can be uniquely extended to an automorphism of $S(\Gamma)$.
(5) N is normal in $S(\Gamma) \rtimes T^*$.

We shall study Seifert fiberings of infra-solvmanifolds. Suppose our model space P itself is a connected, simply connected Lie group; G a connected closed normal subgroup and $W = P/G$. We shall consider the short exact sequence of groups $1 \to G \to P \to W \to 1$ as a principal G-bundle. The group $\mathrm{Diff}_G(P)$ of all weakly G-equivariant smooth diffeomorphisms of P onto itself is exactly the normalizer of $G = \ell(G)$ in $\mathrm{Diff}(P)$, and is equal to $\mathrm{TOP}_G(P) \cap \mathrm{Diff}(P)$. Let $C(W, G)$ be the group of all smooth maps from W to G. Suppose $P \to W$ has a *smooth cross section.* Then we have a short exact sequence

$$1 \to C(W, G) \rtimes \mathrm{Inn}(G) \to \mathrm{Diff}_G(P) \to \mathrm{Out}(G) \times \mathrm{Diff}(W) \to 1.$$

The *affine group* $\mathrm{Aff}(P) = P \rtimes \mathrm{Aut}(P)$ acts on P by: $(p, \gamma) \cdot u = p \cdot \gamma(u)$ for $(p, \gamma) \in \mathrm{Aff}(P)$ and $u \in P$. Note that P acts as left translations. For $g \in G$, we have $(p, \gamma)(g, 1)(p, \gamma)^{-1} = (p\gamma(g)p^{-1}, 1)$. Let us denote the subgroup of $\mathrm{Aut}(P)$ which leaves G invariant by $\mathrm{Aut}(P, G)$. An important fact for us is

$$P \rtimes \mathrm{Aut}(P, G) \subset \mathrm{Diff}_G(P). \tag{11.7.1}$$

This is true because $\mathrm{Diff}_G(P)$ is the normalizer of $\ell(G)$ in $\mathrm{Diff}(P)$, and $\ell(G)$ is normal in $P \rtimes \mathrm{Aut}(P, G)$.

11.7.25 (Proof of Theorem 11.7.20). We go back to our solvable Lie groups. Since N is the nilradical of S, S/N is commutative, say of dimension s. Therefore, we have an exact sequence of groups

$$1 \to N \to S \to S/N = \mathbb{R}^s \to 1$$

On the other hand, since $[S(\Gamma), S(\Gamma)] \subset \Gamma N$ from Property 11.7.24(3) of $S(\Gamma)$, and $[S(\Gamma), S(\Gamma)]$ is connected, we have $[S(\Gamma), S(\Gamma)] \subset N$. Therefore N is normal in $S(\Gamma)$ and $S(\Gamma)/N$ is a commutative Lie group, $\mathbb{R}^s$. Therefore

$$1 \to N \to S(\Gamma) \to S(\Gamma)/N = \mathbb{R}^s \to 1$$

is exact.

Since N is normal in both S and $S(\Gamma)$, the inclusion maps $\Gamma N \hookrightarrow S$ and $\Gamma N \hookrightarrow S(\Gamma)$ induce $\Gamma/(\Gamma \cap N) \hookrightarrow S/N$ and $\Gamma/(\Gamma \cap N) \hookrightarrow S(\Gamma)/N$. By these homomorphisms we identify $S/N = \mathbb{R}^s$ with $S(\Gamma)/N = \mathbb{R}^s$.

The group $\Pi \subset S$ acts on S as left multiplications. Therefore, from the inclusion (11.7.1), we have $\Pi \subset \mathrm{Diff}_N(S)$. In fact, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi \cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi \cap N) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S) & \longrightarrow & \mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1.
\end{array}$$

Similarly, $\Pi \subset S(\Gamma) \rtimes T^* \subset S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma), N)$, because N is normal in $S(\Gamma) \rtimes T^*$. $S(\Gamma) \rtimes T^*$ acts on $S(\Gamma)$ as affine maps which implies that $\Pi \subset \mathrm{Diff}_N(S(\Gamma))$ by $(*)$. We have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi \cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi \cap N) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S(\Gamma)) & \longrightarrow & \mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1.
\end{array}$$

Let us denote $\Pi/(\Pi \cap N)$ simply by Q. Then Q is a free Abelian group of rank s, where $s = \dim(S/N)$. Clearly, $\Gamma/(\Gamma \cap N)$ is a subgroup of Q of finite index, because $M \subset \Gamma$ (so, $\Pi \cap N = \Gamma \cap N$). We shall examine the two actions of Q on S/N and $S(\Gamma)/N$.

The action of Q on S/N is induced by the left translation by Π on S. Therefore, $Q = \mathbb{Z}^s$ acts on $S/N = \mathbb{R}^s$ also as left translations. Moreover, Q is a lattice in S/N.

