CHAPTER 3

Cluster algebras

3.1. Basic definitions and examples

The goal of this chapter is to introduce cluster algebras and to study their basic properties. To make the exposition more transparent, and to emphasize the connections to geometry, we decided to restrict our attention to the so-called cluster algebras of geometric type. However, in the text we have indicated possible extensions to general cluster algebras.

Let us start from several technical definitions.

Definition 3.1. Let B be an $n \times n$ integer matrix. We say that

B is *skew-symmetric* if $b_{ij} = -b_{ji}$ for any $i, j \in [1, n]$;

B is *skew-symmetrizable* if there exists a positive integer diagonal matrix D such that DB is skew-symmetric; in this case D is called the *skew-symmetrizer* of B, and B is called *D-skew-symmetrizable*;

B is *sign-skew-symmetric* if $b_{ij}b_{ji} \le 0$ for any $i, j \in [1, n]$ and $b_{ij}b_{ji} = 0$ implies $b_{ij} = b_{ji} = 0$.

Evidently, each skew-symmetric matrix is skew-symmetrizable, and each skew-symmetrizable matrix is sign-skew-symmetric.

The *coefficient group* $\mathcal{P}$ is a free multiplicative abelian group of a finite rank m with generators $g_1, \dots, g_m$. An *ambient field* in our setting is the field $\mathcal{F}$ of rational functions in n independent variables with coefficients in the field of fractions of the integer group ring $\mathbb{Z}\mathcal{P} = \mathbb{Z}[g_1^{\pm 1}, \dots, g_m^{\pm 1}]$ (here and in what follows we write $x^{\pm 1}$ instead of x, x^{-1}).

Definition 3.2. A *seed* (of *geometric type*) in $\mathcal{F}$ is a pair $\Sigma = (\mathbf{x}, \widetilde{B})$, where $\mathbf{x} = (x_1, \dots, x_n)$ is a transcendence basis of $\mathcal{F}$ over the field of fractions of $\mathbb{Z}\mathcal{P}$ and $\widetilde{B}$ is an $n \times (n + m)$ integer matrix whose principal part B (that is, the $n \times n$ submatrix formed by the columns $1, \dots, n$) is sign-skew-symmetric.

In what follows $\mathbf{x}$ is called a *cluster*, and its elements $x_1, \dots, x_n$ are called *cluster variables*. Denote $x_{n+i} = g_i$ for $i \in [1, m]$. We say that $\widetilde{\mathbf{x}} = (x_1, \dots, x_{n+m})$ is an *extended cluster*, and $x_{n+1}, \dots, x_{n+m}$ are *stable variables*. It will be sometimes convenient to think of $\mathcal{F}$ as of the field of rational functions in $n + m$ independent variables with rational coefficients. The square matrix B is called the *exchange matrix*, and $\widetilde{B}$, the *extended exchange matrix*. Slightly abusing notation, we denote the entries of $\widetilde{B}$ by b_{ij}, $i \in [1, n]$, $j \in [1, n + m]$; besides, we say that $\widetilde{B}$ is skew-symmetric (D-skew-symmetrizable, sign-skew-symmetric) whenever B possesses this property.

Definition 3.3. Given a seed as above, the *adjacent cluster* in direction $k \in [1, n]$ is defined by

$$\mathbf{x}_k = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\},$$

where the new cluster variable x'_k is defined by the *exchange relation*

$$x_k x'_k = \prod_{\substack{1\le i\le n+m\\ b_{ki}>0}} x_i^{b_{ki}} + \prod_{\substack{1\le i\le n+m\\ b_{ki}<0}} x_i^{-b_{ki}}; \tag{3.1}$$

here, as usual, the product over the empty set is assumed to be equal to 1.

REMARK 3.4. It is important to note that most papers use a different correspondence between the (extended) exchange matrix and the exchange relations. Namely, their exchange relations correspond to the *columns* of the matrix, and not to its *rows* as above; respectively, the extended exchange matrix has n columns and $n+m$ rows. Clearly, these two approaches are equivalent up to taking the transpose.

Occasionally we will be interested in separating cluster and stable variables in (3.1) by rewriting it in the form

$$x_k x'_k = p_k^+ \prod_{\substack{1\le i\le n\\ b_{ki}>0}} x_i^{b_{ki}} + p_k^- \prod_{\substack{1\le i\le n\\ b_{ki}<0}} x_i^{-b_{ki}}, \tag{3.2}$$

where the coefficients $p_k^\pm \in \mathcal{P}$ are given by

$$p_k^+ = \prod_{\substack{1\le i\le m\\ b_{kn+i}>0}} x_{n+i}^{b_{kn+i}}, \qquad p_k^- = \prod_{\substack{1\le i\le m\\ b_{kn+i}<0}} x_{n+i}^{-b_{kn+i}}. \tag{3.3}$$

DEFINITION 3.5. Let A and A' be two matrices of the same size. We say that A' is obtained from A by a *matrix mutation* in direction k and write $A' = \mu_k(A)$ if

$$a'_{ij} = \begin{cases} -a_{ij}, & \text{if } i=k \text{ or } j=k;\\ a_{ij} + \dfrac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2}, & \text{otherwise.}\end{cases}$$

It can be easily verified that $\mu_k(\mu_k(A)) = A$.

We say that two matrices $\widetilde{B}$ and $\widetilde{B}'$ are *mutation equivalent*, and write $\widetilde{B} \simeq \widetilde{B}'$, if each of them can be obtained from the other by a sequence of matrix mutations. It is easy to see that the property of a matrix to be sign-skew-symmetric is not necessarily preserved under mutation equivalence. For example, the matrices

$$\begin{pmatrix} 0 & 1 & -2\\ -3 & 0 & 1\\ 1 & -2 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 2\\ 3 & 0 & -5\\ -1 & -1 & 0\end{pmatrix}$$

are connected by a matrix mutation in direction 1. However, the first of these matrices is sign-skew-symmetric, while the second is not. This brings us to the following definition: a matrix is said to be *totally* sign-skew-symmetric if any matrix that is mutation equivalent to it is sign-skew-symmetric.

An important class of totally sign-skew-symmetric matrices is furnished by the following proposition.

PROPOSITION 3.6. *Skew-symmetrizable matrices are totally sign-skew-symmetric.*

PROOF. The proof follows immediately from Definition 3.1 and Definition 3.5. Moreover, it is easy to see that if $\widetilde{B} \simeq \widetilde{B}'$ and DB is skew-symmetric for some integer diagonal matrix D, then DB' is skew-symmetric as well. □

Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$, we say that a seed $\Sigma' = (\mathbf{x}', \widetilde{B}')$ is *adjacent* to Σ (in direction k) if $\mathbf{x}'$ is adjacent to $\mathbf{x}$ in direction k and $\widetilde{B}' = \mu_k(\widetilde{B})$. Two seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds.

To get better acquainted with the introduced objects, let us study the change of the $2n$-tuple of coefficients $\mathbf{p} = (p_1^{\pm}, \dots, p_n^{\pm})$ after a transition to an adjacent seed.

Proposition 3.7. *Let the seeds Σ and Σ' be adjacent in direction k. Then the coefficients $(p_j^{\pm})'$ in the exchange relations of the seed Σ' satisfy relations $(p_k^{\pm})' = p_k^{\mp}$ and*

$$\frac{(p_j^+)'}{(p_j^-)'} = \begin{cases} \dfrac{p_j^+}{p_j^-}(p_k^+)^{b_{jk}}, & \text{if } b_{jk} \geq 0, \\ \dfrac{p_j^+}{p_j^-}(p_k^-)^{b_{jk}}, & \text{if } b_{jk} \leq 0. \end{cases} \tag{3.4}$$

Proof. The case $j = k$ follows immediately from (3.3) and the definition of matrix mutations. Let $j \neq k$, then by (3.3) and Definition 3.5 one has

$$\frac{(p_j^+)'}{(p_j^-)'} = \prod_{1 \leq i \leq m} (x'_{n+i})^{b'_{jn+i}} = \prod_{1 \leq i \leq m} x_{n+i}^{b_{jn+i}} \prod_{1 \leq i \leq m} x_{n+i}^{(|b_{jk}|b_{kn+i} + b_{jk}|b_{kn+i}|)/2},$$

since stable variables in the cluster Σ' remain the same. The first factor in the latter product equals p_j^+/p_j^-. Let $b_{jk} \geq 0$, then the second factor equals

$$\prod_{1 \leq i \leq m} x_{n+i}^{b_{jk}(b_{kn+i} + |b_{kn+i}|)/2} = \left(\prod_{\substack{1 \leq i \leq m \\ b_{kn+i} > 0}} x_{n+i}^{b_{kn+i}} \right)^{b_{jk}} = (p_k^+)^{b_{jk}}.$$

Similarly, for $b_{jk} \leq 0$ the second factor equals $(p_k^-)^{b_{jk}}$, and the result follows. □

Now we are ready to define the main object of our studies.

Definition 3.8. Let $\Sigma = (\mathbf{x}, \widetilde{B})$ be a seed with a skew-symmetrizable matrix $\widetilde{B}$, $\mathbb{A}$ be a subring with unity in $\mathbb{Z}\mathcal{P}$ containing all the coefficients $p_k^{\pm}$ for all seeds mutation equivalent to Σ. The *cluster algebra* (of *geometric type*) $\mathcal{A} = \mathcal{A}(\widetilde{B})$ over $\mathbb{A}$ associated with Σ is the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds mutation equivalent to Σ. The number n of rows in the matrix $\widetilde{B}$ is said to be the *rank* of $\mathcal{A}$. The ring $\mathbb{A}$ is said to be the *ground ring* of $\mathcal{A}$.

In what follows we usually assume that for cluster algebras of geometric type the ground ring is the polynomial ring $\mathbb{Z}[x_{n+1}, \dots, x_{n+m}]$, and do not mention the ground ring explicitly.

A convenient tool in dealing with cluster algebras is the n-regular tree $\mathbb{T}_n$. Its vertices correspond to seeds, and two vertices are connected by an edge labeled by k if and only if the corresponding seeds are adjacent in direction k. The edges of $\mathbb{T}_n$ are thus labeled by the numbers $1, \dots, n$ so that the n edges emanating from each vertex receive different labels. The *distance* $d(\Sigma, \Sigma')$ between the seeds Σ and Σ' is the length of the path between the corresponding vertices in $\mathbb{T}_n$; when this does not lead to a confusion, we will write $d(\mathbf{x}, \mathbf{x}')$ instead of $d(\Sigma, \Sigma')$. Two seeds obtained one from the other by an arbitrary permutation of cluster (or stable) variables and the corresponding permutation of the rows and columns of $\widetilde{B}$ are called *equivalent*. The *exchange graph* of a cluster algebra is defined as the quotient of the tree $\mathbb{T}_n$ modulo this equivalence relation.

EXAMPLE 3.9. Let $n = 1$. The matrix $\widetilde{B}$ is a vector $(b_1, \dots, b_{m+1})$, and the skew-symmetrizability condition boils down to $b_1 = 0$. The initial cluster $\mathbf{x}$ contains one cluster variable x_1, and the unique exchange relation reads $x_1x_1' = p_1^+ + p_1^-$, where $p_1^\pm$ are monomials in stable variables $x_2, \dots, x_{m+1}$ with exponents prescribed by $\widetilde{B}$. The adjacent cluster consists of x_1', and the corresponding exchange relation coincides with the above one, with p_1^+ and p_1^- interchanged. The two adjacent clusters exhaust the class of mutation equivalence of $\mathbf{x}$. The exchange graph coincides with the 1-regular tree $\mathbb{T}_1$ and is the path of length 1.

Though trivial, this example serves for the description of the homogeneous coordinate ring of $G_2(4)$ and of the ring of regular functions on L^{e,w_0} in SL_3. In the former case one takes $m = 4$ and $\widetilde{B} = (0, 1, 1, -1, -1)$, which leads to the Plücker exchange relation from Section 2.1.2. The ring $\mathbb{C}[G_2(4)]$ is obtained by tensoring the corresponding cluster algebra with $\mathbb{C}$. In the latter case one takes $m = 2$ and $\widetilde{B} = (0, 1, -1)$, which leads to relation $M_3M_3' = M_1 + M_2$ from Example 2.17. The ring $\mathcal{O}(L^{e,w_0})$ is obtained by tensoring the corresponding cluster algebra over the ground ring $\mathbb{Z}\mathcal{P}$ with $\mathbb{C}$. Instead of changing the ground ring, one can take the localization of the cluster algebra (over the standard polynomial ring) with respect to the stable variables.

Let $n = 2$. Since $\mathbb{T}_2$ is an infinite path, its quotient can be either an infinite path or a cycle. Instead of trying to provide a complete classification of cluster algebras of rank 2, we give several examples.

EXAMPLE 3.10. Let us start from the matrix

$$\widetilde{B} = \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The two possible exchange relations are $x_1x_1' = x_2x_5 + x_4x_6$ and $x_2x_2' = x_1x_7 + x_3x_6$; recall that the variables $x_3, \dots, x_7$ are stable, so $p_1^+ = x_5$, $p_1^- = x_4x_6$, $p_2^+ = x_3x_6$, $p_2^- = x_7$. Applying matrix mutation in direction 1 we get

$$\widetilde{B}' = \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & -1 \end{pmatrix}.$$

The corresponding cluster variables are $(x_2x_5 + x_4x_6)/x_1$ and x_2.

