CHAPTER 1

Introduction

The aim of this book is to study the connective real K-cohomology $ko^*(BG)$ as a ring and the associated homology $ko_*(BG)$ as a module over the ring, for a range of compact Lie groups G, with both general results and a wide range of specific calculations. We believe that these calculations are only the beginning of what can be done with these methods, and we hope others will find them useful in many more cases.

1.1. Motivation

There are general reasons for interest that would not justify a project of this size. On the one hand, calculations of connective K-theory of a space are illuminating for its homotopy theory; on the other, classifying spaces BG provide both embodiments of group theory and especially accessible infinite spaces.

A more compelling argument comes from three processes for deducing subtle information from cruder information: uncompletion, descent and unlocalization. All three occur rather widely, and most fashionably in the study of elliptic cohomology, but they still have considerable substance even in the more accessible case of K-theory. To explain further, we imagine we live in a perfect world but with imperfect understanding. Thus, although there is an equivariant form of connective real K-theory, we begin with a knowledge of non-equivariant, periodic, complex K-theory alone. Since the Atiyah-Segal theorem states that $KU^*(BG)$ is a completion of KU_G^*, the process of going from a non-equivariant theory to the equivariant theory can be viewed as uncompletion. Since KO is the fixed point spectrum of KU, the process of recovering real from complex K-theory is a form of descent. Finally, the periodic theory KO is obtained from the connective theory ko by inverting the Bott element, so the recovery of ko from KO is an unlocalization. In the forms just described, these processes are well understood theoretically, but if we ask for equivariant forms of descent and unlocalization there is more to be said, and similarly if there is a space involved as well. Finally, we might actually want to use this theoretical understanding to give calculations. The present book could be viewed as a case study in these matters.

Finally, it is worth explaining why we *actually* began this project. When G is a discrete group, $ko_n(BG)$ is the obstruction group for the existence of a positive scalar curvature metric on a spin manifold of dimension $n \geq 5$ and fundamental group G. We will give a brief description of work on the Gromov-Lawson-Rosenberg (GLR) conjecture, referring to [**88**] for a more detailed guide. First, for a spin manifold M of dimension $n \geq 5$, the existence of a positive scalar curvature metric depends only on the bordism class $[M]$, and there are maps

$$\Omega_n^{spin}(BG) \xrightarrow{D} ko_n(BG) \xrightarrow{p} KO_n(BG) \xrightarrow{A} KO_n(C^*_{red}G),$$

with the composite vanishing on $[M]$ when it admits a positive scalar curvature metric, by the Lichnerowicz method. The GLR conjecture for G is that this necessary condition is also sufficient; this is known to be false in general [**89, 34**], but there are no known finite groups for which it fails. In any case, Jung and Stolz [**66, 95**] have shown that the existence of a positive scalar curvature metric depends only on $D([M]) \in ko_n(BG)$, and we write $ko_n^+(BG)$ for the subgroup of $ko_n(BG)$ consisting of indices of manifolds M admitting a metric of positive scalar curvature. The conjecture states that the containment

$$ko_n^+(BG) \subseteq \ker(Ap)$$

is actually an equality. To verify the conjecture for G, it suffices to find a small set of generators of $\ker(Ap)$ and realize them as indices of manifolds with psc metric.

Evidently it is essential to have a precise calculational understanding of the module $ko_*(BG)$, with as much additional structure as possible to minimize the difficult geometric constructions. As a result of our calculations it has been possible to verify the conjecture in a number of new cases, and details will appear elsewhere [**63, 78, 79**]. This gives a compelling reason for interest in $ko_*(BG)$ for discrete groups G; on the other hand the importance of characteristic classes of representations in giving comprehensible answers shows that we should consider compact Lie groups G as well.

1.2. Forms of K-theory

We found ourselves led into considering several different forms of K-theory, and it seems helpful to introduce them by explaining how they came in.

Periodic K-theory. A template for the project comes from Atiyah's periodic theories. Periodic complex K-theory is 2-periodic with coefficient ring

$$KU_* = \mathbb{Z}[v, v^{-1}],$$

where v is the Bott element of degree 2. Periodic real K-theory is 8-periodic, with coefficient ring

$$KO_* = \mathbb{Z}[\alpha, \beta, \beta^{-1}, \eta]/(2\eta, \eta^3, \alpha^2 = 4\beta)$$

where β is the degree 8 Bott element, α is of degree 4, and η is the image of the Hopf map in degree 1. The study of the periodic theories may be viewed as classical.

The connective theories are obtained by taking connective covers. These have coefficient rings

$$ku_* = \mathbb{Z}[v]$$

and

$$ko_* = \mathbb{Z}[\alpha, \beta, \eta]/(2\eta, \eta^3, \alpha^2 = 4\beta)$$

The complex case. In fact our earlier work [**28**] on ku was originally undertaken as a preliminary study, to see the likely shape of the answer, and to identify effective methods in a simpler case. This turned out to have been very wise, since phenomena familiar from the case of ku recur for ko, but often in a significantly more complicated form: it is doubtful that we would have found our way through without the experience gained by looking at ku. However we have been delighted to find that it is not just the methods, but also the calculations themselves that are useful.

***ko*-cohomology.** There is a Bockstein spectral sequence (see Chapter 4) which takes $ku^*(BG)$ as an input, and collapses at E_4 to give $ko^*(BG)$ as an output. Furthermore, the differentials have almost algebraic descriptions, so that it is usually a mechanical and combinatorial process to calculate them. Finally, the filtration on $ko^*(BG)$ only has three terms, and filtrations 1 and 2 lie in the $\mathbb{Z}$-torsion submodule of $ko^*(BG)$. Our experience is that the Bockstein spectral sequence is more effective than a direct Adams spectral sequence calculation of $ko^*(BG)$, especially when coupled with representation theoretic information.

***ko*-homology.** To obtain $ko_*(BG)$, there are two methods, each of which has been useful. On the one hand we may apply the local cohomology spectral sequence (see Chapter 2) to $ko^*(BG)$. On the other hand we may apply the Bockstein spectral sequence to $ku_*(BG)$. The local cohomology spectral sequence has the advantage that it displays the homotopy Gorenstein property of the cohomology ring. This implies a fascinating and precise connection between filtrations of representation rings and the action of the Steenrod algebra on ordinary group cohomology.

