CHAPTER 1

Basic Idealizers

This chapter introduces the idealizer subring $\mathbb{I}_S(A)$ of a right ideal A in a ring S. Its main aim is to investigate, in §4 and §5, the 'basic idealizer' case — when A is not two-sided and $S/A \cong U^{(n)}$ for some n and some simple module U_S — this being the case that underpins much that follows. The preceding sections lead up to this by considering the relationship between S and $\mathbb{I}_S(A)$ under less stringent restrictions on A.

1. Idealizers and Endomorphisms

In this preliminary section we introduce the notion of an idealizer and link this with endomorphisms.

1.1. Definition. Let A be a right ideal of a ring S. The subring $\mathbb{I}(A)$ or, if the ring concerned needs emphasis, $\mathbb{I}_S(A)$ defined by

$$\mathbb{I}(A) = \{x \in S \mid xA \subseteq A\}$$

is called the *idealizer* of A in S. There is a corresponding notion for any left ideal B of S with its *(left) idealizer* being denoted by $\mathbb{I}(B)$, $\mathbb{I}_S(B)$ or $\mathbb{I}({}_SB)$. For example, if

$$S = M_2(\mathbf{Z}), \quad A = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \mathbf{Z} \\ 0 & \mathbf{Z} \end{pmatrix},$$

then A is a right ideal of S, B a left ideal and $\mathbb{I}(A) = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{pmatrix} = \mathbb{I}(B)$.

1.2. Remark. We note two immediate consequences of the definition:
(i) $\mathbb{I}(A)$ is a ring, namely the largest subring of S in which A is an ideal, and the factor ring $\mathbb{I}(A)/A$ is sometimes called the *eigenring* of A;
(ii) if B is an ideal of S with $B \subseteq A$ then $\mathbb{I}_{S/B}(A/B) = (\mathbb{I}_S(A))/B$.

One can identify $\mathrm{End}(S_S)$ with S acting on itself via left multiplication; and then $\mathbb{I}(A) = \{\lambda \in \mathrm{End}(S_S) \mid \lambda(A) \subseteq A\}$. This leads to the following result.

1.3. Lemma. *Let A, B be right ideals of a ring S.*
(i) $\mathbb{I}(A)/A \cong \mathrm{End}_S(S/A)$ *acting via left multiplication.*
(ii) $\{s \in S \mid sA \subseteq B\}/B \cong \mathrm{Hom}_S(S/A, S/B)$ *acting via left multiplication.*

Proof. First we consider (ii). To simplify notation, let $\mathrm{Hom}_S(S/A, S/B) = H$ and $\{s \in S \mid sA \subseteq B\} = C$; so $B \subseteq C$. Given any element $c \in C$, let λ_c denote the endomorphism of S given by left multiplication by c. Now λ_c maps A into B; so λ_c restricts to an element of H. This restriction is the zero homomorphism precisely when $cS \subseteq B$; thus we have obtained an injective map from C/B to H. We wish to show it is surjective. So, let $h \in H$ and suppose that $h(1 + A) = x + B$. Then

$h(s+A) = xs + B$ for all $s \in S$ and in particular, since $h(a+A) = h(0) = 0$ for each $a \in A$, then $xA \subseteq B$. Thus $x \in C$ and h is the restriction of λ_x. Thus the map from C/B to H is indeed surjective.

We note that (i) is the special case of (ii) when $A = B$. It can be checked that the isomorphism given in (i) is a ring isomorphism. □

1.4. COROLLARY. *Let A be a right ideal of any ring S, and let C/A be a fully invariant submodule of $(S/A)_S$. Then $\mathbb{I}_S(A) \subseteq \mathbb{I}_S(C)$.*

PROOF. For any $x \in \mathbb{I}_S(A)$, left multiplication by x is an endomorphism of $(S/A)_S$. Full invariance of C/A therefore implies that $x(C/A) \subseteq C/A$. Since $xA \subseteq A$ we also have $xC \subseteq C$, as desired. □

The existence of a link between idealizers and endomorphisms leads to the Dual Basis Lemma playing an important role in the basic theory of idealizer rings. We therefore include here a brief account of this.

1.5. DEFINITIONS (Duals and related products). Let R be any ring and M a right R-module. The *dual* of M is denoted by M^* , or M^*_R if the ring needs emphasis, and is defined to be $\mathrm{Hom}(M, R)$. Recall from 0.2 that M is an (S, R)-bimodule where $S = \mathrm{End}\, M_R$. Similarly, M^* may be viewed as an (R, S)-bimodule: for $f \in M^*$, $r \in R$, and $\phi \in S$, rf is the map $m \to r{\cdot}f(m)$ in M^*, and $f\phi$ is the map $m \to f(\phi(m))$ in M^*. Verification of 'bimodule associativity', that is $(rf)\phi = r(f\phi)$, is straightforward.

We often make use of two related 'products'. One is $M \otimes_R M^* \to MM^* \subseteq S$: for $m_1 \in M$ and $f \in M^*$, $m_1 f$ is the R-endomorphism given by $m \to m_1{\cdot}f(m)$. The other is the more obvious $M^* \otimes_S M \to M^*M \subseteq R$, defined by $fm_1 = f(m_1)$. As usual, the notation MM^* and M^*M denotes the additive groups generated by the 'monomials' that define them. We note that these products are bimodule maps. Therefore MM^* and M^*M are ideals of the rings S and R respectively. The overall facts about the two rings and the two modules concerned, together with their products, can be summed up by saying that the set of formal 2×2 matrices $\begin{pmatrix} R & M^* \\ M & S \end{pmatrix}$ forms a ring via the given products. This ring is sometimes called the *ring of the Morita context.* (See [**McR 01**, 1.1.6, 1.9.1] for more details.)

1.6. LEMMA (Dual Basis Lemma).

(i) *A module M_R over any ring R is finitely generated and projective if and only if $MM^* = \mathrm{End}(M_R)$ (equivalently, $1_{\mathrm{End}(M_R)} \in MM^*$).*

(ii) *If M_R is projective with a set of n generators, then the same is true of ${}_RM^*$; and $M \cong M^{**}$ via the canonical identification which maps $m \in M$ to the map $M^* \to R$ given by $f \to f(m)$.*

PROOF. (i) Suppose first that $1 \in MM^*$; so we have an expression $1 = \sum_{i=1}^n m_i f_i$ with each $m_i \in M$ and $f_i \in M^*$. Let e_i be the element of $R^{(n)}$ whose i^{th} coordinate is 1 and whose other coordinates are zero. Now consider the map $\beta{:}R^{(n)} \twoheadrightarrow M$ defined by $\beta(e_i) = m_i$. Bearing in mind the equation $m = 1m = \sum_i m_i f_i(m)$, we define $\alpha{:}M \to R^{(n)}$ by $\alpha(m) = \sum_{i=1}^n e_i f_i(m)$ and note that $\beta\alpha = 1_M$. Thus M, being isomorphic to a direct summand of $R^{(n)}$, is projective and is generated by the elements $\{m_i \mid i = 1, \ldots, n\}$.

Conversely, suppose that M is finitely generated and projective, generated by the elements $m_1, \ldots, m_n$. The map $g : R^{(n)} \twoheadrightarrow M$, defined by $g(e_i) = m_i$ with the

e_i as above, is a surjection which is split by some injective map $f : M \hookrightarrow R^{(n)}$. Let f_i: $M \to R$ be the i^{th} coordinate map of f. Then, for each $m \in M$, we have $f(m) = \sum_i e_i f_i(m)$ and therefore

$$m = gf(m) = \sum_i g(e_i) f_i(m) = \sum_i m_i f_i(m).$$

Therefore we have the desired relation $1 = \sum_i m_i f_i$.

(ii) The canonical right R-module homomorphism $\theta : M \to M^{**}$ sends each $m \in M$ to the homomorphism $\theta(m)$: $M^* \to {}_R R$ given by $(f)\theta(m) = f(m)$, (with the map of the left module M^* written on the right). We will show θ is an isomorphism. We know, from (i), that there is an expression $1 = \sum_i m_i f_i \in \operatorname{End}(M_R)$, since M_R is finitely generated and projective. Therefore $m = \sum_i m_i f_i(m)$ for every $m \in M$. If $\theta(m) = 0$, then $(f_i)\theta(m) = f_i(m) = 0$ for every i, and therefore $m = \sum_i m_i f_i(m) = 0$. Thus θ is (1,1).

Next, take any left R-module homomorphism α: $M^* \to {}_R R$. For every $g \in M^*$ we have $g = \sum_i g(m_i) f_i$. Applying α yields

$$(g)\alpha = \sum_i g(m_i)(f_i)\alpha.$$

However, if we let $m = \sum_i m_i (f_i)\alpha$ then

$$(g)\theta(m) = g(m) = \sum_i g(m_i)(f_i)\alpha = (g)\alpha.$$

Thus θ is onto and so is an isomorphism.

Finally, the fact that the left R-module ${}_R M^* = (M_R)^*$ is projective and generated by $f_1, \ldots, f_n$ follows from the left-handed version of (i) applied to M^* and the isomorphism $(M^*)^* \cong M$. □

Indeed, for finitely generated projective right R-modules M, the double duality functor $M \twoheadrightarrow M^{**}$ is equivalent to the identity functor. However, this fact will not be needed here.

1.7. Corollary. *For every finitely generated projective M_R, the 'trace' ideal $\operatorname{tr}_R(M) = M^*M$ of M is idempotent.*

Proof. The associativity of the Morita ring, mentioned in 1.5, provides the equations $(M^*M)(M^*M) = M^*(MM^*)M = M^* \operatorname{End}(M_R)M = M^*M$. □

1.8. Remark. Suppose that R is a field and the elements m_i in the Dual Basis Lemma form a basis of M. Then the 'dual base' $\{m_i\}$ and $\{f_i\}$ above form a pair of dual bases in the sense of linear algebra.

To see this start with the relation $m = \sum_i m_i f_i(m)$, which holds for every $m \in M$. Taking $m = m_j$ for some j yields $m_j = \sum_i m_i f_i(m_j)$. Then linear independence of basis elements of vector spaces yields $f_i(m_j) = \delta_{ij}$, the Kronecker delta, for every i, j.

2. Subidealizers of Generative Right Ideals

The rich theory of idealizers which fills this chapter is built upon the case when the right ideal has certain special properties. This section introduces one of the relevant properties — being generative — and delineates the consequences for the idealizer ring and, usefully, for a more general class of rings. We now give the appropriate definitions.

2.1. Definition. Let A be a right ideal of a ring S. Any subring T such that $\mathbb{I}_S(A) \supseteq T \supset A$ is called a *subidealizer* of A. Hence $1_S \in T$ and A is an ideal of T.

2.2. Definition. A right ideal A of a ring S is said to be *generative* provided that $SA = S$. For example:

(i) if S is a simple ring then every nonzero right ideal is generative;

(ii) if $S = M_n(D)$ for some ring D and $a \in S$ is a matrix which has an entry which is a unit of D then aS is a generative right ideal of S — thus if $S = M_2(\mathbf{Z})$ then $e_{11}S = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & 0 \end{pmatrix}$ is a generative right ideal.