Now the action of Q on $S(\Gamma)/N$ is induced by the affine action of Π on $S(\Gamma)$. The projection $S(\Gamma) \to S(\Gamma)/N$ yields a homomorphism $S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma), N) \to (S(\Gamma)/N) \rtimes \mathrm{Aut}(S(\Gamma)/N)$ naturally. We recall how $S \subset S(\Gamma) \cdot T^*$ of Property 11.7.24(2) was constructed in [**Aus61b**]. S acts on ΓN by conjugation, which extends to an automorphism of $\Gamma M_{\mathbb{R}}$. The latter is trivial on $\Gamma M_{\mathbb{R}}/M_{\mathbb{R}}$, and hence it can be extended to an automorphism of $S(\Gamma)$ by Property 11.7.24(4). Since the S-action on $\Gamma N/N$ is trivial, and $\Gamma N/N = \mathbb{Z}^s$ sits in $\mathbb{R}^s = S(\Gamma)/N$ as a uniform lattice, the action of S on $S(\Gamma)/N$ must be trivial as well. This implies that $S \subset S(\Gamma) \cdot T^* \subset S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma), N) \to (S(\Gamma)/N) \rtimes \mathrm{Aut}(S(\Gamma)/N)$ has image in $S(\Gamma)/N \times \{1\}$. Therefore, $Q = \mathbb{Z}^s$ acts on $S(\Gamma)/N = \mathbb{R}^s$ as left translations.

Moreover, Q is a lattice $S(\Gamma)/N$. We conclude that both actions of $Q = \Pi/\Pi \cap N$ on S/N and $S(\Gamma)/N$ are as left translations.

Furthermore, $z \in \Pi \cap N$ goes into $C(S/N, N) \rtimes \mathrm{Inn}(N)$ and $C(S(\Gamma)/N, N) \rtimes \mathrm{Inn}(N)$ as $(z^{-1}, \mu(z))$, as left translations, where $\mu(z)$ is the conjugation by z so that $\mu(z)(a) = zaz^{-1}$. Actually, $\Pi \cap N \subset N$ sits in $C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N)$ as constant maps.

Choose an N-equivariant diffeomorphism $\tau : S \to S(\Gamma)$. This can be done as follows. Take smooth sections (not homomorphisms) $s_1 : \mathbb{R}^s \to S$ and $s_2 : \mathbb{R}^s \to S(\Gamma)$. With these sections, we define an N-bundle equivalence $\tau : S \to S(\Gamma)$ by $\tau(x \cdot s_1(w)) = x \cdot s_2(w)$ for all $x \in N$ and $w \in \mathbb{R}^s$. Let us denote the representations of Π into $\mathrm{Diff}_N(S)$ and $\mathrm{Diff}_N(S(\Gamma))$ by ψ_1, ψ_2, respectively. More precisely, $\psi_1 : \Pi \to S \subset \mathrm{Diff}_N(S)$, and $\psi_2 : \Pi \to S \subset S(\Gamma) \rtimes T^* \subset S(\Gamma) \rtimes \mathrm{Aut}(S(\Gamma), N) \subset \mathrm{Diff}_N(S(\Gamma))$. Since τ is N-equivariant, $\mu(\tau) \circ \psi_1$ is a representation of Π into $\mathrm{Diff}_N(S(\Gamma))$. This bundle map $\tau : S \to S(\Gamma)$ induces an isomorphism $f \mapsto \tau \cdot f \cdot \tau^{-1}$ of $\mathrm{Diff}_N(S)$ onto $\mathrm{Diff}_N(S(\Gamma))$.

Consider the two representations $\mu(\tau) \circ \psi_1, \psi_2 : \Pi \to \mathrm{Diff}_N(S(\Gamma))$. Since they induce the same maps of the kernel $\Pi \cap N$ into $C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N)$, and of the quotient $\Pi/(\Pi \cap N)$ into $\mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s)$, we can now apply the uniqueness of the Seifert Construction. We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi \cap N & \longrightarrow & \Pi & \longrightarrow & \Pi/(\Pi \cap N) & \longrightarrow & 1 \\
 & & \downarrow & & {\scriptstyle \psi_2}\downarrow{\scriptstyle \mu(\tau)\circ\psi_1} & & \downarrow & & \\
1 & \longrightarrow & C(\mathbb{R}^s, N) \rtimes \mathrm{Inn}(N) & \longrightarrow & \mathrm{Diff}_N(S(\Gamma)) & \longrightarrow & \mathrm{Out}(N) \times \mathrm{Diff}(\mathbb{R}^s) & \longrightarrow & 1.
\end{array}$$

By Theorem 7.3.2, there exists an element $\lambda \in C(\mathbb{R}^s, N)$ which conjugates ψ_2 to $\mu(\tau) \circ \psi_1$. Thus

$$\begin{array}{ccc}
\Pi & \xrightarrow{\psi_1} & \mathrm{Diff}_N(S) \\
{\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \mu(\tau)} \\
\mathrm{Diff}_N(S(\Gamma)) & \xrightarrow{\mu(\lambda)} & \mathrm{Diff}_N(S(\Gamma))
\end{array}$$

is commutative. The map $\mu(\tau^{-1} \circ \lambda)$ sends $\psi_2(\Pi)$ to $\psi_1(\Pi)$ yielding a diffeomorphism from $\Pi \backslash S(\Gamma)$ onto $\Pi \backslash S$. In this argument, the fact that N is a connected, simply connected nilpotent Lie group is essential.