Equivariant forms. Anyone repeating our calculations will swiftly fall into the habit of using representation theoretic names for elements and making character theoretic calculations. This proves very effective, despite the fact that representation theory in $ko^*(BG)$ only enters through a completion. For example, the Atiyah-Segal completion theorem [**14**] shows

$$ko^0(BG) = KO^0(BG) = RO(G)^{\wedge}_J,$$

where $RO(G)$ is the real representation ring and $J = \ker(RO(G) \longrightarrow RO(1) = \mathbb{Z})$ is the augmentation ideal. The fact that uncompleted calculations are effective suggests that there is an uncompleted object to which they apply: in the periodic case, *equivariant* KO-theory gives the uncompleted version of the Borel theory, and in particular

$$KO^*(BG) = (KO^*_G)^{\wedge}_J.$$

Again KO^{-i}_G has the purely representation theoretic description in terms of the real, complex and symplectic representation rings:

$$\begin{aligned} &RO(G),\ RO(G)/RU(G),\ RU(G)/RSp(G),\ 0, \\ &\qquad\qquad\qquad RSp(G),\ RSp(G)/RU(G),\ RU(G)/RO(G),\ 0 \end{aligned}$$

for $i \equiv 0, 1, 2, 3, 4, 5, 6, 7 \bmod 8$, so we obtain the desired representation theoretic description.

To make the best use of our calculations we therefore begin by defining an appropriate equivariant form of ko-theory. Those averse to equivariance can ignore this and complete all calculations, but we believe that they will soon find this sufficiently tiresome to overcome the aversion. Although the equivariant form of ko sees representation theory in uncompleted form, the v-torsion is unchanged. Indeed, the only method we have for calculating this torsion in ko^*_G is to look in $ko^*(BG)$. Hence the necessity for even equivariant fanatics to resort to classical homotopy theory. Using the uncompleted form, we have maps

$$\begin{array}{ccc} ko^*_G & \longrightarrow & KO^*_G \\ \downarrow & & \downarrow \\ ko^*(BG) & \longrightarrow & KO^*(BG), \end{array}$$

and we prove that the square is a pullback (the analogous complex case was proved in [**46**]).

Relationship to periodic forms. In the complex case we found it useful to consider the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku_*^G \longrightarrow QU \longrightarrow 0$$

where

$$QU = \mathrm{im}(ku_*^G \longrightarrow KU_*^G)$$

and TU is the v-power torsion. Remarkably often QU is generated by $1, v$ and the Chern classes (so that it is the *modified Rees ring*) and TU is detected in ordinary cohomology. Our calculations in the real case show that something similar happens: it is useful to consider the exact sequence

$$0 \longrightarrow ST \longrightarrow ko_*^G \longrightarrow QO \longrightarrow 0$$

where

$$QO = \mathrm{im}(ko_*^G \longrightarrow KO_*^G)$$

and ST is the β-power torsion. Remarkably often QO is generated by elements in degrees ≥ 0 together with characteristic classes lifting to ko. Remarkably often ST is detected in ordinary cohomology. It is also sometimes convenient to consider a $\mathbb{Z}$-torsion free version of the representation theoretic part. For this we consider the exact sequence

$$0 \longrightarrow T \longrightarrow ko_*^G \longrightarrow \overline{QO} \longrightarrow 0$$

where

$$\overline{QO} = \mathrm{im}(ko_*^G \longrightarrow KU_*^G = RU(G)[v, v^{-1}]).$$

In particular, T also contains the submodule τ of η-multiples, and we have an exact sequence

$$0 \longrightarrow \tau \longrightarrow T \longrightarrow TO \longrightarrow 0,$$

and in all the examples we calculate, $TO = ST$.

Complex conjugation. In fact our calculations lead us one further step into the equivariant world. In studying the Bockstein spectral sequence we need to consider the action of complex conjugation on $ku^*(BG)$. In other words we are forced to consider equivariant K-theory with reality (in the sense of Atiyah [**12**]). The version of $K\mathbb{R}$ considered by Atiyah-Segal [**14**] is equivariant for a semidirect product $G \rtimes Q$ where Q is a group of order 2 corresponding to complex conjugation, but experience shows ([**37**], [**73**]) that the need to consider subgroups of $G \rtimes Q$ makes it desirable to allow non-split extensions $G.Q$ as in Karoubi [**68**]. We include a brief summary in a form convenient for our use.

Now that we have set the context, we describe our results in more detail.

1.3. The complex case

Before tackling the real case considered in this book, the reader should be generally familiar with the complex case as described in [**28**]; it should be sufficient to look at Pages 1-18. We summarize the contents briefly, so the reader knows where to look for further details.

In [**28**, Chapter 1] there are a number of general results giving a description of the coarser features of $ku^*(BG)$ as a ring in terms of the group theory of G. As might be expected, the answer is that the variety is a mixture of that of ordinary cohomology at various integer primes and that of K-theory. Quillen proved that the mod p cohomology ring has Krull dimension equal to the p-rank of G, whilst periodic K-theory is one dimensional. It follows that $ku^*(BG)$ has dimension equal to the rank of G if G is non-trivial, and the variety is formed by sticking the variety of $KU^*(BG)$ (with components corresponding to conjugacy classes of cyclic subgroups of G) to those of $H^*(BG;\mathbb{F}_p)$ (with components corresponding to conjugacy classes of maximal elementary abelian p-subgroups of G).

In [**28**, Chapter 2], there are calculations of the cohomology $ku^*(BG)$ as a ring for a number of groups of small rank. The additive structure largely follows from the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{*,*}(H^*(ku;\mathbb{F}_p), H^*(BG;\mathbb{F}_p)) \Rightarrow [BG_+, ku_p^\wedge]^* = ku^*(BG)_p^\wedge.$$

The mod p cohomology of ku is free over $\mathcal{A}//E(Q_0,Q_1)$, where $\mathcal{A}$ is the mod p Steenrod algebra and $E(Q_0,Q_1)$ is the exterior algebra on the Milnor primitives Q_0 and Q_1, the E_2-term, so the E_2-term can be calculated from $H^*(BG)$ as a module over $E(Q_0,Q_1)$. Amongst the rank 1 groups treated are the cyclic and generalized quaternion groups. Those of rank 2 are the Klein 4-group, the dihedral groups and the alternating group A_4.

For rank 1 groups, $ku^*(BG)$ is a subring of $KU^*(BG)$, so that once we have identified the generators of $ku^*(BG)$ using the Adams spectral sequence, we can use $RU(G)$ (via equivariant K-theory) to determine the relations. Among higher rank groups, the calculation is complicated by the presence of (p,v)-torsion; although the Adams spectral sequence is less effective, it often shows that there is no (v)-torsion in positive Adams filtration. Accordingly, ordinary mod p cohomology together with representation theory determines the multiplicative structure.

