Preface

This volume, the third in our series, is about Hopf algebras (mostly) and Lie algebras. It is independent of the first two volumes; though, to be sure, there are some references to them, just as there are references to other books and journals.

The first chapter is devoted to Lie algebras. It is a fairly standard concise treatment of the more established parts of the theory with the exceptions that there is a bit more emphasis on Dynkin diagrams (also pictorially) and that the chapter includes a complete treatment of the correspondence, initiated by Peter Gabriel, between representations of quivers whose underlying graph is a Dynkin diagram and representations of the Lie algebra with that Dynkin diagram, Gabriel, [**4**], [**3**]; Bernstein-Gel'fand-Ponomarev, [**1**]; Dlab-Ringel, [**2**]. The treatment is via the very elegant approach by Coxeter reflection functors of [**1**].

The remaining seven chapters are on Hopf algebras.

The first two of these seven are devoted to the basic theory of coalgebras and Hopf algebras paying special attention to motivation, history, intuition, and provenance. In a way these two chapters are primers[1] on their respective subjects. The remaining five chapters are quite different.

Chapter number four is on the symmetric functions from the Hopf algebra point of view. This Hopf algebra is possibly the richest structured object in mathematics and a most beautiful one. One aspect that receives special attention is the Zelevinsky theorem on *PSH* algebras. The acronym *PSH* stands for 'positive selfadjoint Hopf'. What one is really dealing with here is a graded, connected, Hopf algebra with a distinguished basis that is declared orthonormal, such that each component of the grading is of finite rank, and such that multiplication and comultiplication are adjoint to each other and positive. If then, moreover, there is only one distinguished basis element that is primitive, the Hopf algebra is isomorphic to **Symm**, the Hopf (and more) algebra of symmetric functions with the Schur functions as distinguished basis. Quite surprisingly the second (co)multiplication on **Symm**, which makes each graded summand a ring in its own right and which is distributive over the first one (in the Hopf algebra sense), turns up during the proof of the theorem, the Bernstein morphism. This certainly calls for more investigations.

The enormously rich structure of **Symm** is discussed extensively (various lambda ring structures, Frobenius, and Verschiebung morphims, Adams operations, ... etc. Correspondingly a fair amount of space is given to the big Witt vectors.

[1]As a rule a professional mathematician is well advised to stay away from 'primers'; we hope and believe that these are an exception.

Chapter five is on the representations of the symmetric groups; more precisely it is on the direct sum of the RS_n where S_n is the symmetric group on n letters and as an Abelian group RS_n is the Grothendieck group of (virtual) complex representations. The direct sum, with $RS_0 = \mathbf{Z}$ by decree, is given a multiplication and a comultiplication by using induction up from two factor Young subgroups and restriction down to such subgroups. The result is a *PSH* algebra.

That is not at all difficult to see: basically Frobenius reciprocity and the Mackey double coset formula do the job. The distinguished basis is formed by the irreducible representations and there is only one primitive among these, viz the trivial representation of the symmetric group on one letter. It follows that $RS = \bigoplus_{n=0}^{\infty} RS_n$ is a *PSH* algebra and that it is isomorphic to **Symm**.

It also follows that **Symm** itself is *PSH* (with one distinguished primitive). So far that was not clear. At the end of chapter four the situation was that if there were a *PSH* algebra with one distinguished primitive it would be isomorphic to **Symm**, making **Symm** itself also *PSH*.

The stumbling block in proving directly that **Symm** is *PSH* is positivity. There seems to be no direct proof, without going through the representation theory of the symmetric groups, of the fact that the product of two Schur functions is a positive linear combination of (other) Schur functions.

The question of positivity of multiplication and comultiplication, such a vexing matter in the case of **Symm**, becomes a triviality in the case of *RS*. Indeed, the product is describable as follows. Take a representation ρ of S_p and a representation σ of S_q and form the tensor product $\rho \otimes \sigma$ which is a representation of the Young subgroup $S_p \times S_q$ of S_{p+q} and induce this representation up to a representation of all of S_{p+q}. If the two initial representations are real (as opposed to virtual), so is their tensor product and the induced representation of S_{p+q}. Thus multiplication is positive. Comultiplication can be treated in a similar way, or handled via duality.

The isomorphism between **Symm** and *RS* is far more than just an isomorphism of Hopf algebras; it also says things regarding the second (co)multiplications, the various lambda ring structures and plethysms, ... ; see [**7**]. It won't be argued here, but it may well turn out to be the case that among the many Hopf algebras isomorphic to **Symm** the incarnation *RS* is the central one; see loc. cit.

Of course the fact that **Symm** and *RS* are isomorphic (as algebras) is much older than the Zelevinsky theorem; see [**8**], section I.7, for a classical treatment.

Chapter six is about two generalizations of **Symm** that have become important in the last 25 years or so: the Hopf algebra **QSymm** of quasi-symmetric functions and the Hopf algebra **NSymm** of non commutative symmetric functions. They are dual to each other and this duality extends the autoduality of **Symm** via a natural imbedding **Symm** $\subset$ **QSymm** and a natural projection **NSymm** $\longrightarrow$ **Symm**. Both Hopf algebras carry a good deal more structure. The seminal paper which started all this is [**5**], but there, and in a slew of subsequent papers, things were done over a field of characteristic zero and not over the integers as here.

It is somewhat startling to discover how many concepts, proofs, constructions, ... have natural non commutative and quasi analogs; and not rarely these analogs are more elegant than their counterparts in the world of symmetric functions.

Things can be generalized still further to the Hopf algebra *MPR* of permutations of Malvenuto, Poirier, and Reutenauer, [**9**]; [**10**]. This one is twisted autodual and this duality extends the duality between **NSymm** and **QSymm** via a natural imbedding **NSymm** $\longrightarrow MPR$ and a natural projection $MPR \longrightarrow$ **QSymm**. This is the subject matter of chapter seven. There are still further generalizations, complete with duality, [**6**], but the investigation of these has only just started. Like *MPR* they are Hopf algebras of endomorphisms of Hopf algebras.

Finally, chapter eight contains over fifteen relatively short and shorter outlines, heavily bibliographical, on the roles of Hopf algebras in other parts of mathematics and theoretical physics. Much of this material has not previously appeared in the monographic literature.

In closing, we would like to express our cordial thanks to our friends and colleagues (among them, we especially thank S.A.Ovsienko) who have read portions of preliminary versions of this book and offered corrections, suggestions and other constructive comments improving the entire text. We also give special thanks to A. Katkow and V. V. Sergeichuk who have helped with the preparation of this manuscript.

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