Now we show the space $\Pi \backslash S(\Gamma)$ admits a smooth maximal torus action. Let $\mathcal{Z}(\Pi) = \mathbb{Z}^k$ be the center of Π. Since M is the maximal normal nilpotent subgroup of Π, $\mathbb{Z}^k \subset M$. Let $\mathbb{R}^k$ be the smallest connected subgroup of $M_{\mathbb{R}}$ containing $\mathbb{Z}^k$. Since Π commutes with $\mathbb{Z}^k$ and $\Pi \subset \mathrm{Aff}(S(\Gamma))$, Π commutes with $(\mathbb{Z}^k)_{\mathbb{R}} = \mathbb{R}^k$. This means that $\mathbb{R}^k$ lies in the centralizer of Π in $\mathrm{Diff}_N(S(\Gamma))$. Of course, $\mathbb{R}^k \cap \Pi = \mathcal{Z}(\Pi)$. Thus we obtain an action of torus $\mathbb{R}^k/\mathbb{Z}^k$ on the model space $\Pi \backslash S(\Gamma)$. This action is smooth, (actually, it is a group of isometries if we give a left invariant metric on $S(\Gamma)$), and is a maximal torus action on $\Pi \backslash S(\Gamma)$. Now one can pull back this action to a smooth action on $\Pi \backslash S$. This completes the proof of Theorem 11.7.20. □

Corollary 11.7.26 (Mostow). *Let S_1, S_2 be two connected, simply connected solvable Lie groups. Let Γ_i be a lattice in S_i, $i = 1, 2$. Suppose Γ_1 is isomorphic to Γ_2. Then S_1/Γ_1 is diffeomorphic to S_2/Γ_2.*

For Mostow's argument, see [**Rag72**, Theorem 3.6]. We give a different proof. Since $\Gamma_1 \cong \Gamma_2 (= \Pi)$, construct a connected, simply connected solvable Lie group $S(\Gamma)$ on which these groups act. By Theorem 11.7.20, S_i/Γ_i is diffeomorphic to $S(\Gamma)/\Pi$, $i = 1, 2$. Therefore, S_1/Γ_1 is diffeomorphic to S_2/Γ_2.

The following example illustrates the construction employed in the proof of the theorem. Moreover, the example serves to illustrate why one is compelled to look for a larger group than S if one wishes to construct a maximal torus action from the descent of a vector subgroup.

Example 11.7.27. Let $S = \widetilde{E_0(2)} = \mathbb{R}^2 \rtimes \mathbb{R}$ be the universal covering group of the 2-dimensional Euclidean group, where $(0, t)$ acts on $\mathbb{R}^2$ by $x \mapsto e^{2\pi i t} x$, x seen as a complex number. Let Π be the lattice generated by

$$t_1 = \left(\begin{bmatrix}1\\0\end{bmatrix}, 0\right), \quad t_2 = \left(\begin{bmatrix}0\\1\end{bmatrix}, 0\right), \quad \alpha = \left(\begin{bmatrix}0\\0\end{bmatrix}, \frac{1}{2}\right).$$

The subgroup Γ generated by t_1, t_2 and α^2 is a characteristic subgroup of Π, isomorphic to $\mathbb{Z}^3$. Then $S(\Gamma) = \mathbb{R}^3$ and we get an embedding of S into $S(\Gamma) \rtimes S^1 = \mathbb{R}^3 \rtimes \mathrm{SO}(2) \subset \mathbb{R}^3 \rtimes \mathrm{O}(3) = E(3)$. The homomorphism is obvious:

$$\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}, t\right) \mapsto \left(\begin{bmatrix}x_1\\x_2\\t\end{bmatrix}, \begin{bmatrix}\cos 2\pi t & \sin 2\pi t & 0\\ -\sin 2\pi t & \cos 2\pi t & 0\\ 0 & 0 & 1\end{bmatrix}\right).$$

The image of Π in $E(3)$ is the orientable Bieberbach group of dimension 3 with holonomy group $\mathbb{Z}_2$. Clearly, the manifold $\Pi\backslash\mathbb{R}^2 \rtimes \mathbb{R}$ is diffeomorphic to the flat manifold $\Pi\backslash\mathbb{R}^3$, i.e., $\mathfrak{G}_2$. On $\Pi\backslash\mathbb{R}^3$, there is a maximal torus action by S^1, generated by the left translation by $\mathbb{R} = \{[0\ 0\ s]^t : s \in \mathbb{R}\}$. Note that this subgroup $\mathbb{R}$ of $S(\Gamma)$ is not in the image of S. This means that there is no S^1-action on $\Pi\backslash S$ coming from the left translation. In fact, it comes from the right translation by the $\mathbb{R}$-factor of $S = \mathbb{R}^2 \rtimes \mathbb{R}$.