We should note that any generative right ideal A is automatically a generator right module; i.e. the ring is a sum of homomorphic images of A_S. However, a right ideal which is a generator right module is not necessarily generative: any proper nonzero ideal in any commutative principal ideal domain is a counterexample.

2.3. Proposition. *Let A be a generative right ideal of a ring S and let R be a subidealizer of A. Then:*

(i) *S_R and ${}_RA$ are finitely generated projective;*

(ii) *$S \otimes_R S \cong S \otimes_R A \cong S$ via multiplication;*

(iii) *$(S/R) \otimes_R S = 0 = (R/A) \otimes_R A$ and $(S/R) \otimes_R A \cong S/A \cong (R/A) \otimes_R S$.*

Proof. (i) First consider S_R. By the Dual Basis Lemma [1.6] it suffices to show that $1 \in SS^*$ where 1 denotes the identity endomorphism of S_R. The hypothesis $SA = S$ yields an expression $\sum_{i=1}^n s_i a_i = 1$. Since $A \subseteq R$, left multiplication by any element of A is a map in $S^* = \operatorname{Hom}(S_R, R_R)$. Moreover, left multiplication by each product $s_i a_i$ is an endomorphism of S_R. Therefore the expression $\sum_{i=1}^n s_i a_i = 1$ shows that S_R is finitely generated projective.

Likewise right multiplication by any $s \in S$ gives a map in $\operatorname{Hom}({}_RA, {}_RR)$. Therefore the expression $\sum_{i=1}^n s_i a_i = 1$ shows that $1 \in A^*A$ and hence ${}_RA$ is finitely generated projective.

(ii) We note first that, since S_R is projective and thus flat, the embeddings ${}_RA \subseteq {}_RR \subseteq {}_RS$ yield embeddings $S \otimes_R A \subseteq S \otimes_R R \subseteq S \otimes_R S$. Then, viewing all terms involved as subsets of $S \otimes_R S$, we see that

$$S \otimes_R S = SA \otimes_R S = S \otimes_R AS = S \otimes_R A = S \otimes_R AR = SA \otimes_R R = S \otimes_R R.$$

The multiplication map $S \otimes_R S \to S$ when restricted to $S \otimes_R R$ is the canonical isomorphism $S \otimes_R R \cong S$. This, by the equalities above, yields the isomorphisms $S \otimes_R S \cong S \cong S \otimes_R A$.

(iii) The short exact sequence of right R-modules

$$0 \to R \to S \to S/R \to 0$$

when tensored on the right by ${}_RS$, gives the exact sequence

$$R \otimes S \to S \otimes S \to (S/R) \otimes S \to 0.$$

But (ii) shows that $S \otimes S \cong S$ via multiplication, and the image in S of $R \otimes S$ is $RS = S$. Thus $(S/R) \otimes S = 0$.

On the other hand if we had tensored the given short exact sequence by A we would have obtained the exact sequence

$$R \otimes A \to S \otimes A \to (S/R) \otimes A \to 0.$$

In this case, $S \otimes A \cong S$ via multiplication; and the image in S of $R \otimes A$ is $RA = A$. Thus $(S/R) \otimes A \cong S/A$.

The remaining facts are proved similarly using the short exact sequence

$$0 \to A \to R \to R/A \to 0$$

and tensoring it on the right by S or A. □

2.4. Remark. If R is a subring of a ring S, then S is sometimes called a *left localization* of R if $S \otimes_R S \cong S$ and S_R is flat. If, further, S_R is finitely generated projective, S is a *finite left localization* of R. This applies, of course, to the rings R and S in 2.3 above.

2.5. Proposition. *Let R be a subring of a ring S and suppose that $S \otimes_R S \cong S$ via multiplication. Let M and N be right S-modules and L a left S-module. Then:*

(i) *$M \otimes_R S \cong M$ via multiplication;*
(ii) *if M_R is projective then M_S is projective;*
(iii) $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_S(M, N)$*;*
(iv) *if M_R is injective then M_S is injective;*
(v) *$M \otimes_R L \cong M \otimes_S L$.*

Proof. (i) $M \otimes_R S \cong M \otimes_S S \otimes_R S \cong M \otimes_S S \cong M$ via multiplication.

(ii) If M_R is a direct summand of a free right R-module, then tensoring over R by S shows that M_S is a direct summand of a free right S-module.

(iii) Let ϕ: $M \to N$ be an R-homomorphism. It induces the S-homomorphism $\phi' = \phi \otimes 1$ from $M \otimes_R S$ to $N \otimes_R S$. Then after identifying $M \otimes_R S$ and $N \otimes_R S$ with M and N respectively, using (i), we get $\phi' = \phi$. In other words, ϕ is an S-homomorphism.

(iv) Suppose first that M_R is injective. Let I be a right ideal of S and let $\alpha : I \to M$ be an S-homomorphism. To demonstrate that M_S is injective, we need only show that α can be lifted to a homomorphism $S \to M$. Of course α is also an R-module homomorphism which, since M_R is injective, lifts to an R-homomorphism $S \to M$; and this, by (iii), is also an S-homomorphism.

(v) $M \otimes_R L \cong (M \otimes_S S) \otimes_R (S \otimes_S L) \cong M \otimes_S S \otimes_S L \cong M \otimes_S L$. □

Next we apply this to a subidealizer of a generative right ideal of a ring where more can be shown.

2.6. Proposition. *Let A be a generative right ideal of a ring S and let R be a subidealizer of A. Then:*

(i) *all the statements in 2.5 hold;*
(ii) *for each nonzero ideal B of S, $B \cap R \neq 0$;*
(iii) *if M is a right S-module then $M \otimes_R A \cong M$ via multiplication; and M_R is projective if and only if M_S is projective;*
(iv) *if $S \supseteq {}_R X \supseteq A$ then $S \otimes_R X \cong SX = S$; $({}_R X)^* \cong \{s \in S \mid Xs \subseteq R\}$ via right multiplication; and, viewing this isomorphism as an identification, $S \supseteq X^* \supseteq A$;*
(v) *$({}_R A)^* = S$ and $(S_R)^* = A$, in each case acting via multiplication;*

(vi) *if M is a right S-module and L a left S-module then* $\mathrm{Tor}_n^R(M,L) \cong \mathrm{Tor}_n^S(M,L)$ *for all n.*

PROOF. Note first that 2.3 shows not only that $S \otimes_R S \cong S$ via multiplication but also that S_R is projective and hence flat. These facts will be used throughout this proof without further comment.

(i) Clear.

(ii) Note that $AB \subseteq B$ and $SA = S$. Thus $0 \neq B = SB = SAB \subseteq SB = B$ and so $0 \neq AB \subseteq B \cap R$.

(iii) The first claim is clear since $M \otimes_R A \cong M \otimes_S S \otimes_R A \cong M \otimes_S S \cong M$ via multiplication. For the second claim, note that since S_R is projective, any free right S-module will be projective over R and so too is a direct summand such as M_S. The converse is covered directly by 2.5(ii).

(iv) Since S_R is flat, $S \otimes_R X \subseteq S \otimes_R SX$. However

$$S \otimes_R SX \cong S \otimes_R S \otimes_S SX \cong S \otimes_S SX \cong SX$$

via multiplication. Since $S \otimes_R X \twoheadrightarrow SX$ under multiplication, we deduce that this epimorphism is in fact an isomorphism. Finally, $S \supseteq SX \supseteq SA = S$ and so $SX = S$.

Next we turn to $({}_RX)^*$. If $Xs = 0$ for some $s \in S$ then $0 = SXs \supseteq SAs = Ss$ and so $s = 0$. Thus we need only show that each $\phi \in \mathrm{Hom}({}_RX, R)$ is given by right multiplication by some element of S. However, given ϕ, then $1 \otimes \phi : S \otimes_R X \to S \otimes_R R$. Now we have just seen that $S \otimes_R X \cong SX = S$. Also $S \otimes_R R \cong SR = S$. Thus $1 \otimes \phi : S \to S$ and so is given by right multiplication, as required.

(v) It is immediate from (iv) that $({}_RA)^* = S$, acting via right multiplication. Consequently $(S_R)^* = A$ acting via left multiplication.

(vi) We start with any short exact sequence $0 \to K_S \to P_S \to M_S \to 0$ with P_S projective. Since P_S is flat, the long exact Tor sequence (described in 53.17) demonstrates that $\mathrm{Tor}_1^S(M,L) \cong \ker(K \otimes_S L \to P \otimes_S L)$ and likewise, since (iii) shows that P_R is projective and so flat, $\mathrm{Tor}_1^R(M,L) \cong \ker(K \otimes_R L \to P \otimes_R L)$. Hence, using the isomorphisms given by 2.5(v) above, $\mathrm{Tor}_1^S(M,L) \cong \mathrm{Tor}_1^R(M,L)$. Similarly $\mathrm{Tor}_{k+1}^S(M,L) \cong \mathrm{Tor}_k^S(K,L)$ and $\mathrm{Tor}_{k+1}^R(M,L) \cong \mathrm{Tor}_k^R(K,L)$. Induction on k allows us to suppose that $\mathrm{Tor}_k^S(K,L) \cong \mathrm{Tor}_k^R(K,L)$ and the result follows. $\square$

Further results along the lines of 2.5 and 2.6, but under stronger hypotheses, will be found in 4.13.

3. Idealizers of Isomaximal and Semimaximal Right Ideals

In this section we introduce the second property of the right ideal A which will be heavily involved throughout the remainder of the chapter.

3.1. DEFINITIONS. Let A be a right ideal of a ring S. If the right S-module S/A is semisimple (necessarily of finite length) we say A is *semimaximal*. If, further, $S/A \cong U^{(n)}$ for some simple module U_S and some $n \geq 1$ (and so S/A is semisimple isotypic), we say A is *isomaximal of type U*.

We follow the convention that the zero module is semisimple, being the empty direct sum of simple modules. Thus S itself is semimaximal.

For example, if D is a division ring and $S = M_n(D)$, every right ideal is isomaximal and, if nonzero, is also generative. If $S = M_2(\mathbf{Z})$ and

$$A = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 2\mathbf{Z} & 2\mathbf{Z} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ 6\mathbf{Z} & 6\mathbf{Z} \end{pmatrix}, \qquad C = M_2(2\mathbf{Z}) \quad \text{and} \quad D = M_2(4\mathbf{Z}),$$

then A is a generative isomaximal right ideal, B is a generative semimaximal right ideal, C is an isomaximal right ideal which is not generative and D is a right ideal which is neither semimaximal nor generative.

There is a useful criterion for being semisimple isotypic.

