Introduction and Standard Notation

HNP (hereditary Noetherian prime) rings are perhaps the only noncommutative Noetherian rings whose projective modules, both finitely generated and infinitely generated, have nontrivial direct sum behaviour and a structure theorem describing that behaviour. This book provides a full account of that structure and behaviour, as well as much of what is known about nonprojective finitely generated modules, injective modules, and the ring structure that underlies all of this. In doing so, it revisits several areas of theory which are used in the solution and which have previously only been available scattered through a large number of research papers. Happily, but perhaps not unexpectedly, this has allowed substantial simplifications, improved clarity, fresh insights and new results.

History[1]: We start by providing a historical perspective. Steinitz and others, by the early 1900s, had extended the fundamental theorem of abelian groups to a structure theorem for finitely generated modules over commutative Dedekind domains. There are many equivalent definitions of Dedekind domains. The most apt for our purposes is that they are commutative hereditary Noetherian integral domains; but they can equally well be defined, for example, as integrally closed Noetherian domains of Krull dimension 1 or as Noetherian domains whose nonzero ideals are all invertible.

Steinitz's theorems include a precise description of all isomorphism classes of finitely generated torsionfree modules and their direct sums. In modern terminology, the description involves just two invariants, both additive in direct sums: the rank (i.e. the number of indecomposable summands, each isomorphic to an ideal) and an element of an abelian group called the ideal class group of the Dedekind domain. Somewhat more recent results are that direct sums of ideals of a Dedekind domain coincide with its projective modules and that all infinitely generated projective modules are free.

In the light of Goldie's Theorem of 1958, it was natural to seek a similar theory for an appropriate class of noncommutative Noetherian rings which included all Dedekind domains. To a greater or lesser extent, this was considered and accomplished during the 1960s and 1970s for two special types of HNP ring: *classical hereditary orders* (HNP rings that are finitely generated modules over their centre, which is a Dedekind domain) and noncommutative *Dedekind prime rings* (Noetherian prime rings whose nonzero one-sided ideals are not only projective but also generators). However, the theory for a general HNP ring was much more elusive. There was spasmodic progress over a long period by several authors with the final complete solution only coming in 1999/2000 in a series of papers by the authors of this book.

[1] Further historical comments and precise references appear in the sections entitled 'Notes on Chapter(s)'

The structure: Let R be an HNP ring, and P_R a nonzero finitely generated projective module. We now give a description of the structure of P and then outline this book's approach to it. We associate two isomorphism invariants with P: the 'Steinitz class' $\mathcal{S}(P)$, and the 'genus' $\Psi(P)$. The Steinitz class $\mathcal{S}(P)$ is an element of an abelian group $G(R)$ called the 'ideal class group' of R (because it becomes that in the commutative case); and the genus is a noncommutative replacement for rank. It is a function from the prime spectrum of R to the non-negative integers which, in the commutative case, is simply a constant function equal to the rank of P.

The structure theory first determines precisely what functions can occur as genera. Then it shows that these invariants $\mathcal{S}(P)$ and $\Psi(P)$ can be independently assigned, are additive in direct sums and, if $\operatorname{udim}(P) > 1$ (i.e. P is not indecomposable), they determine P up to isomorphism.

Outline: Perhaps surprisingly, the book starts with what might appear a diversion: a relatively complete account in Chapters 1,2 of idealizer subrings of an arbitrary ring, S say. However, idealizers play a crucial role in the theory which follows; so it seems best to meet them first. Some explanation is needed to see why they are relevant. In studying the structure of an HNP ring R and its modules, we need to study all rings S between R and its Goldie quotient ring R_{quo}. The main results depend upon the case when S_R is finitely generated; and we show that, in that situation, R can be obtained from S by iterating the process of forming a special type of idealizer subring that we call a 'basic idealizer'. It turns out that the larger ring here is, in a sense, a localization of the smaller one. The extremely tight relationship between these rings R, S and their simple modules, which is made explicit in §13 and §14, ultimately allows a reduction to the special case when R is a Dedekind prime ring, where the module structure is readily determined.

Chapter 4 introduces the notion of a 'tower', described very roughly as follows. A nonzero extension relation between certain pairs of simple modules over the HNP ring R partitions the simple R-modules into finite sets, each such set being either cyclicly or linearly ordered. We call these sets 'cycle towers' and 'faithful towers', respectively. We omit further comment on these here, except to say that they are involved in either the statement or proof of almost every important result from there on. Then Chapter 5, 'Integral overrings', introduces a noncommutative extension of the familiar concept of the same name in commutative rings. After all this preparation, Chapter 6 studies the isomorphism invariants of our projective module P_R and proves Theorem 35.13, the main structure theorem. The remaining HNP parts of this book can be viewed as applications of its first six chapters.

Some appendices provide material required in the earlier work but not about idealizers or HNP rings. In particular, we give a self-contained account of modules over Artinian serial rings, in a form suitable for our work. This is relevant because all proper factor rings of an HNP ring are Artinian serial rings. Another topic, which has more than one application here, concerns a surjective homomorphism $P \twoheadrightarrow U$ of modules over an arbitrary ring S, with P projective and U of finite length, and deals with questions about the lifting to P of decompositions of U.

0.1. Acknowledgments. Writing this book has diverted the authors from other activities. We both thank Ruth and Lyn for their gracious acceptance of this.

0.2. Standard Notation and Terminology. We list here some notation and terminology that is used throughout the book. See also the index sections at the end of the book; and see [**McConnell-Robson 01**], referred to henceforth as [**McR 01**], for other standard ring-theoretic terminology and notation.

By *ring* we mean a ring with an identity element, and a *subring* shares the same identity element.

An *HNP (hereditary, Noetherian, prime) ring* is a right and left Noetherian prime ring in which every right and every left ideal is a projective module. To avoid trivialities, we also assume that the ring is not Artinian. If, further, it is an integral domain, we term it an HNP domain

Throughout, as in the preceding definition, omission of the term 'right' or 'left' denotes that the ring satisfies both right and left conditions.

$M_n(R)$ denotes the ring of $n \times n$ matrices over a ring R.

Module means 'right module' unless otherwise specified; and M_R denotes a right R-module.

Ideal means 'two-sided ideal' unless otherwise specified.

(S, R)*-bimodule* means a bimodule which is a left S-module and a right R-module.

Maps of right modules (i.e. right module homomorphisms) will be viewed and written as acting on the left. In particular, this makes every M_R into an (S, R)-bimodule where $S = \text{End}(M_R)$, the endomorphism ring of M.

Likewise, if N is a left R-module, denoted N_R, then its maps are viewed as acting from the right and then N becomes an (R,S)-bimodule where $S = \text{End}({}_R N)$.

$\text{rad}(M)$ and $J(M)$ denote the Jacobson radical of a module M and $\text{top}(M)$ denotes $M/\text{rad}(M)$.

$\text{soc}(M)$ and $\text{soc}_R(M)$ denote the socle of an R-module M.

$M^{(n)}$ denotes the direct sum of n copies of M.

ann, ann_R, rann, lann denote, respectively, the annihilator, the annihilator in R, the right annihilator and the left annihilator.

$\lambda(M_R)$ denotes the length of a module M_R of finite length.

$E(M)$ denotes the injective envelope of a module M.

fd = flat dimension and pd = projective dimension.

$\subset$ means strict containment.

$\hookrightarrow$ and $\twoheadrightarrow$ denote maps that are, respectively, (1,1) and onto.