In any case, the outcome of this process gives a short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BG) \longrightarrow QU^\wedge \longrightarrow 0,$$

where QU is the image in periodic K-theory. The image QU contains the Modified Rees ring generated by $1, v$ and the Chern classes, and is often equal to it. The kernel TU is often detected in ordinary cohomology. The equivariant cohomology ring itself can then be calculated from the pullback square

$$\begin{array}{ccc} ku_G^* & \longrightarrow & KU_G^* \\ \downarrow & & \downarrow \\ ku^*(BG) & \longrightarrow & KU^*(BG), \end{array}$$

from [**46**]. This has the effect of replacing the exact sequence above by an uncompleted version

$$0 \longrightarrow TU \longrightarrow ku_G^* \longrightarrow QU \longrightarrow 0,$$

where QU is the image in periodic *equivariant* K-theory (the torsion part TU is unaffected).

In [**28**, Chapter 3] the homology $ku_*(BG)$ is deduced from the cohomology ring $ku^*(BG)$ using the local cohomology spectral sequence:

$$E_2^{*,*} = H_J^*(ku^*(BG)) \Rightarrow ku_*(BG),$$

where H_J^* is Grothendieck's local cohomology functor [**54**]. Because the local cohomology vanishes above the rank of the group, this is a finite spectral sequence. In fact, the appropriate commutative algebra shows that the $ku^*(BG)$ and $ku_*(BG)$

contain the same information up to duality. The local cohomology spectral sequence together with the universal coefficient theorem establishes a remarkable duality property of the ring $ku^*(BG)$. This is a somewhat more elaborate counterpart of the fact that for ordinary mod p cohomology the corresponding duality implies that a Cohen-Macaulay cohomology ring is automatically Gorenstein. The implications for Tate cohomology are also considered.

Finally, [**28**, Chapter 4] gives an account of the group theoretically benign case of elementary abelian 2-groups V. It is chastening that both $ku^*(BV)$ and $ku_*(BV)$ are nonetheless so intricate. Again we consider the extension

$$0 \longrightarrow TU \longrightarrow ku^*(BV) \longrightarrow QU^\wedge \longrightarrow 0$$

of $ku^*(BV)$-modules, where TU is the v-power torsion (in this case annihilated by (p, v)). We view TU as a subset of $H^*(BV; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \ldots, x_r]$; it is the PC-submodule generated by the elements $q_S = Q_1Q_0(\prod_{s\in S} x_s)$ with S a subset of $\{1, 2, \ldots, r\}$, where PC is the polynomial ring $\mathbb{F}_2[y_1, y_2, \ldots, y_r]$ generated by the Chern classes $y_i = x_i^2$. However, the commutative algebra of $ku^*(BV)$ is rather complicated. It turns out that there is a direct sum decomposition $TU = TU_2 \oplus TU_3 \oplus \cdots \oplus TU_r$, and TU_i is of dimension r and depth i. Remarkably, the local cohomology of TU_i is concentrated in degrees i and r. More startling still is the duality: the subquotients of $H^1_I(Q)$ under the 2-adic filtration are the modules $H^i_I(TU_i)$, and the differentials in the local cohomology spectral sequence give the isomorphism. Furthermore the top local cohomology groups pair up $H^r_I(TU_i)^\vee = \Sigma^{-r+4}TU_{r-i+2}$ for $i = 2, 3, \ldots, r-1$ (the exceptional behaviour of TU_r is exactly what is required to lead to a clean final duality statement): there is a natural exact sequence

$$0 \longrightarrow \Sigma^{-4}TU^\vee \longrightarrow ku_*(BV) \longrightarrow \mathbb{Z}[v] \oplus \Sigma^{-1}(2^{r-1}H^1_I(QU)) \longrightarrow 0.$$

After this summary of results for complex connective K-theory, we may turn to the contents of the present book.

1.4. Highlights of Chapter 2

Since we need to consider both real and complex K-theory, we are led to K-theory with reality in the sense of Atiyah. We are really concerned with the K-theory of classifying spaces, but as exemplified by the Atiyah-Segal completion theorem, these are best understood through the representation theory of G. We therefore begin in Chapter 2 by describing the relevant background in representation theory. We want to understand the ring and module structure of KO, so we need a little more detail in the theory of Clifford algebras than is explicit in the literature, so we spend some time describing the theory in a convenient and explicit form. Nonetheless, the periodic theory is well-known, and we are just giving a summary. When it comes to the connective forms, the second author has given a construction in the complex case [**46**], and we present here the obvious counterpart in the real case. Fausk [**38**] has pointed out that there is a t-structure for which this is indeed the connective cover, thereby making the construction seem more natural. In fact we go a little further by generalizing to the case with reality.

This gives the entire array of theories we need. The construction provides a Hasse square for calculating the coefficients, and by using characteristic classes we can show that these coefficient rings are Noetherian. Having written down the equivariant forms, we need to show that their relationship to the homology and cohomology of classifying spaces is as expected, and we do this by proving a completion theorem and a local cohomology theorem for the theory. This is ideally suited to the study of manifolds with positive scalar curvature, since the local cohomology calculation of homology highlights exactly the subgroup one needs to realize to prove the GLR conjecture.

Finally, we end the chapter by showing that for $i \geq -7$, there is a representation theoretic description of ko_i^G. We give examples to show that this regularity vanishes in lower degrees. These results are proved by using the classical Atiyah-Hirzebruch spectral sequence together with the generalization to ko of Bott's original sequence of Ω-spaces defining the spectrum KO.

1.5. Highlights of Chapter 3

In Chapter 3 we turn to twisting, periodicity and descent. In the periodic case, the behaviour was described by Atiyah in terms of his K-theory with reality. Continuing with our convention that Q is the group of order 2 giving complex conjugation, all theories are 8-fold periodic, and K-theory with reality is periodic under suspension by $\mathbb{C} = 1 \oplus \xi$, where ξ is the non-trivial representation of the group Q of order 2 on the real line; this is equivalent to a 4ξ-periodicity equivalence $\Sigma^{4\xi} K\mathbb{R} \simeq \Sigma^4 K\mathbb{R}$. The twisting by other representations give different theories, but they are well understood. When it comes to describing the relationship between real and complex K-theory, the descent theorem is familiar in the periodic case. It states that the homotopy fixed points of complex K-theory under complex conjugation is real K-theory so that there is a spectral sequence $H^*(C_2; KU^*(X)) \Rightarrow KO^*(X)$. We give a brief summary of this behaviour.