3.2. Lemma. *If a nonzero module M has finite length and no proper, fully invariant submodules, then M is semisimple isotypic.*

Proof. Since $\operatorname{soc}(M)$ is fully invariant and nonzero, then $M = \operatorname{soc}(M)$ and so M is semisimple. If M had more than one isotypic component, then each of these would be a proper, fully invariant submodule. Hence the result. □

The next result shows that, in studying the nature of the idealizer of an isomaximal right ideal A, there is no loss in assuming that A is generative, as defined in 2.2.

3.3. Lemma. *Let A be an isomaximal right ideal of a ring S. If A is not generative, then A is an ideal and hence $\mathbb{I}_S(A) = S$.*

Proof. Let $S/A = U^{(n)}$, with U simple. Then either $UA = 0$ or $UA = U$. If $UA = 0$, then $(S/A)A = 0$ and therefore $SA \subseteq A$; that is, A is an ideal. If $UA = U$ then $(S/A)A = S/A$, which implies $SA = S$. □

For future use, we note the following related fact.

3.4. Lemma. *Let A be an isomaximal right ideal of a ring S. Then A is generative if and only if there is an isomaximal right ideal, A' say, with $A' \subset A$.*

Proof. Let $B = \operatorname{ann}(S/A) = \operatorname{ann}(U)$.

Suppose that A is generative. Now $B = \cap C_i$ where the intersection is of all maximal right ideals C_i with $S/C_i \cong U$. Since A is generative, A is not an ideal and so $A \supset B$. Thus there is a maximal right ideal C such that $S/C \cong U$ and $C \not\supseteq A$. We let $A' = A \cap C$.

Conversely, if A is not generative then, by 3.3, A is an ideal and so $A = B$. Evidently $B \subseteq A'$ for every isomaximal right ideal A' of type U. □

The next two results start the process of reducing questions about idealizers of semimaximal right ideals to the generative isomaximal case.

3.5. Lemma. *Let S be a ring and $A \neq S$ a semimaximal right ideal. Then:*

(i) *$\mathbb{I}_S(A)/A$ is a semisimple Artinian ring and is simple if A is isomaximal;*

(ii) *there is a finite set of isomaximal right ideals A_i of distinct types such that $A = \cap A_i$;*

(iii) *A is generative if and only if each A_i is generative.*

Proof. (i) $\mathbb{I}_S(A)/A \cong \operatorname{End}(S/A)$ by 1.3 and hence is semisimple Artinian and is simple if A is isomaximal.

(ii) Let U_i be one of the simple factor modules of S/A and

$$A_i = \cap \{M_S \mid S \supset M \supseteq A,\ \ S/M \cong U_i\}.$$

Then A_i is isomaximal of type U_i and $A = \cap A_i$.

(iii) Evidently, if $SA = S$ then $SA_i = S$. Now suppose that each A_i is generative and yet A is not. We aim at a contradiction. Since $SA \neq S$ then $A \subseteq SA \subseteq M$ for some maximal ideal M of S. Since $(S/A)_S$ has finite length and M is a maximal ideal, S/M is a simple Artinian ring. Thus $(S/M)_S$ is semisimple isotypic. But $M \supseteq A$, so the simple type of $(S/M)_S$ must be that of some A_i. To ease notation, suppose that $i = 1$. Then $M = \operatorname{ann} U_1 \subseteq A_1$ and also $M \neq A_1$ since A_1 is generative. Thus A_1/M is nonzero isotypic of type U_1. This contradicts the fact that A_1/A is semisimple with composition factors coming from $U_2, \dots, U_k$ and so completes the proof. □

3.6. Proposition. *Let S be a ring and let $\{A_i \mid i = 1, \dots, k\}$ be a finite set of isomaximal right ideals of distinct simple type. Let $A = \cap_1^k A_i$ and let A' be the (possibly empty) intersection of all those A_i which are generative. Then A' is a generative semimaximal right ideal and $\mathbb{I}_S(A) = \cap_i \mathbb{I}_S(A_i) = \mathbb{I}_S(A')$.*

Proof. First we show that $\mathbb{I}_S(A) = \cap \mathbb{I}_S(A_i)$. Let $x \in \mathbb{I}_S(A)$. From 1.3 we know that left multiplication by x induces an endomorphism of S/A. Evidently A_i/A is an invariant submodule of S/A; so 1.4 shows that $\mathbb{I}_S(A) \subseteq \mathbb{I}_S(A_i)$ and hence $\mathbb{I}_S(A) \subseteq \cap \mathbb{I}_S(A_i)$. On the other hand, if $x \in \mathbb{I}_S(A_i)$ for each i then $xA_i \subseteq A_i$ and so $x(\cap_i A_i) \subseteq \cap_i A_i$.

Next note, by 3.3, that $\mathbb{I}_S(A_i) = S$ whenever A_i is not generative. Hence $\mathbb{I}_S(A) = \cap\{\mathbb{I}_S(A_i) \mid A_i \text{ generative}\}$ and the latter term equals $\mathbb{I}_S(A')$ by the first paragraph of this proof. □

Since our interest in idealizers lies in the rings they provide, the preceding proposition suggests that we should concentrate on idealizers of generative isomaximal right ideals. That is the focus of the next few sections. In fact we show, in 8.10, that every semimaximal idealizer can also be obtained by forming a succession of idealizers of generative isomaximal right ideals.

4. Basic Idealizers

We now turn to the study of the idealizer R of an isomaximal generative right ideal of a ring S. The section establishes, in 4.4 and 4.8, a remarkably tight connection between the simple modules of R and S, and then explores some of its consequences.

We start with some useful terminology.

4.1. Definition. We say that $R = \mathbb{I}_S(A)$ *is a basic idealizer of type* U if A is a generative isomaximal right ideal of S of type U. Thus $S/A \cong U^{(n)}$ for some $n \geq 1$.

4.2. Lemma. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type U.*

(i) *R/A is a simple Artinian ring. Indeed if $S/A \cong U^{(n)}$ then $R/A \cong M_n(\operatorname{End}(U))$.*
(ii) *A is a maximal ideal of R and is idempotent.*

Proof. (i) This follows from 1.3.
(ii) Note that, since A is generative, $SA = S$; and so $A^2 = ASA = AS = A$. □

Next we demonstrate some symmetry in certain special cases. Note that, if S is simple Artinian, then $\mathbb{I}(A)$ is basic for every right ideal $A \neq 0, S$.

4.3. Lemma. *Let S be a ring and $R = \mathbb{I}_S(A)$ be a basic (right) idealizer.*

(i) *Suppose that S is a simple Artinian ring and $A = eS$ with $e = e^2$. Then $R = \mathbb{I}_S(eS) = \mathbb{I}_S(S(1-e))$ and so is a basic left idealizer from S.*

(ii) *More generally, suppose that $S/\operatorname{ann}(S/A)_S$ is simple Artinian. Then R is a basic left idealizer from S.*

Proof. (i) Let $e' = 1 - e$. From $S = (eSe) \oplus (eSe') \oplus (e'Se) \oplus (e'Se')$, one easily calculates that $\mathbb{I}_S(eS) = eS + Se' = \mathbb{I}_S(Se')$. Also, since $eS \neq 0, S$ then the same is true of Se'.

(ii) Let $C = \operatorname{ann}(S/A)_S$. Evidently $C \subseteq R$ and (as was noted in 1.2) one can check that $R/C = \mathbb{I}_{S/C}(A/C)$. This, by (i), is a basic (left) idealizer for a generative isomaximal left ideal of S/C. This lifts to a left ideal A' with $C \subset A' \subset S$; and then $R = \mathbb{I}(A')$. ☐

The next result describes all simple R-modules; and their isomorphism types are dealt with by 4.8.

4.4. Theorem. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type U, with $S/A \cong U^{(n)}$.*

(i) *If X is a simple S-module not isomorphic to U, then X_R is simple.*

(ii) *U_R has a unique composition series of length 2; and if its top and bottom R-composition factors are named V and W respectively, then $V \ncong W$ and*

$$(S/R)_R \cong V^{(n)} \qquad \text{and} \qquad (R/A)_R \cong W^{(n)}. \tag{4.4.1}$$

(iii) *Every simple right R-module is of the form V, W or X as described in (i) and (ii).*

Proof. We start with some discussion related to both (i) and (ii). First note that 4.2(i) shows that R/A, viewed as a right R-module, is a direct sum of n copies of a simple right R-module, W say; and W is distinguished amongst simple right R-modules by having A as its annihilator.

Next, given any simple right S-module, we write it in the form S/B for some maximal right ideal B. Let C/B be a proper R-submodule of S/B. Since $A \subseteq R$,

$$(CA + B)/B = (C/B)A \subseteq C/B \subset S/B.$$

Thus $(CA + B)/B$ is a proper S-submodule of the simple S-module S/B; hence $(CA + B)/B = 0$ and so $CA \subseteq B$. Thus $C \subseteq D = \{d \in S \mid dA \subseteq B\}$. Moreover, 1.3 shows that, for each $c \in C$, left multiplication by c induces an element of $\operatorname{Hom}_S(S/A, S/B)$.

(i) Suppose now that $S/B = X \ncong U$ and so $\operatorname{Hom}_S(S/A, S/B) = 0$. Then left multiplication by c must induce the zero map; that is, $cS \subseteq B$ for each $c \in C$. Thus $C \subseteq B$ and so $C/B = 0$. Thus S/B has no nonzero proper R-submodules; so it is a simple R-module.

(ii) Suppose next that $S/B \cong U$ and that C/B is a proper R-submodule. The remarks above tell us that $C \subseteq D = \{d \in S \mid dA \subseteq B\}$; and $D \neq S$ since $SA = S \nsubseteq B$. We deduce that D/B is the unique maximal R-submodule of $S/B \cong U$ and so $(S/D)_R$ is simple, say $(S/D)_R \cong V$. Note that $VA = V$ since $SA = S$; so $V \ncong W$.

Next, choose a set of n maximal right ideals B_i with $\cap B_i = A$, and so each $S/B_i \cong U$. For each i, let D_i/B_i be the maximal R-submodule provided by the preceding paragraph. Then

$$\cap D_i = \{d \in S \mid dA \subseteq \cap B_i\} = \{d \in S \mid dA \subseteq A\} = R$$

and so $\oplus_{i=1}^{n} D_i/B_i \cong R/A \cong W^{(n)}$. We see from this that $D_i/B_i \cong W$. Thus we have shown that U has a unique composition series of length 2, and that the two composition factors are nonisomorphic. Since $S/A \cong U^{(n)}$ and $R/A \cong W^{(n)}$ we deduce that $S/R \cong V^{(n)}$.

(iii) Let Y_R be simple. It is enough to show that Y is an R-composition factor of some S-module of finite length, since, using (i) and (ii), every S-composition series can be refined to an R-composition series whose composition factors are as described.

Now $Y \cong R/E$ for some maximal right ideal E of R; and we can assume, in the notation of (ii), that $Y \ncong V$. We know from (ii) that $(S/R)_R$ has finite length, and so too, of course, has R/E. Therefore $(S/E)_R$ has finite length, and has a composition factor isomorphic to Y.

