Contents

Preface ix
References xi

Chapter 1. Lie algebras and Dynkin diagrams 1
1.1. Lie algebras. Definitions and examples 2
1.2. Ideals, homomorphisms and representations 7
1.3. Solvable and nilpotent Lie algebras 11
1.4. Radical of a Lie algebra. Simple and semisimple Lie algebras 15
1.5. Modules for Lie algebras. Weyl's theorem. Ado's theorem 22
1.6. Lie's theorem 29
1.7. The Lie algebra $\mathfrak{sl}(2;k)$. Representation of $\mathfrak{sl}(2;k)$ 32
1.8. The universal enveloping algebra of A lie algebra 35
1.9. Poincaré-Birkhoff-Witt theorem 38
1.10. Free Lie algebras 43
1.11. Examples of simple Lie algebras 45
1.12. Abstract root systems and the Weyl group 47
1.13. Cartan matrices and Dynkin diagrams 57
1.14. Coxeter groups and Coxeter diagrams 61
1.15. Root systems of semisimple Lie algebras 67
1.16. The Weyl group of a quiver 73
1.17. Reflection functors 78
1.18. Coxeter functors and Coxeter transformations 85
1.19. The Gabriel theorem 87
1.20. Generalized Cartan matrices and Kac-Moody Lie algebras 88
1.21. Historical notes 91
References 94

Chapter 2. Coalgebras: motivation, definitions, and examples 99
2.1. Coalgebras and 'addition formulae' 100
2.2. Coalgebras and decompositions 102
2.3. Dualizing the idea of an algebra 103
2.4. Some examples of coalgebras 105
2.5. Sub coalgebras and quotient coalgebras 108
2.6. The main theorem of coalgebras 108
2.7. Cofree coalgebras 109
2.8. Algebra - coalgebra duality 112
2.9. Comodules and representations 121
2.10. Graded coalgebras 123
2.11. Reflexive modules 125
2.12. Measuring 126

2.13. Addition formulae and duality 128
2.14. Coradical and coradical filtration 128
2.15. Coda to chapter 2 129
References 129

Chapter 3. Bialgebras and Hopf algebras. Motivation, definitions, and examples 131
3.1. Products and representations 131
3.2. Bialgebras 133
3.3. Hopf algebras 138
3.4. Some more examples of Hopf algebras 140
3.5. Primitive elements 146
3.6. Group-like elements 149
3.7. Bialgebra and Hopf algebra duality 152
3.8. Graded bialgebras and Hopf algebras 153
3.9. Crossed products 159
3.10. Integrals for Hopf algebras 162
3.11. Formal groups 167
3.12. Hopf modules 169
3.13. Historical remarks 170
3.14. The Hopf algebra of an algebra 171
References 172

Chapter 4. The Hopf algebra of symmetric functions 175
4.1. The algebra of symmetric functions 175
4.2. The Hopf algebra structure 184
4.3. *PSH* algebras 185
4.4. Automorphisms of Symm 193
4.5. The functor of the Witt vectors 194
4.6. Ghost components 197
4.7. Frobenius and Verschiebung endomorphisms 199
4.8. The second multiplication of Symm 202
4.9. Lambda algebras 203
4.10. *Exp* algebras 210
4.11. Plethysm 213
4.12. The many incarnations of Symm 214
References 215

Chapter 5. The representations of the symmetric groups from the Hopf algebra point of view 217
5.1. A little bit of finite group representation theory 217
5.2. Double cosets of Young subgroups 221
5.3. The Hopf algebra 223
5.4. Symm as a PSH algebra 226
5.5. The second multiplication on RS 227
5.6. Remarks and acknowledgements 229
References 229

Chapter 6. The Hopf algebra of noncommutative symmetric functions and the Hopf algebra of quasisymmetric functions 231

6.1. The Hopf algebra NSymm 232
6.2. NSymm over the rationals 234
6.3. The Hopf algebra QSymm 235
6.4. Symm as a quotient of NSymm 238
6.5. More on *Shuffle* and *LieHopf* 241
6.6. The autoduality of Symm 248
6.7. Polynomial freeness of QSymm over the integers 250
6.8. Hopf endomorphisms of QSymm and NSymm 254
6.9. Verschiebung and Frobenius on NSymm and QSymm 255
References 260

Chapter 7. The Hopf algebra of permutations 263
7.1. The Hopf algebra of permutations of Malvenuto, Poirier and Reutenauer 263
7.2. The imbedding of NSymm into MPR 267
7.3. LSD permutations 271
7.4. Second multiplication and second comultiplication 274
7.5. Rigidity and uniqueness of MPR 275
References 276

Chapter 8. Hopf algebras: Applications in and interrelations with other parts of mathematics and physics 277
8.1. Actions and coactions of bialgebras and Hopf algebras 277
8.2. The quantum groups $GL_q(n, \mathbf{C})$ and multiparameter generalizations 285
8.3. The quantum groups $U_q(\mathfrak{sl}(n; k))$ 294
8.4. R-matrices and QIST: Bethe Ansatz, FCR construction and the FRT theorem 296
8.5. Knot invariants from quantum groups. Yang-Baxter operators 311
8.6. Quiver Hopf algebras 324
8.7. Ringel-Hall Hopf algebras 333
8.8. More interactions of Hopf algebras with other parts of mathematics and theoretical physics 347
8.8.1. Capelli identities and other formulas for matrices and determinants with mildly noncommuting entries 347
8.8.2. Quantum symmetry 355
8.8.3. Hopf algebra symmetries in noncommutative geometry 369
8.8.4. Hopf algebras in Galois theory 371
8.8.5. Hopf algebras and renormalization 372
8.8.6. Quantum calculi 372
8.8.7. Umbral calculus and Baxter algebras 376
8.8.8. q-special functions 380
References 381

Index 407