On the next step we mutate the obtained matrix in direction 2, then again in direction 1, and so on. The matrices obtained on the four consecutive steps are

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix},$$

the corresponding pairs of cluster variables are

$$\left(\frac{x_2x_5 + x_4x_6}{x_1}, \frac{x_1x_4x_7 + x_2x_3x_5 + x_3x_4x_6}{x_1x_2}\right),$$
$$\left(\frac{x_1x_7 + x_3x_6}{x_2}, \frac{x_1x_4x_7 + x_2x_3x_5 + x_3x_4x_6}{x_1x_2}\right),$$
$$\left(\frac{x_1x_7 + x_3x_6}{x_2}, x_1\right),$$
$$(x_2, x_1).$$

Observe that the last matrix coincides with the original matrix $\widetilde{B}$ up to the transposition of rows and columns, and that the same is true for the elements of the corresponding cluster. Therefore, the sixth cluster is equivalent to the initial one, and hence the total number of nonequivalent clusters in this case is equal to 5. Comparing the exchange relations with the Plücker relations (2.1), one concludes that the arising cluster algebra tensored with $\mathbb{C}$ is isomorphic to the homogeneous coordinate ring of the Grassmannian $G_2(5)$. The corresponding exchange graph is the Stasheff pentagon described in Section 2.1.3. It is convenient to identify the initial cluster with the triangulation in the lower left corner of the pentagon, and the variables $x_1, \dots, x_7$ with the Plücker coordinates $x_{24}, x_{25}, x_{12}, x_{23}, x_{34}, x_{45}, x_{15}$.

Our next example shows that the number of nonequivalent clusters in a cluster algebra is not necessarily finite.

Example 3.11. Let again $n = 2$ and let

$$\widetilde{B} = B = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}.$$

Here $m = 0$, so we are dealing with a *coefficient-free* cluster algebra (that is, all the coefficients in the exchange relations are equal to 1). The two possible exchange relations are $x_1x_1' = x_2^3 + 1$ and $x_2x_2' = 1 + x_1^3$. Applying matrix mutation in direction 1 we get $B' = -B$. The corresponding cluster variables are $(x_2^3 + 1)/x_1$ and x_2.

On the next step we mutate the obtained matrix in direction 2, then again in direction 1, and so on. The matrices obtained this way are equal to $\pm B$, so the exchange relations remain the same all the time. The new cluster variables are rational functions in x_1 and x_2. Given a rational function f, write it as $f = p/q$ where p and q are polynomials and define $\delta(f) = \deg p - \deg q$ (evidently, $\delta(f)$ depends only on f and does not depend on the specific choice of p and q). Let now $(\bar{x}_1, \bar{x}_2)$ be one of the clusters obtained by the above process. Assume that $\delta(\bar{x}_1) \geq \delta(\bar{x}_2) > 0$; it is easy to see that for $\bar{x}_2' = (1 + \bar{x}_1^3)/\bar{x}_2$ one has $\delta(\bar{x}_2') = 3\delta(\bar{x}_1) - \delta(\bar{x}_2) > \delta(\bar{x}_1)$. Since $\delta(x_1) = \delta(x_2) = 1$, this means that δ is not bounded from above on the set of cluster variables, and hence the number of nonequivalent clusters in the cluster algebra $\mathcal{A}(B)$ is infinite. The corresponding exchange graph is a path infinite in both directions.

Though the cluster algebra in the previous example has an infinite number of nonequivalent clusters, it is not difficult to prove that it is finitely generated; in fact, each cluster algebra of rank 2 is generated by four cluster variables subject to two relations, see Lemma 3.17 below. Our next example shows that already cluster algebras of rank 3 can be much more complicated.

EXAMPLE 3.12. Consider the coefficient-free cluster algebra of rank 3 defined by the matrix

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Similarly to the previous example, the only other matrix that is mutation equivalent to B is $-B$. Therefore, all the exchange relations are the same as in the initial cluster, namely $x_i x_i' = x_j^2 + x_k^2$, where $\{i, j, k\} = \{1, 2, 3\}$.

All cluster variables are rational functions in x_1, x_2, x_3. Let f be an arbitrary rational function in x_1, x_2, x_3, and $\alpha_1, \alpha_2, \alpha_3$ be arbitrary integers. Define the *valuation* $\nu_{\alpha_1\alpha_2\alpha_3}(f)$ as the multiplicity at point 0 of f restricted to the curve $\{x_i = t^{\alpha_i}\}_{i=1,2,3}$. It is easy to see that $\nu_{\alpha_1\alpha_2\alpha_3}$ has the following properties:

$$\begin{aligned} \nu_{\alpha_1\alpha_2\alpha_3}(fg) &= \nu_{\alpha_1\alpha_2\alpha_3}(f) + \nu_{\alpha_1\alpha_2\alpha_3}(g), \\ \nu_{\alpha_1\alpha_2\alpha_3}(f+g) &\ge \min\{\nu_{\alpha_1\alpha_2\alpha_3}(f), \nu_{\alpha_1\alpha_2\alpha_3}(g)\}, \end{aligned} \tag{3.5}$$

provided $f + g$ does not vanish identically; the latter relation turns to an equality if both f and g can be represented as ratios of polynomials with nonnegative coefficients.

Let $\bar{x}_i, \bar{x}_j, \bar{x}_k$ form a cluster $\bar{\mathbf{x}}$, and let $\bar{x}_i'$ belong to the cluster adjacent to $\bar{\mathbf{x}}$ in direction i; we denote $\nu_{\alpha_1\alpha_2\alpha_3}(\bar{\mathbf{x}}) = \min\{\nu_{\alpha_1\alpha_2\alpha_3}(\bar{x}_i) \colon i = 1, 2, 3\}$. For a particular choice of $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2$ one infers from (3.5) by induction on the distance $d(\mathbf{x}, \bar{\mathbf{x}})$ that

$$\nu_{1,1,2}(\bar{x}_i') = 2\min\{\nu_{1,1,2}(\bar{x}_j), \nu_{1,1,2}(\bar{x}_k)\} - \nu_{1,1,2}(\bar{x}_i) \le \nu_{1,1,2}(\bar{x}_i), \tag{3.6}$$

provided $d(\mathbf{x}, \bar{\mathbf{x}}') > d(\mathbf{x}, \bar{\mathbf{x}})$, and the inequality is strict if the path between $\mathbf{x}$ and $\bar{\mathbf{x}}'$ in $\mathbb{T}_3$ contains at least one mutation in direction 3. Define

$$\nu_r = \min\{\nu_{1,1,2}(\bar{\mathbf{x}}) \colon d(\mathbf{x}, \bar{\mathbf{x}}) = r\}.$$

It follows from the above inequality that the sequence $\nu_0, \nu_1, \dots$ is strictly decreasing.

Let us first prove that the cluster algebra $\mathcal{A}(B)$ has an infinite number of clusters. To do this, consider the so-called *Markov equation*

$$t_1^2 + t_2^2 + t_3^2 = 3t_1t_2t_3;$$

positive integer triples (t_1, t_2, t_3) satisfying this equation are called *Markov triples*. [1] It is easy to check that if t_i, t_j and t_k form a Markov triple, then so do t_i, t_k and $3t_it_k - t_j$. Moreover, if $t_i \ge t_j \ge t_k$ then $3t_it_k - t_j > t_i \ge t_k$, and hence proceeding in this way we can get an infinite number of different Markov triples. Finally, Markov equation stipulates $3t_it_k - t_j = (t_i^2 + t_k^2)/t_j$.

Let us assign to each cluster $\bar{\mathbf{x}}$ a triple of integer numbers by expressing cluster variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ via the initial cluster variables x_1, x_2, x_3 and substituting $x_1 = x_2 = x_3 = 1$. Evidently, the triple $(1, 1, 1)$ assigned to the initial cluster is a Markov triple. It follows from the above discussion that the triple of integers assigned to

[1] Markov triples appear mysteriously in different areas of mathematics, such as mathematical physics, algebraic geometry, theory of triangulated categories, etc. Connections of these areas with the cluster algebra theory became a subject of active study in recent years. One of the most intriguing problems involving Markov triples is a Markov conjecture stating that any triple is determined uniquely by its largest element.

any cluster is a Markov triple, and that in such a way we get the same infinite set of Markov triples as above. Therefore, the number of clusters in $\mathcal{A}(B)$ is infinite.

Assume now that $\mathcal{A}(B)$ is finitely generated, and choose r to be the maximum among the distances from $\mathbf{x}$ to generators of $\mathcal{A}(B)$. Then the valuation $\nu_{1-\nu_r,1-\nu_r,2-\nu_r}$ possesses the following properties:

(i) $\nu_{1-\nu_r,1-\nu_r,2-\nu_r}(\bar{\mathbf{x}}) \geq 0$ for any $\bar{\mathbf{x}}$ such that $d(\mathbf{x}, \bar{\mathbf{x}}) \leq r$;

(ii) $\nu_{1-\nu_r,1-\nu_r,2-\nu_r}(\bar{\mathbf{x}}) < 0$ for some $\bar{\mathbf{x}}$ such that $d(\mathbf{x}, \bar{\mathbf{x}}) = r + 1$.

Property (i) together with the inequality in (3.5) implies, in particular, that $\nu_{1-\nu_r,1-\nu_r,2-\nu_r}$ is nonnegative on the whole cluster algebra $\mathcal{A}(B)$, which contradicts property (ii).

Let $\mathcal{A} = \mathcal{A}(\widetilde{B})$ be a cluster algebra of rank n. We can declare some of the cluster variables to be stable. The corresponding exchange relations are discarded. Assuming without loss of generality that only the variables $x_1, \dots, x_k$ remain in the initial cluster for some $k < n$, we get a subalgebra of the original cluster algebra, which is called the *restriction* of $\mathcal{A}$. The corresponding extended exchange matrix is the submatrix of $\widetilde{B}$ in the rows $1, \dots, k$ and columns $1, \dots, n + m$. The tree $\mathbb{T}_k$ is obtained from $\mathbb{T}_n$ by deleting all the edges labeled by $k + 1, \dots, n$ and chosing the connected component containing the initial seed. For example, restricting the algebra from Example 3.10 to any one of its cluster variables we get an algebra from Example 3.9.

In a similar way, we can declare some of the stable variables to be equal to one. Assuming without loss of generality that only the stable variables $x_{n+1}, \dots, x_{n+k}$ remain in the initial extended cluster for some $k < m$, we get a quotient algebra of the original cluster algebra; we say that it is obtained from $\mathcal{A}$ by *freezing* variables $x_{n+k+1}, \dots, x_{n+m}$. The corresponding extended exchange matrix is the submatrix of $\widetilde{B}$ in the rows $1, \dots, n$ and columns $1, \dots, n+k$. For example, the algebra related to L^{e,w_0} in SL_3 is obtained from the algebra related to $G_2(4)$ by freezing stable variables x_3 and x_4, see Example 3.9.

REMARK 3.13. To define a *general* cluster algebra, we have to consider, as a coefficient group, an arbitrary multiplicative abelian group without torsion. A seed is defined as a triple $\Sigma = (\mathbf{x}, \mathbf{p}, B)$, where $\mathbf{x}$ has the same meaning as above, $\mathbf{p} = (p_1^{\pm}, \dots, p_n^{\pm})$, $p_i^{\pm} \in \mathcal{P}$ is a $2n$-tuple of coefficients, B is a totally sign-skew-symmetric exchange matrix. Exchange relations are defined by (3.2). The adjacent seed in a given direction i consists of the new cluster $\mathbf{x}'$ obtained via the corresponding exchange relation, new matrix $B' = \mu_i(B)$ and the new tuple of coefficients $\mathbf{p}'$; the new coefficients are *required* to satisfy relations (3.4) given in Proposition 3.7. Observe that these relations do not define the coefficients uniquely. The way to overcome this problem consists in turning $\mathcal{P}$ into a semifield by introducing an additional operation $\oplus$, and in requiring the *normalization condition* $p_j^+ \oplus p_j^- = 1$ for all j. In case of cluster algebras of geometric type the operation $\oplus$ is given by $\prod_j g_j^{a_j} \oplus \prod_j g_j^{b_j} = \prod_j g_j^{\min\{a_j,b_j\}}$.

3.2. Laurent phenomenon and upper cluster algebras

3.2.1. Laurent phenomenon. Let us look once again at the Example 3.10. We have listed there the cluster variables in all the five distinct clusters expressed via the stable variables and the initial cluster variables. The obtained rational functions possess a remarkable feature: all the denominators are monomials in the initial cluster variables. It turns out that that this feature is not specific for the cluster algebra from Example 3.10, but rather is characteristic for all cluster algebras. More precisely, the following statement, known as the *Laurent phenomenon* holds true.

THEOREM 3.14. *Any cluster variable is expressed via the cluster variables from the initial* (*or any other*) *cluster as a Laurent polynomial with coefficients in the group ring* $\mathbb{Z}\mathcal{P}$.

PROOF. Since the nature of coefficients is irrelevant in the proof, we assume that our cluster algebra is general, as defined in Remark 3.13.

Given a seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$, consider the ring $\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}]$ of Laurent polynomials in variables $x_1, \dots, x_n$. In fact, Theorem 3.14 claims that the whole cluster algebra is contained in the intersection of such rings over all seeds mutation equivalent to Σ. We will prove that under certain mild restrictions the cluster algebra is contained already in the polynomial ring

$$\mathcal{U}(\mathbf{x}) = \mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[\mathbf{x}_1^{\pm 1}] \cap \dots \cap \mathbb{Z}\mathcal{P}[\mathbf{x}_n^{\pm 1}],$$

where $\mathbf{x}_1, \dots, \mathbf{x}_n$ are all the clusters adjacent to $\mathbf{x}$. The latter polynomial ring is called the *upper bound* associated with the cluster $\mathbf{x}$.

