

## Preface

Over the past several decades, algebra has become increasingly important in combinatorial design theory. The flow of ideas has for the most part been from algebra to design theory. Moreover, despite our successes, fundamental algebraic questions in design theory remain open. It seems that new or more sophisticated ideas and techniques will be needed to make progress on these questions. In the meantime, design theory is a fertile source of problems that are ideal for spurring the development of algorithms in the active field of computational algebra.

We hope that this book will encourage the investigation, by researchers at all levels, of the algebraic questions posed by design theory. To this end, we provide a large selection of the algebraic objects and applications to be found in design theory. We also isolate a small number of problems that we think are important.

This book is a technical work that takes an unusually abstract approach. While the approach is non-standard, it offers uniformity and enables us to highlight the principal themes in such a way that they can be studied for their own sake, rather than as a means to an end in special cases.

Everything begins with the following notion of orthogonality. Fix an integer  $b > 1$ , and a non-empty set  $\mathcal{A}$  (an ‘alphabet’) excluding zero. Let  $\Lambda$  be a set (an ‘orthogonality set’) of  $2 \times b$  arrays whose non-zero entries come from  $\mathcal{A}$ . Much of design theory is concerned with instances of the question

When does there exist a  $v \times b$  array  $D$  such that every  $2 \times b$  subarray of  $D$  is in  $\Lambda$ ?

If  $D$  exists, then we say that its rows are *pairwise  $\Lambda$ -orthogonal*. Since essentially combinatorial constraints are being placed on pairs of distinct rows, and because of antecedents in the design of experiments, we call  $D$  a *pairwise combinatorial design*, or  $\text{PCD}(v, \Lambda)$  for short. Chapter 2 describes families of widely-studied pairwise combinatorial designs. These designs are of interest in diverse fields including electrical engineering, statistical analysis, and finite geometry.

This book develops a theory of *square* pairwise combinatorial designs, i.e., those with  $v = b$ . For such designs we use the abbreviated notation  $\text{PCD}(\Lambda)$ . Each of the principal design-theoretic themes finds expression. The ‘ambient rings’ introduced in Chapter 5 allow the free interplay of these themes: orthogonality, equivalence, transposability, composition, transference, the proliferation of inequivalent designs, the automorphism group, and links to group ring (norm) equations.

We pay particular attention to designs that possess a type of *regular* group action. The acting group has a certain central subgroup  $Z$ , and the corresponding 2-cocycles with coefficients in  $Z$  have a significant influence on properties of the design. Such a design is said to be *cocyclic*. This book contains a general theory for

cocyclic pairwise combinatorial designs, plus many case studies. Along the way, we encounter numerous classical designs and other well-known mathematical objects.

This is a book of ideas. It is our opinion that design theory is still—even now—in its infancy. Thus, at this stage, ideas are more valuable than a compendium of our present state of knowledge (which will keep growing rapidly beyond the confines of a single volume). We have aimed to stimulate a creative reader rather than to be encyclopedic.

With respect to cocyclic designs, the chief omissions from our book are Noboru Ito's work on Hadamard groups; and work by Kathy Horadam, her colleagues, and her students.

Our book covers some of Ito's results, but from a different perspective. Starting in the 1980s, Ito produced a sequence of papers identifying regular group actions on the expanded design of a Hadamard matrix. We are content to refer the reader to those papers.

The first author, together with Horadam, founded the theory of cocyclic designs in the early 1990s. Horadam and her school have since published many results focusing on Hadamard, complex Hadamard, and generalized Hadamard matrices. That material is covered in Horadam's engaging book [87]. There one will find topics such as shift equivalence of cocycles, equivalence classes of relative difference sets, and the connection between generalized Hadamard matrices and presemifields, that are not in this book.

We have tried to make the book as accessible as possible; we especially hope that our treatment of the new ideas is welcoming and open-ended. Proofs are given for nearly all results outside of the 'algebraic primer' chapter and the chapter on Paley matrices. The book also contains a wealth of examples and case studies which should persuade the reader that the concepts involved are worthy of pursuit.

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