The situation in the connective case, for twisting, periodicity and descent is considerably more complicated, and we spend some time making it explicit. It turns out that $k\mathbb{R}$ is connectively 8-fold periodic, but not quite 4ξ-periodic. On the other hand, the Borel theory is 4ξ-periodic, so it is still quite practical to make calculations. One of the most useful facts for us is that there is an equivalence

$$(S^{-\xi} \wedge k\mathbb{R})^Q \simeq \Sigma ko$$

even though the corresponding statement is false before passage to fixed points (note the negative suspension on the left and the positive suspension on the right). We provide a chart of the entire $RO(Q)$-graded homotopy of $k\mathbb{R}$, and describe all twists of the descent spectral sequence. It is illuminating to see how these periodicities emerge from successive truncations of the descent spectral sequence.

1.6. Highlights of Chapter 4

The results of Chapter 3 show that the descent spectral sequence does not give ko from ku. It turns out that this is really because the wrong truncation was chosen in moving from the periodic case. Indeed, the η-Bockstein spectral sequence does give a method, taking the form

$$E^1_{*,*} = ku_*(X)[\tilde{\eta}] \Rightarrow ko_*(X);$$

we describe it in detail in Chapter 4. Perhaps the most familiar construction begins from a cofibre sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku,$$

but after the construction through K-theory with reality, it is in some ways more natural to give an equivariant approach through the cofibre sequence

$$F(S^{\xi}, k\mathbb{R}) \longrightarrow k\mathbb{R} \longrightarrow F(Q_{+}, k\mathbb{R})$$

of Q-spectra. We describe both approaches and show that they agree up to the expected completion, both in the connective and periodic cases. All of this has a G-equivariant version.

In any case, since $\eta^3 = 0$ in K-theory, the Bockstein spectral sequences all collapse at the E_4-page. If we apply the spectral sequences for the periodic theories to the spaces BG, the complete behaviour of the spectral sequence is known. This gives a convenient packaging of the relationship between self-conjugate and anti-self-conjugate K-theory on the one hand, and real, complex and quaternionic K-theory on the other. The fact that this spectral sequence agrees with the descent spectral sequence in the periodic case is also interesting; for example, the fact that the descent spectral sequence collapses at E_4 is not obvious a priori from its construction.

Turning to the connective case, since the construction is compatible with the Bott element, the periodic case gives extensive information about the connective case. In particular, we may consider the exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*_G \longrightarrow QU \longrightarrow 0$$

as an exact sequence of chain complexes under d^1. The known behaviour in the periodic case means that the d^1 differential on the QU part is $1 \pm \tau$, with the sign depending on the degree mod 4. When TU is detected by mod 2 cohomology, d^1 is given by Sq^2 on the TU part. The connecting homomorphism appears on the charts as a d^1, and we typically display Bockstein spectral sequence charts at the $E^{1\frac{1}{2}}$-page (i.e., after the d^1 on the TU and QU parts, but before the connecting homomorphism has been taken into account as a late d^1).

1.7. Highlights of Chapter 5

Our primary focus is on finite groups. Nonetheless, when calculating group cohomology, characteristic classes of representations are extremely valuable. Since representation theory is relatively easy to understand, characteristic classes give a large collection of cohomology classes we can name and manipulate. In particular, we can follow them through change of groups homomorphisms and changes of cohomology theory. In the best behaved cases (exemplified by Chern classes in complex oriented theories) we obtain formulae for the values on sums of representations, or even for tensor products. Of course the basis of this is knowing the values of a cohomology theory on the classical groups, and obtaining universal formulae for the structure maps.

Accordingly, we spend Chapter 5 studying ko^*_G for the groups $O(n)$, $U(n)$ and $Sp(n)$, which serve as universal examples for characteristic classes of real, complex and quaternionic representations. The Yoneda Lemma tells us that the value of $ko^*_{G(n)}$ or $ku^*_{G(n)}$ gives all characteristic classes in ko^* or ku^* for $G(n)$ representations.

The symplectic groups give Pontrjagin classes: these are the best behaved, and we have a complete calculation. Indeed,

$$ko^*_{Sp(1)^n} = ko^*[z_1, \ldots, z_n]$$

where z_i is the first Pontrjagin class of the natural representation of the ith factor, and

$$ko^*_{Sp(n)} = ko^*[z_1, \ldots, z_n]^{\Sigma_n} = ko^*[p_1, \ldots, p_n].$$

This allows us to define Pontrjagin classes of symplectic representations and gives a symplectic splitting principle. Accordingly, we obtain numerous useful formulae and we can easily relate the classes to counterparts in periodic theories and in representation theory.

We have made one departure from convention in our definition of Pontrjagin classes. Classically, $p_i \mapsto \sigma_i(y_1^2, \ldots, y_n^2)$ under the restriction $H^*(BSp(n); \mathbb{Z}) \longrightarrow H^*(BT^n; \mathbb{Z})$ to the maximal torus. Noting that this factors through restriction to $Sp(1)^n$, and that the generators z_i of $E^*Sp(1)^n$ (for complex oriented E) map to $y_i[-1](y_i)$ we contend that, in general, we should have

$$p_i \mapsto \sigma_i(y_1[-1](y_1), \ldots, y_n[-1](y_n)).$$

For cohomology, this specializes to $p_i \mapsto \sigma_i(-y_1^2, \ldots, -y_n^2)$ so that our p_i is the traditional $(-1)^i p_i$. This simplifies a number of formulae and has the merit of being the $v = 0$ case of a formula which has clear meaning in terms of the formal group law for K-theory.

The unitary groups are well behaved for ku, but have quite complicated relations and torsion for ko. The orthogonal groups are more complicated, and we only give complete calculations in small dimensions.

1.8. Highlights of Chapter 6

In Chapter 6 we make the first calculations of $ko^*(BG)$ for finite groups G. Naturally we begin with the easiest groups, which are those of rank 1, namely cyclic and generalized quaternion, together with the Klein 4-group and A_4. Even in these cases, there is some complexity.

Our method has already been layed out. We take ku^*_G from [**28**] or elsewhere, and use the Bockstein spectral sequence. Working from the exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*_G \longrightarrow QU \longrightarrow 0,$$

we must find the E^2-term by calculating d^1. The work falls naturally into two parts. First, a calculation of the self-conjugate (QU_{sc}) and anti-self-conjugate (QU_{asc}) parts of QU, together with $H = QU_{sc}/v^{-1}(1+\tau)QU$ gives the representation theoretic contibution. Second, a calculation of the Sq^2-cycles and homology of TU gives the torsion contribution. The connecting homomorphism may be treated as a late d^1 in the spectral sequence.