The main idea of the proof is as follows. It is easy to see that the cluster algebra is contained in the subalgebra of $\mathcal{F}$ generated by the union of upper bounds over all seeds. Therefore, it suffices to prove that under certain mild conditions all upper bounds coincide. It is worth to note that all the computations are done for cluster algebras of ranks 1 and 2.

We start from analyzing the following simple situation. Consider an arbitrary cluster algebra of rank 1. Let $\mathbf{x} = (x_1)$ be the initial cluster and $\mathbf{x}_1 = (x_1')$ be the adjacent cluster. The exchange relation is $x_1 x_1' = P_1$, where $P_1 \in \mathbb{Z}\mathcal{P}$. Observe that even in this simple case P_1 is a sum of two elements of $\mathcal{P}$, and hence itself does not belong to $\mathcal{P}$.

LEMMA 3.15. $\mathbb{Z}\mathcal{P}[x_1^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(x_1')^{\pm 1}] = \mathbb{Z}\mathcal{P}[x_1, x_1']$.

PROOF. The "$\supseteq$" inclusion is obvious. To prove the opposite inclusion, consider an arbitrary $y \in \mathbb{Z}\mathcal{P}[x_1^{\pm 1}]$ and observe that each monomial in y can be written as $c_k x_1^k = c_k'/(x_1')^k$ with $c_k \in \mathbb{Z}\mathcal{P}$ and $c_k' = c_k P_1^k$. Additional condition $y \in \mathbb{Z}\mathcal{P}[(x_1')^{\pm 1}]$ implies $c_k' \in \mathbb{Z}\mathcal{P}$, which means that c_{-k} is divisible by P_1^k if $k \geq 0$ (since $P_1 \notin \mathcal{P}$). Evidently, for any $k \geq 0$, both $c_k x_1^k$ and $c_{-k}' P_1^k / x_1^k = c_{-k}'(x_1')^k$ belong to $\mathbb{Z}\mathcal{P}[x_1, x_1']$, and hence the same holds for y. □

REMARK 3.16. Observe that we can add an arbitrary tuple $\mathbf{z}$ of independent variables to both sides of the above relation; in other words, the following is true:

$$\mathbb{Z}\mathcal{P}[x_1^{\pm 1}, \mathbf{z}] \cap \mathbb{Z}\mathcal{P}[(x_1')^{\pm 1}, \mathbf{z}] = \mathbb{Z}\mathcal{P}[x_1, x_1', \mathbf{z}].$$

The proof follows immediately from Lemma 3.15 by changing the ground ring from $\mathbb{Z}\mathcal{P}$ to $\mathbb{Z}\mathcal{P}[\mathbf{z}]$.

As a next step, consider an arbitrary cluster algebra of rank 2. Let $\mathbf{x} = (x_1, x_2)$ be the initial cluster, $\mathbf{x}_1 = (x_1', x_2)$ and $\mathbf{x}_2 = (x_1, x_2')$ be the two adjacent clusters (in directions 1 and 2, respectively), and $\mathbf{x}_{1,2} = (x_1', x_2'')$ be the cluster adjacent to $\mathbf{x}_1$ in direction 2.

LEMMA 3.17. $\mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2'] = \mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2'']$.

PROOF. It is enough to show that $x_2'' \in \mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2']$. Consider the 2×2 exchange matrix B corresponding to $\mathbf{x}$.

Assume first that $b_{12} = b_{21} = 0$. Then the exchange relations in direction 2 for the clusters $\mathbf{x}$ and $\mathbf{x}_1$ are $x_2 x_2' = p_2^+ + p_2^-$ and $x_2 x_2'' = (p_2^+)' + (p_2^-)'$, with $(p_2^+)'/(p_2^-)' = p_2^+/p_2^-$. Therefore, $x_2'' = p x_2'$ for some $p \in \mathcal{P}$, and we are done.

Assume now that $b_{12} = b$ and $b_{21} = -c$, so that $bc > 0$. Without loss of generality, we may assume that both b and c are positive (the opposite case is symmetric to this one). The exchange relations for the cluster $\mathbf{x}$ are $x_1 x_1' = p_1^+ x_2^b + p_1^-$ and $x_2 x_2' = p_2^- x_1^c + p_2^+$ for some $p_1^\pm, p_2^\pm \in \mathcal{P}$. Similarly, the exchange relations for cluster $\mathbf{x}_1$ are $x_1' x_1 = p_1^+ x_2^b + p_1^-$ and $x_2 x_2'' = (p_2^+)'(x_1')^c + (p_2^-)'$, where by Proposition 3.7 $(p_2^+)'/(p_2^-)' = p_2^+/p_2^-(p_1^-)^c$. Therefore,

$$\begin{aligned} x_2'' &= \frac{(p_2^+)'(x_1')^c + (p_2^-)'}{x_2} = \frac{(p_2^+)'(x_1')^c(x_2 x_2' - p_2^- x_1^c)}{p_2^+ x_2} + \frac{(p_2^-)'}{x_2} \\ &= \frac{(p_2^+)'(x_1')^c x_2'}{p_2^+} - \left(\frac{(p_2^+)' p_2^-}{p_2^+} \cdot \frac{(x_1 x_1')^c}{x_2} - \frac{(p_2^-)'}{x_2} \right) \\ &= \frac{(p_2^+)'(x_1')^c x_2'}{p_2^+} - \frac{(p_2^-)'}{(p_1^-)^c} \cdot \frac{(p_1^+ x_2^b + p_1^-)^c - (p_1^-)^c}{x_2}. \end{aligned}$$

Evidently, both terms in the last expression belong to $\mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2']$, and the Lemma is proved. □

In fact, Lemma 3.17 completes the proof of the Theorem for cluster algebras of rank 2. Indeed, an immediate corollary of Lemma 3.17 is that the whole cluster algebra is generated by x_1, x_1', x_2, x_2'. However, to get the proof for cluster algebras of an arbitrary rank, we have to further study cluster algebras of rank 2.

In the same situation as above, let P_1 and P_2 denote the right hand sides of the exchange relations corresponding to the initial cluster $\mathbf{x}$.

LEMMA 3.18. *Assume that P_1 and P_2 are coprime in $\mathbb{Z}\mathcal{P}[x_1, x_2]$. Then*

$$\mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[x_1^{\pm 1}, x_2, x_2'] = \mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2'].$$

PROOF. Consider the same two cases as in the proof of Lemma 3.17.

In the first case, both P_1 and P_2 lie in $\mathbb{Z}\mathcal{P}$. Therefore,

$$\mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2'] = \mathbb{Z}\mathcal{P}[x_1, x_2] + \mathbb{Z}\mathcal{P}[x_1, x_2'] + \mathbb{Z}\mathcal{P}[x_1', x_2] + \mathbb{Z}\mathcal{P}[x_1', x_2'].$$

Applying the same reasoning as in the proof of Lemma 3.15, we see that any $y \in \mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2']$ is the sum of monomials of the four following types:

(i) $c_{k,l} x_1^k x_2^l$;

(ii) $c_{k,-l} x_1^k x_2^{-l}$ with $c_{k,-l}$ divisible by P_2^l;

(iii) $c_{-k,l} x_1^{-k} x_2^l$ with $c_{-k,l}$ divisible by P_1^k;

(iv) $c_{-k,-l} x_1^{-k} x_2^{-l}$ with $c_{-k,-l}$ divisible by $P_1^k P_2^l$.

In all the listed cases $k, l \geq 0$.

On the other hand, the proof of Lemma 3.15 together with Remark 3.16 provides a similar description for the monomials of any element in $\mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[x_1^{\pm 1}, x_2, x_2']$. The only difference is that the coefficient of $x_1^{-k} x_2^{-l}$ should be divisible by P_1^k and P_2^l. However, this condition is equivalent to divisibility by $P_1^k P_2^l$, since P_1 and P_2 are coprime.

In the second case, let us start with proving that

$$\mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}] = \mathbb{Z}\mathcal{P}[x_1, x_1', x_2, x_2'] + \mathbb{Z}\mathcal{P}[x_1, x_2^{\pm 1}]. \tag{3.7}$$

As before, the "$\supseteq$" inclusion is obvious. To prove the opposite inclusion, observe that each monomial $y \in \mathbb{ZP}[x_1, x_1', x_2^{\pm 1}]$ can be written as

$$y = c_{k_1,k_2,k_3}(p_1^+ x_2^b + p_1^-)^{k_1} x_1^{k_2} x_2^{k_3},$$

where $k_1 \geq 0$, $k_1 + k_2 \geq 0$, while k_2, k_3 have arbitrary signs. If $k_3 \geq 0$ then $y \in \mathbb{ZP}[x_1, x_1', x_2, x_2']$. If $k_3 < 0$ and $k_2 \geq 0$ then $y \in \mathbb{ZP}[x_1, x_2^{\pm 1}]$. In the remaining case $k_2 < 0$, $k_3 < 0$, and y can be rewritten as

$$y = c_{k_1,k_2,k_3}(p_1^+ x_2^b + p_1^-)^{k_1+k_2}(x_1')^{-k_2} x_2^{k_3}.$$

It is therefore enough to prove that

$$(x_1')^{-k_2} x_2^{k_3} \in \mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2^{\pm 1}]. \tag{3.8}$$

Indeed, the exchange relation for x_2 implies $(1-t)/x_2 \in \mathbb{ZP}[x_1, x_2']$, where $t = -(p_2^-/p_2^+)x_1^c \in \mathbb{ZP}[x_1, x_2']$. Multiplying $(1-t)/x_2$ by t^i and summing the obtained products for $i = 0, \dots, -k_2 - 1$ we get $(1 - t^{-k_2})/x_2 \in \mathbb{ZP}[x_1, x_2']$. Raising the left hand side of this inclusion to power $-k_3$ we get $x_2^{k_3} - x_1^{-k_2} w \in \mathbb{ZP}[x_1, x_2']$, where $w \in \mathbb{ZP}[x_1, x_2^{-1}]$. It remains to multiply the left hand side of the latter inclusion by $(x_1')^{-k_2}$ and to use the exchange relation for x_1 to get (3.8), and hence (3.7).

It is now easy to complete the proof of the lemma. Indeed, by (3.7) and an obvious inclusion $\mathbb{ZP}[x_1, x_1', x_2, x_2'] \subset \mathbb{ZP}[x_1^{\pm 1}, x_2, x_2']$ we get

$$\mathbb{ZP}[x_1, x_1', x_2^{\pm 1}] \cap \mathbb{ZP}[x_1^{\pm 1}, x_2, x_2'] = \mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2^{\pm 1}] \cap \mathbb{ZP}[x_1^{\pm 1}, x_2, x_2'].$$

It is easy to see that the second term in the above sum equals $\mathbb{ZP}[x_1, x_2, x_2'] \subset \mathbb{ZP}[x_1, x_1', x_2, x_2']$, and the result follows. □

The last statement concerning cluster algebras of rank 2 claims that under the coprimality condition, the intersection of three Laurent polynomial rings defined by a cluster and its two neighbors does not depend on the cluster. More precisely, recall that the upper bound associated with $\mathbf{x}$ is

$$\mathcal{U}(\mathbf{x}) = \mathbb{ZP}[\mathbf{x}^{\pm 1}] \cap \mathbb{ZP}[\mathbf{x}_1^{\pm 1}] \cap \mathbb{ZP}[\mathbf{x}_2^{\pm 1}],$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the clusters adjacent to $\mathbf{x}$. Besides, let P_1, P_2 be the exchange polynomials for $\mathbf{x}$, and P_1', P_2' be the exchange polynomials for $\mathbf{x}_1$.

Lemma 3.19. *Assume that P_1 and P_2 are coprime in $\mathbb{ZP}[x_1, x_2]$ and P_1' and P_2' are coprime in $\mathbb{ZP}[x_1', x_2]$. Then $\mathcal{U}(\mathbf{x}) = \mathcal{U}(\mathbf{x}_1)$.*

Proof. First of all, using Lemma 3.15 and Remark 3.16 we get

$$\mathcal{U}(\mathbf{x}) = \mathbb{ZP}[x_1, x_1', x_2^{\pm 1}] \cap \mathbb{ZP}[x_1^{\pm 1}, x_2, x_2'],$$

which by Lemma 3.18 is equal to $\mathbb{ZP}[x_1, x_1', x_2, x_2']$. Similarly,

$$\mathcal{U}(\mathbf{x}_1) = \mathbb{ZP}[x_1, x_1', x_2, x_2''].$$

The latter two expressions are equal by Lemma 3.17, which completes the proof. □

The statement of Lemma 3.19 can be immediately extended to cluster algebras of an arbitrary rank by applying the argument of Remark 3.16 to Lemmas 3.17

and 3.18. Assuming that all the exchange polynomials $P_1, \dots, P_n$ are pairwise coprime in $\mathbb{Z}\mathcal{P}[\mathbf{x}]$, and the same is true for $P_1', \dots, P_n'$ in $\mathbb{Z}\mathcal{P}[\mathbf{x}_1]$, we get

$$\begin{aligned}
\mathcal{U}(\mathbf{x}) &= \bigcap_{i=1}^{n} \left(\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap\ Z\mathcal{P}[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, (x_i')^{\pm 1}, x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}]\right) \\
&= \bigcap_{i=1}^{n} \mathbb{Z}\mathcal{P}[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i, x_i', x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}] \\
&= \bigcap_{i=2}^{n} \left(\mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}, \dots, x_n^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i, x_i', x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}]\right) \\
&= \bigcap_{i=2}^{n} \mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i, x_i', x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}] \\
&= \bigcap_{i=2}^{n} \mathbb{Z}\mathcal{P}[x_1, x_1', x_2^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i, x_i'', x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}] = \mathcal{U}(\mathbf{x}_1).
\end{aligned}$$

As it is explained at the beginning of the proof, the claim of the theorem follows immediately from this equality.