Next consider the R-module ES/E which is a submodule of S/E and so has finite length. Now $ES/E \cong E \otimes_R (S/R)$ as right R-modules. However, $E \otimes_R (S/R)$ is a sum of right R-submodules of the form $e \otimes (S/R)$, where $e \in E$. Each of these is a homomorphic image of (S/R); i.e. by (ii), of $V^{(n)}$. Hence all composition factors of ES/E are isomorphic to V which, by hypothesis, is not isomorphic to Y.

Finally, we note that $S/ES \cong (S/E)/(ES/E)$. Since S/E has an R-composition factor isomorphic to Y and ES/E does not, S/ES is the desired S-module of finite length that has an R-composition factor isomorphic to Y. □

4.5. Notation. To avoid repetition, when $R = \mathbb{I}_S(A)$ is a basic idealizer of type U and so U_R is uniserial of length 2 with composition factors V, W, we will simply say that *$R = \mathbb{I}_S(A)$ is a basic idealizer of type $U = [VW]$* and say that *R slices U* into $[VW]$.

The next few results concern the 'new' simple modules V and W.

4.6. Proposition. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$. Then, viewing W as an R-submodule of U_S,*

$$R = \{s \in S \mid Ws \subseteq W\}, \quad A = \operatorname{ann}_S(W) \quad \textit{and} \quad W = \operatorname{ann}_U(A).$$

Proof. It is enough to prove these assertions with W replaced by $W^{(n)}$, regarded as an R-submodule of $U_S^{(n)}$, with $n \geq 1$ chosen so that $S/A \cong U^{(n)}$. However, there is a unique R-submodule of S/A that corresponds to $W^{(n)}$ under any isomorphism $(S/A)_R \cong U^{(n)}$, namely R/A, since $V \ncong W$ [4.4(ii)]. So we may replace $W^{(n)}$ by R/A. However, if $(R/A)s \subseteq R/A$ then $s \in R$, as desired. The second assertion is proved similarly. For the third assertion, suppose that $(s + A)A = 0$ in S/A. Then $sA \subseteq A$ and therefore $s \in \mathbb{I}_S(A) = R$. Thus $s + A \in R/A$, as desired. □

Keep the notation in 4.6; and recall [2.6] that, for any S-module X we have $X \otimes_R S \cong X \otimes_R A \cong X$. Thus we already know the effect of such tensoring by S and A on all simple R-modules other than V and W.

4.7. Proposition. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$.*

(i) $VA = V$ *and* $WA = 0$.
(ii) $V \otimes_R S = 0$ *and* $V \otimes_R A \cong U$.
(iii) $W \otimes_R S \cong U$ *and* $W \otimes_R A = 0$.
(iv) $\operatorname{pd}(V_R) \leq 1$ *and* $\operatorname{pd}(W_R) \leq \operatorname{pd}(U_S)$.

Proof. (i) 4.6 shows that $WA = 0$. To see that $VA = V$, note that $S/R \cong V^{(n)}$, by 4.4, and $SA = S$; so $V^{(n)}A = V^{(n)}$.

(ii) 2.3 shows that $(S/R) \otimes S = 0$ and hence $V \otimes S = 0$. The same result also shows that $(S/R) \otimes A \cong S/A$. But $S/A \cong U^{(n)}$ and so $V^{(n)} \otimes A \cong U^{(n)}$, by (4.4.1). Hence the result holds.

(iii) This is proved in a similar fashion.

(iv) Since S_R is projective, by 2.3, $(S/R)_R \cong V^{(n)}$ has projective dimension at most 1; so the same is true of V_R. Next, since any S-projective resolution of U_S is also an R-projective resolution, we see that $\mathrm{pd}(U_R) \leq \mathrm{pd}(U_S)$. The existence of the short exact sequence $0 \to W \to U \to V \to 0$ now implies the result, using, e.g., [**McR 01**, 7.1.6]. □

4.8. Corollary. *Let $R = \mathbb{I}_S(A)$, a basic idealizer of type $U = [VW]$. Then:*

(i) *V is the unique simple R-module with the property that $V \otimes_R S = 0$;*
(ii) *W is the unique simple R-module with the property that $W \otimes_R A = 0$;*
(iii) *the distinct simple R-isomorphism classes are precisely the simple S-isomorphism classes but with that of U replaced by those of V and W.*

Proof. Let $X \not\cong U$ be a simple S-module. Recall [4.4] that every simple R-module is either isomorphic to such an X or to V or W. Recall, from 2.6(i),(iii), that $X \otimes_R S \cong X \otimes_R A \cong X$. However, 4.7 asserts that $V \otimes_R S = 0$ and $W \otimes_R S \cong U$, proving (i), and that $V \otimes A \cong U$ and $W \otimes_R A = 0$, proving (ii).

By (i) and (ii), we know that no two of V, W and X are isomorphic. Therefore, to prove (iii), it is enough to prove that if Y_S is simple and $Y_S \not\cong X_S$ then $Y_R \not\cong X_R$; and this holds since $\mathrm{Hom}_R(X,Y) = \mathrm{Hom}_S(X,Y) = 0$ [2.6(i) and 2.5(iii)]. □

4.9. Lemma. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$ and let W' be an R- submodule of $U^{(a)}$, for some a, with $W' \cong W$. Then W' is the R-socle of some S-submodule U' of $U^{(a)}$ with $U' \cong U$.*

Proof. The nonzero S-submodule $U' = W'S$ of $U^{(a)}$ generated by W' is a homomorphic image of the simple S-module $W \otimes_R S \cong U$. Hence $U' \cong U$, as desired. □

4.10. Lemma. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$. If M is an R-submodule of some S-module, there exists a commutative diagram with exact rows*

$$\begin{array}{ccccc} M/MA & \hookrightarrow & MS/MA & \twoheadrightarrow & MS/M \\ \downarrow{\scriptstyle(\cong)} & & \downarrow{\scriptstyle(\cong)} & & \downarrow{\scriptstyle(\cong)} \\ W^{(a)} & \hookrightarrow & U^{(a)} & \twoheadrightarrow & V^{(a)} \end{array} \tag{4.10.1}$$

where the vertical maps are isomorphisms and a is a cardinal number; and a is finite if M_R is finitely generated.

Proof. We have a short exact sequence $MR/MA \hookrightarrow MS/MA \twoheadrightarrow MS/MR$ of R-modules, where we have written MR in place of M for emphasis. Note that $(MS/MA)_S \cong M \otimes_R (S/A)$. This is a sum of S-submodules of the form $m \otimes (S/A)$, with m ranging over the members of a generating set for M_R; and each of these is a homomorphic image of $U^{(n)}$ where $S/A \cong U^{(n)}$. Thus $(MS/MA)_S \cong U^{(a)}$ as S-modules, and hence as R-modules, for some a which is finite if M_R is finitely generated. Similarly, with the help of (4.4.1), we get $MR/MA \cong W^{(c)}$ and

$MS/MR \cong V^{(b)}$ as R-modules for suitable cardinal numbers b, c. Substituting into the short exact sequence above yields a short exact sequence of R-modules:

$$0 \to W^{(c)} \to U^{(a)} \to V^{(b)} \to 0.$$

Since the map $U^{(a)} \to V^{(b)}$ is a surjection and its kernel contains no copies of the top composition R-factor V of U, we have $b = a$. Similarly, since the image of this map contains no copies of W, the kernel must be the entire R-socle $W^{(a)}$ of $U^{(a)}$. Thus $c = a$. The rest now follows. □

We note one consequence.

4.11. PROPOSITION. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$. If $S_R \supseteq X_R \supseteq R_R$ then X_R is finitely generated projective.*

PROOF. We know from 4.7 that V has projective dimension at most 1. Since S_R is projective [2.3], the short exact sequence

$$0 \to X \to S \to V^{(k)} \to 0$$

shows that X_R is projective. Moreover, $X/R \subseteq S/R$ which has finite length [4.4] and so X_R is finitely generated. □

4.12. THEOREM. *If $R = \mathbb{I}_S(A)$ is a basic idealizer then ${}_RS$ is flat.*

PROOF. We have seen, in 2.6(vi) , that if M is a right S-module and N a left S-module then $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^S(M, N)$.

Next we show that $\mathrm{Tor}_1^R(S/R, S) = 0$. We consider the short exact sequence

$$0 \to R \to S \to S/R \to 0.$$

When tensored over R with S this gives us [see 53.17.1] the long exact sequence

$$0 \to \mathrm{Tor}_1^R(S/R, S) \to R \otimes_R S \to S \otimes_R S \to (S/R) \otimes_R S \to 0$$

using the fact that $\mathrm{Tor}_1^R(S, S) = 0$ since S_R is flat. The isomorphism of $R \otimes_R S$ and $S \otimes_R S$ under the given map implies that $\mathrm{Tor}_1^R(S/R, S) = 0$, as desired.

Next, let B be any right ideal of R. By 4.10, BS/B is a direct sum of copies of V and hence is a direct summand of a direct sum of copies of S/R. Hence, from above, we deduce that $\mathrm{Tor}_1^R(BS/B, S) = 0$. But we know, as noted above, that $\mathrm{Tor}_1^R(S/BS, S) = \mathrm{Tor}_1^S(S/BS, S)$; and the latter term is zero since ${}_SS$ is flat. We deduce that $\mathrm{Tor}_1^R(S/B, S) = 0$. Finally, the short exact sequence

$$0 \to B \to S \to S/B \to 0$$

yields the long exact sequence

$$\ldots \to \mathrm{Tor}_1^R(S/B, S) \to B \otimes_R S \to S \otimes_R S \to (S/B) \otimes_R S \to 0$$

and so the map $B \otimes_R S \to S \otimes_R S$ is an embedding. The same is therefore true of the map $B \otimes_R S \to R \otimes_R S$. By [**Anderson and Fuller 92**, 19.17], this is equivalent to ${}_RS$ being flat. □

This result, combined with 2.3(ii), shows that a basic idealizer satisfies the hypotheses of the next proposition which gives further results along the lines of 2.6.

4.13. PROPOSITION. *Let R be a subring of a ring S such that $S \otimes_R S \cong S$ via multiplication and ${}_RS$ is flat — as is true when R is a basic idealizer from S.*

(i) *If M and N are R-submodules of right S-modules then $M \otimes_R S \cong MS$ via multiplication and* $\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_S(MS, NS)$.

(ii) *If M is a right S-module then M_R is injective if and only if M_S is injective.*

(iii) *If M and N are right S-modules then* $\operatorname{Ext}^1_R(M, N) = \operatorname{Ext}^1_S(M, N)$*, when viewed as equivalence classes of short exact sequences; (and if R is a basic idealizer from S, then* $\operatorname{Ext}^n_R(M, N) \cong \operatorname{Ext}^n_S(M, N)$ *for each n, via the forgetful functor from S to R.)*

(iv) *$I = (I \cap R)S$ for every right ideal I of S.*

PROOF. (i) Note first that, since ${}_RS$ is flat, then $M \otimes_R S \subseteq MS \otimes_R S$. However

$$MS \otimes_R S \cong MS \otimes_S S \otimes_R S \cong MS \otimes_S S \cong MS$$

via multiplication. Since $M \otimes_R S \twoheadrightarrow MS$ under multiplication, we deduce that this epimorphism is in fact an isomorphism.