If we consider just the subgroup Γ, it is even clearer what the theorem says. The solvmanifold $\Gamma\backslash\mathbb{R}^2 \rtimes \mathbb{R}$ is diffeomorphic to the torus $\Gamma\backslash\mathbb{R}^3$. On the latter torus, there is a standard T^3-action as translations. However, no vector subgroup in S descends to give a maximal torus action on $\Gamma\backslash\mathbb{R}^2 \rtimes \mathbb{R}$.

We now turn to general Lie groups. Little is known for the existence of a maximal torus action on general double coset spaces. Under some strong conditions, we can show an aspherical double coset space of a Lie group admits a maximal torus action.

Theorem 11.7.28. *Let G be a connected, simply connected Lie group, and let R be its radical. Suppose $S = G/R$ does not contain any normal compact factor. Let K be a maximal compact subgroup of G and Γ a torsion-free cocompact lattice in G such that $(\Gamma \cap R, R)$ has the unique automorphism extension property. If $\exp : \mathcal{R} \to R$ is surjective, then the double coset space $\Gamma\backslash G/K$ admits a smooth maximal torus action.*

Proof. Let $G = R \rtimes S$ be the Levi decomposition of G. Let $A = \{a \in R | (a, u) \in \mathcal{Z}(\Gamma)$ for some $u \in S\}$. Let $(a, u) \in \mathcal{Z}(\Gamma)$. Then for any $(z, 1) \in \Gamma_R = \Gamma \cap R$, $(z, 1)(a, u) = (a, u)(z, 1)$. This implies that ${}^{u}z = a^{-1} z a$. Since $(\Gamma \cap R, R)$ has UAEP, the two automorphisms u and $\mu(a^{-1})$ induce the same automorphisms on R. Therefore, ${}^u x = a^{-1}xa$ for all $x \in R$. Moreover, for any $(b, v) \in \Gamma$, we have ${}^v a = a$. Now it is easy to see that A is a commutative subgroup of R.

Choose generators $(a_i, u_i), i = 1, 2, \ldots, k$ for $\mathcal{Z}(\Gamma)$. We define a homomorphism $\phi_R : \mathbb{R}^k \to R$ as follows: Since $\exp : \mathcal{R} \to R$ is onto, log is defined on R. Let $A_i = \log a_i$. Then ϕ_R is the composite $\mathbb{R}^k \to \mathcal{R} \xrightarrow{\exp} R$, where the first map is the linear transformation from $\mathbb{R}^k$ to $\mathcal{R}$ sending the standard basis to $A_1, A_2, \ldots, A_k$. Since $[A_i, A_j] = 0$, the image of $\mathbb{R}^k$ in $\mathcal{R}$ is a commutative Lie subalgebra, and hence the exponential map restricted to this subalgebra is a homomorphism. Consequently, ϕ_R is a homomorphism.

Next, we define $\phi_S : \mathbb{R}^k \to S$ as follows: Let $S = S_1 \times S_2 \times \cdots \times S_r$, where each S_i is a simple group. For each i, let S_i^* denote the adjoint form of S_i, and choose a maximal compact subgroup of S_i^*. This maximal compact subgroup is of the form either $S^1 \times H$ or H, where H does not have a circle factor, depending on whether S_i has infinite center or not. This determines a subgroup $\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i \subset S_i$, where $\widetilde{H}_i$ is compact, and $\epsilon_i = 1$ or 0, depending on whether S_i has infinite center or not. In the former case, $\mathbb{R}$ contains the infinite summand of the center of S_i. Then $K = \Pi\widetilde{H}_i$ is a maximal compact subgroup of S.

Consider the map $\mathcal{Z}(\Gamma) \to \Pi(\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i) \to \Pi\mathbb{R}^{\epsilon_i} \subset \Pi S_i$, where $\Pi(\mathbb{R}^{\epsilon_i} \times \widetilde{H}_i) \to \Pi\mathbb{R}^{\epsilon_i}$ is a projection. We extend this to a homomorphism $\phi_S : \mathbb{R}^k \to \Pi\mathbb{R}^{\epsilon_i} \subset S$. Note that $\phi_S(\mathcal{Z}(\Gamma))$ differs from $p(\mathcal{Z}(\Gamma))$ by elements in $\Pi\widetilde{H}_i \subset K$.

Note that K commutes with $\mathbb{R}^{\epsilon_1} \times \mathbb{R}^{\epsilon_2} \times \cdots \times \mathbb{R}^{\epsilon_r}$. Thus we have an induced action of $\mathbb{R}^k$ on G/K. The action of $\mathbb{R}^k$ on G/K will not be effective in general, because $\mathcal{Z}(\Gamma) \to \Pi\mathbb{R}^{\epsilon_i} \subset S$ may have a nontrivial kernel. Even though the actions by $\mathbb{Z}^k \subset \mathbb{R}^k$ and by $\mathcal{Z}(\Gamma)$ are different on S, they induce the same one over S/K.

