# Preface

Random matrices is an active field of mathematics and physics. Initiated in the 1920s–1930s by statisticians and introduced in physics in the 1950s–1960s by Wigner and Dyson, the field, after about two decades of the "normal science" development restricted mainly to nuclear physics, has became very active since the end of the 1970s under the flow of accelerating impulses from quantum field theory, quantum mechanics (quantum chaos), statistical mechanics, and condensed matter theory in physics, probability theory, statistics, combinatorics, operator theory, number theory, and theoretical computer science in mathematics, and also telecommunication theory, qualitative finances, structural mechanics, etc. In addition to its mathematical richness random matrix theory was successful in describing various phenomena of these fields, providing them with new concepts, techniques, and results.

Random matrices in statistics have arisen as sample covariance matrices and have provided unbiased estimators for the population covariance matrices. About twenty years later physicists began to use random matrices in order to model the energy spectra of complex quantum systems and later the systems with complex dynamics. These, probabilistic and spectral, aspects have been widely represented and quite important in random matrix theory until the present flourishing state of the theory and its applications to a wide variety of seemingly unrelated domains, ranging from room acoustics and financial markets to zeros of the Riemann $\zeta$-function.

One more aspect of the theory concerns integrals over matrix measures defined on various sets of matrices of an arbitrary (mostly large) dimension. Matrix integrals proved to be partition functions of models of quantum field theory and statistical mechanics and generating functions of numerical characteristics of combinatorial and topological objects; they satisfy certain finite-difference and differential identities connected to many important integrable systems. However, the matrix integrals themselves, their dependence on parameters, etc., can often be interpreted in spectral terms related to random matrices whose probability law is a matrix measure in the integral.

Thus, random matrix theory can be viewed as a branch of random spectral theory, dealing with situations where operators involved are rather complex and one has to resort to their probabilistic description. It is worth noting that approximately at the same time as Wigner and Dyson, i.e., in the 1950s, Anderson, Dyson, and Lifshitz proposed to use finite-difference and differential operators with random coefficients, i.e., again certain random matrices, to describe the dynamics of elementary excitations in disordered media (crystals with impurities, amorphous substances), thereby creating another branch of random spectral theory, known now as random operator theory (see e.g. [**396**]) and its theoretical physics counterpart, the theory of disordered systems (see e.g. [**345**]). The statistical approach in both cases goes a step further from that of quantum statistical mechanics, where traditionally the operators (Hamiltonians and observables) are not random but the quantum states are random and their probability law (Gibbs measure) is determined by the corresponding Hamiltonian. Note that even this tradition was broken in the 1970s, when the intensive studies of disordered magnets, spin glasses in particular, began and the random statistical mechanics Hamiltonians, hence the randomized Gibbs measures, were introduced; see e.g. [**93, 357**].

However, as in statistical mechanics, the infinite size limit and related asymptotic regimes play a quite important role in random spectral theory, random matrix theory in particular. This is also in agreement with principal settings of probability theory, since, according to the classics, "... the epistemological value of the theory of probability is revealed only by limit theorems. Moreover, without limit theorems it is impossible to understand the real content of the primary concept of all our sciences – the concept of probability" [**238**, Preface].

By the way, the large size asymptotic regimes, which are used almost everywhere in this book, can also be applied to draw a borderline between random operators and random matrices. In our opinion, this can be inferred from the large-$n$ behavior of the number $\nu_n$ of the entries of the same order of magnitude of an $n \times n$ matrix on its principal and adjacent diagonals (these matrices are known as the band matrices). If $\nu_n/n$ has a finite integer limit as $n \to \infty$, then there exists a limiting object, a random operator in $l^2(\mathbb{Z})$. In particular, in the case of hermitian $n \times n$ matrices, if the limit is an odd positive integer $2p+1$, then we have a hermitian finite-difference operator of order $2p$ with random coefficients and the spectral properties of this "limiting" operator are strongly related to those of its "finite box" restriction. This approach to the spectral analysis of selfadjoint operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$, $d \geq 1$, dates back to the work by H. Weyl of the 1910s and has proved to be quite efficient since then. If, however, $\nu_n/n \to \infty$, $n \to \infty$, then we have a "genuine" random matrix and have to deal with various asymptotic regimes or just estimates, despite the fact that many of them can be used to characterize certain infinite-dimensional operators, as, for example, in the quantum chaos studies since the 1970s or in recent studies of asymptotic eigenvalue spacing as possible tools to distinguish between the pure point and the absolutely continuous spectrum of random operators. Besides, there exists a variety of results in both theories which allow one to say, by using the terminology of statistical mechanics, that random matrix theory can often be viewed as the mean field version of random operator theory.