Finally, to lift the coprimality assumption we use the following argument: we regard the coefficients $p_i^{\pm}$ of the exchange polynomials P_i as indeterminates. Then all the coefficients in all exchange polynomials become "canonical" (i.e. independent of the choice of $\mathcal{P}$) polynomials in these indeterminates, with positive integer coefficients. $\square$

For the cluster algebras of geometric type one can sharpen Theorem 3.14 as follows.

PROPOSITION 3.20. *Any cluster variable in a cluster algebra of geometric type is expressed via the cluster variables from the initial* (*or any other*) *cluster as a Laurent polynomial whose coefficients are polynomials in stable variables.*

PROOF. Fix an arbitrary stable variable x. Let $\bar{x}$ be a cluster variable belonging to a seed $\bar{\Sigma}$. We will use induction on $d(\bar{\Sigma}, \Sigma)$ to show that $\bar{x}$, as a function of x, is a polynomial whose constant term is a subtraction-free rational expression in $\mathbf{x}$ and all stable variables except for x.

This statement is trivial when $d(\bar{\Sigma}, \Sigma) = 0$. If $d(\bar{\Sigma}, \Sigma) > 0$, then, by the definition of the distance, $\bar{x}$ can be expressed via the cluster variables belonging to a seed whose distance to Σ is smaller than $d(\bar{\Sigma}, \Sigma)$. Therefore, by the inductive assumption, $\bar{x}$ is a ratio of two polynomials in x. Moreover, since x enters at most one of the two monomials in the exchange relation (3.1), both these polynomials have nonzero constant term. It remains to notice that if a ratio of two polynomials with nonzero constant terms a and b is a Laurent polynomial, then it is, in fact, a polynomial, and its constant term equals a/b. $\square$

3.2.2. Upper cluster algebras. Let $\mathcal{A} = \mathcal{A}(\widetilde{B})$ be a cluster algebra of geometric type. The *upper cluster algebra* $\mathcal{U} = \mathcal{U}(\widetilde{B})$ is defined as the intersection of the upper bounds $\mathcal{U}(\mathbf{x})$ over all clusters $\mathbf{x}$ in $\mathcal{A}$. Upper cluster algebras are especially easy to describe in the case of extended exchange matrices of the maximal rank.

THEOREM 3.21. *Let $\widetilde{B}$ be an $n \times (n+m)$ skew-symmetrizable matrix of rank n. Then the upper bounds do not depend on the choice of $\mathbf{x}$, and hence coincide with the upper cluster algebra $\mathcal{U}(\widetilde{B})$.*

PROOF. For cluster algebras of geometric type, the extension of Lemma 3.19 to cluster algebras of an arbitrary rank discussed in Section 3.2.1 can be summarized as follows.

COROLLARY 3.22. *If the exchange polynomials $P_1, \dots, P_n$ are coprime in the ring $\mathbb{Z}[x_1, \dots, x_{n+m}]$ for any cluster $\mathbf{x}$ in $\mathcal{A}(\widetilde{B})$ then the upper bounds do not depend on the choice of $\mathbf{x}$, and hence coincide with the upper cluster algebra $\mathcal{U}(\widetilde{B})$.*

Therefore, to prove the theorem it suffices to check the coprimality condition of Corollary 3.22. We start from the following observation.

LEMMA 3.23. *If* $\operatorname{rank} \widetilde{B} = n$*, and $\widetilde{B}'$ is obtained from $\widetilde{B}$ by mutation in direction i then* $\operatorname{rank} \widetilde{B}' = n$*.*

PROOF. Indeed, apply the following sequence of row and column operations to the matrix $\widetilde{B}$. For any l such that $b_{il} < 0$, subtract the ith column multiplied by b_{il} from the lth column. For any k such that $b_{ki} > 0$, add the ith row multiplied by b_{ki} to the kth row. Finally, multiply the ith row and column by -1. It is easy to see that the result of these operations is exactly $\widetilde{B}'$, and the lemma follows since the elementary row and column operations do not change the rank of a matrix. $\square$

It remains to check that the maximal rank condition implies the coprimality condition.

LEMMA 3.24. *Exchange polynomials $P_1, \dots, P_n$ are coprime in $\mathbb{Z}[x_1, \dots, x_{n+m}]$ if and only if no two rows of $\widetilde{B}$ are proportional to each other with the proportionality coefficient being a ratio of two odd integers.*

PROOF. Let $\widetilde{B}_i$ and $\widetilde{B}_j$ be two rows of $\widetilde{B}$ such that $\widetilde{B}_i = \pm\frac{b}{a}\widetilde{B}_j$ with a and b odd coprime positive integers. Then $P_i = M_1^a + M_2^a$, $P_j = M_1^b + M_2^b$ for some monomials $M_1, M_2 \in \mathbb{Z}[x_1, \dots, x_{n+m}]$, and so $M_1 + M_2$ is a common factor of P_i and P_j.

Assume now that P_i and P_j are not coprime. Let $P_i = M_1 + M_2$. Since M_1 and M_2 do not share common factors, we can rename the variables entering M_1 as $y_1, \dots, y_p$, and those entering M_2 as $z_1, \dots, z_q$, so that $M_1 = \prod_{s=1}^p y_s^{d_s}$ and $M_2 = \prod_{r=1}^q z_r^{\delta_r}$. Clearly, d_s and δ_r constitute positive and negative elements of B_i. Assume that P_i can be factored as $P_i = P'P''$, then each of P' and P'' contain exactly one monomial in y-variables and exactly one monomial in z-variables. Consider P' and denote these monomials M_1' and M_2', and the corresponding exponents d_s' and δ_r', respectively. We want to prove the following fact:

$$d_s'/d_s = \delta_r'/\delta_r = c(P_i) \qquad \text{for any } 1 \le s \le p,\ 1 \le r \le q. \tag{3.9}$$

Let us assign to the variables $y_1, \dots, y_p$ and $z_1, \dots, z_q$ nonnegative weights $w_1, \dots, w_p$ and $\omega_1, \dots, \omega_q$, respectively. The weight of a monomial is defined as the sum of the weights of the variables; for example, the weight of M_1 is equal to $\sum_{s=1}^p w_s d_s$. A polynomial is called *quasihomogeneous* with respect to the weights as above if the weights of all monomials are equal. It is easy to see that if a polynomial is quasihomogeneous with respect to certain weights then all its factors are quasihomogeneous with respect to the same weights as well. To prove (3.9) assign weights $w_s = 1/d_s$, $\omega_r = 1/\delta_r$, $w_{\bar{s}} = \omega_{\bar{r}} = 0$ for any $\bar{s} \neq s$ and $\bar{r} \neq r$. This makes P_i into a quasihomogeneous polynomial, hence P' is quasihomogeneous as well. The weights of the monomials M_1' and M_2' are d_s'/d_s and δ_r'/δ_r, respectively, and (3.9) follows with $c(P_i) \neq 0$. Consequently, the row B_i of $\widetilde{B}$ can be restored up to the sign from the vector of exponents of the variables in P' by dividing it by $c(P_i)$.

Assume now that P' is a common factor of P_i and P_j. Then the row B_j can be restored up to the sign from the same vector by dividing it by $c(P_j)$. Therefore, $c(P_i)B_i = \pm c(P_j)B_j$. Represent the rational number $c(P_j)/c(P_i)$ as b/a, where a and b are coprime positive integers. It remains to prove that a and b are both odd.

Indeed, $P_i = M_3^a + M_4^a$ and $P_j = M_3^b + M_4^b$ for some monomials M_3 and M_4. Consider polynomials $t^a + 1$ and $t^b + 1$ in one variable. If either a or b is even then $t^a + 1$ and $t^b + 1$ are coprime in $\mathbb{Z}[t]$, and hence there eixst $f, g \in \mathbb{Z}[t]$ such that $f(t)(t^a + 1) + g(t)(t^b + 1) = 1$. Substituting $t = M_4/M_3$ we obtain $F(M_3^a + M_4^a) + G(M_3^b + M_4^b) = M_3^c$ in $\mathbb{Z}[x_1, \dots, x_{n+m]}$ for some nonnegative integer c. Therefore the common factor of P_i and P_j is a monomial, a contradiction. □

□

It follows from the discussion in the previous section that $\mathcal{A}(\widetilde{B}) \subseteq \mathcal{U}(\widetilde{B})$. The following example shows that in general these two algebras are different.

Example 3.25. Consider the coefficient-free cluster algebra $\mathcal{A}$ introduced in Example 3.12 and the corresponding upper cluster algebra $\mathcal{U}$. This time we make use of valuation $\nu_{\alpha_1\alpha_2\alpha_3}(f)$ for $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Similarly to (3.6) we can prove that

$$\nu_{1,1,1}(\bar{x}_i') = 2\min\{\nu_{1,1,1}(\bar{x}_j), \nu_{1,1,1}(\bar{x}_k)\} - \nu_{1,1,1}(\bar{x}_i) = \nu_{1,1,1}(\bar{x}_i),$$

and hence $\nu_{1,1,1}(x) = 1$ for any cluster variable x. This defines a grading on $\mathcal{A}$, and the zero-degree component in this grading consists of the elements of the ground ring.

On the other hand, consider the element

$$y = \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2} \in \mathcal{F}.$$

Using the exchange relations, one can rewrite it as

$$y = \frac{x_1 + x_1'}{x_2} = \frac{x_2 + x_2'}{x_1} = \frac{(x_1^2 + x_2^2)(x_1^2 + x_2^2 + (x_3')^2)}{x_1 x_2 (x_3')^2},$$

so $y \in \mathcal{U}$. Clearly, $\nu_{1,1,1}(y) = 0$ and y does not lie in the ground ring, so $y \notin \mathcal{A}$, and hence $\mathcal{A} \neq \mathcal{U}$.

3.3. Cluster algebras of finite type

We say that a cluster algebra $\mathcal{A}$ is of *finite type* if the number of nonequivalent clusters in $\mathcal{A}$ is finite, or, which is the same, if the exchange graph of $\mathcal{A}$ is finite. Surprisingly, the classification of these algebras is very similar to the Cartan–Killing classification of semisimple Lie algebras. To formulate a precise statement we need to introduce several definitions.

Let A be an integer symmetrizable square matrix. We say that A is *quasi-Cartan* if all the diagonal entries of A are equal to 2, and *Cartan* if, in addition, all off-diagonal entries of A are non-positive. A (quasi)-Cartan matrix is called *positive* if all its principal minors are positive.

Given an integer skew-symmetrizable square matrix B, its *quasi-Cartan companion* is a quasi-Cartan matrix A such that $|a_{i,j}| = |b_{i,j}|$. Observe that B has 2^N quasi-Cartan companions, where N is the number of nonzero entries in B lying above the main diagonal. Exactly one of these quasi-Cartan companions is a Cartan matrix, called the *Cartan companion* of B.

Let B be an integer skew-symmetrizable square matrix. We say that B is 2-*finite* if for any matrix B' mutation equivalent to B one has

$$|b'_{ij}b'_{ji}| \leq 3 \tag{3.10}$$

for all pairs of indices i, j.

THEOREM 3.26. *Let $\widetilde{B}$ be a sign-skew-symmetric matrix and B be its principal part, then the following statements are equivalent.*

(1) *The cluster algebra $\mathcal{A}(\widetilde{B})$ is of finite type.*

(2) *B is 2-finite.*

(3) *There exists a matrix B' mutation equivalent to B such that the Cartan companion of B' is positive.*

Furthermore, the Cartan–Killing type of the Cartan matrix in (3) *is uniquely determined by B.*

REMARK 3.27. As follows from the definitions above, it is implicit in (2) and (3) that B is skew-symmetrizable.

3.3.1. Proof of the implication (1) $\Longrightarrow$ (2). Assume to the contrary that there exist B' mutation equivalent to B and indices i, j such that $|b'_{ij}b'_{ji}| \geq 4$. Without loss of generality, we may restrict ourselves to the case $B' = B$ and $i = 1$, $j = 2$. We want to prove that already the rank 2 subalgebra of $\mathcal{A}(\widetilde{B})$ obtained by the restriction to the first two cluster variables possesses an infinite number of clusters.

Let $b = |b_{12}|$, $c = |b_{21}|$.

The case $b, c \geq 2$, $bc \neq 4$ is quite similar to Example 3.11 above. Indeed, let (y_1, y_2) be a cluster obtained from the initial one by applying mutations in directions 2 and 1 alternatively, the total number of mutations being even. Write y_1, y_2 as rational functions in the initial cluster variables x_1, x_2 and define $\delta(y_1), \delta(y_2)$ as in Example 3.11. Assume that $\delta(y_1) \geq \delta(y_2) > 0$; after applying the mutation in direction 2 followed by the mutation in direction 1 we get a cluster (z_1, z_2) such that

$$\delta(z_1) = bc\delta(y_1) - b\delta(y_2) - \delta(y_1), \qquad \delta(z_2) = c\delta(y_1) - \delta(y_2).$$

It is easy to see that $\delta(z_1) > \delta(z_2)$ provided $bc > b + c$, and $\delta(z_1) > \delta(y_1)$ provided $bc > b + 2$. In our case both inequalities are satisfied, so we get at infinite number of clusters.

The case $b = c = 2$ follows immediately from the Example 3.12 above. Indeed, it suffice to notice that that all the Markov triples obtained from $(1, 1, 1)$ by the process described in Example 3.12 are of the form $(t_1, t_2, 1)$.