Next, we note that every $\phi \in \operatorname{Hom}_R(M, N)$ induces $\phi \otimes 1\colon M \otimes_R S \to N \otimes_R S$ which, by the preceding paragraph, we may consider to be an S-homomorphism $MS \to NS$.

(ii) The case when M_R is injective is covered by 2.5(iv). So we suppose that M_S is injective. Let I be a right ideal of R and let $\alpha : I \to M$ be an R-homomorphism. We need only show that α can be lifted to a homomorphism $R \to M$. We form tensor products by S over R. Since ${}_RS$ is flat, we get $I \otimes_R S$ embedded in $R \otimes_R S$ and also $\alpha \otimes 1 : I \otimes_R S \to M \otimes_R S$. Using (i), $M \otimes_R S \cong MS = M$, $R \otimes_R S \cong S$ and $I \otimes_R S \cong IS$, all via multiplication. Since M_S is injective, the map $\alpha \otimes 1$ from IS to M lifts to a map from S to M. Restricted to R, this is the required lifting of α.

(iii) Let $\mathcal{E} : 0 \to N \to X \to M \to 0$ be a short exact sequence of R-modules. Since ${}_RS$ is flat, tensoring by S gives a short exact sequence of right S-modules

$$\mathcal{E}' : 0 \to N \otimes_R S \to X \otimes_R S \to M \otimes_R S \to 0.$$

The multiplication maps $N \otimes_R S \to N$ and $M \otimes_R S \to M$ are isomorphisms; so we deduce that $X \cong X \otimes_R S$ and that $\mathcal{E}$ was already a short exact sequence of S-modules. We know, from 2.5(iii), that $\operatorname{Hom}_R = \operatorname{Hom}_S$ for right S-modules. The process of forming Ext from short exact sequences, which is described in 53.11, is now readily seen to be identical whether one considers R-modules or S-modules. Hence $\operatorname{Ext}^1_R(M, N) = \operatorname{Ext}^1_S(M, N)$.

Suppose that R is a basic idealizer from S. This, by 2.6(iii), gives the additional property that projective S-modules are also projective over R. We now proceed to prove, by induction on n, that $\operatorname{Ext}^n_R(M, N) \cong \operatorname{Ext}^n_S(M, N)$, the cases $n = 0, 1$ being already known. Consider the start of an S-projective resolution of M_S, say $0 \to K \to P \to M \to 0$. If we apply $\operatorname{Hom}_S(-, N)$ to this, we get $\operatorname{Ext}^{n+1}_S(M, N) \cong \operatorname{Ext}^n_S(K, N)$ for $n \geq 1$, using 53.8. As noted above, P is also projective over R and so, if we apply $\operatorname{Hom}_R(-, N)$ instead, we get $\operatorname{Ext}^{n+1}_R(M, N) \cong \operatorname{Ext}^n_R(K, N)$. The inductive hypothesis applied to K_S tells us that $\operatorname{Ext}^n_R(K, N) \cong \operatorname{Ext}^n_S(K, N)$. Hence $\operatorname{Ext}^{n+1}_R(M, N) \cong \operatorname{Ext}^{n+1}_S(M, N)$ as required.

(iv) Note that $I/(I \cap R) \cong (I + R)/R \subseteq S/R$. Hence $(I/(I \cap R)) \otimes_R S = 0$ and so $I = IS = (I \cap R)S$. □

Next we obtain some results directly involving the right ideal A.

4.14. PROPOSITION. *Let $R = \mathbb{I}(A)$ be a basic idealizer from S (or, more generally, let A be a generative right ideal of S and R be a subidealizer of A such that*

$_RS$ is flat). If $S \supseteq Y_R \supseteq R$ then $(Y_R)^* \cong \{s \in S \mid sY \subseteq R\}$ via left multiplication and, if we view this isomorphism as an identification, then $R \supseteq Y^* \supseteq A$.

PROOF. For the nontrivial part of this, take $\phi \in Y^* = \mathrm{Hom}_R(Y, R)$. Since Y is an R-submodule of S_S, 2.3(ii)(a) shows that ϕ extends to an element of $\mathrm{Hom}_S(YS, RS) = \mathrm{Hom}_S(S_S, S_S)$, and hence ϕ equals left multiplication by some element of S, as desired. The inclusion $Y^* \subseteq R$ holds because $1 \in R \subseteq Y$. □

Using this, we now demonstrate that in a basic idealizer situation, S is a minimal extension of R in the following sense.

4.15. PROPOSITION. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type U. Then there are no rings strictly between R and S.*

PROOF. Let T be a ring with $R \subseteq T \subseteq S$. From 4.11 we see that T_R is finitely generated projective and 4.14 shows that $T^* \cong \{r \in R \mid rT \subseteq R\}$. Now $R \supseteq T^* \supseteq A$ and, moreover, T^* is an ideal of R. Since A is a maximal ideal of R, we deduce that $T^* = A$ or $T^* = R$. Since T_R is projective, $T^{**} = T$. However, $A^* = S$ and $R^* = R$. □

Next we give a left-handed version of 4.11.

4.16. COROLLARY. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer. If $A \subseteq Y \subseteq S$ with Y a left R-submodule of S, then $_RY$ is flat; and if $A \subseteq Y \subseteq R$ then $_RY$ is finitely generated projective.*

PROOF. We know that $_RA$ is projective and so $\mathrm{pd}(_R(R/A)) \leq 1$. It follows that, if W' is the simple left R-module annihilated by A, then $\mathrm{pd}(_RW') \leq 1$ and so $\mathrm{fd}(_RW') \leq 1$. The same is then true of S/Y since this is a direct sum of copies of W'. Then the short exact sequence $0 \to Y \to S \to S/Y \to 0$ shows that $_RY$ is flat.

Finally, suppose that $Y \subseteq R$. Note that each of R/Y and Y/A is a finite direct sum of copies of W'. So $_RY$ is projective and, since $_RA$ is finitely generated, $_RY$ is finitely generated. □

4.17. COROLLARY. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer. Let M, N be right S-modules and K an R-submodule of M such that $M/K \cong N$ as R-modules. Then K is an S-submodule of M.*

PROOF. We know from 2.3 and 2.5 that $M \otimes_R S \cong M$ via multiplication, and likewise for N. The flatness of $_RS$ given by 4.12 shows that tensoring the exact sequence $0 \to K \to M \to N \to 0$ by S yields the exact sequence $0 \to K \otimes_R S \to M \to N \to 0$. Identifying $K \otimes S$ with its image in M therefore yields $K = KS$, as desired. □

The final two results of this section show, respectively, how module-theoretic and ring-theoretic properties pass up or down in the basic idealizer situation.

4.18. COROLLARY. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer, and let M be an S-module.*

(i) *M_S has finite length if and only if M_R has finite length.*
(ii) *M_S is Noetherian if and only if M_R is Noetherian.*
(iii) *If M_S is Noetherian then M_S is uniserial if and only if M_R is uniserial.*

Proof. (i) 4.4 shows that an S-composition series for M can be refined to an R-composition series of no more than double the length; and the rest is clear.

(ii) Evidently, if M_R is Noetherian then so too is M_S. Conversely, suppose that M_S is Noetherian. Let N be any R-submodule of M. Then $NS \supseteq N \supseteq NA$. Each of NS and NA are finitely generated over S and hence over R. Now, by 4.10, $NS/N \cong V^{(a)}$, $N/NA \cong W^{(a)}$ and $NS/NA \cong U^{(a)}$ for some a. Since $(NS/NA)_S$ is Noetherian, a must be finite; and then $(N/NA)_R$ has finite length and N_R is finitely generated.

(iii) Say M_R is uniserial and N_1 and N_2 are two S-submodules of M_S. They are also R-submodules and so must be comparable. Hence M_S is uniserial.

Conversely, say M_S is uniserial. Choose any $m \in M$. Now mS/mA is an S-homomorphic image of S/A which, in turn, is isomorphic to $U^{(n)}$, and yet mS/mA is uniserial. Hence either $mS/mA = 0$ or else $mS/mA \cong U$, which, as an R-module, is uniserial of length 2. Thus each cyclic R-submodule of M is either an S-submodule or is the unique R-submodule lying between two consecutive S-submodules. Hence, any two cyclic R-submodules are comparable and their sum is the larger. Consequently, each finitely generated R-submodule is cyclic and any two are comparable. □

Note that right serial rings are defined at the start of §50.

4.19. Theorem. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer of type $U = [VW]$.*

(i) *S is right Artinian if and only if R is right Artinian.*
(ii) *S is right Noetherian if and only if R is right Noetherian.*
(iii) *S is right hereditary if and only if R is right hereditary.*
(iv) *S is right Artinian, right serial if and only if R is right Artinian, right serial.*

Proof. We review some facts that this proof uses repeatedly, in addition to those in 4.18. Since A is generative, S_R is finitely generated, and S-modules are projective if and only they are projective as R-modules [2.6(iii)]. In particular, S_R is projective.

(i) Suppose that S is right Artinian, so S_S has finite length. Then S_R has finite length, and so its submodule R_R has finite length. Conversely, if R is right Artinian, then the finitely generated right R-module S has finite length. Hence the same is true of S_S.

(ii) If R is right Noetherian then so is the finitely generated right R-module S_R. Therefore S_S is Noetherian too. Conversely, if S is right Noetherian then 4.18(ii) shows that S_R is Noetherian and so too is its R-submodule R_R.

(iii) Suppose first that R is right hereditary. Let C be a right ideal of S. Since R is right hereditary, the submodule C_R of the projective R-module S_R is again projective, and therefore, as noted above, C_S is projective.

Conversely, suppose that S is right hereditary, let B be a right ideal of R, and consider the following short exact sequence of R-modules.

$$0 \to B \to BS \to BS/B \to 0$$

Since S is right hereditary, $(BS)_S$ is projective, and hence $(BS)_R$ is projective. Thus the short exact sequence is the start of a projective resolution of BS/B. However, $BS/B \cong V^{(a)}$ for some a [4.10] and $\mathrm{pd}(V) \leq 1$ [4.7]; so $\mathrm{pd}(BS/B) \leq 1$. Hence B_R is projective.

(iv) The Artinian property is dealt with by (i). First suppose that R is right serial and let B be an indecomposable direct summand of S_S. Say $B_R = C \oplus D$. Since $SA = S$, we have $B = BA = CA \oplus DA$. Since CA and DA are S-submodules of B, one of them must be zero; say $CA = 0$. But then $B = DA \subseteq D$, and hence $C = 0$. Thus we see that B_R is indecomposable.