We now comment on basic terminology, conventions, and the contents of the book. Since random matrix theory is largely asymptotic theory, it deals not with random matrices of a fixed size $n$, but rather with sequences of random matrices defined for all positive integer $n$'s, despite the fact that we write quite often, together with the random matrix community, "random matrix", i.e., the singular form of our main object. Moreover, to make our formulas, often long, more readable, we do not write the subindex $n$ in matrices to indicate their size, excepting the cases where it can lead to misunderstandings. It is always understood that we deal with $n \times n$ matrices and that we are interested mostly in the large-$n$ behavior of their spectral characteristics.

Next is the term *ensemble* or *random matrix ensemble*, whose meaning is sometimes a bit vague in random matrix texts. The term seems to be borrowed from the early days of probability theory and statistical mechanics (where it is widely used until now). We use the term just to designate the sequence of matrix probability laws determining the random matrix in the above meaning. We also use quite often the term *spectrum*, while discussing the large size limit of random matrices, despite the fact that according to the above, there is no limiting operator, as is the case in random operator theory. In this book the term is just a synonym for the support of the limiting Normalized Counting (or empirical) Measure of eigenvalues of the random matrix in question. This has to be compared with random operator theory that deals with differential and finite-difference operators with random ergodic coefficients and where also there exists the limiting Normalized Counting Measure of eigenvalues of the finite box restrictions of corresponding operators. Here the spectrum of the "limiting" operator does indeed coincide with probability 1 with the support of the limiting measure [**396**, Sections 4.C and 5.C]. Moreover, there exist certain families of random ergodic operators interpolating between the two cases; see e.g. Section 17.3 of the book.

Now we comment on the contents of the book. The detailed contents can be seen from the Contents and from the introductions to the chapters. We will therefore restrict ourselves to general remarks. The book treats three main themes: the existence and the properties of the nonrandom limiting Normalized Counting Measure of eigenvalues, the fluctuation laws of linear eigenvalue statistics, and the local regimes. The first two themes are often referred to as the global (or macroscopic) regime and require the scaling of matrix entries (or the spectral axis) guaranteeing the existence of the well-defined limit of the Normalized Counting Measure of eigenvalues in question. The themes are similar to those in probability theory on the Law of Large Numbers and the Central Limit Theorem for the sums of independent or weakly dependent random variables. The main difference here is that the eigenvalues of a random matrix are strongly dependent even if its entries are independent; thus one needs new techniques or, at least, appropriately extended and treated versions of existing probabilistic techniques.

The third theme is entirely the random matrix theme. It is on the local regimes, i.e., on the statistics of eigenvalues falling into intervals, whose length is of order of magnitude of typical spacing between eigenvalues and thus tends to zero with an appropriate rate as the size of the matrix tends to infinity.

In treating the above themes, we confine ourselves to the normal random matrices, more precisely, to real symmetric, hermitian, orthogonal, and unitary matrices. Random matrix theory studies also quaternion real and symplectic matrices, which are, roughly speaking, the hermitian and unitary matrices with quaternion ($2 \times 2$ matrix) entries. They possess a number of interesting properties that can be found in [**217**] and [**356**] and references therein.

We do not consider complex matrices, the random matrix jargon term for real, but not real symmetric or orthogonal, and complex, but not hermitian or unitary, matrices. This is a big and fast developing field with a lot of interesting recent results and it deserves a separate book.

The book consists of an introduction and three parts. In the introduction we discuss first the archetype *Gaussian Ensembles* of random matrix theory, deriving them from the requirements of orthogonal or unitary invariance (for the real symmetric and hermitian matrices, respectively). We also discuss briefly other widely used (but not all) ensembles. We then introduce certain notions, objects, and settings of random matrix theory by using an elementary example of diagonal matrices with i.i.d. random diagonal entries, i.e., in fact, the standard probabilistic set up. In particular, we introduce the main asymptotic regimes of the theory.

Part 1 is devoted to classical ensembles, i.e., the Gaussian Ensembles for real symmetric and hermitian matrices, introduced by Wigner in the 1950s, the Wishart Ensemble for the real symmetric matrices, well known in statistics since the late 1920s, its hermitian analog, known as the Laguerre Ensemble, and the ensembles of real symmetric, hermitian, orthogonal, and unitary matrices whose randomness is due to the classical groups (orthogonal, unitary) and related symmetric spaces, seen as the matrix probability spaces, with the normalized to unity Haar measure or its restrictions.

We first study in detail the global regime. This is carried out by using basically two technical tools: certain versions of integration by parts, which we call the differential formulas, and the Poincaré-type inequalities, providing an efficient bound for the variance of relevant random objects. In particular, the inequalities lead almost immediately to the bound of the variance of linear eigenvalue statistics, which is of the order $O(1)$ as $n \to \infty$, unlike $O(n)$ for i.i.d. Gaussian random variables. This is a first manifestation of strong statistical dependence of eigenvalues, one of the principal sources of new and often highly nontrivial results of random matrix theory.

We then pass to the local (bulk and edge) regimes and establish basic facts about them, thereby presenting a considerable part of the random matrix "arsenal" both for random matrix theory itself and for numerous applications.