The remaining case $b = 1$, $c \geq 4$ (or $b \geq 4$, $c = 1$) is similar to the first one, though slightly more subtle. Consider (y_1, y_2) and (z_1, z_2) as above, and, in addition, the cluster (w_1, w_2) obtained from (z_1, z_2) by the mutation in direction 2 followed by the mutation in direction 1. It is easy to see that

$$\delta(z_1) = c\delta(y_1) - \delta(y_1) - \delta(y_2), \qquad \delta(z_2) = c\delta(y_1) - \delta(y_2);$$

assume that

$$\delta(z_1) > \delta(y_1) > 0, \qquad \delta(z_2) > \delta(y_2) > 0. \tag{3.11}$$

The first inequality in (3.11) is equivalent to $(c-2)\delta(y_1) > \delta(y_2)$, the second one, to

$$c\delta(y_1) > 2\delta(y_2). \tag{3.12}$$

Consequently, the first inequality in (3.11) follows from (3.12), provided $c-2 \geq c/2$, which is equivalent to $c \geq 4$.

Let us check that $c\delta(z_1) > 2\delta(z_2)$, which by the above is equivalent to

$$c(c-3)\delta(y_1) > (c-2)\delta(y_2).$$

Indeed, by (3.12),

$$c(c-3)\delta(y_1) > 2(c-3)\delta(y_2) \geq (c-2)\delta(y_2),$$

since $c \geq 4$. We therefore see that (3.11) implies $\delta(w_1) > \delta(z_1) > 0$, $\delta(w_2) > \delta(z_2) > 0$. To get an infinite number of distinct clusters, it suffices to check (3.11) for $(y_1, y_2) = (x_1, x_2)$. In this case we have $\delta(y_1) = \delta(y_2) = 1$, $\delta(z_1) = c-2$, $\delta(z_2) = c-1$, and the result follows from $c \geq 4$. In the case $b \geq 4$, $c = 1$ it suffices to interchange the order of mutations.

Now, when the inequality (3.10) in the definition of 2-finiteness has been already proved, it remains to check that B is skew-symmetrizable. It is known that a sign-skew-symmetric matrix B is skew-symmetrizable if

$$a_1 a_2 \cdots a_k = (-1)^k b_1 b_2 \cdots b_k \tag{3.13}$$

for any $k \times k$ principal submatrix

$$\widehat{B} = \begin{pmatrix} 0 & a_1 & * & \dots & * & b_k \\ b_1 & 0 & a_2 & \dots & * & * \\ * & b_2 & 0 & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & a_{k-1} \\ a_k & * & * & \dots & b_{k-1} & * \end{pmatrix}$$

of B (recall that we do not distinguish between matrices obtained by a simultaneous permutation of rows and columns).

For $k = 3$ we have, in fact, to check a statement concerning 3×3 integer matrices. By (3.10), the number of such matrices is finite (and does not exceed 140), so this can be done by direct computer-aided inspection. This inspection reveals that the only cases when a 3×3 sign-skew-symmetric integer matrix together with three adjacent matrices satisfy (3.10) are

$$\pm\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}. \tag{3.14}$$

Clearly, all these matrices satisfy (3.13) with $k = 3$.

Assume now that (3.13) is already established for $k = 3, \dots, l-1$, and we want to check it for $k = l$. We may assume that all a_i's and b_i's are nonzero, since otherwise (3.13) holds trivially. Suppose that $\hat{b}_{ij} = c \neq 0$, where $1 < j - i < l - 1$, and let $d = \hat{b}_{ji}$. Consider two submatrices of $\widehat{B}$, one formed by rows and columns $1, \dots, i, j, \dots, l$, and the other by rows and columns $i, \dots, j$. Then by induction

$$\begin{aligned} a_1 \cdots a_{i-1} c a_j \cdots a_l &= (-1)^{l+i-j+1} b_1 \cdots b_{i-1} d b_j \cdots b_l, \\ a_j \cdots a_{j-1} d &= (-1)^{j-i+1} b_j \cdots b_{j-1} c, \end{aligned}$$

and (3.13) follows since $cd \neq 0$.

Finally, if all the asterisques in $\widehat{B}$ are equal to zero, we apply the following procedure called *shrinking of cycles*. Consider the mutation in direction j, $1 < j < l$. It is easy to see that all the entries off the principal 3×3 submatrix in the rows and columns $j-1, j, j+1$ remain intact, while this submatrix is transformed as follows:

$$\begin{pmatrix} 0 & a_{j-1} & 0 \\ b_{j-1} & 0 & a_j \\ 0 & b_j & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -a_{j-1} & (|a_{j-1}|a_j + a_{j-1}|a_j|)/2 \\ -b_{j-1} & 0 & -a_j \\ (|b_{j-1}|b_j + b_{j-1}|b_j|)/2 & -b_j & 0 \end{pmatrix}.$$

It follows that if a_{j-1} and a_j have the same sign then (3.13) for $\widehat{B}$ follows from (3.13) for the $(l-1) \times (l-1)$ submatrix of $\mu_j(\widehat{B})$ obtained by deleting the jth row and column. Finally, if the signs of a_j's alternate, it suffices to apply first μ_{j-1} (or μ_{j+1}) followed by μ_j.

3.3.2. Proof of the implication (2) $\Longrightarrow$ (3). The main tool we are going to use in this section is the weighted directed graph $\Gamma(B)$ called the *diagram* of B. It is defined for a skew-symmetrizable $n \times n$ matrix B as follows: $\Gamma(B)$ has n vertices, and an edge goes from i to j if and only if $b_{ij} > 0$; the weight w_{ij} of this edge is set to be $\sqrt{|b_{ij} b_{ji}|}$. The weights thus defined are strictly positive. Slightly abusing notation, we will say that the weight w_{ij} of an edge from i to j is negative (and equals $-w_{ji}$) whenever the edge is directed from j to i, and equals zero whenever there are no edges between i and j in either direction.

The following proposition describes how $\Gamma(B)$ changes with matrix mutations.

LEMMA 3.28. *Let B be skew-symmetrizable, $\Gamma = \Gamma(B)$ and $\Gamma' = \Gamma(\mu_k(B))$. Then Γ' is obtained from Γ as follows:*

(1) *the orientations of all edges incident to k are reversed, their weights remain intact;*

(2) *for any vertices i and j such that both w_{ik} and w_{kj} are positive, the direction of the edge between i and j in Γ' and its weight are uniquely determined by the rule*

$$w_{ij} + w'_{ij} = w_{ik} w_{kj};$$

(3) *the rest of the edges and their weights remain intact.*

PROOF. Let D be a skew-symmetrizer for B, then $S(B) = D^{1/2} B D^{-1/2}$ is evidently skew-symmetric. Moreover, it is easy to check that its entries can be written as $s_{ij} = \operatorname{sgn} b_{ij} \sqrt{|b_{ij} b_{ji}|} = \operatorname{sgn}(b_{ij})|w_{ij}|$. Therefore the diagram $\Gamma(S(B))$ coincides with $\Gamma(B)$, including the weights of the edges. Besides, $S(\mu_k(B)) = \mu_k(S(B))$ for any k, since mutation rules are invariant under conjugation by a diagonal matrix with positive entries. Translating this statement into the language of diagrams we obtain all three assertions of the Lemma. $\square$

Let us consider abstract diagrams (weighted directed graphs), without relating them to skew-symmetrizable matrices. In view of Lemma 3.28, it is natural to define the *diagram mutation* in direction k by $\mu_k(\Gamma) = \Gamma(\mu_k)$. The notions of mutation equivalence and 2-finiteness are readily carried to diagrams.

To proceed with the proof, recall that indecomposable positive Cartan matrices are classified via their Dynkin diagrams (see Section1.2.2). Given a Dynkin diagram Π, the corresponding *Dynkin graph* $G(\Pi)$ is an undirected weighted graph on the same set of vertices. Two vertices i and j of the graph $G(\Pi)$ are connected by an edge if and only if (i, j) or (j, i) is an edge in Π. The weight of an edge in $G(\Pi)$ equals the square root of the number of edges between the corresponding vertices of Π. Figure 3.1 below presents the list of Dynkin diagrams for indecomposable positive Cartan matrices together with the corresponding Dynkin graphs; all unspecified weights are equal to 1. Observe that the same Dynkin graph corresponds to Dynkin diagrams B_n and C_n.

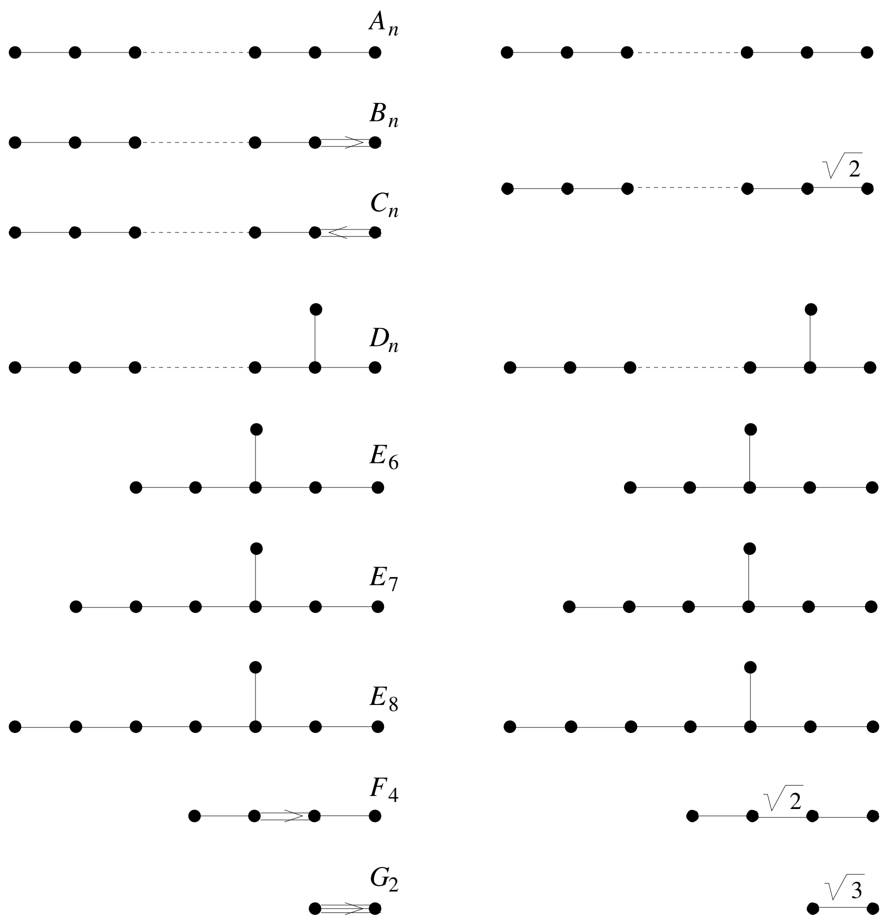

FIGURE 3.1. Dynkin diagrams and Dynkin graphs

By Lemma 3.28, all we need to prove is the following result.

THEOREM 3.29. *Any connected 2-finite diagram is mutation equivalent to an orientation of a Dynkin graph. Furthermore, all orientations of the same Dynkin graph are mutation equivalent to each other.*

PROOF. Let us introduce several useful notions. Recall that a subgraph is called *induced* if it is obtained from the original graph by deleting an arbitrary subset of vertices together with all the edges incident to these vertices. When referring to diagrams, we will write *subdiagram* instead of induced subdiagram. It is clear that any subdiagram of a 2-finite diagram is 2-finite as well.

We say that a diagram is *tree-like* if it is an orientation of a tree. An edge e of a connected diagram Γ is called a *tree edge* if the diagram $\Gamma \setminus \{e\}$ has two connected components and at least one of them is tree-like.

LEMMA 3.30. *Let Γ be a connected diagram, e be a tree edge, and Γ' be the diagram obtained from Γ by reversing the direction of e. Then Γ' and Γ are mutation equivalent.*

PROOF. Let Γ_1 be a tree-like component of $\Gamma \setminus \{e\}$, and $i \notin \Gamma_1$, $j \in \Gamma_1$ be the endpoints of e. Without loss of generality assume that e is directed from i to j. Denote by V_+ (resp., V_-) the set of vertices $k \in \Gamma_1$ such that the last edge in the unique path from j to k is directed from k (resp., to k). To reverse the direction of e it suffices to apply first mutations at the vertices of V_- in the decreasing order of the distance to j, then the mutation at j, and then mutations at the vertices of V_+ in the increasing order of the distance to j (in both cases ties are resolved arbitrarily). Indeed, to see that no new edges arise and that the existing edges do may only change direction, it is enough to notice that each mutation is applied at a vertex that currently has no outcoming edges, and to use Lemma 3.28. Moreover, each edge except for e retains its direction, since we apply mutation once at each vertex of Γ_1. □

Observe that Lemma 3.30 immediately implies the second assertion of Theorem 3.29.

We say that a diagram is *cycle-like* if it is an orientation of a cycle.

LEMMA 3.31. (i) *Any 2-finite cycle-like diagram is an oriented cycle.*

(ii) *The only 2-finite cycle-like diagrams with weights distinct from 1 are the three-cycle with weights $(\sqrt{2}, \sqrt{2}, 1)$ and the four-cycle with weights $(\sqrt{2}, 1, \sqrt{2}, 1)$.*

PROOF. The proof of this Lemma is basically a translation of the proof of (3.13) to the language of diagrams. The cycles of length 3 correspond to the matrices listed in (3.14), and the validity of the assertions for these matrices follows from inspection. The shrinking of cycles described in the proof of (3.13) immediately implies the first assertion of the Lemma. Next, the product of the edge weights along the cycle, called the *total weight* of the cycle, is preserved under shrinking, so (3.14) imply that this product equals either 1 or 2. In the latter case the initial cycle-like diagram contains two edges of weight $\sqrt{2}$. If the number of edges in the cycle is at least 5, the shrinking process leads to the four-cycle $(\sqrt{2}, \sqrt{2}, 1, 1)$, which is discarded by applying mutation at the vertex incident to both edges of weight $\sqrt{2}$. If there are four edges, the only other possibility is the four-cycle described in Lemma. □

Observe that the property of a diagram to be tree-like or cycle-like is not preserved under diagram mutations.