Since B_S is finitely generated projective, so is B_R. But every finitely generated indecomposable projective right module over the right Artinian right serial ring R is uniserial (by the Krull-Schmidt theorem, applied to free R-modules). Therefore B_R is uniserial, and hence B_S is too. Thus S is a right serial ring.

Conversely, suppose that S is right serial, and let $J = J(S)$. Since A is isomaximal and generative we have $A \supset J$. Therefore A/J has an idempotent generator in S/J; and this lifts to an idempotent, e say, in S such that $A = eS + J$. Let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of 1 into a sum of orthogonal primitive idempotents of S such that $e = e_1 + e_2 + \cdots + e_m$ for some $m < n$. We now show that every $e_i \in R$.

If $1 \leq i \leq m$, we have $e_i = ee_i \in eS \subseteq A \subseteq R$. If $m + 1 \leq i \leq n$, we have $e_i e \in J \subseteq A$; and hence $e_i A = e_i(eS + J) \subseteq J \subseteq A$. Therefore $e_i \in \mathbb{I}_S(A) = R$.

Thus we see that $R = e_1 R \oplus \ldots \oplus e_n R$. Since each $e_i S$ is uniserial over S it is also uniserial over R; hence so is its R-submodule $e_i R$. □

4.20. Remark. We will see, in 14.6, that S is an HNP ring if and only if R is an HNP ring.

5. Extensions of Simple Modules in Idealizers

In this section, given a basic idealizer R from a ring S, we will describe all endomorphisms of simple R-modules and all extensions of simple R-modules by simple R-modules. The description will be in terms of information about simple modules over S.

5.1. Notation. Let $R = \mathbb{I}_S(A)$, a basic idealizer of type $U = [VW]$. The embedding of W in U gives a short exact sequence $W \overset{\alpha}{\hookrightarrow} U \overset{\beta}{\twoheadrightarrow} V$. We will use this notation throughout this section.

General facts about Ext from §53 will be used in this section, often without further comment.

5.2. Proposition. *Let $R = \mathbb{I}_S(A)$ be a basic idealizer.*

(i) *Let H be a right R-module such that* $\operatorname{Hom}_R(H, N) = 0$ *for all N_S; then* $\operatorname{Ext}^n_R(H, N) = 0$ *for all N_S.*

(ii) *Let H be a right R-module such that* $\operatorname{Hom}_R(M, H) = 0$ *for all M_S; then* $\operatorname{Ext}^n_R(M, H) = 0$ *for all M_S.*

Proof. (i) We proceed by induction. We consider the start $0 \to N \to I \to K \to 0$ of an S-injective resolution of N_S and apply $\operatorname{Hom}_R(H, -)$. All the Hom terms are zero; and so too are all the $\operatorname{Ext}^n_R(H, I)$, since, by 4.13(ii), I_R is injective. Hence $\operatorname{Ext}^{n+1}_R(H, N) \cong \operatorname{Ext}^n_R(H, K)$ for all $n \geq 0$. However, the latter is zero by the induction hypothesis applied to K.

(ii) This is proved similarly, using the start of an S-projective resolution of M. □

CHAPTER 6

Invariants for Finitely Generated Projective Modules

This chapter describes two independent invariants, Genus and Steinitz class, which are additive in direct sums and which, together, completely determine the isomorphism class of any finitely generated projective right R-module of udim ≥ 2.

The first section concerns the rank of a finitely generated projective module at an unfaithful simple module. This is then used in §33 to provide a generalization of the classical notion of 'genus', phrased in a way that avoids reference to classical localization since that is not available here. Then, after discussing direct-sum cancellation (§34), we proceed, in §35, to our generalizations of 'ideal class group' and 'Steinitz class' and the definitive result promised above.

32. Rank and Merging

In this section[1] we define the ranks of a finitely generated projective module at unfaithful simple modules and, in preparation for the following sections, we investigate how these react to merging.

32.1. Definitions. The notation modspec(R) denotes a set, the *module spectrum*, consisting of the zero R-module together with a set $\mathcal{W}$ of representatives of the isomorphism classes of unfaithful simple (right) R-modules. We note that, since R is an HNP ring, spec(R) comprises simply the zero ideal together with the nonzero maximal ideals. Thus replacing each $W \in \mathcal{W}$ by the nonzero maximal ideal $M = \text{ann}_R(W)$, and the zero module by the zero ideal converts the module spectrum, modspec(R), into spec(R).

Let P_R be finitely generated projective. We define the *rank* of P at W — equivalently, at M — to be:

(32.1.1) $$\rho(P, W) = \rho(P, M) = \lambda(P/PM)$$

where λ denotes composition length. We define the *rank* of P at a tower $\mathcal{C}$ to be:

(32.1.2) $$\rho(P, \mathcal{C}) = \sum\{\rho(P, W) \mid W \in \mathcal{W} \cap \mathcal{C}\}.$$

Thus, if $\mathcal{T}$ is a faithful tower, $\rho(P, \mathcal{C})$ ignores the faithful module in $\mathcal{C}$. By slight abuse of notation, we also write (32.1.2) in the form

(32.1.3) $$\rho(P, \mathcal{C}) = \sum\{\rho(P, M) \mid M \in \mathcal{C}\}.$$

We note that the definition of the rank of P at W extends the definition of the rank of a ring T at W given in 9.2. Thus 20.5, 20.6 and 25.27 provide examples showing that the ranks of P at the various unfaithful members of a tower can be complicated.

[1]In this section R denotes an HNP ring unless the contrary is specified.

Since R/M is a simple Artinian ring, P/PM is a direct sum of $\rho(P, W)$ copies of W. It is worth noting explicitly the following consequence.

32.2. Lemma. *Let W be a simple unfaithful R-module. Then $\rho(P, W)$ is the largest n such that P can be mapped onto $W^{(n)}$.* □

In view of 32.2, it is tempting to define $\rho(P, W)$, where W_R is faithful and simple, to be the maximum n such that P maps onto $W^{(n)}$. However, this maximum never exists for HNP rings; see 49.12 or the comment after 20.3.

32.3. Definition. Let σ be a function from (isomorphism classes of) finitely generated projective R-modules to an abelian group G. We say that σ is *additive on direct sums,* if $\sigma(P \oplus Q) = \sigma(P) + \sigma(Q)$.

32.4. Lemma. *Rank is additive on direct sums of modules.*

Proof. Clear. □

32.5. Lemma. *Let P_R be finitely generated projective, $\mathcal{C}$ a cycle of maximal ideals, and I the intersection of the maximal ideals in $\mathcal{C}$. Then $\rho(P, \mathcal{C}) = \lambda(P/PI)$.*

Proof. Let $M_1, \ldots, M_n$ be the maximal ideals in $\mathcal{C}$. By the Chinese Remainder Theorem we have $R/I \cong \oplus_i R/M_i$ as both rings and R/I-modules. Since rank is additive in direct sums, the lemma follows from the natural identification $P/PI = P \otimes_R R/I$. □

32.6. Definition. Let P_R be finitely generated projective. We say that P has *standard rank* at an unfaithful simple module W or at a cycle tower $\mathcal{C}$, respectively, if:

$$\rho(P, W) = \rho(R, W)\frac{\operatorname{udim}(P)}{\operatorname{udim}(R)} \quad \text{or} \quad \rho(P, \mathcal{C}) = \rho(R, \mathcal{C})\frac{\operatorname{udim}(P)}{\operatorname{udim}(R)}. \tag{32.6.1}$$

32.7. Lemma. *For each unfaithful simple module W and each tower $\mathcal{C}$, having standard rank at W, or at $\mathcal{C}$, is preserved under direct sums. In particular, all free modules of finite rank have standard rank at every W and every $\mathcal{C}$.*

Proof. Clear. □

32.8. Theorem (Almost standard rank). *Let P be a finitely generated projective R-module. Then P has standard rank at W for almost all (i.e. for all but finitely many) isomorphism classes of unfaithful simple modules W. (We say, more briefly: 'P has almost standard rank'.)*

Proof. We start by proving that if E, F are uniform right ideals of R then $\rho(E, M) = \rho(F, M)$ for almost all maximal ideals M.

By symmetry it suffices to prove $\rho(E, M) \geq \rho(F, M)$ for almost all M. By 12.4, every uniform right ideal of R is isomorphic to a submodule of every other uniform right ideal. Thus we may suppose that $E \subseteq F$. Then the R-module F/E has finite length [12.17]. Let M be any maximal ideal that does not annihilate any of the finitely many composition factors of F/E. (We are disregarding only a finite number of maximal ideals.) However, M annihilates all composition factors of F/FM. Hence $F/(E + FM) = 0$; i.e. $F = E + FM$.

Therefore there are maps

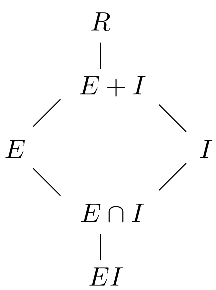
$$E/EM \twoheadrightarrow E/(E \cap FM) \cong (E + FM)/FM = F/FM,$$

and the inequality $\rho(E, M) \geq \rho(F, M)$ follows.

To make use of this, write P and R as direct sums of $a = \operatorname{udim}(P)$ and $b = \operatorname{udim}(R)$ uniform right ideals, respectively. Choose any maximal ideal M at which all of these uniform right ideals have the same rank, say r. Then $\rho(P, M) = ar$ and $\rho(R, M) = br$ by 32.4. Therefore, if W is the unfaithful simple module with $\operatorname{ann} W = M$ then P has standard rank at W. Since this is true for almost all $W \in \mathcal{W}$, the result is proved. ☐

32.9. THEOREM (Cycle standard rank). *Each finitely generated projective module P_R has standard rank at every cycle tower.*

PROOF. Standard rank at $\mathcal{C}$ is preserved by direct sums [32.4]. Since every finitely generated projective R-module is isomorphic to a direct sum of uniform right ideals, it is sufficient to show that every uniform right ideal H has standard rank at every $\mathcal{C}$.



Say $\operatorname{udim}(R) = r$; so R is a direct sum of r uniform right ideals. Since every uniform right ideal is isomorphic to a necessarily essential submodule of every other, it follows that $H^{(r)}$ is isomorphic to an essential right ideal of R. It therefore suffices to show that every essential right ideal E has standard rank at every $\mathcal{C}$.

Let I be the intersection of the maximal ideals in $\mathcal{C}$. Then I is invertible, by 22.9; and $\rho(E, \mathcal{C}) = \lambda(E/EI)$, by 32.5. In particular, $\rho(R, \mathcal{C}) = \lambda(R/I)$. Moreover, R obviously has standard rank at $\mathcal{C}$. So it is enough to show that $\lambda(E/EI) = \lambda(R/I)$.