This part of the book is rather traditional. We only mention that our presentation of the global regime is based on the systematic use of the Stieltjes and the Fourier transforms of the Normalized Counting Measure, providing the links with the resolvent and the unitary group for the matrices in question, and is rather efficient in the context. The main technical tool for the local regimes here is the orthogonal polynomial techniques, introduced in random matrix theory by Gaudin, Mehta, Wigner, and Dyson, based in fact on observations of analysts of the nineteenth century.

Part 2 is on the Matrix Models (known also as the invariant ensembles) of hermitian and real symmetric matrices. This class of random matrix ensembles shares with the Gaussian Ensembles the property of invariance with respect to orthogonal or unitary transformations; however their entries are strongly dependent, unlike those for the Gaussian Ensembles. The main technical tools here are variational methods and so-called determinantal formulas for marginal densities (*correlation functions* in statistical mechanics) of the joint eigenvalue distribution whose essential ingredients are special orthogonal polynomials, known as the polynomials orthogonal with respect to varying weights. This leads to important representations for relevant spectral characteristics, quite convenient for the large-$n$ asymptotic analysis of local regimes. One can then use the asymptotics of orthogonal polynomials to complete the analysis. This strategy is used in Part 1, where one deals with the classical polynomials whose asymptotics are well known. To use the strategy in the case of Matrix Models, one needs the asymptotics for the polynomials orthogonal with respect to varying weights. They are obtained and applied to the study of the local regime of the Matrix Models in a series of recent works (see e.g. [**152, 154, 162**] and Chapter 14). We use these new asymptotics to study the fluctuations of linear eigenvalue statistics of Matrix Models in Chapter 14. As for their local regimes, we carry out a direct analysis of determinantal formulas based essentially on spectral properties of Jacobi matrices, associated with the corresponding orthogonal polynomials, rather than on their asymptotics.

Part 3 deals with ensembles determined by independent but not necessarily Gaussian random variables, mostly with real symmetric and hermitian Wigner matrices, whose upper triangular entries are independent, and with sample covariance matrices, for which the corresponding data matrices have independent entries. We study in detail the existence and properties of the limiting Normalized Counting Measure of eigenvalues and the fluctuation of linear eigenvalue statistics by using the differentiation formulas, martingale-type bounds (instead of Poincaré-type inequalities of Part 1), and an *"interpolation trick"*, allowing us to use results on the classical ensembles. As for the local regime, where considerable progress has been achieved recently, we present a brief review of results obtained and methods used in Sections 18.7 and 19.3.

Random matrix theory is the result of a nontrivial synthesis of ideas and constructions from several branches of mathematics and physics. Therefore it employs a wide range of often specialized concepts and methods belonging to various fields that have been traditionally only very tenuously related. For the same reason, it attracts the interest of scientists from a number of branches of mathematics and related sciences. Finally, the theory has accumulated a good deal of profound facts and interrelations between them, some of which have not yet been rigorously proved in the generality in which they are believed to be true. Because of the above and the wide variety of recent developments, it seems hardly possible to present the essentials of the theory in a book of reasonable size by using the traditional style of mathematical writing, where everything is proved in detail, thus comprising a reasonably complete and self-contained text. We therefore depart sometimes from this style, basically in two cases. The first case is where we need certain results of analysis, probability, operator theory, etc. They are formulated without proof or with just the sketch of a proof, however with the appropriate references. Such statements are called propositions, in contrast to theorems and lemmas, which are proved in full. Other results, especially those obtained quite recently, are also just formulated or described, and their proofs, which are as a rule cumbersome and technically complicated, are replaced by discussions of the main ideas involved. Results of this type are either presented as remarks, comments, sometimes problems and special sections that are more survey-like: for example, Comments 1.3.1 and 7.6.1, Section 18.7, and Problem 2.4.13. Note also that the importance and driving force of random matrix theory are mostly due to its numerous and diverse applications. Their sufficiently comprehensive description requires much more space and expertise than we possess. This is why we mention this or that application and/or link and provide a selection of references (mostly recent) after the presentation of the corresponding result.

We are aware that this type of presentation may not satisfy everyone, but we hope that our intention of giving a comprehensive impression of the subject will serve as at least a partial justification.

Likewise, we did our best to write a book that is of interest to a sufficiently wide audience, but we could not avoid being subjective in the choice of results and references, determined to a large extent by our points of view and our works (mostly due to spectral theory and mathematical physics) and the lack of space. We apologize strongly for not including or/and mentioning many important contributions.

Our final remark concerns notation: throughout the book we write the integral without limits for the integral in the whole line and $C$, $C_1$, etc., and $c$, $c_1$, etc., for generic quantities which do not depend on the matrix size, special parameters, etc., but whose values may be different in different formulas.

We would also like to thank the coauthors of our joint papers and numerous colleagues with whom many ideas and results were obtained and discussed.