A desired action of $\mathbb{R}^k$ on $G/K = R \cdot S/K$ is then given by

$$\phi(t)(x, w) = (x \cdot \phi_R(t), w \cdot \phi_S(t)).$$

Since Γ acts on G as left multiplications, it commutes with the $\mathbb{R}^k$-action defined above. Moreover, we have $\mathbb{R}^k \cap \Gamma = \mathcal{Z}(\Gamma)$ on G/K. Consequently, we have obtained a smooth action of $T^k = \mathcal{Z}(\Gamma)\backslash\mathbb{R}^k$ on $\Gamma\backslash G/K$. □

THEOREM 11.7.29 ([**LR91**]). *Let G be a connected, simply connected Lie group without any normal compact factors in its semisimple part. Let Γ be a torsion-free cocompact lattice and K a maximal compact subgroup of G. Then there is a smooth manifold M, which is homotopy equivalent to the double coset space $\Gamma\backslash G/K$, admitting a smooth maximal torus action.*

PROOF. We may assume that $\Gamma = \pi_1(\Gamma\backslash G/K)$. Let R be the radical of G. Then $G = R \rtimes S$. Let $p : G \to S$ be the projection, and let $\mathcal{Z}(\Gamma)$ denote the center of Γ. Let $\widetilde{\Gamma} = \Gamma_R \cdot \mathcal{Z}(\Gamma)$, where $\Gamma_R = \Gamma \cap R$. It is poly-{cyclic or finite} since $1 \to \Gamma_R \to \Gamma_R \cdot \mathcal{Z}(\Gamma) \to p(\mathcal{Z}(\Gamma)) \to 1$ is exact, Γ_R is a lattice of R, and $p(\mathcal{Z}(\Gamma))$ is a finitely generated Abelian group. Such a group $\widetilde{\Gamma}$ contains a characteristic subgroup Γ' of finite index which is a Mostow-Wang group (see Definition 9.5.1), with ${}^n\Gamma' = {}^n\widetilde{\Gamma}$, where ^{n} denotes the discrete nilradical. Now $\widetilde{\Gamma}$ contains a characteristic subgroup $\widehat{\Gamma}$ of finite index which is predivisible and ${}^n\widehat{\Gamma} = {}^n\Gamma'$. This implies that $\widehat{\Gamma}/{}^n\widehat{\Gamma}$ is free Abelian, say $\mathbb{Z}^m$.

Let $Q = \Gamma/\widehat{\Gamma}$ and $S^* = S/p(\mathcal{Z}(\Gamma))$. Note that S^* is not necessarily the adjoint form of S. Let K^* be a maximal compact subgroup of S^*. Note that K^* is a finite quotient of $T \times K$, where T is a torus generated by free Abelian factors of $p(\mathcal{Z}(\Gamma))$.

Now $\Gamma/\widetilde{\Gamma} = \Gamma/\Gamma_R \cdot \mathcal{Z}(\Gamma)$ acts on S^*/K^* with compact quotient. Therefore $Q = \Gamma/\widehat{\Gamma}$ acts on S^*/K^* with compact quotient via the homomorphism $\Gamma/\widehat{\Gamma} \to \Gamma/\widetilde{\Gamma}$. Let us denote ${}^{n}\widehat{\Gamma}$ by Δ. Since ${}^{n}\widehat{\Gamma}$ is characteristic in $\widehat{\Gamma}$, it is normal in Γ. Consider the exact sequences $1 \to \Delta \to \Gamma \to \Gamma/\Delta \to 1$ and $1 \to \mathbb{Z}^m \to \Gamma/\Delta \to Q \to 1$. We get the latter exact sequence from the fact that $\widehat{\Gamma}$ is predivisible. We do the Seifert space construction with the latter exact sequence and the action of Q on the space S^*/K^* to obtain an action of Γ/Δ on $\mathbb{R}^m \times S^*/K^*$. Now we do a Seifert space construction with the first exact sequence and the action of Γ/Δ on $\mathbb{R}^m \times S^*/K^*$. Consequently we obtain an action of Γ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times S^*/K^*$. Let $\mathbb{Z}^k$ be the center of Γ. It lies in the center of Δ. Since the center of Δ lies in the center of $\Delta_{\mathbb{R}}$, there is a unique subgroup $\mathbb{R}^k$ in the center of $\Delta_{\mathbb{R}}$ containing $\mathbb{Z}^k$ as a uniform lattice. The action of $\mathbb{R}^k$ on $\Delta_{\mathbb{R}} \times \mathbb{R}^m \times S^*/K^*$, by left multiplication on the first factor, commutes with the action of Γ. Therefore, it induces an action of torus $\mathbb{R}^k/\mathbb{Z}^k$ on $M = \Gamma\backslash(\Delta_{\mathbb{R}} \times \mathbb{R}^m \times S^*/K^*)$ (which has the same homotopy type as $\Gamma\backslash G/K$). Clearly, this is a smooth maximal torus action. □

REMARK 11.7.30. In [**FJ98**], Farrell and Jones, using surgery theory, show that any closed manifold M^n homotopically equivalent to $\Gamma\backslash G/K$ is homeomorphic to it provided G has a faithful representation into $\mathrm{GL}(m, \mathbb{R})$ for some m, $n \neq 3, 4$. Thus when G, in the theorem above, has a faithful linear representation, then $\Gamma\backslash G/K$ has a maximal torus action. Of course there are simply connected G without faithful representations.