Knowing the answer in positive degrees agrees with KO^*_G with a known representation theoretic description, we may usually deduce the remaining differentials (since d^2 and d^3 must cancel everything above the 3-line). Finally, we attempt to give a representation theoretic description of the d^2 and d^3-cycles in QU_{sc}: in many cases (all except A_4 in this chapter) we observe it is generated by Chern and and Pontrjagin classes.

In any case, we describe our conclusions in terms of the short exact sequence

$$0 \longrightarrow ST \longrightarrow ko^*_G \longrightarrow QO \longrightarrow 0.$$

Of course QO agrees with KO^G_* in degrees ≥ 0. In negative degrees, the torsion free quotient is often generated by characteristic classes of self-conjugate representations in degrees which are multiples of 4, together with anti-self-conjugate classes in degrees which are 2 mod 4. The η-multiples are eventually 8-periodic, but may or may not be bounded below. The torsion TO is zero above degree -8 and it is elementary abelian in our examples, with growth equal to the rank of the group.

1.9. Highlights of Chapter 7

In Chapter 7 we apply the local cohomology spectral sequence to determine $ko_*(BG)$ for the same collection of groups as in Chapter 6. This takes the form

$$H_I^*(ko^*(BG)) \Rightarrow ko_*(BG)$$

where H_I^* denotes Grothendieck's local cohomology with respect to the augmentation ideal $I = \ker(ko^*(BG) \longrightarrow ko^*)$. Because the local cohomology vanishes above the rank of the group, this is a finite spectral sequence.

The first step here is to calculate the local cohomology modules. Using the exact sequences

$$0 \longrightarrow T \longrightarrow ko_G^* \longrightarrow \overline{QO} \longrightarrow 0$$

and

$$0 \longrightarrow \tau \longrightarrow T \longrightarrow TO \longrightarrow 0,$$

this is broken into three parts: calculations for TO, for τ and for $\overline{QO}$. The calculation of $H_I^*(TO)$ tends to be straightforward, since TO is finite over a polynomial subring of $H^*(BG;\mathbb{F}_2)$, often one generated by Pontrjagin classes: by observation TO is always of depth ≥ 2, so the local cohomology is in degrees 2 and above.

If τ is bounded below, then $H_I^*(\tau) = H_I^0(\tau) = \tau$. If τ is unbounded, its local cohomology is in degrees 0 and 1. In large degrees, it is very regular, but there are often edge effects near degree 0. Calculations can be made with characters, using the connection between QO and representation theory.

Finally, there is $H_I^*(\overline{QO})$, which tends to be the most laborious. First, it is easy to see that $H_I^0(\overline{QO})$ consists of characters supported at the identity, which is therefore $\mathbb{Z}$ in degrees divisible by 4 and ≥ 0. Next, we have found in every example we have calculated that the augmentation ideal is the radical of a principal ideal (p_*), so that local cohomology may be calculated using the ideal (p_*). Very often p_* can be taken to be the Pontrjagin class of some representation of G, and often its degree d is 4 or 8. Accordingly local cohomology vanishes above cohomological degree 1 and we have an exact sequence

$$0 \longrightarrow H_I^0(\overline{QO}) \longrightarrow \overline{QO} \longrightarrow \overline{QO}[\frac{1}{p_*}] \longrightarrow H_I^1(\overline{QO}) \longrightarrow 0.$$

One finds that multiplication by p_* becomes an isomorphism $\overline{QO}_n \longrightarrow \overline{QO}_{n-d}$ for sufficiently negative n, giving a calculation of $\overline{QO}[\frac{1}{p_*}]$. Thus, to calculate local cohomology in degree n, we choose k large enough that $p_*^k : \overline{QO}_n \longrightarrow \overline{QO}_{n-kd}$ has codomain at the stable value of $\overline{QO}[\frac{1}{p_*}]_n$ in this congruence class mod d, and then uses

$$H_I^1(\overline{QO})_n = \mathrm{cok}(p_*^k : \overline{QO}_n \longrightarrow \overline{QO}_{n-kd}).$$

This gives the E_2-term of the local cohomology spectral sequence. In rank 1 there are no differentials (except for the d^1 corresponding to the connecting homomorphism), and in rank 2 there is only one. This is very often forced by the fact that $ko_*(BG)$ is connective, together with the module structure.

In very general terms, all the answers have the following form. The torsion free part consists of a copy of $\mathbb{Z}$ in each degree $4k \geq 0$. Corresponding to the η-multiples in cohomology there is a copy of RO/RU in large degrees 1 and 2 mod 8 (and a subgroup of this in low degrees), and similarly a copy of RSp/RU in large degrees 5 and 6 mod 8 (and a subgroup of it in low degrees). Next, in odd degrees there is a sum of cyclic groups in odd degrees with orders increasing as the degree increases by 8. In degrees $4k-1$ the number of factors is one less than the rank of RU_{sc}, and in degrees $4k-3$ the number of factors is the rank of RU_{asc}. Finally, there is a an elementary abelian summand, with growth rate equal to the rank of the group.

1.10. Highlights of Chapter 8

In Chapter 8 we take a step into a more complicated realm by moving to an infinite class of groups of rank more than 1: we apply our methods to calculate $ko^*(BD)$ and $ko_*(BD)$ where D is a dihedral 2-group. It is convenient to relate these to the answers for the group $O(2)$, because it neatly encodes the maps induced by the inclusions of groups, and because it highlights the role of representation theory and characteristic classes. We used a more ad hoc basis in [**28**], so the present approach to $ku^*(BD)$ and $ku_*(BD)$ gives new information.

From $ku^*(BD)$ we deduce $ko^*(BD)$ using the Bockstein spectral sequence, but there is a single additive extension in degree -6 for which we needed to appeal directly to the Adams spectral sequence. We then apply the local cohomology theorem to give $ku_*(BD)$ and $ko_*(BD)$. In the complex case, we have short exact sequences

$$0 \longrightarrow TU \longrightarrow ku^*_D \longrightarrow QU \longrightarrow 0$$

where QU is the Modified Rees ring and TU is a free module over the double of $H^*(BD;\mathbb{F}_2)$ on a class of degree -6. Similarly in the real case we have a short exact sequence

$$0 \longrightarrow TO \longrightarrow ko^*_D \longrightarrow QO \longrightarrow 0$$

so that $ST = TO$. It turns out that in this case, TO is a proper submodule of the Sq^2-cyles ZTU; indeed there is a short exact sequence

$$0 \longrightarrow TO \longrightarrow ZTU \longrightarrow \Sigma^{-12}\mathbb{F}_2[p_2] \longrightarrow 0.$$

We have $QO = \overline{QO} \oplus \tau$ and it is notable that the group τ of η-multiples is unbounded. Since the size of TU is independent of the order of D, it is the representation theory that is hardest to understand explicitly. In cohomology this means we do not give the Chern filtration precisely, but in homology it means that, even for the dihedral group of order 8, we only give the order of $ko_i(BD)$ if i is 3 or 7 mod 8, and do not determine the exact structure.