EXAMPLE 3.32. For any positive integer p, s and any tree-like diagrams T_1, T_2 denote by C_{s,p,T_1,T_2} the diagram on Figure 3.2,a. Applying mutations at vertices $a_1, a_2, \dots, a_s, b_1, b_2, \dots, b_p$ we arrive at the tree-like diagram T_{s,p,T_1,T_2}, see Figure 3.2,b. Moreover, if an additional leave a_0 is connected by a directed edge to a_1, then the same sequence of mutations results in T_{s,p,T_1,T_2} with a_0 connected to b_p.

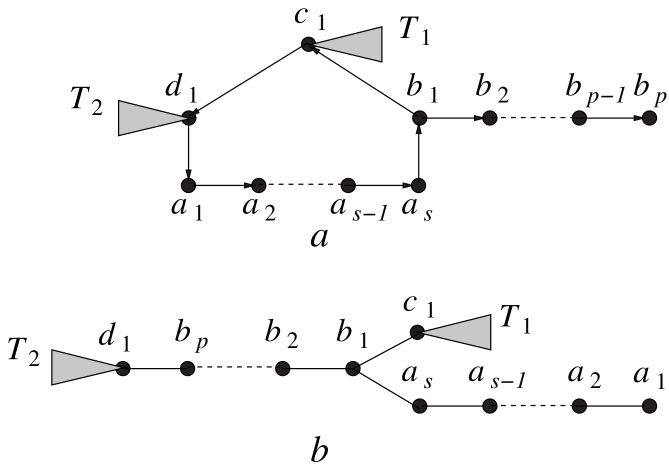

FIGURE 3.2. Transforming a cycle of total weight 1

EXAMPLE 3.33. For any tree-like diagrams T_1, T_2 denote by C_{p,T_1,T_2} the diagram on Figure 3.3,a. Applying mutations at vertices $b_1, b_2, \dots, b_p$ we arrive at the tree-like diagram T_{p,T_1,T_2}, see Figure 3.3,b.

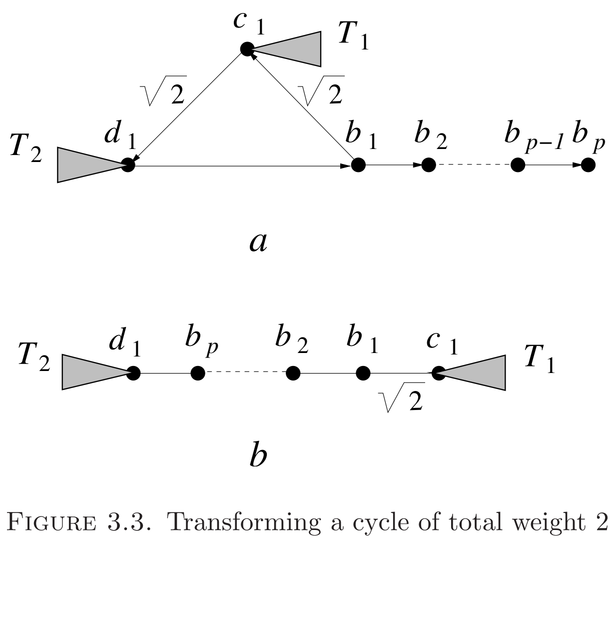

FIGURE 3.3. Transforming a cycle of total weight 2

We can now prove the following weaker version of Theorem 3.29.

LEMMA 3.34. *Any 2-finite tree-like diagram is mutation equivalent to an orientation of a Dynkin graph.*

PROOF. A weighted tree is called *critical* if it is not a Dynkin graph itself, and each of its proper induced subgraphs is a disjoint union of Dynkin graphs. The complete list of critical trees is given on Figure 3.4; here w can take any value in $\{1, \sqrt{2}, \sqrt{3}\}$.

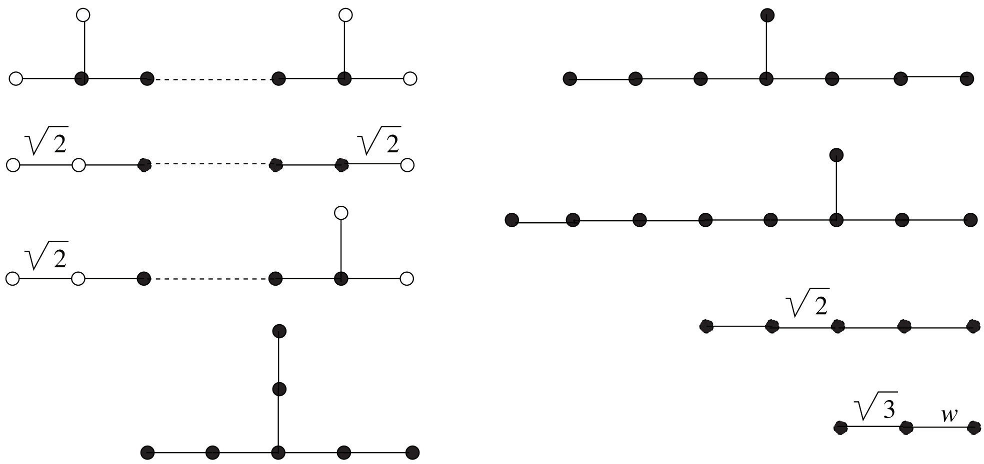

FIGURE 3.4. Critical trees

To prove the Lemma, it is enough to check that any orientation of a critical tree is 2-infinite. By Lemma 3.30, it suffices to consider one particular orientation. For the first three critical trees, direct the edges top-down or leftwards, and apply mutations consecutively at all non-leaves from right to left.

For the first tree, the diagram induced by the white vertices is a non-oriented cycle, in a contradiction with Lemma 3.31. For the second tree, the diagram induced by the white vertices is the three-cycle $(\sqrt{2}, \sqrt{2}, 2)$, in a contradiction with Lemma 3.31. For the third tree, uppend the above sequence by the mutation at the rightmost vertex; the diagram induced by the white vertices is the four-cycle $(\sqrt{2}, \sqrt{2}, 1, 1)$, in a contradiction with Lemma 3.31.

The forth tree is $T_{2,2,P_2,P_1}$, where P_i is a path on i vertices. By Example 3.32, any its orientation is mutation equivalent to $C_{2,2,P_2,P_1}$, which contains an instance of the first critical tree as an induced subgraph (for a reference see vertices a_s, b_1, b_2, c_1, d_1 and a neighbour of c_1 in T_1 on Figure 3.2).

The fifth tree is $T_{1,2,P_3,P_2}$. By the same example, any its orientation is mutation equivalent to $C_{1,2,P_3,P_2}$, which contains an instance of the fourth critical tree as an induced subgraph (for a reference see vertices b_1, b_2, c_1, d_1, d_2 and a path of length 2 in T_1 starting at c_1 on Figure 3.2).

The sixth tree is $T_{1,3,P_2,P_3}$. By the same example, its every orientation is mutation equivalent to $C_{1,3,P_2,P_3}$, which contains an instance of the fourth critical tree as an induced subgraph (for a reference see vertices b_1, b_2, b_3, c_1, d_1, a neighbor of c_1 in T_1 and a path of length 2 in T_2 starting at d_1 on Figure 3.2).

For the remaining two trees the claim follows easily from inspection. □

To complete the proof of Theorem 3.29 we need the following statement.

LEMMA 3.35. *Any connected 2-finite diagram is mutation equivalent to a tree-like diagram.*

PROOF. Assume that assertion of the Lemma is wrong, and let Γ be a counterexample with the minimal number of vertices. This number is at least 4, since for diagrams on three vertices the assertion can be checked easily. Let v be a vertex in Γ such that $\Gamma' = \Gamma \setminus \{v\}$ is connected. We may assume that Γ' is tree-like, and hence, by Lemma 3.34, is an orientation of a Dynkin graph. In particular, Γ' contains at most one edge of weight $\sqrt{2}$, which we denote by e.

If the degree of v in Γ equals 1, then Γ is already tree-like.

If the degree of v in Γ is at least 3, direct all the edges of the Dynkin graph corresponding to Γ' leftwards and top-down (cp. Fig. 3.1). It is easy to see that we immediately get a non-oriented cycle, in a contradiction with Lemma 3.31.

Let the degree of v in Γ be 2, so that Γ contains a unique cycle C passing through v. By Lemma 3.31, the total weight of C is either 1, or 2.

Assume first that the total weight of C equals 1. If C is a three-cycle, the mutation at v turns Γ into a tree-like diagram. Otherwise, Γ is an instance of C_{s,p,T_1,T_2} (possibly, with an additional edge incident to a_1) for some suitable choice of the parameters. By Example 3.32, any such diagram is mutation equivalent to a tree-like diagram.

Assume now that the total weight of C equals 2. By Lemma 3.31, C is either the three-cycle with weights $(\sqrt{2}, \sqrt{2}, 1)$, or the four-cycle with weights $(\sqrt{2}, 1, \sqrt{2}, 1)$. In the former case, Γ is an instance of C_{p,T_1,T_2} for some suitable choice of the parameters; by Example 3.33, any such diagram is mutation equivalent to a tree-like diagram. To discard the remaining case, it suffices to notice that e belongs to C, and that connecting a new vertex by an edge of weight 1 with a vertex of the four-cycle $(\sqrt{2}, 1, \sqrt{2}, 1)$ immediately produces an instance of the first critical tree (cp. Figure 3.4). □

Now the proof of Theorem 3.29 is complete. □

3.3.3. Proof of the implication (3) $\Longrightarrow$ (1). We start from the following general construction. Let $\mathfrak{P}$ be a simple convex polytope in $\mathbb{R}^n$. Assume that the vertices of $\mathfrak{P}$ are labeled by n-tuples of elements of some finite ground set in such a way that i-dimensional faces of $\mathfrak{P}$ correspond bijectively to maximal sets of n-tuples having exactly $n - i$ elements in common. Besides, an integer $n \times n$ sign-skew-symmetric matrix B_v is attached at each vertex v of $\mathfrak{P}$. The rows and columns of B_v are labeled by the elements of the n-tuple at v. Moreover, for every edge $(v, \bar{v})$ of $\mathfrak{P}$, the matrices B_v and $B_{\bar{v}}$ satisfy the following condition: let γ and $\bar{\gamma}$ be the unique elements of the labels at v and $\bar{v}$, respectively that are not common to these two labels, the $B_{\bar{v}}$ is obtained from B_v by a matrix mutation in direction γ followed by replacing the label γ by $\bar{\gamma}$.

Next, fix an arbitrary 2-dimensional face F of $\mathfrak{P}$. The label at any vertex in F contains exactly two elements that are not common to the labels of all other vertices in F; denote these element α and β. It is trivial to check that the integer $|b_{\alpha\beta} b_{\beta\alpha}|$ does not depend on the choice of the vertex in F; we call this number the *type* of F.

Proposition 3.36. *Let $\mathfrak{P}$ be a simple convex polytope and $\{B\}$ be a family of matrices as above. Assume that 2-dimensional faces of $\mathfrak{P}$ are quadrilaterals, pentagons, hexagons and octagons of types* 0, 1, 2 *and* 3, *respectively. Then the cluster algebra $\mathcal{A} = \mathcal{A}(\widetilde{B})$ is of finite type for any vertex v of $\mathfrak{P}$ and any $\widetilde{B}$ such that its principal part coincides with B_v.*

Proof. We will prove that the 1-skeleton of $\mathfrak{P}$ covers the exchange graph of $\mathcal{A}$. Let $\Sigma = (\mathbf{x}, \widetilde{B})$ be a seed of $\mathcal{A}$ and let B be the principal part of $\widetilde{B}$. It is convenient to regard the rows and columns of B as being labeled by the entries of $\mathbf{x}$. An *attachment* of Σ at v is a bijection between the labels at v and the cluster variables entering $\mathbf{x}$; such an attachment provides an identification of B and B_v. The *transport* of a seed attachment along and edge $(v, \bar{v})$ is defined as follows: the seed $\bar{\Sigma}$ attached at $\bar{v}$ is obtained from Σ by the mutation in direction $x(\gamma)$, and the new bijection at $\bar{v}$ is uniquely determined by $\bar{x}(\alpha) = x(\alpha)$ for all $\alpha \neq \bar{\gamma}$. Clearly, the transport of $\bar{\Sigma}$ attached at $\bar{v}$ backwards to v recovers the original seed attachment.

Let us fix an arbitrary vertex v of $\mathfrak{P}$ and attach a seed of $\mathcal{A}$ at v. Clearly, we can transport the initial attachment to any other vertex v' along a path from v to v'. To show that the resulting attachment does not depend on the choice of a path, it suffice to prove that transporting the attachment along a loop brings it back unchanged. Since all loops in $\mathfrak{P}$ are generated by 2-dimensional faces, it is enough to prove this fact for quadrilaterals, pentagons, hexagons and octagons of types 0, -1, -2 and -3, respectively. This is a simple exercise on cluster algebras of rank 2, which we leave to the interested reader.

Take an arbitrary seed of $\mathcal{A}$. It is obtained from the seed attached at v by a sequence of mutations. This sequence is uniquely lifted to a path on $\mathfrak{P}$ such that transporting the initial seed attachment at v along the edges of this path produces the chosen sequence of mutations. Therefore, the vertices of $\mathfrak{P}$ are mapped surjectively onto the set of all seeds of $\mathcal{A}$.