11.8. Toral rank of spherical space forms

11.8.1. The *toral rank* of a space X is the dimension of the largest torus that acts effectively on X. It is also called the *toral degree of symmetry* of X (by [**Hsi75**]). We have seen that the only connected Lie groups that act on admissible manifolds are tori (Theorem 3.2.2) and they must act injectively. Therefore, an upper bound on the toral rank of an admissible manifold is the rank of the center of its fundamental group. When the admissible manifold admits a maximal torus action, then this upper bound is the toral rank. In this section, we will study the toral rank of manifolds covered by the sphere, and in particular, determine the toral rank of spherical space forms.

A finite group that acts freely on S^{n-1} has periodic cohomology with the minimum period dividing n. This can be seen, in the linear or simplicial case, by splicing copies of the chain complex for the sphere together to get a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$:

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(S^{n-1}) \leftarrow C_1(S^{n-1}) \leftarrow \cdots \leftarrow C_{i-1}(S^{n-1}) \leftarrow C_i(S^{n-1}) \leftarrow \cdots .$$

A finite group satisfies the *pq-condition* (p, q are primes) if every subgroup of order pq is cyclic. A finite group F has periodic cohomology if and only if F satisfies all the p^2-*conditions* [**CE56**, XII-§11]. A group F that acts freely and orthogonally on a sphere S^{2n-1} satisfies all pq-conditions (p may be equal to q). In this case, $F\backslash S^{2n-1}$ is a spherical space form (see subsections and examples, 4.5.16–4.5.18) and F has periodic cohomology a divisor of $2n$. Conversely, if F is solvable and satisfies all pq-conditions, free linear actions do exist [**Wol77**, §6.1.11]. Though in general, all the pq-conditions are not sufficient for F to act freely and linearly on some sphere. However, Milnor has shown that if F acts freely and topologically on a sphere, it satisfies all the $2q$-*conditions* [**Mil57**].

The following, proved by surgery theory, characterizes those groups that can act freely and topologically on some sphere.

THEOREM 11.8.2 ([**MTW76**]). *A finite group F can act freely on a sphere if and only if it satisfies all $2p$- and p^2-conditions.*

For example, all the groups $\mathrm{SL}(2,p)$, p an odd prime, can act freely on a sphere. But it is only $\mathrm{SL}(2,3)$ and $\mathrm{SL}(2,5)$ that can act freely and orthogonally on a sphere. Note $\mathrm{SL}(2,2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ does not satisfy the $2p$-condition but it does satisfy all the p^2-conditions.

In the theorem, the actions produced are not necessarily on spheres S^{n-1}, where n is the minimal period of the cohomology of F. However, the discrepancy is at most a factor of 2. In the examples constructed by T. Petrie [**Pet71**], with $F = \mathbb{Z}_m \rtimes \mathbb{Z}_q$, m, q odd and $\mathbb{Z}_q \to \mathrm{Aut}(\mathbb{Z}_m)$ faithful, the minimal period is $2q$ and the free topological actions are on S^{2q-1}. Not all are linear; for example, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ acts freely on the 5-sphere, does not satisfy the pq-condition, and does not act linearly.

The groups that act freely and orthogonally on a sphere are divided into six types I, II, ..., VI. If a group F is of type I, II, III, or V and its minimal period is $2n$, then F admits a free orthogonal actions on S^{2n-1}. But there are some in type IV and VI where the minimal sphere is S^{4n-1} and the period is $2n$.

11.8.3. If a finite group F acts freely on a cohomology $(n-1)$-manifold over $\mathbb{Z}$, having the $\mathbb{Z}$-cohomology of S^{n-1}, then the cohomology of F is periodic with period dividing n. For an easy proof in this context, consider the Borel space $\Sigma_F = EF \times_F \Sigma$. This fibers over $F\backslash\Sigma$ with contractible fiber EF and also fibers over BG with fiber Σ. Now use the spectral sequence $''E$ of Chapter 10 associated with the second fibering. For facts concerning cohomology manifolds; see Section 1.8.