1.11. Highlights for elementary abelian groups

Finally, we devote four entire chapters (Chapters 9 to 12) to elementary abelian 2-groups $V(r)$, where r is the rank. This may seem excessive, but they give the simplest family of increasing rank, and therefore provide an essential test of our methods as well as a basis for future calculations. In fact, we give a complete calculation of the ring $ko^*(BV(r))$ and the module $ko_*(BV(r))$. Additively the homology calculation is due to Cherng Yih Yu [**99**], but our method gives more information about naturality and the module structure. Those familiar with the complex case from Chapter 4 of [**28**] will be prepared for some complexity, but the intricate additional complications in the real case may still be some surprise.

On the other hand, the entire calculation is done from the complex case purely algebraically. Indeed, we begin with the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BV(r)) \longrightarrow QU^\wedge \longrightarrow 0,$$

where QU is the Rees ring, and $TU \subseteq H^*(BV(r))$ is a submodule over the polynomial ring PC on Chern classes, understood in detail from [**28**]. No other input is required, though this relies on some numerical good fortune.

1.11.A. Highlights of Chapter 9. In Chapter 9 we use the Bockstein spectral sequence to deduce $ko^*(BV(r))$ from $ku^*(BV(r))$: perhaps surprisingly, it is the combinatorial complexity of the action of Sq^2 on TU that needs most work, and the details necessary to give precise numerical values for dimensions are not worked through until Chapter 11. The chapter ends with Bockstein spectral sequence charts for ranks ≤ 8, but the summary of general conclusions in Section 9.12 may be of particular interest. In effect we identify the exact sequence

$$0 \longrightarrow ST \longrightarrow ko_*^{V(r)} \longrightarrow QO \longrightarrow 0.$$

We find in this case that $QO \cong \overline{QO} \oplus \tau$, and $ST = TO = ZTU = \ker(Sq^2 : TU \longrightarrow TU)$, where τ consists of the η-multiples (and is bounded below by roughly $-2r$) and $\overline{QO}$ is generated by $1, v$, the Pontrjagin classes and JU^4. It is notable that $ko^G_{-8} = JU^4$ and $ko^G_{-9} = 0$, which contrasts with the fact that in the complex case ku^G_{-6} has torsion for $r \geq 2$ and $ku^G_{-7} \neq 0$ for $r \geq 3$.

1.11.B. Highlights of Chapter 10. In Chapter 10 we apply the Bockstein spectral sequence to deduce $ko_*(BV(r))$ from $ku_*(BV(r))$ (the spectral sequences are displayed in a range for $r \leq 6$). It is rather surprising that it is possible to understand the entire behaviour, and indeed, we only came to this method at a late stage because we incorrectly expected it to be impractical. It is only possible because of the detailed structural information we have about the complex case. In fact the additive extensions in degrees congruent to 3 mod 8 involve special consideration, and we only resolve them by using the local cohomology spectral sequence and some arithmetic good fortune. Once again the answers are made numerically explicit by appeal to Chapter 11. In the complex case we have the exact sequence

$$0 \longrightarrow \mathrm{Start}(2)TU^\vee \longrightarrow \widetilde{ku}_*(BV(r)) \longrightarrow \mathrm{Start}(1)2^{r-1}H^1_I(QU) \longrightarrow 0$$

In the real case it is a little more complicated, but we do have an isomorphism

$$ko_*(BV(r))/\eta \cong ku_{\equiv 0(4)} \oplus D_{\equiv 1(8)} \oplus D_{\equiv 2(8)} \oplus \mathrm{Start}(2)(TU/Sq^2)^\vee \oplus \\ \mathrm{Start}(1)2^{r-1}H^1_I(QU)_{\equiv 0(4)},$$

where D is concentrated in degrees 1 and 2 mod 8, where it is elementary abelian and of rank $2^r - 1$ in large degrees. The η-multiples are also identified explicitly.

1.11.C. Highlights of Chapter 11. Chapter 11 is a combination of combinatorics and commutative algebra. It makes explicit how Sq^2 acts on TU, and shows how it is controlled by the combinatorics of pairs of subsets of $\{1, 2, \ldots, r\}$. In order to understand the Hilbert series explicitly, and to make calculations of local cohomology possible, we also give explicit resolutions of $TO = ZTU = \ker(Sq^2 : TU \longrightarrow TU)$ as modules over the polynomial algebra PP on the Pontrjagin classes. The process by which these are formed, starting from the PC-resolutions of TU, and putting a compatible action of Sq^2, passing to cycles and then adding generators to remove homology may be of some interest. Due to the combinatorial intricacy we will not attempt a detailed summary here.

1.11.D. Highlights of Chapter 12. Finally, in Chapter 12 we calculate $ko_*(BV(r))$ using the local cohomology spectral sequence for $r = 1, 2, 3$ and 4. This alternative calculation gives additional information, and makes certain duality properties explicit. Unfortunately it seems unlikely this method can be extended to arbitrary rank: although Chapter 11 reduces the local cohomology calculation to combinatorics and linear algebra, it still involves calculating with rather large matrices with entries in the polynomial ring PP. In rank 3 the calculation could be done by hand, but computer assistance seems essential even in rank 4, and the complexity it reveals suggests this is unavoidable. On the other hand, the calculations do reveal striking behaviour: firstly the E^∞ term is concentrated on the H^0_I, H^1_I and H^r_I columns, and secondly almost all differentials come from the H^1_I column, with the finitely many exceptions coming from the H^0_I column. Furthermore the differentials from the H^1_I all seem to be of maximal rank. This conclusion shows $ko_*(BV(r))$ has a filtration of length 2: the top quotient comes from the coefficients, the middle section is purely representation theoretic and the bottom piece comes from mod 2 homology. The last piece is by far the most complicated, and is the only piece really new in rank r, but the relationship to mod 2 homology makes it rather accessible. The way in which differentials are forced by the bare requirement that $ko_*(BV(r))$ is connective is astonishing, and the required cancellation establishes remarkable relationships between local cohomology groups.