However, since I is invertible, $\lambda(R/E) = \lambda(I/EI)$. Therefore, we see from the diagram that $\lambda\big(R/(E + I)\big) = \lambda\big((E \cap I)/EI\big)$ and hence that $\lambda(R/I) = \lambda(E/EI)$, as desired. ☐

32.10. COROLLARY.

(i) *Standard rank is an integer for almost all $W \in \mathcal{W}$.*
(ii) *Cycle standard rank is always an integer.*

PROOF. These are immediate consequences of 32.8 and 32.9. ☐

We apply these theorems to add to the description of an integral overring S of R in terms of $\mathcal{Z}_R(S)$, as defined in 29.3. Parts of the next result have counterparts in 29.5 that refer to $\mathcal{Z}_R(Q)$ where Q is any integral overring of R, and not necessarily R-projective.

32.11. LEMMA.

(i) *Let Q be a nonzero, finitely generated, projective R-module and M_R any right R-module. Then $\mathcal{Z}(M \oplus Q) \subseteq \mathcal{Z}(Q)$ and both these sets are finite and contain no cycle tower.*
(ii) *Let $\mathcal{F}$, $\mathcal{F}'$ be finite subsets of $\mathcal{W}$ which contain no cycle tower. Then there is a unique right finite overring $S(\mathcal{F})$ such that $\mathcal{Z}_R(S) = \mathcal{F}$; and $\mathcal{F}' \subset \mathcal{F} \Leftrightarrow S(\mathcal{F}') \subset S(\mathcal{F})$.*

PROOF. (i) Evidently $\mathcal{Z}(M \oplus Q) \subseteq \mathcal{Z}(Q)$ since any simple image of Q is also a simple image of $M \oplus Q$. Almost standard rank [32.8] shows that the set $\mathcal{Z}(Q)$

is finite. Cycle standard rank [32.9] shows that every cycle tower contains at least one X such that $\rho(Q,X) \neq 0$. Therefore $\mathcal{Z}(Q)$ contains no cycle tower.

(ii) This is immediate from 29.5. $\square$

The next few results concern descending chains of three finitely generated projective modules whose composition factors reflect consecutive terms in a tower (as in (32.12.1) below). It is helpful to set up some notation.

32.12. Notation. Consider a nonsplit short exact sequence $W \hookrightarrow U \twoheadrightarrow V$ where W_R, V_R are simple and W is unfaithful: i.e. W is the unfaithful successor of V. Let $A = \mathrm{ann}_R(W)$ and $S = O_r(A)$. Then 22.1(i) shows that $A = \mathrm{ann}(W)$ is an idempotent maximal ideal of R and 14.9 shows that S is an overring of R satisfying $R = \mathbb{I}_S(A)$, a basic idealizer of type U. Moreover U_S is simple and $U_R \cong [VW]$, a uniserial R-module of length 2.

We study the existence and ranks of finitely generated projective modules P_R, P'_R, P''_R such that:

$$P \supset P' \supset P'' \quad \text{with} \quad P/P'' \cong U, \quad P/P' \cong V, \quad P'/P'' \cong W. \tag{32.12.1}$$

32.13. Lemma. *Let $P_R \neq 0$ be finitely generated projective and X_R simple.*

(i) *If X is faithful then there exists a submodule $P' \subset P$ with $P/P' \cong X$.*

(ii) *If X is unfaithful, then there exists a submodule $P' \subset P$ with $P/P' \cong X$ if and only if $\rho(P,X) \neq 0$.*

Proof. (i) It suffices to find a nonzero map $g\colon P \to X$, since X is simple. Choose a nonzero element $p_0 \in P$. Since P is a direct summand of a free module, there is a map $\pi\colon P \to R$ such that $\pi(p_0) \neq 0$. Since X is faithful, there exists $x_0 \in X$ such that $x_0\pi(p_0) \neq 0$. The desired g is the map $p \to x_0\pi(p)$.

(ii) This follows directly from the definitions. $\square$

32.14. Lemma. *Let W be the unfaithful successor of V in some R-tower.*

(i) *Let $P_R \neq 0$ be finitely generated projective and suppose that either V is faithful or $\rho(P,V) \neq 0$. Then there exist P', P'' such that* (32.12.1) *holds.*

(ii) *Let P'_R be finitely generated projective with $\rho(P',W) \neq 0$. Then there exist finitely generated projective modules P_R, P''_R such that* (32.12.1) *holds.*

Proof. We use the notation of 32.12.

(i) The existence of a submodule P' with $P/P' \cong V$ is shown by 32.13. We now have two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & P' & \to & P & \xrightarrow{\pi} & V & \to & 0 \\ & & & & \downarrow\beta & & \downarrow(=) & & \\ 0 & \to & W & \to & U & \xrightarrow{\alpha} & V & \to & 0 \end{array}$$

Since P is projective, there is a map $\beta : P \to U$ such that $\pi = \alpha\beta$. It follows that $\mathrm{im}\,\beta \not\subseteq W$ and hence $\mathrm{im}\,\beta = U$. So if we let $P'' = \ker(\beta)$, then $P/P'' \cong U$. Also $P'' = \ker(\beta) \subseteq \ker(\pi) = P'$. Since $P/P' \cong V$ and U_R is uniserial of length 2, then $P'/P'' \cong W$.

(ii) Consider the inclusions $P'S \supseteq P' \supseteq P'A$. Let $n = \rho(P',W) \neq 0$, so $P'/P'A \cong W^{(n)}$. Since we are working with a basic idealizer, 4.10 shows:

$$(P'S/P'A)_S \cong U^{(n)}, \quad (P'S/P')_R \cong V^{(n)} \quad \text{and} \quad (P'/P'A)_R \cong W^{(n)}$$

where the latter two isomorphisms are restrictions of the first. Thus we have an S-homomorphism ϕ: $P'S \twoheadrightarrow U^{(n)}$ with kernel $P'A$ such that $\phi(P') = W^{(n)}$.

Let $Y = U \oplus W^{(n-1)} \subseteq U^{(n)}$ and $P = \phi^{-1}(Y)$. Then

$$P/P' \cong \phi(P)/\phi(P') = Y/W^{(n)} = (U \oplus W^{(n-1)})/W^{(n)} \cong V$$

as desired. Next let $Z = 0 \oplus W^{(n-1)} \subset Y$ and let $P'' = \phi^{-1}(Z)$. Then, as above,

$$\frac{P}{P''} \cong \frac{Y}{Z} = \frac{U \oplus W^{(n-1)}}{0 \oplus W^{(n-1)}} \cong U \quad \text{and} \quad \frac{P'}{P''} \cong \frac{W^{(n)}}{Z} = \frac{W^{(n)}}{0 \oplus W^{(n-1)}} \cong W$$

as desired. □

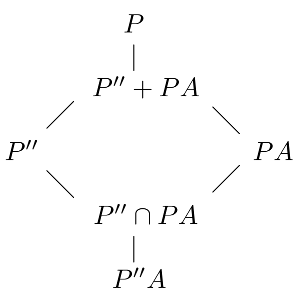
32.15. Lemma. *Let W be the unfaithful successor of V in some R-tower and let $P \supset P' \supset P''$ be finitely generated projective R-modules such that $P/P'' \cong U$, $P/P' \cong V$ and $P'/P'' \cong W$, as in* (32.12.1). *Then:*

(i) $\rho(P', V) = \rho(P, V) - 1$ *if V is unfaithful;*
(ii) $\rho(P', W) = \rho(P, W) + 1$*;*
(iii) $\rho(P', X) = \rho(P, X)$ *for all other unfaithful simple R-modules X.*

Proof. All undefined notation comes from 32.12.

(i) By 22.1, the nonzero maximal ideal that annihilates V, B say, is idempotent. Since $P/P' \cong V$, we have $PB \subseteq P' \subseteq P$. Hence $PB^2 = PB \subseteq P'B \subseteq PB$ and so $PB = P'B$. It follows, again since $P/P' \cong V$, that the composition length of $P'/P'B = P'/PB$ is one less than that of P/PB, as desired.

(ii) We can reason, as in (i), that since A is the nonzero, idempotent annihilator of $P'/P'' \cong W$, then $\lambda(P''/P''A) = \lambda(P''/P'A) = \lambda(P'/P'A) - 1$. In other words, if $\rho(P'', W) = k$ then $\rho(P', W) = k+1$. Thus it is sufficient to show that $\rho(P, W) = \rho(P'', W) = k$.



The diagram shows the submodules of P that interest us. Consider the short exact sequence

$$(W^{(k)} \cong)\ P''/P''A \hookrightarrow P/P''A \twoheadrightarrow P/P''\ (\cong U).$$

Since $\operatorname{Ext}^1_R(U, W) = 0$, by 5.8(ii), the short exact sequence splits. Hence $P/P''A \cong W^{(k)} \oplus U$. Multiplying this by A shows that $PA/P''A \cong 0 \oplus UA$. Since $R = \mathbb{I}_S(A)$ is a basic idealizer, $SA = S$. However U is an S-module, so $UA = USA = U$, which is isomorphic to P/P''. Thus $PA/P''A \cong P/P''$, and so we see from the diagram (arguing as in the last paragraph of 32.9) that $\lambda(P/PA) = \lambda(P''/P''A) = k$; that is, $\rho(P, W) = k$, as desired.

(iii) X is not the unfaithful successor of V in the tower that contains V. Therefore, by 15.1 and 15.2, $\operatorname{Ext}^1_R(V, X) = 0$. Let $N = \operatorname{ann}(X)$; then $P'/P'N \cong X^{(t)}$ where $t = \rho(P', X)$. The inclusions $P'N \subseteq P' \subset P$ show that $P/P'N$ is the middle term of some element of $\operatorname{Ext}^1_R(P/P', P'/P'N) = \operatorname{Ext}^1_R(V, X^{(t)}) = 0$. Therefore $P/P'N \cong V \oplus X^{(t)}$. Since the maximal ideal N is not the annihilator of the simple module V we have $VN = V$ and therefore

$$\frac{P}{PN} \cong \frac{P/P'N}{PN/P'N} \cong \frac{V \oplus X^{(t)}}{(V \oplus X^{(t)})N} \cong X^{(t)} \cong \frac{P'}{P'N}$$

as desired. □

It is convenient to blend facts from the preceding lemmas into a form suitable for later application.

32.16. LEMMA. *Let V_R be the predecessor of the unfaithful simple module W_R in some nontrivial tower of simple R-modules and let Q_R be nonzero, finitely generated projective.*

(i) *If $\rho(Q, W) \neq 0$, then there exists a finitely generated projective $Q'_R \supset Q$ such that $Q'/Q \cong V$, $\rho(Q', W) = \rho(Q, W) - 1$ and, if V is unfaithful, $\rho(Q', V) = \rho(Q, V) + 1$.*

(ii) *If V is faithful, or is unfaithful with $\rho(Q, V) \neq 0$, then there exists a finitely generated projective $Q' \subset Q$ such that $Q/Q' \cong V$, $\rho(Q', W) = \rho(Q, W)+1$ and, if V is unfaithful, $\rho(Q', V) = \rho(Q, V) - 1$.*

In each situation, $\rho(Q', X) = \rho(Q, X)$ for all unfaithful simple modules X_R other than V and W.