LEMMA 11.8.4 ([**CR69**, 4.15]). *Let F act freely on an $(2n-1)$-dimensional $\mathbb{Z}$-cohomology manifold Σ^{2n-1} having the $\mathbb{Z}$-cohomology of S^{2n-1}. Suppose the minimal period of the cohomology of F is $2n$. If S^1 acts effectively on $M = F\backslash\Sigma^{2n-1}$, then $M^{S^1} = \emptyset$.*

PROOF. Suppose $x \in M^{S^1}$. Choose $\widehat{x} \in \Sigma^{2n-1}$ as a base point in Σ^{2n-1} over $x \in M$. The S^1-action can be lifted to Σ^{2n-1} and $\widehat{x}$ is fixed. Now $(\Sigma^{2n-1})^{S^1} = \Sigma'$ is a closed sphere-like cohomology manifold of odd dimension $< 2n-2$; see Subsection 1.8.1. Furthermore, if C is the connected component of M^{S^1} containing x, $\nu^{-1}(C) = \Sigma'$, where $\nu : \Sigma^{2n-1} \to M$ is the orbit mapping of the F-action, see Corollary 2.3.6. Since F is invariant on Σ', and dimension of Σ' is less than $2n-2$, the period of F must be less than $2n$. This is a contradiction.

Let us also observe that M^{S^1} is connected (when nonempty) and therefore, it is the same as C. For, M is rationally an odd dimensional cohomology sphere and the fixed set M^{S^1} is an odd dimensional rational cohomology manifold with the rational cohomology of an odd dimensional sphere. □

PROPOSITION 11.8.5. *Let F act freely and topologically on a sphere S^{2n-1}. Let $2d$ be the minimal period of the cohomology of F and suppose there is a free topological action of F on S^{2d-1}. Then $n = kd$ and the toral rank of $M = F\backslash S^{2n-1}$ is $\leq k$.*

PROOF. We show first that if T^k acts effectively on M, it cannot have any fixed points. We shall use the following fact: If K is normal in G, and G acts on X, then $\mathrm{Fix}(G, X) = \mathrm{Fix}(G/K, \mathrm{Fix}(K, X))$. Now suppose $\mathrm{Fix}(T^k, M) \neq \emptyset$. Then $\mathrm{Fix}(T^{k-1}, \mathrm{Fix}(T^1, M)) \neq \emptyset$, where $T^{k-1} = T^k/T^1$.

If we lift the T^1-action to S^{2kd-1}, we get $\mathrm{Fix}(T^1, S^{2kd-1}) = \Sigma^s$, an odd dimensional compact $\mathbb{Z}$-cohomology s-manifold having the integral cohomology of the s-sphere with $s \leq 2n-3$. Since s is odd, Σ^s is connected. As T^1 commutes with the covering F-action, the period $2d$ divides $s+1$. Thus $s = 2dr_1 - 1$, where $r_1 \leq k-1$. Thus we have $\mathrm{Fix}(T^k, M) = \mathrm{Fix}(T^{k-1}, F\backslash\Sigma^{2dr_1-1})$. Continuing inductively, we have $\mathrm{Fix}(T^k, M) = \mathrm{Fix}(T^{k-1}, F\backslash\Sigma^{2dr_1-1}) = \cdots = \mathrm{Fix}(T^{k-s}, F\backslash\Sigma^{2dr_s-1}) \neq \emptyset$ with $r_s < r_{s-1} < \cdots < r_1 \leq k-1$. Thus $r_s \leq k-s$. To avoid a contradiction, we have $r_{k-1} = 1$. Then we have T^1-lifting to Σ^{2d-1} and acting with fixed points. The fixed set is again a sphere-like cohomology manifold of dimension less than $2d-1$. But again, F is invariant on this fixed set which yields a contradiction to Lemma 11.8.4. So the action of T^k on M is without fixed points.

Now suppose there is an effective (T^{k+1}, M)-action. There are a finite number of distinct isotropy groups each having rank at most $k-1$. Then there is a circle subgroup S^1 in T^{k+1} which is not completely contained in any of the isotropy groups. Therefore, this S^1 acts with only finite isotropy subgroups. Now $H^*(M;\mathbb{Q}) \cong H^*(S^{2kd-1};\mathbb{Q})$ and the cohomology of the orbit space $S^1\backslash M$ is rationally like the cohomology of $\mathbb{C}P_{kd-1}$. T^k acts on $S^1\backslash M$ and $\mathrm{Fix}(T^k, S^1\backslash M) \neq \emptyset$ because $\chi(S^1\backslash M) \neq \emptyset$. Then, each S^1-orbit over a fixed $w \in \mathrm{Fix}(T^k, S^1\backslash M)$ is invariant under the T^{k+1}-action. Thus, there is a subgroup of rank k which fixes the S^1-orbit. This contradicts Lemma 11.8.4. □

11.8.6. If G is a connected Lie group acting effectively on X and $\pi_1(X)$ is finite, then the finite covering group G' of G corresponding to the kernel of $\mathrm{ev}_*^x : \pi_1(G, 1) \to \pi_1(X, x)$ lifts to an effective action on the universal covering of X.