1.12. Conclusions

The specific calculations feeding into other applications, and the specific structural results are discussed at the appropriate point. Here, we record the general conclusion, that $ku^*(BG), ku_*(BG), ko^*(BG)$ and $ko_*(BG)$ are remarkably computable. Indeed, providing both the group cohomology and representation theory are accessible, there is a good chance of a complete calculation of all four.

Perhaps more interesting is that in most cases the only homotopical input required is $ku^*(BG)$, with the rest being encoded by Sq^2 alone, together with representation theory and commutative algebra. As for methods of calculation for $ko_*(BG)$, it seems that for groups of small rank it is best to go via $ko^*(BG)$, but that for larger rank it is best to go via $ku_*(BG)$.

More striking perhaps are the remarkable phenomena that the calculations illustrate. Firstly, we were constantly struck by the intimacy of the relationship between ordinary cohomology and representation theory that is necessary for the calculations to be consistent. Secondly, the Gorenstein-type duality given by combining the local cohomology theorem with the Universal Coefficient Theorem continues to be very striking for ko.

1.13. History and comparisons with other methods

The following historical remarks are intended to give a sense of the way in which the subject has developed, but are in no sense a comprehensive survey of all relevant work. Notably omitted is the interesting work on operations and cooperations in connective K-theories. Our goal here is simply to put the present volume into context.

Historically, the first substantial calculation involving connective K-theory was Adams's 1961 determination of $H^*(ku)$ in his work on the integrality of Chern classes [**1**], although this was formulated as the calculation of the cohomology of Postnikov covers of BU in the stable range. In 1963, Stong [**97**] completed the calculation and also calculated the cohomology of the Postnikov covers of BO. At this point, it would have been possible to make easy Adams spectral sequence calculations of connective real and complex K-theory in a number of simple cases. Nonetheless, the first explicit appearance of this method came in Don Davis's Stanford thesis [**31, 32**]. Davis credits Milgram with having explained to him the relevance of Stong's results to the calculation of connective K-theory [**33**]. Although no mention of connective K-theory can be found there, the 1968 paper by Gitler, Mahowald and Milgram [**40**] contains Adams spectral sequence charts for the real connective K-theory of stunted projective spaces. Mahowald [**76**] said that "By the late 60s this sort of thing was well known." However, the absence of any explicit reference in the literature to Adams spectral sequence methods for calculating connective K-theory until [**31, 32, 77**] suggests that it was only well known to a small circle of experts.

The rest of the work involving connective K-theory until the middle 1970s was concerned with the relation to complex cobordism and the beginnings of what we now recognize as chromatic phenomena, as in the work of Conner and Floyd [**29**], Novikov [**86**], Conner and Smith [**30**], Smith [**94**] and Johnson and Smith [**64**].

In the 1970s and into the 1980s quite a lot of work appeared using connective K-theory to study the immersion problem for projective spaces. This involved computing, among other things, $ko_*(BC_2)$ and $ko_*(BC_2 \times BC_2)$, as well as the ko homology of subquotients of these classifying spaces. Results in the odd primary case at around this time were proved by Hashimoto [**55**]. In this period, homology theories had come to be accepted as the preferred algebraic context, in order to avoid the complications arising from the inevitable completions and topological algebra occuring in the generalized cohomology of infinite complexes. (Adams, [**2**, pp. 50-55], gives a very nice account of the reasons for this switch in emphasis.) An exception to this preference for ko_* over ko^* is in the 1981 work of Kane [**67**]. Another, in 1988, was Ossa's [**87**] calculation of $ku^*(BV)$ and $ko^*(BV)$, where V is elementary abelian, using the Adams spectral sequence, the earlier calculations for cyclic groups and the relation to representation theory.

In the early 1990s, the role of $ko_*(BG)$ as the recipient of the $\widehat{A}$-genus obstruction to the existence of metrics of positive scalar curvature (Sections 1.1 and 2.7) brought the problem of calculating $ko_*(BG)$ for many more groups to the attention of the first author. The work by his student Bayen [**17**] started by calculating $ko_*(BG)$ using the Adams spectral sequence. However, they soon realized [**18**] that the ring structure of $ko^*(BG)$ and $ku^*(BG)$ was crucial. In particular, for periodic groups, the Tate theory and the ko and ku homology are straightforward to deduce from the cohomology by periodicity, while the cohomologies $ko^*(BG)$ and $ku^*(BG)$ can be calculated by using the ring structure to propagate the relations deduced from periodic K-theory. For non-periodic groups, once the ring structure is known, we may use the local cohomology spectral sequence to give a more structured understanding of the homology groups $ko_*(BG)$ and $ku_*(BG)$. The Adams spectral sequence alone gives very little information about the ring structure, since the relations involve terms of mixed Adams filtration in all but the simplest cases. However, the Adams spectral sequence often implies that the crucial relations occur in dimensions in which the Bott map acts monomorphically and this shows that the relations are accessible through the representation rings [**28**]. This gives us the ability to determine the exact ring structure, and often provides the simplest derivation of Adams spectral sequence differentials as well. Already for C_{p^2}, D_8 or Q_8 there are multiplicative extensions which are obscure in the Adams spectral sequence [**28**]. Even for a group as simple as Q_{16}, there are additive extensions which are unclear in the Adams spectral sequence.

In the end, representation theoretic means have supplanted the use of the Adams spectral sequence in most parts of our work. However, the Adams spectral sequence still remains a useful way to do some accounting: it can tell us when we have enough generators and what sort of relations to look for. It also provides a different filtration than either the Bockstein spectral sequence or the local cohomology spectral sequence, so can help resolve extensions which are obscure in one or another of these. As is always the case, having multiple methods of getting information about the objects of interest gives a check on each and together they can give more information than any one of them alone; we have corrected many errors by these comparisons.

In the present century, more work has used equivariant methods, for example in the work of the second author [**45, 46**] and in the present volume. It is worth also noting Dugger's equivariant Atiyah-Hirzebruch spectral sequence [**35**].

Finally, we want to point out that, conceptually, the Atiyah-Hirzebruch and Bockstein spectral sequences can be seen as associated graded versions of the Adams spectral sequence, drawing in the structure in stages, rather than all at once, as the Adams spectral sequence does. To be precise, we can think of the passage from mod 2 cohomology to ko as taking place by three Bockstein spectral sequences:

(i) the Q_0, or p-Bockstein spectral sequence, computes integral cohomology,
(ii) the Q_1, or v-Bockstein spectral sequence, then gives complex connective K-theory, and finally
(iii) the Sq^2, cR or η-Bockstein spectral sequence then gives real connective K-theory.