Let now v' and v'' be two vertices of $\mathfrak{P}$ such that their labels contain a common element α. Clearly, they can be connected by a path such that each vertex on this path possesses the same property. Therefore, $x'(\alpha) = x''(\alpha)$, and hence the attachment of cluster variables to the elements of the ground set does not depend on the choice of the vertex. It follows from above that this attachment is a surjection. Thus, $\mathcal{A}$ is of finite type, since the ground set is finite. □

The proof of the implication (3) $\Longrightarrow$ (1) consists of the construction of the polytopes $\mathfrak{P}$ for all each of the Cartan–Killing types. In the case of A_n we are in fact done for the following reasons. Let $\mathfrak{P}$ be the associahedron described in Section 2.1.4. It was already explained there that $\mathfrak{P}$ is a simple convex polytope, and that its 2-dimensional faces are quadrilaterals and pentagons. Moreover, let B' be the matrix mentioned in statement (3), then $|b'_{ij}b'_{ji}|$ is either 0, or 1. Recall that the rows and columns of B' are labeled by the cluster variables, that is, by the diagonals of R_m. It follows immediately from the structure of the Plücker relations that $b'_{ij} = 0$ if the diagonals corresponding to i and j are disjoint, and $b'_{ij} = \pm 1$ otherwise. Therefore, quadrilateral 2-faces of $\mathfrak{P}$ are of type 0, while pentagonal 2-faces are of type 1. So, the associahedron satisfies all the assumptions in Proposition 3.36, and hence the implication (3) $\Longrightarrow$ (1) holds true for Cartan–Killing type A_n.

The above construction for type A_n can be described in the following terms. Let Φ be the corresponding root system, Δ be the set of simple roots, Φ_+ be the set of all positive roots (see Section 1.2.2). Define the set of *almost positive roots* $\Phi_{\geq -1}$ as the union of $-\Delta$ and Φ_+. The elements of $\Phi_{\geq -1}$ correspond to the proper diagonals of R_{n+3}, and hence to the cluster variables of the corresponding cluster algebra. Indeed, consider the snake-like triangulation of R_{n+3} formed by the diagonals $(1, n+2), (2, n+2), (2, n+1), (3, n+1), \dots$, see Fig. 2.4 in Section 2.1.4. This diagonals are identified with $-\alpha_1, -\alpha_2, \dots$ for $\alpha_i \in \Pi$. The diagonal intersecting initial diagonals labeled by $-\alpha_j, -\alpha_{j+1}, \dots, -\alpha_k$ is identified with the positive root $\alpha_j + \alpha_{j+1} + \dots + \alpha_k$. A collection of almost positive roots is called *compatible* if the corresponding collection of diagonals is non-crossing. So, the faces of the associahedron are labeled by collections of compatible almost positive roots. In particular, the facets are labeled by the almost positive roots themselves, and the vertices are labeled by maximal compatible collections.

The same construction goes through for other Cartan–Killing types as well. The two main problems are to define the notion of compatibility and to prove that the resulting abstract polytope can be realized as a simple convex polytope satisfying the assumptions of Proposition 3.36. Consequently, the cluster variables of the cluster algebra are identified with the almost positive roots of the corresponding root system. The concrete details of the construction rely on intricate combinatorial properties of finite root systems. Since we will not need these results in the following chapters, they are not going to be reproduced here.

3.3.4. Proof of the uniqueness. Let B'' be another matrix mutation equivalent to B such that its Cartan companion is positive. Denote the Cartan companions of B' and B'' and the corresponding root systems by A', A'' and Φ', Φ'', respectively. It follows from the explanations in the previous section that the sets $\Phi'_{\geq 0}$ and $\Phi''_{\geq 0}$ are isomorphic, hence Φ' and Φ'' have the same rank and the same cardinality. A direct check shows that the only different Cartan–Killing types with this property are B_n and C_n for all $n \geq 3$, and also E_6, which has the same data as B_6 and C_6. To distinguish between these types, note that mutation-equivalent matrices share the same skew-symmetrizer D. Furthermore, D is a skew-symmetrizer for a skew-symmetrizable matrix, it is also a symmetrizer for its Cartan companion. Therefore, the diagonal entries of D are given by $d_i = (\alpha_i, \alpha_i)$, see Section 1.2.2. Since the root system of type B_n has one short simple root and $n-1$ long ones, while that of type C_n has one long root and $n-1$ short ones, the corresponding matrices cannot be mutation equivalent. The same is true for E_6 and B_6 or C_6 since all simple roots for E_6 are of the same length. Therefore, A' and A'' have the same Cartan–Killing type.

Let now B' and B'' be two sign-skew-symmetric matrices such that their Cartan companions A' and A'' are of the same Cartan–Killing type. Clearly, the corresponding diagrams $\Gamma(B')$ and $\Gamma(B'')$ are tree-like. By Lemma 3.30, we may assume that $\Gamma(B') = \Gamma(B'') = \Gamma$. It is easy to see that any D that serves a symmetrizer for A' (and hence for A'') is also a skew-symmetrizer for B' and B''. By the proof of Lemma 3.28, the skew-symmetric matrices $D^{1/2}B'D^{-1/2}$ and $D^{1/2}B''D^{-1/2}$ have the same diagram Γ, hence they coincide. Therefore, B' and B'' coincide as well.

3.4. Cluster algebras and rings of regular functions

As we have seen in Chapter 2, one can find cluster-like structures in the rings of functions related to Schubert varieties, such as the ring of regular functions on a double Bruhat cell, or a homogeneous coordinate ring of a Grassmannian. The goal of this section is to establish a corresponding phenomenon in a context of general Zariski open subsets of the affine space.

Let V be a Zariski open subset in $\mathbb{C}^{n+m}$, $\mathcal{A}_\mathbb{C}$ be a cluster algebra of geometric type tensored with $\mathbb{C}$. We assume that the variables in some extended cluster are identified with a set of algebraically independent rational functions on V. This allows us to identify cluster variables in any cluster with rational functions on V as well, and thus to consider $\mathcal{A}_\mathbb{C}$ as a subalgebra of $\mathbb{C}(V)$. Finally, we denote by $\mathcal{A}_\mathbb{C}^V$ the localization of $\mathcal{A}_C$ with respect to the stable variables that do not vanish on V; assuming that the latter variables are $x_{n+1}, \dots, x_{n+k}$, taking the localization is equivalent to changing the ground ring $\mathbb{A}$ from $\mathbb{Z}[x_{n+1}, \dots, x_{n+m}]$ to $\mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_{n+k}^{\pm 1}, x_{n+k+1}, \dots, x_{n+m}]$.

Proposition 3.37. *Let V and $\mathcal{A}$ as above satisfy the following conditions:*

(i) *each regular function on V belongs to $\mathcal{A}_\mathbb{C}^V$;*

(ii) *there exists an extended cluster $\widetilde{\mathbf{x}} = (x_1, \dots, x_{n+m})$ in $\mathcal{A}_\mathbb{C}$ consisting of algebraically independent functions regular on V;*

(iii) *any cluster variable x'_k, $k \in [1, n]$, obtained by the cluster transformation* (3.1) *applied to $\widetilde{\mathbf{x}}$ is regular on V.*

Then $\mathcal{A}_\mathbb{C}^V$ coincides with the ring $\mathcal{O}(V)$ of regular functions on V.

Proof. All we have to prove is that any element in $\mathcal{A}_\mathbb{C}^V$ is a regular function on V. The proof consists of three steps.

Lemma 3.38. *Let $\widetilde{\mathbf{z}} = (z_1, \dots, z_{n+m})$ be an arbitrary extended cluster in $\mathcal{A}_\mathbb{C}$. If a Laurent monomial $M = z_1^{d_1} \cdots z_{n+m}^{d_{n+m}}$ is regular on V then $d_i \geq 0$ for $i \in [1, n]$.*

Proof. Indeed, assume that $d_k < 0$ and consider the cluster $\widetilde{\mathbf{z}}_k$. By (3.1), M can be rewritten as $M = M'(z'_k)^{-d_k}/B^{-d_k}$, where M' is a Laurent monomial in common variables of $\widetilde{\mathbf{z}}$ and $\widetilde{\mathbf{z}}_k$, and B is the binomial (in the same variables) that appears in the right hand side of (3.1). By condition (i) and Theorem 3.14, M can be written as a Laurent polynomial in the variables of $\widetilde{\mathbf{z}}_k$. Equating two expressions for M, we see that B^{-d_k} times a polynomial in variables of $\widetilde{\mathbf{z}}_k$ equals a Laurent monomial in the same variables. This contradicts the algebraic independence of variables in $\widetilde{\mathbf{z}}_k$, which follows from the algebraic independence of variables in $\widetilde{\mathbf{x}}$. □

Lemma 3.39. *Let z be a cluster variable in an extended cluster $\widetilde{\mathbf{z}}$, and assume that z is a regular function on V. Then z is irreducible in the ring of regular functions on V.*

Proof. Without loss of generality, assume that $z_{n+1} = x_{n+1}, \dots, z_{n+m'} = x_{n+m'}$ do not vanish on V, and $z_{n+m'+1} = x_{n+m'+1}, \dots, z_{n+m} = x_{n+m}$ may vanish on V. Moreover, assume to the contrary that $z = fg$, where f and g are non-invertible regular functions on V. By condition (i) and Proposition 3.20, both f and g are Laurent polynomials in $z_1, \dots, z_{n+m'}$ whose coefficients are polynomials in $z_{n+m'+1}, \dots, z_{n+m}$. Applying the same argument as in the proof of Lemma 3.38, we see that both f and g are, in fact, Laurent monomials in $z_1, \dots, z_{n+m}$ and that $z_{n+1}, \dots, z_{n+m'}$ enter both f and g with a non-negative degree. Moreover, by Lemma 3.38, each cluster variable $z_1, \dots, z_n$ enters both f and g with a nonnegative degree. This can only happen if one of f and g is invertible in $\mathcal{O}(V)$, a contradiction. □

Denote by $U_0 \subset V$ the locus of all $t \in V$ such that $x_i(t) \neq 0$ for all $i \in [1, n]$. Besides, denote by $U_k \subset V$ the locus of all $t \in V$ such that $x_i(t) \neq 0$ for all $i \in [1, n] \setminus k$ and $x'_k(t) \neq 0$. Note a similarity between sets U_0, U_i and sets $U_{\mathbf{i}}, U_{n,\mathbf{i}}$ defined in Lemma 2.11. The following statement is a straightforward generalization of the codimension 2 result implicitly comprising the first part of Theorem 2.16.

Lemma 3.40. *Let* $U = \cup_{i=0}^n U_i$, *then* $\operatorname{codim} V \setminus U \geq 2$.

Proof. Follows immediately from Lemma 3.39 and conditions (ii) and (iii). □

We can now complete the proof of Proposition 3.37. Inclusion $\mathcal{A}_{\mathbb{C}}^V \subseteq \mathcal{O}(U)$ is an immediate corollary of Proposition 3.20. The rest of the proof follows literally the proof of Corollary 2.18. □

In some situations condition (i) of Proposition 3.37 is difficult to check. In this case one may attempt prove a weaker result, with cluster algebras replaced by upper cluster algebras, by imposing additional conditions on the variables x_i, $i \in [1, n+m]$ and x'_k, $k \in [1, n]$. As an example, consider reduced double Bruhat cells $L^{u,v}$ studied in Section 2.2. In Section 2.2.3 we associated to each reduced word $\mathbf{i}$ a symmetric $(l(u)+l(v)) \times (l(u)+l(v))$ matrix $C = C_{\mathbf{i}}$ and a directed graph $\Sigma_{\mathbf{i}}$. Based on $C_{\mathbf{i}}$ and $\Sigma_{\mathbf{i}}$, we define a matrix $\widetilde{B} = \widetilde{B}_{\mathbf{i}}$ as follows. The rows of $\widetilde{B}$ correspond to $\mathbf{i}$-bounded indices, and the columns of $\widetilde{B}$ correspond to all indices. For an $\mathbf{i}$-bounded index k and an arbitrary index l, $b_{kl} = c_{kl}$ if $(k \to l) \in \Sigma_{\mathbf{i}}$ and $b_{kl} = -c_{kl}$ if $(l \to k) \in \Sigma_{\mathbf{i}}$; all the other entries of $\widetilde{B}$ are equal to zero. So, $\widetilde{B}$ can be considered as a submatrix of the incidence matrix of $\Sigma_{\mathbf{i}}$ weighted by the corresponding elements of $C_{\mathbf{i}}$.

Proposition 3.41. *The matrix* $\widetilde{B}_{\mathbf{i}}$ *satisfies the assumptions of Theorem 3.21.*

Proof. Clearly, $\widetilde{B}_{\mathbf{i}}$ is skew-symmetric. To see that it has a full rank, consider the square submatrix B of $\widetilde{B}_{\mathbf{i}}$ formed by the columns indexed by all p such that $p = q^-$ for some $\mathbf{i}$-bounded index q. Let us rearrange the rows of B in such a way that its diagonal represents the one-to-one correspondance $q \mapsto q^-$. It follows immediately from Definition 2.7 that the rearranged B is upper-triangular and all its diagonal entries equal ± 1. □

Choose functions M_i, $1 \leq i \leq l(u) + l(v)$, discussed in Lemma 2.11 (and described explicitly in the SL_n case immediately after that Lemma) as an initial cluster and denote by $\mathcal{U}_{\mathbf{i}}$ the corresponding upper cluster algebra $\mathcal{U}(\widetilde{B}_{\mathbf{i}})$. In these terms, Corollary 2.18 can be restated as follows.