In Subsection 4.5.18, we saw that for a spherical space form, the group of diagonal unitary matrices D in $U(n)$ with constant entries descends to an effective *unitary* action of $D/(D \cap F)$ on M.

PROPOSITION 11.8.7 ([**Kah70**]). *If F acts freely and unitarily on S^{2n-1}, where $2n$ is the minimal period of the cohomology of F, then the descent of D to $M = F\backslash S^{2n-1}$ is the only effective "unitary" circle action on M.*

PROOF. This is a corollary of Proposition 11.8.5, where $k = 1$. Suppose there is an effective circle action C on M. Then a covering C' of C lifts to an effective action of C' on S^{2n-1} which commutes with F. Suppose, in addition, that C' commutes with D. This will be the case if C' is a unitary action on S^{2n-1}. If $C' \neq D$, then C' and D generate a 2-dimensional torus T which commutes with F. This torus then descends to an effective torus, modulo a finite subgroup, acting on M. This contradicts Proposition 11.8.5. □

11.8.8. Let F act freely and unitarily on some sphere S^{2n-1}. The representation then can be conjugated, modulo an automorphism of F, into $U(n)$. This representation splits into k irreducible complex representations $V_1 \oplus \cdots \oplus V_k$ of constant degree d, independent of n. It follows that each V_i is isomorphic to $\mathbb{C}^d$, and the unit sphere in V_i is homeomorphic to S_i^{2d-1}. Thus, F acts freely on the join $S^{2d-1} \circ S^{2d-1} \circ \cdots \circ S^{2d-1}$ by direct sum representations. Therefore $n = kd$ for some k. Consequently, by the comment in Subsection 4.5.18 and [**Wol77**, §7.4],

$2d-1$ is the minimal dimension of a sphere for which there is a free unitary and, hence, free orthogonal action of F,

On each summand $\mathbb{C}_i^d$, there is the circle action D which commutes with the action of F. We can extend this action to act trivially on the other factors. In this way, we get a k-torus acting on $\sum_{i=1}^k \mathbb{C}_i^d$ and by restriction on S^{2n-1} which commutes with the action of F on S^{2n-1}. This T^k-action descends to give us an effective unitary T^k-action on $F\backslash S^{2n-1}$. Therefore, we have

Theorem 11.8.9 (cf. [**Kah70**]). *If $M = F\backslash S^{2n-1}$ is a spherical space form and d is the (constant) minimal degree of any free irreducible unitary representation of F, then M has an effective unitary action of T^k, where $kd = n$. Moreover, if the minimal period of the cohomology of F is $2d$, then the toral rank of M is exactly k.*

Remark 11.8.10. (1) Proposition 11.8.5 has much wider applicability than just spherical space forms M. C.T.C. Wall and others have shown that if M is a spherical space form of dimension greater than 4, then there is an infinite number of topologically distinct M' h-cobordant to M all covered by the sphere. Thus for each M', the toral rank of M' is less than or equal to k, where M, a spherical space form, satisfies the hypothesis of Proposition 11.8.5.

(2) In [**MTW83**], it is shown that if a finite group F satisfies all p^2 and all $2p$-conditions, then there exists a free and smooth action on the sphere S^{2kd-1} with the standard differential structure. Here $2d$ is the period of F and k can be taken to be 1 or 2. Even when one must choose 2, this result, except for one class of groups, is, geometrically the best possible result. Proposition 11.8.5 gives us an upper bound for the toral ranks of these manifolds. It would be interesting to have better bounds on the torus ranks of those manifolds that do not satisfy the conditions of Theorem 11.8.9.

Note that Proposition 11.8.5 can also be formulated in terms of sphere-like integral cohomology manifolds. The proof is essentially as given in Proposition 11.8.5.

(3) Note for a $(2n-1)$-dimensional lens space $F\backslash S^{2n-1}$, the period of F is 2 and consequently the toral rank is exactly n. Using the join construction as in Subsection 11.8.8, it is easy to construct an infinite number of distinct linear T^n-actions on each lens space. The infinite number of distinct effective T^2-actions on the 3-sphere are all weakly equivalent. The T^2-actions on 3-dimensional lens spaces are classified in [**OR70a**]. All the other 3-dimensional space forms have period 4 and admit a unique S^1-action up to equivalence. This action is the descent of the unique unitary action of Proposition 11.8.7 (cf. Chapters 14 and 15).

(4) Kahn in Proposition 11.8.7 assumed that $2n$ was the minimal degree for which F has a free unitary representation instead of our slightly different assumption of $2n$ being the minimal period of the cohomology of F. His proof of the proposition is explicit and computational and differs from what we gave. As a by-product, he showed that $D \cap \pi_1(M) = D \cap F = \text{Center}(\pi_1(M))$ and the action of $D/D\cap F$ is locally injective and effective on M. Consequently, we may conclude that the *linear* $D/D\cap F$-action lifts to a principal S^1-bundle over $\mathbb{C}P_{n-1}$ whose first Chern class or Euler class is $\pm|D\cap F|$ with $S^1 = D/D\cap F$.