If we combine the first two steps, we have the Adams spectral sequence proceeding from the mod 2 cohomology as a module over $E(Q_0, Q_1)$ directly to ku theory, while if we combine all three, we use the $\mathcal{A}(1)$-module structure to go directly to ko theory. It is no surprise that all these approaches taken together yield better information than any one alone can.

1.14. Comparisons with other theories

One might also compare the calculations of the connective K-theory of BG with what we know for other theories. The simplest answers are for periodic K-theory in the real and complex form. There one has a purely algebraic answer in terms of representation theory. The connective theories are a hybrid of this and the less predictable ordinary cohomology, an infinitesimal extension of the modified Rees ring.

The next comparison is with bordism. On the one hand connective K-theory is Noetherian and of finite Krull dimension, but on the other it usually has $\mathbb{Z}$-torsion and nilpotents. This should be contrasted with bordism, which is not Noetherian, but is often $\mathbb{Z}$-torsion free.

The usual approach to get better approximations to bordism whilst keeping Noetherian rings is to use higher chromatic approximations. The periodic forms are the Johnson-Wilson theories $E(n)$ or their completed Lubin-Tate counterparts. The results of Hopkins-Kuhn-Ravenel [**57, 58**] give a version of character theory and suggest one may hope for a higher version of representation theory. As for K-theory, one would expect this to come from an equivariant version of the theory. In higher Krull dimensions, there are many connective theories with the same periodic theory and one needs to understand the geometry to see which connective form is appropriate. In any case, at the superficial level one will again expect a hybrid of the representation theory and cohomology, perhaps again an infinitesimal extension of a modified Rees ring. Work of Strickland [**98**] on $BP\langle n\rangle$ for abelian groups already shows the complexity one may expect.

In forming K-theory with reality we used the group Q of order 2 since the only global automorphism of the multiplicative group is negation. For higher chromatic levels, this still gives an automorphism, so one may again seek versions with reality. This was done by Landweber [**74**] and Araki [**8, 9**] for bordism. Hu and Kriz [**60, 61**] used this to introduce real oriented versions of the Johnson-Wilson theories and Kitchloo and Wilson have investigated them further [**70, 71, 72**]. For completed or localized theories we expect more automorphisms of the associated geometry, and hence a variety of analogues of the real form.

As usual in higher dimensions, one cannot proceed by simple analogy: one needs to be guided by a detailed geometric understanding. Accordingly, the correct picture is only likely to emerge when we are able to follow the pattern established by Hopkins, Lurie and others in the construction of tmf. The complex periodic form comes from some family of geometric objects, which gives a moduli problem, and one hopes to construct a moduli stack in the world of E_∞-ringed spaces. In the case of K-theory there is only one multiplicative group, corresponding to KU. By considering abelian compact Lie groups one has a model for the abelian equivariant theory. From these one assembles a general equivariant theory from the category of subgroups. In the case of K-theory this gives the familiar equivariant KU. Finally, one has the global sections of this E_∞-ringed moduli stack. In the case of K-theory this gives KO. The connective versions are obtained by careful examination of the geometry. Thus the meaning of 'reality' will depend on the automorphisms of various objects, and the meaning of 'connective' on the geometry of the moduli stack.

In any case, one may hope that one may be able to adapt the methods of this book to extend the work of Hill [**56**] on $tmf^*(B\Sigma_3)$ a little further, though the complexity of the calculation of the coefficient ring of tmf shows that this will be a major undertaking.

1.15. Prerequisites

Although we have made some effort to summarize things about K-theory that we need, we have assumed that the reader is familiar with classifying spaces of compact Lie groups and the usual tools of ordinary cohomology, including cohomology operations and the Steenrod algebra. Similarly, we have assumed the reader is familiar with basic representation theory of finite groups in characteristic zero, has some knowledge of the classical Lie groups. We use a little of the language of commutative algebra, and we assume the reader is fluent with the language of spectral sequences. This standard material is available in many places. May's "Concise Course" [**82**] is a convenient source for the algebraic topology, Serre's book [**93**] on representation theory, Bröcker and tom Dieck [**22**] for the structure and representation theory of compact Lie groups. For commutative algebra we recommend the books of Matsumura [**81**] and Bruns and Herzog [**26**], and on spectral sequences McCleary's book [**84**].

Going further, there are several places where we have used stable homotopy theory in the sense that cohomology theories are represented. We have also used the Adams spectral sequence as a tool for calculating connective K-theory. All this is well covered in Adams's Chicago Lectures [**3**]. There are many useful techniques for using and displaying the Adams spectral sequence; the first author is preparing a systematic account [**25**], but for the present it is necessary to explain as we go along. In particular, we often need to discuss modules over the $\mathcal{A}(1) = \langle Sq^1, Sq^2 \rangle$ and the exterior algebra $E(1)$, and we summarize the essentials in Appendix A.

There are also a few places where we have used the technology of highly structured ring spectra. The main point is that this gives a triangulated homotopy category of modules over a commutative ring spectrum with a symmetric monoidal tensor product. The second author has attempted to give an introduction to this in [**48**], but the original sources of [**36, 59**] should be consulted for details.

1.16. Reading this book

Assuming the prerequisites described above and some familiarity with the case of *complex* connective K-theory, the reader should read Chapter 2 to set the context. It is also almost essential to be familiar with the Bockstein spectral sequence as described in Chapter 4. Apart from this, the reader should be able to select the calculations of interest. In particular, it is not necessary to read Chapter 5 on characteristic classes before reading about the calculations for finite groups, though its contents permit a more coherent overview.

For calculations of ko homology or cohomology, the reader should first be familiar with the corresponding calculation with ku. For calculations of ko-homology the reader should look at the corresponding calculation of ko-cohomology first. It would be wise do look first at Chapter 6 to see how things work out in fairly simple examples.

Many readers interested in elementary abelian groups will not need to read Chapter 12, and readers only interested in small ranks or low degrees may be happy to check results from first principles rather than read Chapter 11 in detail.

In some of the diagrams, there is more information than fits easily into the space available. Rather than omit large numbers, we have sometimes left them in, hoping that the fact that they overflow the bounds of boxes will be a vivid indication of their size. Unfortunately this means that beauty and readability are sometimes compromised.

1.17. Thanks

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