Theorem 3.42. *For any reduced word* $\mathbf{i}$ *for the pair* (u, v), *the ring* $\mathcal{O}(L^{u,v})$ *of regular functions on the reduced double Bruhat cell* $L^{u,v}$ *coincides with the upper cluster algebra* $\mathcal{U}_{\mathbf{i}}$ *tensored with* $\mathbb{C}$.

Proof. Indeed, by Corollary 2.18, $\mathcal{O}(L^{u,v})$ coincides with the ring of regular functions on $U = U_{\mathbf{i}} \cup U_{1,\mathbf{i}} \cup \dots \cup U_{n,\mathbf{i}}$. The latter is, by definition, the upper bound corresponding to the initial cluster, which in view of Theorem 3.21 coincides with the upper cluster algebra $\mathcal{U}_{\mathbf{i}}$. □

The case of double Bruhat cells $\mathcal{G}^{u,v}$ can be treated in a similar way. More exactly, given a reduced word $\mathbf{i}$ for (u, v), define $\bar{\mathbf{i}}$ as $\mathbf{i}$ written in the reverse direction appended with the word $-1 \ -2 \ \cdots -r$, where r is the rank of $\mathcal{G}$. Define the $(l(u)+l(v)+r)\times(l(u)+l(v)+r)$ matrix $C_{\bar{\mathbf{i}}}$ and the directed graph $\Sigma_{\bar{\mathbf{i}}}$ on $l(u)+l(v)+r$ vertices similarly to $C_{\mathbf{i}}$ and $\Sigma_{\mathbf{i}}$. The graph $\Sigma'_{\mathbf{i}}$ is obtained from $\Sigma_{\bar{\mathbf{i}}}$ via the following modifications:

(i) erase all inclined edges with both endpoints lying in $[l(u) + l(v) + 1, l(u) + l(v) + r]$;

(ii) reverse the direction of all remaining edges;

(iii) reverse the ordering of the vertices as folows: $k \mapsto \bar{k}$ with

$$\bar{k} = \begin{cases} l(u) + l(v) + 1 - k & \text{if } k \in [1, l(u) + l(v)], \\ l(u) + l(v) - k & \text{if } k \in [l(u) + l(v) + 1, l(u) + l(v) + r]. \end{cases}$$

As a result, the vertex set of $\Sigma'_{\mathbf{i}}$ is $[-r, -1] \cup [1, l(u) + l(v)]$.

EXAMPLE 3.43. Let $\mathbf{i} = (1\,2\,1\,-1\,-2\,-1)$. Then $\bar{\mathbf{i}} = (-1\,-2\,-1\,1\,2\,1\,-1\,-2)$, and the corresponding graph $\Sigma_{\bar{\mathbf{i}}}$ is shown on Fig. 3.5a. To get the graph $\Sigma'_{\mathbf{i}}$ we delete the edge between the vertices 7 and 8, reverse the directions of eges and reorder the vertices. Since the vertices in $\Sigma_{\bar{\mathbf{i}}}$ are ordered from left to right, the latter operation can be represented as the symmetry with respect to the vertical axes. The resulting graph $\Sigma'_{\mathbf{i}}$ is shown on Fig. 3.5b.

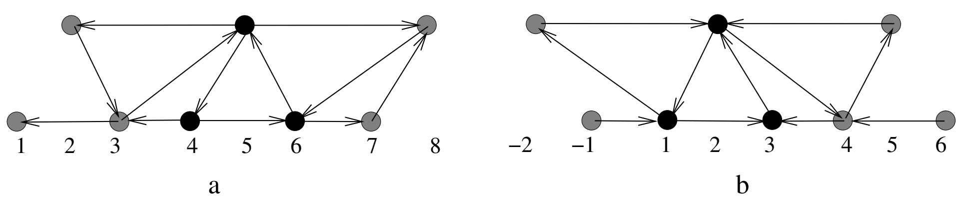

FIGURE 3.5. Transforming $\Sigma_{\bar{\mathbf{i}}}$ to $\Sigma'_{\mathbf{i}}$

An index $\bar{k}$ is called $\mathbf{i}$-*exchangeable* if the corresponding index k is $\mathbf{i}$-bounded. Based on this combinatorial data, define the matrix $\widetilde{B}'_{\mathbf{i}}$ similarly to $\widetilde{B}_{\mathbf{i}}$; its rows correspond to $\mathbf{i}$-exchangeable indices among in $[1, l(u) + l(v)]$, and its columns correspond to all indices. It follows immediately from Proposition 3.41 that $\widetilde{B}'_{\mathbf{i}}$ satisfies the assumptions of Theorem 3.21.

For $k \in [1, l(u) + l(v)]$ denote

$$u_{\le k} = u_{\le k}(\mathbf{i}) = \prod_{l=1,\dots,k} s_{|i_l|}^{\frac{1-\theta(l)}{2}}, \qquad v_{>k} = v_{>k}(\mathbf{i}) = \prod_{l=l(u)+l(v),\dots,k+1} s_{|i_l|}^{\frac{1+\theta(l)}{2}} \tag{3.15}$$

(see Remark 1.1 for notation). Besides, for $k \in -[1, r]$ set $u_{\le k}$ to be the identity and $v_{>k}$ to be equal to v^{-1}. For $k \in [-r, -1] \cup [1, l(u) + l(v]$ put

$$\Delta(k; \mathbf{i}) = \Delta_{u_{\le k}\omega_{|i_k|}, v_{>k}\omega_{|i_k|}}, \tag{3.16}$$

where the right hand side is a generalized minor defined by (1.11). Take the functions $\Delta(k; \mathbf{i})$ as the initial cluster and define the corresponding upper cluster algebra $\mathcal{U}'_{\mathbf{i}} = \mathcal{U}(\widetilde{B}'_{\mathbf{i}})$. An analog of Theorem 3.42 can be stated as follows.

THEOREM 3.44. *For any reduced word* $\mathbf{i}$ *for the pair* (u, v), *the ring* $\mathcal{O}(\mathcal{G}^{u,v})$ *of regular functions on the double Bruhat cell* $\mathcal{G}^{u,v}$ *coincides with the upper cluster algebra* $\mathcal{U}'_{\mathbf{i}}$ *tensored by* $\mathbb{C}$.

3.5. Conjectures on cluster algebras

Besides the general results on cluster algebras presented in the previous two sections, there are also many general conjectures. One can divide the conjectures into several groups.

The conjectures in the first group deal with the structure of the exchange graph of a cluster algebra.

CONJECTURE 3.45. *The exchange graph of a cluster algebra depends only on the initial exchange matrix* B.

The vertices and the edges of the exchange graph are conjecturally described as follows.

CONJECTURE 3.46. (i) *Every seed is uniquely defined by its cluster; thus, the vertices of the exchange graph can be identified with the clusters, up to a permutation of cluster variables.*

(ii) *Two clusters are adjacent in the exchange graph if and only if they have exactly* $n - 1$ *common cluster variables.*

The last conjecture in the first group says that if two clusters contain the same variable then they can be reached one from the other without changing this variable.

CONJECTURE 3.47. *For any cluster variable* x, *the seeds whose clusters contain* x *form a connected subgraph of the exchange graph.*

The second group of conjectures treats in more detail the Laurent phenomenon described in Section 3.2. Recall that by Theorem 3.14, any cluster variable x can be uniquely expressed via the initial cluster variables $x_1, \dots, x_n$ as

$$x = \frac{P(x_1, \dots, x_n)}{x_1^{d_1} \dots x_n^{d_n}}, \tag{3.17}$$

where P is a polynomial not divisible by any of x_i's. The exponent d_i as defined above may depend on the choice of the cluster containing x and on the choice of the initial cluster. The first conjecture in the second group says that there is no such dependence.

CONJECTURE 3.48. *The exponent* d_i *depends only on* x *and* x_i.

The following conjecture deals with the properties of the exponents d_i.

CONJECTURE 3.49. (i) *The exponents* d_i, $i \in [1, n]$, *are nonnegative for any cluster variable* $x \neq x_i$.

(ii) *The exponent* d_i *vanishes if and only if there exists a cluster containing both* x *and* x_i.

The third, and the most interesting group of conjectures deal with cluster monomials. A *cluster monomial* is a monomial in cluster variables all of which belong to the same cluster; in other words, if $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$ is a cluster then $\prod_{i=1}^n \bar{x}_i^{a_i}$ is a cluster monomial for any choice of nonnegative exponents a_i, $i \in [1, n]$.

Conjecture 3.50. *Cluster monomials in any cluster algebra are linearly independent over the ground ring.*

Writing down each variable in a cluster monomial in form (3.17), one can extend the definition of exponents d_i to cluster monomials.

Conjecture 3.51. *If the exponents d_i for two cluster monomials coincide for all $i \in [1, n]$ then the monomials themselves coincide.*

We say that a nonzero element in a cluster algebra $\mathcal{A}$ is *positive* if its Laurent expansion in terms of the variables from any seed in $\mathcal{A}$ has nonnegative coefficients. Further, a positive element is called *indecomposable* if it cannot be written as a sum of two positive elements.

Conjecture 3.52. *Every cluster monomial is an indecomposable positive element.*

Conjectures 3.45–3.50 are known to be true for the algebras of finite type. In what follows we exhibit several other classes of cluster algebras for which some of these conjectures hold. Conjectures 3.51 and 3.52 are even less accessible. They are known to be true for cluster algebras of classical type, that is, for the cluster algebras of finite type corresponding to classical semisimple Lie algebras.

3.6. Summary

- A seed of geometric type is a pair $(\mathbf{x}, \widetilde{B})$, where $\mathbf{x}$ is an $(n+m)$-tuple of variables called an extended cluster and $\widetilde{B}$ is an $n \times (n+m)$ integer matrix such that its principal part $\widetilde{B}[n; n]$ is skew-symmetrizable. The first n entries of $\widetilde{\mathbf{x}}$ are called cluster variables, and the last m entries are called stable variables.
- Two seeds $(\mathbf{x}, \widetilde{B})$ and $(\mathbf{x}', \widetilde{B}')$ are adjacent in direction k if x_k and x'_k are related by exchange relation (3.1), $x_i = x'_i$ for $i \neq k$, and $\widetilde{B}'$ is obtained from $\widetilde{B}$ by the matrix mutation in direction k. By iterating this procedure we obtain an n-regular tree $\mathbb{T}_n$ whose vertices correspond to seeds and edges connect adjacent clusters.
- The cluster algebra of geometric type $\mathcal{A}(\widetilde{B})$ over a ground ring $\mathbb{A}$ is the $\mathbb{A}$-subalgebra of the ambient field generated by all cluster variables at all the vertices of $\mathbb{T}_n$.
- Two seeds are equivalent if they are obtained one from another by a permutation of cluster variables and the corresponding permutation of rows and columns of the exchange matrix. The exchange graph of a cluster algebra is the quotient of $\mathbb{T}_n$ by this equivalence relation.
- Any cluster variable is a Laurent polynomial in the initial cluster variables (the Laurent phenomenon, Theorem 3.14).
- The ring of functions that can be written as Laurent polynomials in the variables of a given cluster $\mathbf{x}$, as well as in the variables of any of the adjacent clusters, is called the upper bound associated with $\mathbf{x}$. The upper cluster algebra $\mathcal{U}$ is the intersection of all upper bounds. If the exchange matrix $\widetilde{B}$ has a full rank then the upper bounds do not depend on the choice of $\mathbf{x}$, and hence coincide with $\mathcal{U}(\widetilde{B})$ (Theorem 3.21). The upper cluster algebra $\mathcal{U}(\widetilde{B})$ contains $\mathcal{A}(\widetilde{B})$, and in general these two algebras do not coincide (Example 3.25).

- The classification of cluster algebras having a finite exchange graph (called algebras of finite type) coincides with the Cartan-Killing classification of semisimple Lie algebras (Theorem 3.26).

Bibliographical notes

Our exposition in this chapter is based on the three consecutive papers [**FZ2, FZ4, BFZ2**], which laid the foundation of the theory of cluster algebras.

3.1. For the complete classification of cluster algebras of rank 2, see [**FZ2, ShZ**]. Example 3.12 involving Markov triples is borrowed from [**BFZ2**]. For relations of Markov numbers to various mathematical problems see [**CuFl**] and references therein. The definition of general cluster algebras in Remark 3.13 follows [**FZ2**]. The approach based on normalization condition is suggested in [**FZ4**].

3.2. Theorem 3.14 was proved in [**FZ2**] (see also [**FZ3**] for an extended discussion of this subject, going beyond the context of cluster algebras). Our proof here follows, with minor modifications, that of [**BFZ2**]. To lift the coprimality assumption at the end of the proof, we use the argument suggested in [**FZ2**], p.507. Proposition 3.20 can be found in [**FZ4**]. The treatment of upper cluster algebras follows [**BFZ2**]. Lemma 3.23 is borrowed from [**GSV2**].

3.3. The classification theorem for cluster algebras of finite type was obtained in [**FZ4**].

Exposition in Section 3.3.1 follows [**BaGZ**] with some deviations. For condition (3.13) see, e.g., Theorem 4 in [**PY**] or Theorem 4 in [**Ma**]. Section 3.3.2 follows [**BaGZ**] with certain modifications. For the complete proof of the implication (3)$\Longrightarrow$(1) in Theorem 3.26 based on the notion of generalized associahedra, see [**FZ4**]. Theory of generalized associahedra was developed in [**FZ5, CFZ**]. For further generalizations of associahedra see [**FRe, HLT**] and references therein.

3.4. Proposition 3.37 can be found in [**GSV7**]. Theorem 3.42, apparently, has not appeared in literature in this form. Theorem 3.44 is proved in [**BFZ2**] based on an analog of Lemma 2.11.

3.5. The list of conjectures is borrowed from [**FZ6**].