PROOF. (i) Apply 32.14 with Q in place of P', getting inclusions $Q' \supset Q \supset Q''$ such that $Q'/Q'' \cong U$, $Q'/Q \cong V$, and $Q/Q'' \cong W$ as in (32.12.1). Then apply 32.15 with P, P', P'' replaced by Q', Q, Q'' respectively.

(ii) Apply 32.14 with Q in place of P, getting inclusions $Q \supset Q' \supset Q''$ such that $Q/Q'' \cong U$, $Q/Q' \cong V$ and $Q'/Q'' \cong W$ as in (32.12.1). Then apply 32.15 with Q, Q', Q'' in place of P, P', P'' respectively. □

32.17. LEMMA. *Let S be the right finite overring of R determined by merging a segment V, W of an R-tower into a simple S-module U. Let $A = \operatorname{ann}_R(W)$, and let P be a finitely generated projective R-module.*

(i) *If Y is an S-module and $Q_R \subset PS$ with $PS/Q \cong Y$ (as R-modules) then Q is an S-submodule of PS.*

(ii) *Let $\rho_R(P, W) = t$; then $PS/PA \cong U^{(t)}$ as S-modules and $PS/P \cong V^{(t)}$ and $P/PA \cong W^{(t)}$ as R-modules.*

(iii) *The R-socle of PS/PA equals P/PA, and is isomorphic to $W^{(t)}$.*

(iv) *If U_S, or equivalently V_R, is unfaithful then $\rho_S(PS, U) = \rho_R(PS, V) = \rho_R(P, V) + \rho_R(P, W)$.*

(v) *For all unfaithful simple S-modules $X \ncong U$, $\rho_S(PS, X) = \rho_R(P, X)$.*

PROOF. Since S merges the single 2-element segment V, W, 28.10 shows that R is a basic idealizer of type U from S. Then 4.8 shows that $R = \mathbb{I}_S(A)$ and $S/A \cong U^{(n)}$ for some n.

(i), (ii) These are immediate from 4.17 and 4.10 respectively.

(iii) This follows from (ii) since the R-socle of $U^{(t)}$ is the unique R-submodule of $U^{(t)}$ that is isomorphic to $W^{(t)}$.

(iv) First we show that $\rho_S(PS, U) = \rho_R(PS, V)$. Note that $\rho_S(PS, U) = \lambda(PS/K) = a$ say, where $K = \cap\{Q \subset PS \mid PS/Q \cong U\}$. Similarly $\rho_R(PS, V) = \lambda(PS/K')$ where $K' = \cap\{Q' \subset PS \mid PS/Q' \cong V\}$. Let Q' be such that $PS/Q' \cong V\}$. By 32.14(i) applied to PS, there exists $Q'' \subset Q'$ with $PS/Q'' \cong U$. We deduce that $K \subseteq K'$. However, since $PS/K \cong U^{(a)}$, it has a submodule J/K with $PS/J \cong V^{(a)}$. So $J = K'$ and $\rho(PS, V) = a$ as claimed.

It remains to show that $\rho_R(PS, V) = \rho_R(P, V) + \rho_R(P, W)$. We know from (ii) that there is a chain of R-modules $P = P_0 \subset P_1 \subset P_2 \ldots \subset P_t = PS$ with each factor isomorphic to V where $t = \rho(P, W)$. We apply 32.15 iteratively to each pair

$P_i \subset P_{i+1}$. It asserts that $\rho(P_{i+1}, V) = \rho(P_i, V) + 1$ and $\rho(P_{i+1}, W) = \rho(P_i, W) - 1$. Thus, after the t steps involved, we get $\rho(PS, V) = \rho(P, V) + \rho(P, W)$, as required.

(v) Since S was determined by merging V, W we know [28.6] that X remains simple as an R-module. Evidently X_R is unfaithful. By 2.6(i), for every a, the S-homomorphisms of PS onto $X^{(a)}$ coincide with the R-homomorphisms of PS onto $X^{(a)}$. Therefore $\rho_S(PS, X) = \rho_R(PS, X)$ by 32.2. With the notation of (iv) above, we see from 32.15 that $\rho_R(P_{i+1}, X) = \rho_R(P_i, X)$ for each i. Hence $\rho_R(PS, X) = \rho_R(P, X)$. □

We now extend some of these results from segments of length two to segments of arbitrary length. Recall that, in any segment of a tower, all simple modules, except possibly the first, will be unfaithful.

32.18. Lemma. *Let P be a finitely generated projective R-module with a maximal submodule P'. Suppose that P/P' is a member of a segment $\mathcal{C} = W_a, \dots, W_b$ of some tower, and let U be the uniserial module associated with $\mathcal{C}$. Then there exist finitely generated projective R-modules P_a, P_{b+1} with $P_a \supseteq P \supset P' \supseteq P_{b+1}$ such that $P_a/P_{b+1} \cong U$.*

Proof. By repeated use of 32.14 we obtain a chain of modules

$$P_a \supset \cdots \supset P_i \supset P_{i+1} \supset \cdots \supset P_{b+1}$$

including P and P' such that each $P_i/P_{i+1} \cong W_i$ $(a \le i \le b)$ and P_i/P_{i+2} is uniserial of length 2 $(a \le i \le b-1)$. The latter condition implies, by 16.1, that P_a/P_{b+1} is uniserial. Then 28.12 shows that $P_a/P_{b+1} \cong U$ since the composition factors of P_a/P_{b+1} enumerate $\mathcal{C}$. □

32.19. Theorem. *Let S be an integral overring of R, P a finitely generated projective R-module, U an unfaithful simple S-module, and $W_1, \dots, W_n$ the R-composition factors of U (and so a segment of an R-tower). Then*

$$\rho_S(PS, U) = \sum \{\rho(P, W_i) \mid 1 \le i \le n\}. \tag{32.19.1}$$

Proof. (a) First suppose that $n = 1$; so U_R is also simple and unfaithful. Let $c = \rho_S(PS, U)$ and $d = \rho_R(P, U)$. We need to show that $c = d$. By 32.2, there is a surjection $P \twoheadrightarrow U^{(d)}$. Tensored with S, this gives a surjection $PS \cong P \otimes_R S \twoheadrightarrow U^{(d)}$ since $U \otimes_R S \cong U$; therefore, by 32.2 again, $c \ge d$.

Conversely, there is a surjection $PS \twoheadrightarrow U^{(c)}$ with kernel K, say. We claim that $P + K = PS$. For suppose that $PS \supset P + K$. We choose $q \in PS - (P + K)$ and note that $qR + P/P$ has finite length. (12.13 proves this for the case $P = R$; and it easily extends to a direct summand of a free R-module of finite rank.) Further, by 13.7 and 13.8, the simple R-composition factors of $(qR + P)/P$ do not include U. Hence the same is true of $(qR + P + K)/(P + K)$. However $PS/(P + K) \cong U^{(c')}$ for some $c' \le c$. From this contradiction, we deduce that $P + K = PS$ as desired. Hence $P/(P \cap K) \cong PS/K \cong U^{(c)}$ and so $c \le \rho_R(P, U) = d$. Thus $c = d$.

(b) Now suppose $n \ge 2$; we proceed by induction on n. Let T be the overring of R determined by merging W_{n-1} and W_n into W'_{n-1}, say. By 32.17((iv), $\rho_T(PT, W'_{n-1}) = \rho_R(P, W_{n-1}) + \rho_R(P, W_n)$. Note that each W_i with $i \ge 2$ is a simple T-module. Moreover, by part (a) of this proof, $\rho_T(PT, W_i) = \rho_R(P, W_i)$ for $i = 1, \dots, n-2$.

Note next that $W_1, \dots, W_{n-2}, W'_{n-1}$ are the T-composition factors of U_T. So, by induction on n, we may assume that

$$\rho_S(PS, U) = \rho_T(PT, W_1) + \cdots + \rho_T(PT, W_{n-2}) + \rho_T(PT, W'_{n-1});$$

and the result follows directly from this and the preceding two equations. □

33. Genus

This section[1] begins by defining the genus of a finitely generated projective R-module as a type of function from modspec(R) (and so implicitly from spec(R)) to the nonnegative integers. Then we give a description of those functions that occur as genera of finitely generated projective R-modules. The characterisation given is that the pair of conditions, 'almost standard rank' and 'cycle standard rank', described in §32 is necessary and sufficient.

33.1. Definitions. Let P_R be finitely generated projective. We define the *genus* of P to be the function $\Psi = \Psi(P)$ from modspec(R) to the nonnegative integers whose values are given by:

$$\begin{cases} \Psi_0 = \operatorname{udim}(P) \\ \Psi_W = \rho(P, W) \quad \text{for all } W \in \mathcal{W} \end{cases} \tag{33.1.1}$$

and we sometimes refer to udim(P) as the *rank of P at zero.* If Q is another finitely generated module with $\Psi(Q) = \Psi(P)$, we will describe Q as being *in the genus of* P and write $Q \in \Psi(P)$.

Note. For a commutative Noetherian ring S, the genus of a finitely generated S-module P is usually defined to be the family of all finitely generated S-modules Q such that, localizing at each maximal ideal $\mathfrak{q}$ of S, $Q_\mathfrak{q} \cong P_\mathfrak{q}$. If S is a Dedekind domain and P is projective, this is easily seen to agree with the statement that $Q \in \Psi(P)$; i.e. $\Psi(Q) = \Psi(P)$.

33.2. Lemma. *Genus is additive on direct sums of modules.*

Proof. Clear since rank is additive [32.4]. □

33.3. Corollary. *If R is a Dedekind prime ring then the genus of any finitely generated projective R-module P is determined by its uniform dimension.*

Proof. Let P_R, Q_R be finitely generated projective modules with udim(P) = udim(Q). We need to prove that $\rho(P, W) = \rho(Q, W)$ for each $W \in \mathcal{W}$. Since all towers in Dedekind prime rings are trivial [23.6], the unfaithful W is the unique member of its tower, $\mathcal{T}$ say, and $\mathcal{T}$ is a cycle tower. The desired result follows from cycle standard rank [32.9]. □

The converse, that if the genus is determined by uniform dimension, then R is a Dedekind prime ring, is clear. For if R were not a Dedekind prime ring then it would have an unfaithful simple module W in a nontrivial tower whose annihilator, M say, is a nonzero idempotent maximal ideal. Then udim(M) = udim(R) but $\Psi(M) \neq \Psi(R)$ since $\rho(M, W) = 0 \neq \rho(R, W)$.

33.4. Proposition. *Let P, Q be finitely generated projective right R-modules with $\Psi(P) = \Psi(Q)$ and S be an integral extension of R. Then $\Psi(PS_S) = \Psi(QS_S)$.*

[1]In this section R denotes an HNP ring unless the contrary is specified.